A POLYNOMIAL LINEAR SEARCH ALGORITHM FOR THE n-DIMENSIONAL KNAPSACK PROBLEM

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Abstract:

We present a Linear Search Algorithm which decides the n-dimensional knapsack problem in $n^4\log(n) + O(n^3)$ steps. This algorithm works for inputs consisting of n numbers for some arbitrary but fixed integer n. This result solves an open problem posed for example in [6] and [7] by Dobkin / Lipton and A.C.C. Yao, resp.. It destroys the hope of proving large lower bounds for this NP-complete problem in the model of Linear Search Algorithms.

Introduction: A Linear Search Algorithm (LSA) is an abstraction of a Random Access Machine (RAM) (see [1]). Whereas the RAM's we consider are assumed to work with integer inputs the LSA gets real ones. When dealing with LSA's one doesn't take into consideration the amount of time necessary for arithmetic and storage allocation, but only for branchings

"If $f(\bar{x})>0$ then goto α, else goto β."

Here $f:\mathbb{R}^n \to \mathbb{R}$ is a affine function, i.e. $f(\bar{x}) = \bar{a}\cdot\bar{x} - b := \sum_{i=1}^{n} a_i x_i - b$, where $\bar{a} = (a_1,\dots,a_n)$, $\bar{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, $b \in \mathbb{R}$. Although it is not true that during a computation of a RAM always affine functions of the input are computed, LSA's are a realistic model of computation in the sense that several lower bounds for LSA's can be extended to RAM's, for example the $\Omega(n \log(n))$ lower bound for sorting ([2], [3]) and the $\Omega(n^2)$ lower bound for the n-dimensional knapsack problem, i.e. the problem to decide

$K_n := \{\bar{x} \in \mathbb{R}_+^n,\ \exists\ I \subset \{1,\dots,n\}$ with $\sum_{i \in I} x_i = 1\}$ (see [4],[5]).

It is well known that $K = \bigcup_{n \geq 1} K_n$ is NP-complete (see [1]). In this paper we shall see that for every fixed n, K_n can be decided in polynomial time, namely we present a LSA which decides K_n in $O(n^4\log(n))$ steps.

This solves one of the central problems of the theory of LSA's as stated for example in [6] or [7], and destroys the hope to prove large lower bounds for this NP-complete problem in the computational model of LSA's.

The above result is a conclusion of the construction of a LSA which decides a set $(\bigcup_{i=1}^{k} H_i) \cap C$ where the H_i's are hyperplanes in R^n and C is a cube in R^n. The time it needs is polynomial in n and log $(\frac{a}{r})$, where a is the edge length of C and r is the "density" of $\{H_1,\ldots,H_k\}$. This value measures how close the hyperplanes lie in R^n, that means how close any two affine subspaces are, which are intersections of some of the H_i's, and which do not intersect each other.

This algorithm is presented in chapter 2 after having introduced basic definitions from linear algebra in chapter 1. Here also an exact definition of LSA's can be found. In chapter 3 we relate the density of $\{H_1,\ldots,H_k\}$ to the coefficients of the H_i's. Here we extensively use ideas from [8] where the volume of a polytope is related to the coefficients of its bounding hyperplanes in order to estimate the running time of Khachiyan's algorithm for linear programming.

In the last chapter the results of chapter 1 and 2 are applied to achieve the LSA for the n-dimensional knapsack problem mentioned above.

Chapter 1: Definitions and Notations.

In this chapter we define LSA's and introduce some notations from linear algebra. We assume the reader to be familiar with the basic concepts of this discipline as affine, linear and convex subspaces of R^n, dimensions of such spaces, and determinants of matrices etc. All definitions and lemmas in the sequel are formulated relative to R^n, but they can in a natural way be transfered to statements relative to some n-dimensional affine subspace of some R^m, $m \geq n$. This will often be done without comment.

A LSA consists of a finite set of labelled instructions of the forms

1) α: If $f(\bar{x})>0$ then goto β, else goto γ
2) α: accept
3) α: reject

where $f:R^n \to R$ is an affine function.

