A Polynomial Linear Search Algorithm for the *n*-Dimensional Knapsack Problem

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Abstract. A linear search algorithm that recognizes the n-dimensional knapsack problem in $2n^4\log n + O(n^3)$ steps is presented. This algorithm works for inputs consisting of n numbers for some arbitrary but fixed integer n. This result solves an open problem posed by Dobkin/Lipton and A.C.C. Yao, among others, and it destroys the hope of proving nonpolynomial lower bounds for this NP-complete problem in the model of linear search algorithms.

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1. *Introduction*

A linear search algorithm (LSA) is an abstraction of a *random access machine* (RAM) (see [1]). Although the inputs of the RAMs we consider are assumed to be integers, inputs for LSAs are real numbers. When dealing with LSAs, one does not take into consideration the amount of time necessary for arithmetic and storage allocation, but only the amount of time necessary for branchings of the form, "If $f(\bar{x}) > 0$ then goto α, else goto β." Here f: $R^n \to R$ is an affine function; that is, $f(\bar{x}) = \bar{a}\cdot\bar{x} - b := \sum_{i=1}^n a_i x_i - b$, where $\bar{a} = (a_1, \ldots, a_n)$, $\bar{x} = (x_1, \ldots, x_n) \in R^n$, $b \in R$. Although it is not true that, during a computation of a RAM, affine functions of the input are always computed, LSAs are realistic models of computation in the sense that several lower bounds for LSAs have been extended to RAMs. Examples of this are the $\Omega(n \log n)$ lower bound for sorting [2, 3] and the $\Omega(n^2)$ lower bound for the n-dimensional knapsack problem; that is, the problem of recognizing $K_n := \{\bar{x} \in R_+^n, \exists I \subset \{1, \ldots, n\}$ with $\sum_{i\in I} x_i = 1\}$ (see [4] and [5]).

It is well known that $K = \bigcup_{n\geq 1}(K_n \cap Q^n)$ is NP complete (see [1]). In this paper we see that for every fixed n, K_n can be recognized in polynomial time, namely, we present an LSA that recognizes K_n in $O(n^4\log n)$ steps.

This solves one of the central problems of the theory of LSAs (see, e.g., [6] or [7]) and destroys the hope of proving nonpolynomial lower bounds for this NP-complete problem in the computational model of LSAs.

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The above result follows from the construction of an LSA that recognizes a set $(\bigcup_{i=1}^{k} H_i) \cap C$ where the H_i's are hyperplanes in R^n and C is a cube in R^n. The time it takes is polynomial in n and $\log (a/r)$, where a is the edge length of C and r is the *coarseness* of $\{H_1, \ldots, H_k\}$. This value measures how close together the hyperplanes lie in R^n; that is, the closeness of any two affine subspaces that are intersections of some of the H_i's and that do not intersect each other.

The algorithm is presented in Section 3 after the introduction of basic definitions from linear algebra in Section 2. A precise definition of LSAs is also given in Section 2. In Section 4 we relate the coarseness of $\{H_1, \ldots, H_k\}$ to the coefficients of the H_i's. Here we extensively use ideas from [8] where the volume of a polytope is related to the coefficients of its bounding hyperplanes to estimate the running time of Khachiyan's algorithm for linear programming.

In the last section the results of Sections 2 and 3 are used to obtain the LSA for the n-dimensional knapsack problem mentioned above.

2. *Definitions and Notations*

In this chapter we define LSAs and introduce some notations from linear algebra. We assume the reader is familiar with the basic concepts of this discipline, including affine, linear, and convex subspaces of R^n; dimensions of such spaces; and determinants of matrices. All definitions and lemmas in the sequel are formulated relative to R^n, but they can be transferred naturally to statements relative to n-dimensional affine subspaces of some R^m for $m \geq n$. This will often be done without comment.

An LSA consists of a finite set of labeled instructions of the form

(1) α: If $f(\bar{x}) > 0$, then goto β, else goto γ
(2) α: accept
(3) α: reject

where $f: R^n \to R$ is an affine function.

The language L recognized by an LSA is the set of inputs $\bar{x} \in R^n$, such that the LSA started with $\bar{x}$ computes "accept". The number of steps the LSA takes is the maximum number of instructions executed during any computation started with an input from R^n.

