Some Lie Theory of Semigroups based on Examples

J. Hilgert

The purpose of this note is to explain some of the basic concepts and problems in the Lie theory of semigroups. We will start from a class of examples, certain contraction semigroups, that allows us to avoid the technical complications in the definitions of a general setup as given in [HL83] and [HHL87]. For these semigroups we define and calculate the tangent objects which play the role of the Lie algebra. Finally we describe how in some cases it is possible to recover a semigroup from its tangent object.

1. Contraction semigroups

Let V be a set and G a group acting on V. Further let $(A, \leq)$ be a partially ordered set and $f: V \to A$ an arbitrary map. Consider the set

(1) $$S_f = \{g \in G: f(g\cdot v) \leq f(v)\ \forall v \in V\}.$$

Note that the transitivity of $\leq$ implies that S_f is a subsemigroup of G.

We give a few examples in order to show the broad range of this construction.

1.1. Example. Let $(V, \langle\,|\,\rangle)$ be a Hilbertspace, and G a group of linear transformations on V. Further let $A = \mathbb{R}$ with the usual order and $f(v) = \langle v \mid v\rangle$. Then S_f is the semigroup of contractions in G.

1.2. Example. (*Invariant sets*) Let M be any subset of V and $\chi: V \to \mathbb{R}$ the characteristic function of M. If $f = 1 - \chi$ then $S_f = \{g \in G: g\cdot M \subseteq M\}$.

(a) ***Positive maps***: Let V be a C^*-algebra, G a group of linear transformations on V and $V^+ = \{v \in V: \mathrm{spec}(v) \subseteq \mathbb{R}^+\}$ be the set of positive elements in V. Then S_f is the semigroup of all order preserving elements of G.

(b) *Nonnegative matrices*: Let $V = \mathbb{R}^n$ and $G = \mathrm{Gl}(n,\mathbb{R})$ with the natural action. If $M = (\mathbb{R}^+)^n$ then $S_f = \{g \in \mathrm{Gl}(n,\mathbb{R}): g{\cdot}M \subseteq M\} = \{g \in \mathrm{Gl}(n,\mathbb{R}): g_{ij} \geq 0\}$.

(c) *Admissible translations*: Let G be a locally compact group and $M_b(G)$ the set of bounded Radon measures on G. The group G acts on $M_b(G)$ by left translations (convolution with point measures). For a given $\mu \in M_b(G)$ let $M = \{\nu \in M_b(G): \nu \prec \mu\}$ where $\nu \prec \mu$ means that ν is absolutely continuous with respect to μ. Then $S_f = \{g \in G: g{\cdot}\mu \prec \mu\}$ since $g{\cdot}\nu \prec g{\cdot}\mu$ for all $\nu \in M$. In other words S_f is the semigroup of all admissible translations for μ.

Let us note that we can describe one and the same semigroup by very different f's.

1.3. Example. Let $V = \mathbb{R}^2$, $G = \mathrm{Gl}(2,\mathbb{R})$ with the natural action, $f_1: V \to \mathbb{R}$ the characteristic function of the unit ball and $f_2: V \to \mathbb{R}$ given by $f_2(r,s) = -(r^2 + s^2)$, then the two semigroups S_{f_1} and S_{f_2} agree.

It is clear that we could change the whole setup in order to describe subsemigroups of associative algebras in which case we would not have to ask for the invertibility of the operators in question, but after all the author is primarily interested in subsemigroups of Lie *groups*.

There are two subgroups naturally associated to any subsemigroup S of a group G, namely the largest group $H(S)$ contained in S and the subgroup $G(S)$ of G generated by S. For the contraction semigroups as defined above it is easy to describe $H(S)$ in terms of f, whereas such a description is not available for $G(S)$. We have