The language L decided by a LSA is the set of inputs $\bar{x} \in R^n$, such that the LSA started with $\bar{x}$ computes "accept". The number of steps the LSA requires is the maximum number of instructions executed during some computation started with some input from R^n.

A hyperplane H in R^n is a (n-1)-dimensional subspace of R^n, i.e. $H:=\{\bar{x} \in R^n,\ \bar{a}\cdot\bar{x}-b=0\}$ for some $\bar{a} \in R^n, b \in R$. $H^+(H^-)$ is the left (right) halfspace of H, $H^+(H^-):=\{\bar{x} \in R^n,\ \bar{a}\cdot\bar{x}-b<(>)0\}$. Two hyperplanes $H=\{\bar{x} \in R^n,\ \bar{a}\cdot\bar{x}=b\}$ and $H'=\{\bar{x} \in R^n,\ \bar{a}'\cdot\bar{x} = b'\}$ are parallel if $\bar{a} = \bar{a}'$ and $b \neq b'$. The distance between H and H' is $\min\{d(\bar{x},\bar{y}),\ \bar{x} \in H,\ \bar{y} \in H'\}$, where $d(\bar{x},\bar{y}):=(\sum_{i=1}^{n}(x_i-y_i)^2)^{1/2}$ is the Euclidian distance between $\bar{x}$ and $\bar{y}$.

If we consider an instruction of type 1 for LSA's we say that the hyperplane $H=\{\bar{x} \in R^n,\ f(\bar{x})=0\}$ defines this instruction and often we represent an instruction by its defining hyperplane. This can be done in several ways. For example, if L is a (n-2)-dimensional space and $\bar{y} \notin L$, then the affine hull of $\bar{y}$ and L, $\mathrm{Aff}(\bar{y},L):=\{\lambda\bar{y}+(1-\lambda)\bar{x}, \bar{x} \in L, \lambda \in R\}$ is a hyperplane.

Now let $S = \{H_1,\dots,H_k\}$ be a set of hyperplanes in R^n. Then the connected components of $R^n \setminus (\bigcup_{i=1}^{k} H_i)$ are the components of S.

Each of them is a (convex) polytope P, i.e. the intersection of left and right halfspaces of the H_i's. Let $\bar{P}$ be the closure of P. Then the H_i's for which $H_i \cap \bar{P}$ is a (n-1)-dimensional convex set are the bounding hyperplanes of P. If for some $I \subset \{1,\dots,k\}$, $\bigcap_{i \in I} H_i = \{\bar{x}\}$ and $\bar{x} \in \bar{P}$, then $\bar{x}$ is a vertex of P. Let P be a bounded polytope with vertices $\{\bar{x}_1,\dots,\bar{x}_p\}$. It is well known (see for example [9]) that

$$P = \mathrm{conv}(\bar{x}_1,\dots,\bar{x}_p) := \{\sum_{i=1}^{p} \lambda_i \bar{x}_i, \lambda_i > 0, i=1,\dots,n, \sum_{i=1}^{n} \lambda_i = 1\}.$$

The ball B in R^n with center $\bar{y} \in R^n$ and radius $r>0$ is the set $\{\bar{x} \in R^n, d(\bar{x},\bar{y})<r\}$. The inner radius of a polytope p is the maximum radius of some ball contained in P.

Finally we introduce two special types of polytopes. A cube C with edge-length $a>0$ is the (unique) bounded component of $\{C_1,\dots,C_{2n}\}$, where for $i=1,\dots,n$, $C_i=\{\bar{x} \in R^n, x_i=d_i\}$, $C_{i+n}=\{\bar{x} \in R^n, x_i=d_i+a\}$ for some $d_1,\dots,d_n \in R$. In other words, $C=\mathrm{conv}(\times_{i=1}^{n} \{d_i, d_i+a\})$.