A hyperplane H in R^n is an $(n-1)$-dimensional subspace of R^n; that is, $H := \{\bar{x} \in R^n, \bar{a}\cdot\bar{x} - b = 0\}$ for some $\bar{a} \in R^n$, $b \in R$. The left (right) half space of H is the set $\{\bar{x} \in R^n, \bar{a}\cdot\bar{x} - b < (>) 0\}$. Two hyperplanes $H = \{\bar{x} \in R^n, \bar{a}\cdot\bar{x} = b\}$ and $H' = \{\bar{x} \in R^n, \bar{a}'\cdot\bar{x} = b'\}$ are parallel if $\bar{a} = \bar{a}'$ and $b \neq b'$. The distance between H and H' is $\min\{d(\bar{x}, \bar{y}), \bar{x} \in H, \bar{y} \in H'\}$, where $d(\bar{x}, \bar{y}) := (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}$ is the Euclidean distance between $\bar{x}$ and $\bar{y}$.

If we consider an instruction of type 1 for LSAs, we say that the hyperplane $H = \{\bar{x} \in R^n, f(\bar{x}) = 0\}$ defines this instruction; often we represent an instruction by its defining hyperplane. This can be done in several ways. For example, if L is an $(n-2)$-dimensional space and $\bar{y} \notin L$, then the affine hull of $\bar{y}$ and L, $\mathrm{Aff}(\bar{y}, L) := \{\lambda\bar{y} + (1-\lambda)\bar{x}, \bar{x} \in L, \lambda \in R\}$ is a hyperplane.

Now let $S = \{H_1, \ldots, H_k\}$ be a set of hyperplanes in R^n. Then the connected components of $R^n \setminus (\bigcup_{i=1}^{k} H_i)$ are called the components of S.

Each of them is a polytope P; that is, the intersection of left and right half spaces of the H_i's. Let $\bar{P}$ be the closure of P. Then the H_i's for which $H_i \cap \bar{P}$ is an $(n-1)$-dimensional convex set are the bounding hyperplanes of P. If for some $I \subset \{1, \ldots, k\}$, $\bigcap_{i\in I} H_i = \{\bar{x}\}$ and $\bar{x} \in \bar{P}$, then $\bar{x}$ is a vertex of P. Let P be a bounded polytope with vertices $\{\bar{x}_1, \ldots, \bar{x}_p\}$. It is well known (see, e.g., [9]) that

$$P = \mathrm{Conv}(\bar{x}_1, \ldots, \bar{x}_p)$$
$$:= \left\{ \sum_{i=1}^{p} \lambda_i \bar{x}_i,\ \lambda_i > 0,\ i = 1, \ldots, p,\ \sum_{i=1}^{p} \lambda_i = 1 \right\}.$$

For $I \subset \{1, \ldots, k\}$, let B_I be the intersection of the left half spaces of the H_i's with $i \in I$ and the closures of the right half spaces of the other H_i's. Then the nonempty B_I's build a disjoint partition of R^n into the so-called improper components of S, each of which consists of one component of S and some parts of its boundary.

The ball B in R^n with center $\bar{y} \in R^n$ and radius $r > 0$ is the set $\{\bar{x} \in R^n, d(\bar{x}, \bar{y}) < r\}$. The inner radius of a polytope P is the maximum of the radii of balls contained in P.

Finally we introduce two special types of polytopes. A cube C with edge length $a > 0$ is the (unique) bounded component of $\{C_1, \ldots, C_{2n}\}$, where for $i = 1, \ldots, n$,

$$C_i = \{\bar{x} \in R^n,\ x_i = d_i\}, \qquad C_{i+n} = \{\bar{x} \in R^n,\ x_i = d_i + a\}$$

for some $d_1, \ldots, d_n \in R$. In other words, $C = \mathrm{conv}(\prod_{i=1}^{n} \{d_i, d_i + a\})$.

Let H be a hyperplane, $A \subset H$ an $(n-1)$-dimensional polytope, and suppose $\bar{y} \notin H$. Then $P(\bar{y}, A) := \{\lambda \bar{y} + (1-\lambda)\bar{x},\ \bar{x} \in A,\ \lambda < 1\}$ is called an (unbounded) pyramid with top $\bar{y}$ and base A. If $A_1, \ldots, A_q$ are the ($(n-2)$-dimensional) bounding hyperplanes of A on H, then $P(\bar{y}, A)$ is a component of $\{\mathrm{Aff}(\bar{y}, A_1), \ldots, \mathrm{Aff}(\bar{y}, A_q)\}$.