$$H(S_f) = \{g \in G: f(g{\cdot}v) = f(v)\ \forall v \in V\}. \tag{2}$$

In the Lie theory of semigroups one associates, similar to the case of Lie groups, with a subsemigroup S of a Lie group G a subset of the Lie algebra $\mathbf{L}(G)$ either via a geometric "tangent - construction" or by considering one-parameter semigroups. Unfortunately, it is necessary to assume that $G(S)$ carries an analytic structure, that is, has a Lie group topology which possibly is finer than the induced topology, and to consider one-parameter semigroups in the closure $\overline{S}$ of S in $G(S)$. This difficulty vanishes if we restrict our attention to semigroups which are closed in G from the beginning. In order to assure that a contraction semigroup S_f is closed we assume that V is a topological space and that the map $f: V \to A$ is *semicontinuous*, by which we mean that for any $v_o \in A$ the set $(f_{v_o} \downarrow) = \{v \in V: f(v) \leq f(v_o)\}$ is closed in V. In fact we have:

1.4. Remark. Let $g \mapsto g{\cdot}v$ be continuous for any $v \in V$ and $f: V \to A$ be semicontinuous then S_f is closed in G.

Proof. Note first that for any $v_o \in V$ and any $s \in S_f$ we have $s{\cdot}v_o \in \{v \in V: f(v) \leq f(v_o)\}$ which is a closed subset of V. The continuity of the action now shows that $g{\cdot}v_o \in \{v \in V: f(v) \leq f(v_o)\}$ for any g in the closure $\overline{S}_f$ of S_f. Since v_o was chosen arbitrarily this just shows that $\overline{S}_f = S_f$. ∎

2. Tangent wedges

We associate with any *closed* subsemigroup S of a Lie group G a tangent object at the identity via

$$\mathbf{L}(S) = \{x \in \mathbf{L}(G): \exp tx \in S \ \ \forall t \in \mathbb{R}^+\}. \tag{3}$$

Using the Trotter product formula $\exp t(x+y) = \lim(\exp \frac{tx}{n} \exp \frac{ty}{n})^n$ (cf. [He78,Chap.II,§2,Lemma 4.4]) we see that $\mathbf{L}(S)$ is a wedge, that is, a closed convex set which is also closed under multiplication by positive scalars. Moreover $W = \mathbf{L}(S)$ satisfies

$$e^{\operatorname{ad} x} W = W \quad \forall x \in W \cap -W, \tag{LIE}$$

since $\exp(e^{\operatorname{ad} x} ty) = \exp x \exp ty \exp -x$ (cf. [He78,Chap.II,§5]). We call any wedge W satisfying (LIE) a *Lie wedge.*

For the special case of closed linear contraction semigroups on finite dimensional vector spaces it is possible to give an explicit .description of $\mathbf{L}(S)$ in terms of f and the action of $\mathbf{L}(G)$ on V associated with the action of G on V via $x\cdot v = \frac{d}{dt}|_{t=0}((\exp tx)\cdot v)$. But first we need to introduce the set of *subtangent vectors* $\mathbf{L}_v(M)$ of a subset M of a Banach space V in a point $v \in M$. We set

$$\mathbf{L}_v(M) = \{w \in V: w = \lim r_n(v_n - v); v_n \in M; \lim r_n = \infty\}. \tag{4}$$

2.1. Theorem. *We assume that V is finite dimensional and $f: V \to A$ is semicontinuous. Then*

$$\mathbf{L}(S_f) = \{x \in \mathbf{L}(G): x\cdot v \in \mathbf{L}_v(f_v \downarrow) \ \forall v \in V\}. \tag{5}$$

Proof. We first prove the theorem for the special case that $A = \{0,1\}$ and $1 - f$ is the characteristic function of a closed subset M of V. In this case the upper semicontinuity of f is a consequence of the closedness of M. Example 1.2 shows that $S_f = \{g \in G: \rho(g)(M) \subseteq M\}$. Moreover we have $(f_v \downarrow) = M$ for all $V \in M$ and $(f_v \downarrow) = V$ for all $v \in V \setminus M$. Consequently we have $\mathbf{L}_v(f_v \downarrow) = \mathbf{L}_v(M)$ for all $v \in M$ and $\mathbf{L}_v(f_v \downarrow) = V$ for all $v \in V \setminus M$. Thus in this case it suffices to show $\mathbf{L}(S_f) = \{x \in \mathbf{L}(G): x\cdot v \in \mathbf{L}_v(M) \ \forall v \in M\}$. But this is an immediate consequence of the following lemma on the invariance of sets under flows applied to the vector fields $X(v) = x\cdot v$.