Let H be a hyperplane and $A \subset H$ a (n-1)-dimensional polytope and $\bar{y} \notin H$. Then $P(\bar{y},A):=\{\lambda\bar{y}+(1-\lambda)\bar{x}, \bar{x} \in A, \lambda \le 1\}$ is a pyramid with top $\bar{y}$ and base A. If $A_1,\dots,A_q$ are the ((n-2)-dimensional) bounding hyperplanes of A on H, then $P(\bar{y},A)$ is a component of $\{\mathrm{Aff}(\bar{y},A_1),\dots,\mathrm{Aff}(\bar{y},A_q)\}$. Note that $P(\bar{y},A)$ is unbounded.

Chapter 2: A LSA for deciding a set of hyperplanes.

Let C by a cube and $S=\{H_1,\dots,H_k\}$ a set of hyperplanes in R^n. In this chapter we construct a LSA which decides S in C on R^n, i.e. which decides a language $L \subset R^n$ where $L \cap C = (\bigcup_{i=1}^{k} H_i) \cap C$.

The idea of this LSA is to partition C to small cubes, such that the hyperplanes from S which intersect one of them have a common, non-empty intersection. We shall see that the problem to decide such a set of hyperplanes can be reduced to an anologous problem in a (n-1)-dimensional space, and thus can be solved recursively. How to apply such LSA's for (n-1)-dimensional problems to n-dimensional ones is shown in the following lemma.

<u>Lemma 1</u>: Let $S:=\{H_1,\dots,H_k\}$ be a set of hyperplanes in R^n, $L:=\bigcap_{i=1}^{k} H_i \neq \emptyset$. Let A be a polytope on a hyperplane H, $L \not\subset H$, $\bar{y} \in L \setminus H$. If $S':=\{H_1 \cap H,\dots,H_k \cap H\}$ can be decided by a LSA in A on H in t steps, then S can be decided by a LSA in $P(\bar{y},A)$ on R^n in t steps, too.

Proof: Let a LSA be given which decides S' in A on H. Now replace its instructions as follows: if some of them is defined by the (n-2)-dimensional hyperplane H' on H, replace it by the instruction which is defined by the hyperplane $Aff(\bar{y},H')$ in R^n. Clearly the new LSA decides S in $P(\bar{y},A)$ on R^n. ☒

In order to apply this lemma we have to partition the cube C to smaller cubes, such that for each of them the hyperplanes which intersect it have a non-empty intersection.

For this purpose we call a number r>0 a density of $S:=\{H_1,\dots,H_k\}$ (on R^n), if for every ball B with radius r it holds that if for some $I \subset \{1,\dots,k\}$, $H_i \cap B \neq \emptyset$ for all $i \in I$, then $\bigcap_{i\in I} H_i \neq \emptyset$.

In the next chapter we shall see that such a density exists for every S. We assume this for a moment.

Lemma 2: Let r>0 be a density of $S=\{H_1,\dots,H_k\}$ on R^n, then r is also a density of $S'=\{H_2 \cap H_1,\dots,H_k \cap H_1\}$ on H_1.

Proof: Suppose that r is no density of S' on H_1. Then there is a ball B' on H_1 with the radius r and center $\bar{y} \in H_1$, say, such that for some $I \subset \{2,\dots,k\}$, $H_i \cap B' \neq \emptyset$ for $i \in I$ and $\bigcap_{i\in I}(H_i \cap H_1)=\emptyset$.
But this would mean, that the ball B on R^n with radius r and center $\bar{y}$ fulfils:
$H_i \cap B \neq \emptyset$ for $i \in I \dot{\cup} \{1\}$ and
$\bigcap_{i\in I\cup\{1\}} H_i = \bigcap_{i\in I}(H_i \cap H_1) = \emptyset$,
which contradicts the fact that r is a density of S on R^n. ☒