3. *An LSA for Recognizing a Set of Hyperplanes*

Let C be a cube and $S = \{H_1, \ldots, H_k\}$ a set of hyperplanes in R^n. In this section we construct an LSA that recognizes S in C on R^n; that is, an LSA that recognizes a language $L \subset R^n$ where $L \cap C = (\bigcup_{i=1}^{k} H_i) \cap C$.

The idea of this LSA is to partition C to small cubes, such that for each cube the hyperplanes from S that intersect it have nonempty common intersection. We shall see that the problem of recognizing such a set of hyperplanes can be reduced to an analogous problem in an $(n-1)$-dimensional space and thus can be solved recursively. How to apply such LSAs for $(n-1)$-dimensional problems to n-dimensional ones is shown in the following lemma.

LEMMA 1. *Let $S := \{H_1, \ldots, H_k\}$ be a set of hyperplanes in R^n, $L := \bigcap_{i=1}^{k} H_i \neq \emptyset$. Let A be a polytope on a hyperplane H, $L \not\subset H$, $\bar{y} \in L \backslash H$. If $S' := \{H_1 \cap H, \ldots, H_k \cap H\}$ can be decided by an LSA in A on H in t steps, then S can be decided by an LSA in $P(\bar{y}, A)$ on R^n in t steps, too.*

PROOF. Let an LSA that recognizes S' in A on H be given. Now replace its instructions as follows: If the instruction is defined by the $(n-2)$-dimensional hyperplane H' on H, replace it by the instruction that is defined by the hyperplane $\mathrm{Aff}(\bar{y}, H')$ in R^n. The new LSA recognizes S in $P(\bar{y}, A)$ on R^n, because a point $\bar{x} \in P(\bar{y}, A)$ belongs to $\bigcup_{i=1}^{k} H_i$, if and only if $\bar{z}$ belongs to $\bigcup_{i=1}^{k} (H_i \cap H)$, where $\bar{z}$ is the point of intersection of the line from $\bar{y}$ to $\bar{x}$ with A. □

To apply this lemma, we have to partition the cube C into smaller cubes, such that for each cube the hyperplanes that intersect it have nonempty intersection.

For this purpose we call a number $r > 0$ *a coarseness of* $S := \{H_1, \ldots, H_k\}$*(on* R^n*)*, if, for the every ball B with radius r, it holds that if, for some $I \subset \{1, \ldots, k\}$, $H_i \cap B \neq \emptyset$ for all $i \in I$, then $\bigcap_{i \in I} H_i \neq \emptyset$.

In the next section, we shall see that such a coarseness exists for every S. We assume this for the moment.

LEMMA 2. *Let $r > 0$ be a coarseness of $S = \{H_1, \ldots, H_k\}$ on R^n, then r is also a coarseness of $S' = \{H_2 \cap H_1, \ldots, H_k \cap H_1\}$ on H_1.*

PROOF. Suppose that r is not a coarseness of S' on H_1. Then there is ball B' on H_1 with the radius r and center $\bar{y} \in H_1$, say, such that for some $I \subset \{2, \ldots, k\}$, $H_i \cap B' \neq \emptyset$ for $i \in I$ and $\bigcap_{i \in I} (H_i \cap H_1) = \emptyset$.

But this would mean, that the ball B on R^n with radius r and center $\bar{y}$ satisfies $H_i \cap B \neq \emptyset$ for $i \in I \cup \{1\}$ and $\bigcap_{i \in I \cup \{1\}} H_i = \bigcap_{i \in I} (H_i \cap H_1) = \emptyset$, which contradicts the fact that r is a coarseness of S on R^n. □

Now we are able to describe an LSA that recognizes S in C on R^n. Let $C_1, \ldots, C_{2n}$ be the bounding hyperplanes and $a > 0$ the edge length of C, and $r > 0$ a coarseness of $S \cup \{C_1, \ldots, C_{2n}\}$. Furthermore, let $T(n, a, r)$ be the maximal number of steps that optimal LSAs need in order to recognize any S in any cube C with edge length a, if r is a coarseness of $S \cup \{C_1, \ldots, C_{2n}\}$. Then a simple divide-and-conquer algorithm guarantees that

$$T(1, a, r) \leq \left\lceil \log\left(\frac{a}{r} + 1\right)\right\rceil + 3 \quad \text{(Footnote 1).}$$