Now we are able to describe a LSA which decides S in C on R^n. Let $C_1,\dots,C_{2n}$ be the bounding hyperplanes and a>0 the edge-length of C, and r>0 a density of $S \cup \{C_1,\dots,C_{2n}\}$. Furthermore let T(n,a,r) be the maximal number of steps which an optimal LSA needs in order to decide some S in some cube C with edge-length a, if r is a density for $S \cup \{C_1,\dots,C_{2n}\}$. Then a simple divide-and conquer algorithm quarantees that $T(1,a,r) \leq \lceil \log(\frac{a}{r}+1)\rceil + 3$ (*) :
Subdivide the cube (i.e. the intervall of length a) in $\lceil\frac{a}{r}\rceil$ intervalls of length at most r. Clearly there are only $\lceil \log(\frac{a}{r}+1)\rceil$ steps necessary to decide to which of these intervalls some input $\bar{x}$ belongs. As only one of the hyperplanes (which are single points) can intersect such an intervall, as its length is the density of $S \cup \{C_1,C_2\}$, there are only three further instructions necessary to decide whether $\bar{x} \in \bigcup_{i=1}^{k} H_i$, two for asking whether $\bar{x}$ lies on the hyperplane of S which belongs to this intervall, the third to accept or to reject.
Now let n>1.
Let $d:=\lceil\frac{a\cdot\sqrt{n}}{r}\rceil$,
$D=\{D_1^1,\dots,D_d^1,\dots,D_1^n,\dots,D_d^n\}$ a set of hyperplanes, such that for $i=1,\dots,n$, $j=1,\dots,d$, D_j^i is parallel to C_i and C_{i+n}, the distance between D_j^i and D_{j+1}^i is $\frac{r}{\sqrt{n}}$, and these hyperplanes partition C in cubes with edge-length $\frac{r}{\sqrt{n}}$ (resp. somewhat smaller at the boundaries

(*) All logarithms in this paper are to the base 2.

of C). Note that these cubes are not open, but contain some parts of their boundaries. But this doesn't disturb what follows. The LSA now begins as follows:

Part 1: Determine in which of the cubes defined by D the input $\bar{x}$ lies.

Remark 1: This can be done in $n\cdot\lceil \log(\frac{a\sqrt{n}}{r}+1)\rceil$ steps by using a divide-and-conquer algorithm for each set $\{D_1^i,\ldots,D_d^i\}$ of parallel hyperplanes, $i=1,\ldots,n$.

Remark 2: Suppose $\bar{x}$ is determined to lie in the cube C' with edge-length at most $\frac{r}{\sqrt{n}}$. As this cube is contained in a ball with radius r, the set $I=\{i \in \{1,\ldots,k\},\ H_i \cap C' \neq \emptyset\}$ fulfils that $L:=\bigcap_{i\in I} H_i \neq \emptyset$ or $I=\emptyset$.

Let $\bar{y} \in L$ and $C_i, i \in I' \subset \{1,\ldots,2n\}$ be those bounding hyperplanes of C with $\bar{y} \in C_i$. Let $F_1,\ldots,F_s$ be the (n-2)-dimensional intersections of two of the C_i's, $i \in I'$, each. Then for $j \in \{1,\ldots,s\}$, $F_j' := \mathrm{Aff}(\bar{y},F_j)$ is a hyperplane in R^n.

Part 2: Determine in which component of $\{F_1',\ldots,F_s'\}$ $\bar{x}$ lies. (The components contain parts of their boundaries.)

Remark 3: $s \leq 2n(n-1)$, because each $C_i, i=1,\ldots,2n$ has a non-empty, i.e. (n-2)-dimensional intersection with 2(n-1) many other C_j's. As thus we have counted each (n-2)-dimensional intersection twice, $s \leq \frac{1}{2}\cdot 2n \cdot 2(n-1) = 2n(n-1)$. Thus part 2 can be executed in 2n(n-1) steps.

Remark 4: Suppose that $\bar{x}$ lies in the component Q of $\{F_1',\ldots,F_s'\}$. Then Q is a pyramid with top $\bar{y}$, a base of which is a subset of some (n-1)-dimensional cube $C_i \cap \bar{C}$ with edge-length a on C_i for some $i \in I'$.

Part 3: Determine whether $\bar{x}$ lies on some of the hyperplanes from S, if $Q \cap C \neq \emptyset$. Otherwise reject.

Remark 5: By Lemma 1, part 3 can be executed as fast as deciding $S'=\{H_j \cap C_i,\ j \in I\}$ (I is defined in remark 2, i in remark 4) in A on C_i. A is contained in a cube on C_i with edge-length a, and by Lemma 2, r is a density for $S' \cup \{C_j \cap C_i,\ j=1,\ldots,2n\}$. Thus part 3 needs at most T(n-1,a,r) steps. Clearly the above algorithm is correct and we obtain

$$T(1,a,r) \leq \lceil \log(\tfrac{a}{r}+1)\rceil + 3, \text{ and for } n>1$$
$$T(n,a,r) \leq n\lceil \log(\tfrac{a\sqrt{n}}{r}+1)\rceil + 2n(n-1) + T(n-1,a,r).$$

Therefore, $T(n,a,r) \leq n^2 \log(\frac{a\cdot\sqrt{n}}{r}) + 2n^3$.

Theorem 1: Let $S=\{H_1,\ldots,H_k\}$ be a set of hyperplanes and C a cube in R^n with edge-length $a>0$ and bounding hyperplanes $\{C_1,\ldots,C_{2n}\}$. Let $r>0$ be a density of $S \cup \{C_1,\ldots,C_{2n}\}$. Then S can be decided by a LSA in C on R^n in $n^2 \log(\frac{a\sqrt{n}}{r})+2n^3$ steps.

Chapter 3: Determining a Density of a Set of Hyperplanes.

In order to apply Theorem 1 to concrete problems we have to determine the density of a set $S=\{H_1,\ldots,H_k\}$ of hyperplanes in R^n. The first step in this direction is to relate the density of S to the inner radii of its components.

Lemma 3: The minimum inner radius of the components of S is a density of S.

Proof: First we prove the lemma for the case that $k=n+1$ and S has a bounded component. In this case, S has exactly one bounded component P, which is a simplex, i.e. which has $n+1$ vertices. Thus each intersection of n of the hyperplanes of S intersects in exactly one point. Now suppose that B, a ball with radius r and center $\bar{y} \in R^n$, is intersected by a set of hyperplanes which has an empty intersection. As mentioned above this set must be S. If no hyperplane from S separates $\bar{y}$ from P, then $\bar{y} \in P$ and as B intersects all bounding hyperplanes of P, its radius r is larger than the inner radius of P. If there is a hyperplane, say H_i, from S which separates $\bar{y}$ from P, let Q be the pyramid with bounding hyperplanes H_j, $j=1,\dots,n+1, j \neq i$, which contains $\bar{y}$. Let $\bar{x}$ be the top of Q. Then $\bar{x}$ and $\bar{y}$ are separated by H_i, because $\bar{x}$ is a vertex of P. Now let $\bar{y}'$ be a point on the straight line between $\bar{x}$ and $\bar{y}$, such that $\bar{y}'$ and $\bar{y}$ are separated by H_i and $d(\bar{y}',H_i)<r$. Let B' be the ball with radius r and center $\bar{y}'$ then H_i intersects B' as $d(\bar{y}',H_i)<r$ and all other H_j's intersect B', too, because they intersect B and Q tapers to $\bar{x}$. But $\bar{y}'$ is neither separated from P by H_i nor by any H_j from which B is not separated. Repeating this process until we have found a ball with radius r whose center belongs to P, we have proved that r is larger then the inner radius of P.

Now let k be arbitrary and let $S=\{H_1,\dots,H_k\}$ be any set of hyperplanes in R^n. Let B be a ball with center $\bar{y}$ and radius r, for which the hyperplanes from S which intersect B have an empty intersection. Let $I \subset \{1,\dots,k\}$ have minimum cardinality such that $H_i \cap B \neq \emptyset$ for $i \in I$ and $\bigcap_{i\in I} H_i=\emptyset$. Let $R:=\bigcap_{i\in I} L(H_i)$ (*), then L is a linear subspace of R^n with dimensional p, say.