Subdivide the cube (i.e., the interval of length a) in $\lceil a/r \rceil$ intervals of length at most r. Clearly, there are only $\lceil \log (a/r + 1) \rceil$ steps necessary to decide to which of these intervals some input $\bar{x}$ belongs. Only one of the hyperplanes (which are single points) can intersect such an interval, as its length is a coarseness of $S \cup \{C_1, C_2\}$. Therefore, only three further instructions are necessary to decide whether $\bar{x} \in \bigcup_{i=1}^{k} H_i$; two for asking whether $\bar{x}$ lies on the hyperplane of S, which belongs to this interval; the third to accept or to reject.

Now let $n > 1$. Let $d := \lceil a \cdot \sqrt{n}/r \rceil$, $D = \{D_1^1, \ldots, D_d^1, \ldots, D_1^n, \ldots, D_d^n\}$ a set of hyperplanes, such that for $i = 1, \ldots, n$, $j = 1, \ldots, d$, D_j^i is parallel to C_i and C_{i+n}, the distance between D_j^i and D_{j+1}^i is $r/\sqrt{n}$, and their improper components are (nonopen) cubes with edge length $r/\sqrt{n}$ (respectively, somewhat smaller at the boundaries of C).

The LSA now begins as follows:

Part 1. Determine in which of the improper components of D the input $\bar{x}$ lies, and reject if $\bar{x} \notin C$.

Remark 1. This can be done in $n \cdot \lceil \log ((a\sqrt{n}/r) + 1) \rceil$ steps by using a divide-and-conquer algorithm for each set $\{D_1^i, \ldots, D_d^i\}$ of parallel hyperplanes, $i = 1, \ldots, n$.

Remark 2. Suppose $\bar{x}$ is determined to lie in the cube C' with edge length at most $r/\sqrt{n}$. Since this cube is contained in a ball with radius r, the set $I = \{i \in \{1, \ldots, k\}, H_i \cap C' \neq \emptyset\}$ has $L := \bigcap_{i \in I} H_i \neq \emptyset$ or $I = \emptyset$.

[1] All logarithms in this paper are to the base 2.

Let $\bar{y} \in L$ and let $F_1, \ldots, F_s$ be those $(n-2)$-dimensional intersections of two of the C_i's, $i \in \{1, \ldots, 2n\}$ for each, on which $\bar{y}$ does not lie. Then, for $j \in \{1, \ldots, s\}$, $F_j' := \text{Aff}(\bar{y}, F_j)$ is a hyperplane in R^n.

Part 2. Determine in which improper component of $\{F_1', \ldots, F_s'\}$ $\bar{x}$ lies.

Remark 3. $s \leq 2n(n-1)$, because each C_i, $i = 1, \ldots, 2n$, has a nonempty (i.e., $(n-2)$-dimensional) intersection with $2(n-1)$ many other C_j's. Since we have counted each $(n-2)$-dimensional intersection twice, $s \leq \frac{1}{2} \cdot 2n \cdot 2(n-1) = 2n(n-1)$. Thus, Part 2 can be executed in $2n(n-1)$ steps.

Remark 4. Suppose that $\bar{x}$ lies in the improper component Q of $\{F_1', \ldots, F_s'\}$. Then Q is a pyramid with top $\bar{y}$ and a base that is a subset of some $(n-1)$-dimensional cube $C_i \cap \bar{C}$ with edge length a on C_i for some $i \in \{1, \ldots, 2n\}$.

Part 3. Determine whether $\bar{x}$ lies on some of the hyperplanes from S.