We claim that $\#I=n-p+1$.

As I is chosen minimally, it holds for every $i \in I$ that $R_j:=\bigcap_{\substack{i\in I\\ i\neq j}} H_i \neq \emptyset$. Let $j \in I$ be fixed. As $R_j \cap H_j = \emptyset$ we obtain that $L(R_j) \subset L(H_j)$. This implies that

$$L(R_j) = L(R_j) \cap L(H_j) = L(\bigcap_{\substack{i\in I\\ i\neq j}} H_i) \cap L(H_j)$$

$$= \bigcap_{\substack{i\in I\\ i\neq j}} L(H_i) \cap L(H_j) = R.$$

Thus R_j has dimension p and therefore $\#(I\setminus\{j\}) \geq n-p$ which implies $\#I \geq n-p+1$.

Now suppose that $\#I > n-p+1$.

For some $j \in I$ let $J \subset I$ be chosen minimally such that $R_j=\bigcap_{i\in J} H_i$. Then $\#J=n-p$ and $j \notin J$. Let $I'=J\cup\{j\}$ then $\#I'=n-p+1$ and

$$\bigcap_{i\in I'} H_i = R_j \cap H_j = (\bigcap_{\substack{i\in I\\ i\neq j}} H_i) \cap H_j = \emptyset.$$ Thus

we obtain a contradiction to the minimalli-

(*) For some affine subspace A in R^n, L(A) denotes the linear subspace parallel to A, $L(A):=\{\bar{x}-\bar{y} \mid \bar{x} \in A\}$ for some $\bar{y} \in A$.

ty of I.

Let A be the (n-p)-dimensional affine subspace of R^n which contains $\bar{y}$ and is orthogonal to R. Then $B'=B \cap A$ is a ball on A with radius r which is intersected by every $H'_{j'}:=H_{j'} \cap A$, $j' \in I$. This is true because the shortest connection between $\bar{y}$ and some H_i, $i \in I$ is orthogonal to H_i and therefore is contained in A, because A is orthogonal to a subspace of H_i. Thus $H_i \cap B \neq \emptyset$ implies $H_i \cap A \cap B \neq \emptyset$. As clearly $\{H_i \cap A,\ i \in I\}$ has a bounded component P on A we know from the beginning of this proof that r is larger then the inner radius of P. But then it is also larger than the inner radius of the component P' of $\{H_i,\ i \in I\}$ in R^n, which contains P, and therefore it is larger than the inner radius of any of the components of S which are subsets of P'. ☒

Now we have restricted our problem of determining a density of S to bounding the inner radii of its components.

This will be done by relating them to the coefficients of the hyperplanes of S. Let for $i=1,\dots,k, H_i:=\{\bar{x} \in R^n,\ \bar{a}_i \cdot \bar{x} = b_i\}$, $\bar{a}_i=(a_{i1},\dots,a_{in}) \in Z^n$, $b_i \in Z$. (Z is the set of integers). Then we say that S has integer coefficients and define

$m(S):=\max\{|a_{ij}|, i=1,\dots,k,\ j=1,\dots,n\}$ and

$M(S):=\max(\{|b_i|,\ i=1,\dots,k\} \cup \{m(S)\})$.

The following two lemmas 4 and 5 and the corollary 1 are almost identical to the lemmas 1 and 2 and the corollary 1 from [8].

Lemma 4: Every vertex of some component of S can be represented as $(\frac{p_1}{q},\dots,\frac{p_n}{q})$ with $p_1,\dots,p_n,q \in Z, |q| \le m(S)^n\, n^{\frac{1}{2}n}$, $|p_1|,\dots,|p_n| \le M(S)^n\, n^{\frac{1}{2}n}$.