Remark 5. By Lemma 1, Part 3 can be executed as fast as recognizing $S' := \{H_j \cap C_i, j \in I\}$ (I is defined in Remark 2, i in Remark 4) in A on C_i. A is contained in a cube on C_i with edge length a, and by Lemma 2, r is a coarseness of $S' \cup \{C_j \cap C_i, j = 1, \ldots, 2n\}$. Thus Part 3 takes at most $T(n-1, a, r)$ steps. Clearly the above algorithm is correct and we obtain

$$T(1, a, r) \leq \left\lceil \log\left(\frac{a}{r} + 1\right) \right\rceil + 3,$$

and for $n > 1$

$$T(n, a, r) \leq n \left\lceil \log\left(\frac{a\sqrt{n}}{r} + 1\right) \right\rceil + 2n(n-1) + T(n-1, a, r).$$

Therefore,

$$T(n, a, r) \leq n^2 \log\left(\frac{a \cdot \sqrt{n}}{r}\right) + 2n^3.$$

THEOREM 1. *Let $S = \{H_1, \ldots, H_k\}$ be a set of hyperplanes and C a cube in R^n with edge length $a > 0$ and bounding hyperplanes $\{C_1, \ldots, C_{2n}\}$. Let $r > 0$ be a coarseness of $S \cup \{C_1, \ldots, C_{2n}\}$. Then S can be recognized in C on R^n in $n^2 \log(a\sqrt{n}/r) + 2n^3$ steps.*

4. *Determining a Coarseness of a Set of Hyperplanes*

In order to apply Theorem 1 to concrete problems, we have to determine a coarseness of a set $S = \{H_1, \ldots, H_k\}$ of hyperplanes in R^n. The first step in this direction is to relate the coarseness of S to the inner radii of its components.

LEMMA 3. *The minimum of the inner radii of the components of S is a coarseness of S.*

PROOF. First we prove the lemma for the case that $k = n + 1$ and S has a bounded component. In this case, S has exactly one bounded component P, which is a simplex; that is, which has $n + 1$ vertices. Thus each intersection of n of the hyperplanes of S intersects in exactly one point. Now suppose that B, a ball with radius r and center $\bar{y} \in R^n$, is intersected by a set of hyperplanes that has an empty intersection. As mentioned above, this set must be S. If no hyperplane from S

separates $\bar{y}$ from P, then $\bar{y} \in P$ and as B intersects all bounding hyperplanes of P, its radius r is larger than the inner radius of P. This can be proved as follows:

First we observe

(*) Let $\bar{x}, \bar{x}_1, \bar{x}_2 \in R^n$ be distinct points such that $\bar{x}_1$ belongs to the line segment from $\bar{x}$ to $\bar{x}_2$. Let H be a hyperplane, $\bar{x} \in H$, $\bar{x}_2 \notin H$. Then $d(\bar{x}_2, H) > d(\bar{x}_1, H)$.

Now suppose that there is a ball B_1 with radius r and center $\bar{y}_1$; say, with $B_1 \subset P$. Then the line from $\bar{y}$ to $\bar{y}_1$ has exactly two points of intersection with the boundary of P, namely, $\bar{z}_1$ and $\bar{z}_2$. Let $\bar{y}$, belong to the line segment from $\bar{z}_1$ to $\bar{y}$ and H be a bounding hyperplane of P that contains $\bar{z}_1$. Then by the definition of B and by (*), $d(\bar{y}_1, H) < d(\bar{y}, H) < r$, which contradicts the fact that $B_1 \subset P$.

If there is a hyperplane, say H_i, from S that separates $\bar{y}$ from P, let Q be the pyramid with bounding hyperplanes H_j, $j = 1, \ldots, n+1$, $j \neq i$, that contains $\bar{y}$. Let $\bar{x}$ be the top of Q. Then $\bar{x}$ and $\bar{y}$ are separated by H_i, because $\bar{x}$ is a vertex of P. Now let $\bar{y}'$ be a point on the line segment from $\bar{x}$ to $\bar{y}$, such that $\bar{y}'$ and $\bar{y}$ are separated by H_i and $d(\bar{y}', H_i) < r$. Let B' be the ball with radius r and center $\bar{y}'$. Then H_i intersects B' since $d(\bar{y}', H_i) < r$ and all other H_j's intersect B' too, because, for $j \neq i$, by (*), we have $d(\bar{y}', H_j) < d(\bar{y}, H_j)$, since $x \in H_j$. But $\bar{y}'$ is separated from P neither by H_i nor by any H_j from which y is not separated. Repeating this process until we have found a ball with radius r whose center belongs to P, we have proved that r is larger then the inner radius of P.

Now let k be arbitrary and let $S = \{H_1, \ldots, H_k\}$ be any set of hyperplanes in R^n. Let B be a ball with center $\bar{y}$ and radius r, for which the hyperplanes from S that intersect B have an empty intersection. Let $I \subset \{1, \ldots, k\}$ have minimum cardinality such that $H_i \cap B \neq \emptyset$ for $i \in I$ and $\bigcap_{i \in I} H_i = \emptyset$. Let $R := \bigcap_{i \in I} L(H_i)$;[2] then R is a linear subspace of R^n with dimension p.

We claim that $\#I = n - p + 1$. As I is chosen minimally, it holds for every $i \in I$ that $R_j := \bigcap_{i \in I, i \neq j} H_i \neq \emptyset$.

Let $j \in I$ be fixed. Since $R_j \cap H_j = \emptyset$, we obtain that $L(R_j) \subset L(H_j)$. This implies that

$$\begin{aligned} L(R_j) &= L(R_j) \cap L(H_j) \\ &= L \bigcap_{\substack{i \in I \\ i \neq j}} H_i \cap L(H_j) \\ &= \bigcap_{\substack{i \in I \\ i \neq j}} L(H_i) \cap L(H_j) = R. \end{aligned}$$

Thus R_j has dimension p and therefore $\#(I \backslash \{j\}) \geq n - p$, which implies $\#I \geq n - p + 1$.

Now suppose that $\#I > n - p + 1$. For some $j \in I$, let $J \subset I$ be chosen minimally such that $R_j = \bigcap_{i \in J} H_i$. Then $\#J = n - p$ and $j \notin J$. Let $I' = J \cup \{j\}$. Then $\#I' = n - p + 1$ and

$$\bigcap_{i \in I'} H_i = R_j \cap H_j = \bigcap_{\substack{i \in I \\ i \neq j}} H_i \cap H_j = \emptyset.$$

Thus we obtain a contradiction to the minimality of I.

Let A be the $(n-p)$-dimensional affine subspace of R^n that contains $\bar{y}$ and is orthogonal to R. Then $B' = B \cap A$ is a ball on A with radius r that is intersected

[2] For any affine subspace A in R^n, $L(A)$ denotes the linear subspace parallel to A; that is, $L(A) := \{\bar{x} - \bar{y} \mid \bar{x} \in A\}$ for some $\bar{y} \in A$.

by every $H'_{j'} := H_{j'} \cap A$, $j' \in I$. This is true because the shortest connection between $\bar{y}$ and some H_i, $i \in I$, is orthogonal to H_i and therefore contained in A because A is orthogonal to a subspace of H_i. Thus $H_i \cap B \neq \emptyset$ implies $H_i \cap A \cap B \neq \emptyset$. As $\{H_i \cap A, i \in I\}$ has a bounded component P with vertices $R_j \cap A$, $j \in I$, on A, we know from the beginning of this proof that r is larger then the inner radius of P. But then it is also larger than the inner radius of the component P' of $\{H_i, i \in I\}$ in R^n, which contains P, and therefore it is larger than the inner radius of any of the components of S which are subsets of P'. □

Now we have restricted our problem of determining a coarseness of S to bounding the inner radii of its components.

This is done by relating the radii to the coefficients of the hyperplanes of S. For $i = 1, \ldots, k$, let $H_i := \{\bar{x} \in R^n, \bar{a}_i \cdot \bar{x} = b_i\}$, $\bar{a}_i = (a_{i1}, \ldots, a_{in}) \in Z^n$, $b_i \in Z$ (Z is the set of integers). Then we say that S has integer coefficients and define

$$m(S) := \max\{|a_{ij}|, i = 1, \ldots, k, j = 1, \ldots, n\},$$
$$M(S) := \max(\{|b_i|, i = 1, \ldots, k\} \cup \{m(S)\}).$$

Lemmas 4 and 5 and Corollary 1 are almost identical to [8, lemmas 1 and 2 and corollary 1].

LEMMA 4. *Every vertex of a component of S can be represented as $(p_1/q, \ldots, p_n/q)$ with $p_1, \ldots, p_n, q \in Z$, $|q| \le m(S)^n \cdot n^{(1/2)n}$, $|p_{1n}, \ldots, |p_n| \le M(S)^n \cdot n^{(1/2)n}$*

PROOF. A vertex $\bar{x} = (x_1, \ldots, x_n)$ of a component of S is the intersection of n hyperplanes from S, without loss of generality of $H_1, \ldots, H_n$. By Cramer's rule, we know that for $i = 1, \ldots, n$, $x_i = \det(D_i)/\det(D)$, where D consists of the columns $(a_{i1}, \ldots, a_{in})^T$ for $i = 1, \ldots, n$ and D_i arises from D by replacing the ith column by $(b_1, \ldots, b_n)^T$.