Proof: A vertex $\bar{x}=(x_1,\dots,x_n)$ of some component of S is the intersection of n hyperplanes from S, wlog. of $H_1,\dots,H_n$. By Cramer's Rule we know that for $i=1,\dots,n$, $x_i=\frac{\det(D_i)}{\det(D)}$, where D consists of the colums $(a_{i1},\dots,a_{in})^T$ for $i=1,\dots,n$ and D_i arises from D by replacing its i'th column by $(b_1,\dots,b_n)$.

As $\det(D) \neq 0$ and $|\det(D)|$ is the volume of the hyperparallelepiped spanned by its column vectors, we may conclude:

$$|\det(D)| \le \prod_{i=1}^{n} d(0,a_i) \le (n\cdot m(S)^2)^{\frac{1}{2}n} = n^{\frac{1}{2}n}\cdot m(S)^n.$$

Analogously we obtain

$$|\det(D_i)| \le n^{\frac{1}{2}n}\cdot M(S)^n.$$ ☒

Corollary 1: Let C be the cube with bounding hyperplane $C_i=\{\bar{x} \in R^n,\ x_i=c\}$, $C_{i+n}=\{\bar{x} \in R^n,\ x_i=-c\}$ for $i=1,\dots,n$, where $c=\lceil n^{\frac{1}{2}n}\cdot M(S)^n\rceil+1$. Then

a) each component of S has a non-empty intersection with C, and

b) each vertex of $S \cup \{C_1,\dots,C_{2n}\}$ can be represented as a vector of rational numbers with common denominator at most $n^{\frac{1}{2}n}\cdot M(S)^n$ in absolute value.

Proof: Let $E_i=\{\bar{x} \in R^n,\ x_i=0\}$ for $i=1,\dots,n$ and let $S'=S \cup \{E_1,\dots,E_n\}$. Then each component of S' has at least one vertex. Thus by lemma 4 it has a non-empty intersection with C, because M(S')=M(S). To verify b) we again apply lemma 4 and notice that $m(S)=m(S \cup \{c_1,\dots,c_{2n}\})$. ∎

Lemma 5: The volume of each component of S is at least $(n! \cdot n^{\frac{1}{2}n} \cdot m(S)^{n^2})^{-1}$.

Proof: By corollary 1 a) it suffices to prove the assertion above for the bounded components of $S \cup \{c_1,\dots,c_{2n}\}$. As each of these components has at least n+1 vertices, its volume is at least the volume of $P=\mathrm{Conv}(v_0,\dots,v_n)$, where $v_0,\dots,v_n$ are n+1 of the above vertices which do not lie on one hyperplane. As P is a simplex, its volume v(P) fulfils

$$v(P) = \frac{1}{n!} \left| \det\begin{pmatrix} 1 & \cdots & 1 \\ v_0 & \cdots & v_n \end{pmatrix} \right| > 0,$$

where the v_i's are column vectors. For each i=1,...,n we know from corollary 1 that its components have the same denominator q_i where $|q_i| \leq n^{\frac{1}{2}n} \cdot m(S)^n$.

Thus

$$v(P) = \frac{1}{n!} \cdot \frac{1}{|q_0|\cdots|q_n|} \cdot \left| \det\begin{pmatrix} q_0 & & q_n \\ v_0\, q_0 & \cdots & v_n\, q_n \end{pmatrix} \right|$$

As the matrix above only has integer coefficients and as its determinant is unequal to zero, its absolute value is at least one. Therefore we obtain

$$v(P) \geq \frac{1}{n!} \frac{1}{|q_0|\cdots|q_n|} \geq \frac{1}{n!\; n^{\frac{1}{2}n^2}\; m(S)^{n^2}}. \;∎$$

Now we are able to relate the inner radii of the components of S to M(S).

Lemma 6: The inner radius of each component of S is at least

$$(M(S)^{2n^2} \cdot n^{3n^2})^{-1}.$$

Proof: Again it suffices to prove the lemma for a bounded polytope $P=\mathrm{Conv}(v_0,\dots,v_n)$ as in the proof of lemma 5. We first bound v(P) from above in terms of M(S), n and the so-called thickness d of P, i.e. the minimum distance of two parallel hyperplanes, between which P lies. Let H_1, H_2 be these hyperplanes. As $P \subset C$, we know that $P \subset \mathrm{Conv}(H_1,H_2) \cap C$.