Since $\det(D) \neq 0$ and $|\det(D)|$ is the volume of the hyperparallelepiped, spanned by its column vectors, we may conclude:

$$|\det(D)| \le \prod_{i=1}^{n} d(\bar{0}, \bar{a}_i) \le (n \cdot m(S)^2)^{(1/2)n} = n^{(1/2)n} \cdot m(S)^n.$$

Analogously we obtain

$$|\det(D_i)| \le n^{(1/2)n} \cdot M(S)^n. \qquad \square$$

COROLLARY 1. *Let C be the cube with bounding hyperplanes $C_i = \{\bar{x} \in R^n, x_i = c\}$, $C_{i+n} = \{\bar{x} \in R^n, x_i = -c\}$ for $i = 1, \ldots, n$, where $c = \lceil n^{(1/2)n} \cdot M(S)^n \rceil + 1$. Then*

(*a*) *each component of S has nonempty intersection with C, and*
(*b*) *each vertex of $S \cup \{C_1, \ldots, C_{2n}\}$ can be represented as a vector of rational numbers with common denominator at most $n^{(1/2)n} \cdot M(S)^n$ in absolute value.*

PROOF. Let $E_i = \{\bar{x} \in R^n, x_i = 0\}$ for $i = 1, \ldots, n$ and let $S' = S \cup \{E_1, \ldots, E_n\}$. Then each component of S' has at least one vertex. Thus by Lemma 4 each component of S has a nonempty intersection with C, because $M(S') = M(S)$. To verify (b) we again apply Lemma 4 and notice that $m(S) = m(S \cup \{C_1, \ldots, C_{2n}\})$. □

LEMMA 5. *The volume of each component of S is at least $(n! \cdot n^{(1/2)n(n+1)} \cdot m(S)^{n(n+1)})^{-1}$.*

PROOF. By Corollary 1(a), it suffices to prove the assertion above for the bounded components of $S \cup \{C_1, \ldots, C_{2n}\}$. Since each of these components has at least $n + 1$ vertices, its volume is at least the volume of $P = \mathrm{Conv}(\bar{v}_0, \ldots, \bar{v}_n)$, where $\bar{v}_0, \ldots, \bar{v}_n$ are $n + 1$ of the above vertices that do not lie on one hyperplane. Since P is a simplex, its volume $v(P)$ satisfies

$$v(P) = \frac{1}{n!} \left| \det \begin{pmatrix} 1 & \cdots & 1 \\ (\bar{v}_0)^{\mathrm{T}} & \cdots & (\bar{v}_n)^{\mathrm{T}} \end{pmatrix} \right| > 0.$$

For each $i = 1, \ldots, n$, we know from Corollary 1 that the coordinates of $\bar{v}_i$ can be written as rationals with the same denominator q_i such that $|q_i| \leq n^{(1/2)n} \cdot m(S)^n$. Thus

$$v(P) = \frac{1}{n!} \cdot \frac{1}{|q_0| \cdots |q_n|} \cdot \left| \det \begin{pmatrix} q_0 & \cdots & q_n \\ (q_0\bar{v}_0)^{\mathrm{T}} & \cdots & (q_n\bar{v}_n)^{\mathrm{T}} \end{pmatrix} \right|$$

Since the matrix above only has integer coefficients and since its determinant is not zero, its absolute value is at least one. Therefore, we obtain

$$v(P) \geq \left(\frac{1}{n!}\right)\left(\frac{1}{|q_0| \cdots |q_n|}\right) \geq \frac{1}{n!\; n^{(1/2)n(n+1)} m(S)^{n(n+1)}}. \qquad \square$$

Now we are able to relate the inner radii of the components of S to $M(S)$.

Lemma 6. *The inner radius of each component of S is at least $(M(S)^{2n^2} \cdot n^{2n^2})^{-1}$.*

Proof. Again it suffices to prove the lemma for a bounded polytope $P = \mathrm{Conv}(\bar{v}_0, \ldots, \bar{v}_n)$ as in the proof of Lemma 5. We first bound $v(P)$ from above in terms of $M(S)$, n and the so-called thickness d of P; that is, the minimum distance of two parallel hyperplanes, between which P lies. Let H_1, H_2 be these hyperplanes. Since $P \subset C$, we know that $P \subset \mathrm{Conv}(H_1, H_2) \cap C$. Since the edge length of C is $2c$ (compare Corollary 1), we obtain

$$v(P) \leq v(\mathrm{Conv}(H_1, H_2) \cap C) \leq (\sqrt{n} \cdot 2c)^{n-1} \cdot d.$$

Applying Lemma 5 it follows:

$$d \geq (n! \cdot n^{(1/2)n(n+1)} \cdot m(S)^{n(n+1)} \cdot \sqrt{n}^{n-1} \cdot (2c)^{n-1})^{-1}.$$

Now we apply a theorem [10] that says that the inner radius of a polytope with thickness d is at least $d/(n + 1)$. Combining this with a rough estimate completes the proof of the lemma. $\square$

Now we can bound the complexity of the LSA from Section 3.

Theorem 2. *Let $S = \{H_1, \ldots, H_k\}$ be a set of hyperplanes with integer coefficients and C a cube with edge length $a \in Z$, $a > 0$ and bounding hyperplanes $C_1, \ldots, C_{2n}$. Let $b := M(S \cup \{C_1, \ldots, C_{2n}\})$. Then S can be recognized in C on R^n in $2n^4 \log n + n^2 \log a + 2n^4 \log b + O(n^3)$ steps.*

Proof. By Lemma 3, each coarseness of S is bounded by the minimum of the inner radii of the components of S. Inserting the bound for it from Lemma 6 in Theorem 1 yields Theorem 2. $\square$

5. *An LSA for the n-Dimensional Knapsack Problem*

We now apply Theorem 2 to the n-dimensional knapsack problem; that is, we want to recognize $K_n := \{\bar{x} \in R^n_+, \exists I \subset \{1, \ldots, n\}$ with $\sum_{i \in I} x_i = 1\}$.

Theorem 3. *K_n can be recognized by an LSA in R^n_+ in $2n^4 \log(n) + O(n^3)$ steps.*

PROOF. Let $C_i := \{\bar{x} \in R^n, x_i = 0\}$, $C_{i+n} := \{\bar{x} \in R^n, x_i = 1\}$ for $i = 1, \ldots, n$ be the bounding hyperplanes of the cube C with edge length 1. As $M(K_n \cup \{C_1, \ldots, C_{2n}\}) = 1$, we know from Theorem 2 that K_n can be recognized in C on R^n in $2n^4 \log n + O(n^3)$ steps. Moreover, in each component of $\{C_1, \ldots, C_{2n}\}$ except C, every element $\bar{x}$ has some coordinate $x_i > 1$. Thus, in such components, the problem reduces to recognizing $K_{n'}$ in $R_+^{n'}$ for some $n' < n$, since we only need consider those n' components of $\bar{x}$ with $x_i \leq 1$. This holds, since it is impossible that $\sum_{i \in I} x_i = 1$ if there is a $j \in I$ with $x_j > 1$, because $x_i \geq 0$ for $i = 1, \ldots, n$. Therefore, the following LSA recognizes K_n on R_+^n.

If $n = 1$, then K_n consists of one point and can be recognized in three steps. Let $n > 1$. Then we apply the following algorithm:

Part 1. Determine in which improper component of $\{C_1, \ldots, C_{2n}\}$ $\bar{x}$ lies and accept if it lies on $C_{n+1}, \ldots, C_{2n}$.

Part 2. If $\bar{x}$ lies in C, then use the algorithm from Section 1 for it. If $\bar{x}$ lies in an other component, use this algorithm recursively to recognize $K_{n'}$ in $R_+^{n'}$ for the appropriate $n' < n$ as described above.

Let $T(n)$ be the time this algorithm needs. Then $T(1) = 3$ and for $n > 1$

$$T(n) \leq 3n + \max\{T(n-1), 2n^4 \log n + O(n^3)\}.$$

This implies that $T(n) \leq 2n^4 \log n + O(n^3)$. □

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