Let $c = 2(\lceil n^{\frac{1}{2}n} M(S)^n \rceil + 1)$ be the edge-length of C then we obtain:

$$v(P) \leq v(\mathrm{Conv}(H_1,H_2) \cap C) \leq (\sqrt{n}\cdot c)^{n-1} \cdot d.$$

Applying lemma 5 it follows:

$$d \geq (n! \cdot n^{\frac{1}{2}n^2} \cdot m(S)^{n^2} \cdot \sqrt{n}^{\,n-1} \cdot c^{n-1})^{-1}.$$

Now we apply a theorem due to Blaschke [10] which says that the inner radius of a polytope with thickness d is a least $\frac{d}{n+1}$.

This theorem and a ruff estimation prove the lemma. ∎

Now we can bound the complexity of the LSA from chapter 2.

Theorem 2: Let $S = \{H_1,\dots,H_k\}$ be a set of hyperplanes with integer coefficients, and C a cube with edge-length $a \in \mathbb{Z}$, $a > 0$ and bounding hyperplanes $c_1,\dots,c_{2n}$, $b := M(S \cup \{c_1,\dots,c_{2n}\})$.

Then S can be decided in C on R^n in $3n^4 \log(n) + n^2 \log(a) + 2n^4 \log(b) + O(n^3)$

steps.

Proof: By lemma 3, each density of S is bounded by the minimum inner radius of the components of S. Inserting the bound for it from lemma 6 in theorem 1 yields theorem 2. ⊠

Chapter 4: A LSA for the n-Dimensional Knapsack Problem.

We now apply theorem 2 to the n-dimensional knapsack problem, i.e. we want to decide $K_n := \{\bar{x} \in R^n_+,\ \exists I \subset \{1,\dots,n\} \text{ with } \sum_{i \in I} x_i = 1\}$.

Theorem 3: K_n can be decided by a LSA in R^n_+ in $n^4 \log(n) + O(n^3)$ steps.

Proof: Let

$C_i := \{\bar{x} \in R^n, x_i = 0\}$, $C_{i+n} := \{\bar{x} \in R^n, x_i = 1\}$ for $i = 1,\dots,n$ be the bounding hyperplanes of the cube C with edge-length 1. As $M(K_n \cup \{C_1,\dots,C_{2n}\}) = 1$, we know from theorem 2 that K_n can be decided in C on R^n in $n^4 \log(n) + O(n^3)$ steps. But for each component of $\{C_1,\dots,C_{2n}\}$ except C, each element $\bar{x}$ of it has a component $x_i > 1$. Thus in such components we have to decide $K_{n'}$ in $R^{n'}_+$ for some $n' < n$, where we only have to consider those n' components of $\bar{x}$ with $x_i \leq 1$. This holds as it is impossible that $\sum_{i \in I} x_i = 1$ if an $i \in I$ exists with $x_i > 1$, because $x_i \geq 0$ for $i=1,\dots,n$.

Therefore, the following LSA decides K_n on R^n_+.

If n=1 then B_n consists of one point and can be decided in 3 steps.

Let n > 1. Then we apply the following algorithm.

Part 1: Determine in which component of $\{C_1,\dots,C_{2n}\}$ $\bar{x}$ lies and accept if it lies on $C_{n+1},\dots,C_{2n}$.

Part 2: If $\bar{x}$ lies in C, then use the algorithm from the first chapter for it. If $\bar{x}$ lies in an other component, use this algorithm recursively to decide $K_{n'}$ in $R^{n'}_+$ for the appropiate n' < n as described above.

Let T(n) be the time this algorithm needs. Then T(1)=3 and for n>1

$T(n) \leq 2n + \max\{T(n-1),\ n^4 \log(n) + O(n^3)\}$.

This implies that $T(n) \leq n^4 \log(n) + O(n^3)$. ⊠

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