2.2. Lemma. (Bony-Brezis, cf. [HHL87]) *Let V be a finite dimensional vector space and M a closed subset of V. If $X: U \to V$ is a locally Lipshitz continuous vector field for some open U in V containing M, then the following statements are equivalent:*

(A) *Every integral curve $\gamma: [0,T] \to U$ of $\gamma'(t) = X(\gamma(t))$ with $\gamma(0) \in M$ is contained in M.*

(B) *For all $v \in M$ we have $X(v) \in \mathbf{L}_v(M)$.* ∎

Now we reduce the general case to the special case we just considered. To do this we note first that we have $f(g{\cdot}v) \leq f(v)$ for all $v \in V$ if and only if for all $v \in V$ the set $(f_v \downarrow)$ is invariant under g. In fact, if $f(g{\cdot}v) \leq f(v)$ for all $v \in V$ and $w \in (f_v \downarrow)$ then $f(g{\cdot}w) \leq f(w) \leq f(v)$. Conversely, since $v \in (f_v \downarrow)$ for any $v \in V$ the invariance of $(f_v \downarrow)$ shows that $f(g{\cdot}v) \leq f(v)$ for all $v \in V$. Thus the semigroup S_f is equal to the intersection of all semigroups $S_{f,v}$ of the form $\{g \in G: g{\cdot}(f_v \downarrow) \subseteq (f_v \downarrow)\}$ where v is a fixed but arbitrary element of V. Recall that by hypothesis the sets $(f_v \downarrow)$ are closed, so that $S_{f,v}$ is equal to the closed semigroup $S_{1-\chi_v}$ where χ_v is the charactistic function of $(f_v \downarrow)$. Thus our special case tells us how to calculate $\mathbf{L}(S_{1-\chi_v})$. Note that the definition of the tangent object $\mathbf{L}(S)$ for a closed semigroup S immediately shows that the tangent object of an intersection of closed semigroups is the intersection of the corresponding tangent objects. This shows that

$$\mathbf{L}(S_f) = \bigcap_{v \in V} \mathbf{L}(S_{1-\chi_v}). \tag{6}$$

But $\mathbf{L}(S_{1-\chi_v}) = \{x \in \mathbf{L}(G): x{\cdot}w \in \mathbf{L}_w(f_v \downarrow)\ \forall w \in (f_v \downarrow)\}$. It is clear that the right hand side of (6) is contained in $\{x \in \mathbf{L}(G): x{\cdot}v \in \mathbf{L}_v(f_v \downarrow)\ \forall v \in V\}$. But if $x \in \mathbf{L}(G)$ is contained in this set we also have $x{\cdot}w \in \mathbf{L}_w(f_w \downarrow) \subseteq \mathbf{L}_w(f_v \downarrow)$ for all $w \in (f_v \downarrow)$ since for these w one has $(f_w \downarrow) \subseteq (f_v \downarrow)$. This proves the theorem. ■

We record the special case from the proof of Theorem 2.1 as a separate corollary.

2.3. Corollary. *Let M be closed subset of V and $S = \{g \in G: g{\cdot}M \subseteq M\}$ then*

$$\mathbf{L}(S) = \{x \in \mathbf{L}(G): x{\cdot}v \in \mathbf{L}_v(M)\ \forall v \in M\}. \tag{7}$$

■

It should be noted that it is by no means clear from (5) that $\mathbf{L}(S_f)$ is a wedge, let alone a Lie wedge.

2.4. Example. Let $V = \mathbb{R}^2$, $G = \mathrm{Gl}(V)$ with the natural action and $f = 1-\chi$ where χ is the charactistic function of $M = (1,0) + (\mathbb{R}^+{\cdot}(1,0) \cup \mathbb{R}^+{\cdot}(0,1))$. Then $L_v(M)$ is equal to $(\mathbb{R}^+{\cdot}(1,0) \cup \mathbb{R}^+{\cdot}(0,1))$ for $v = (1,0)$, equal to $\mathbb{R}{\cdot}(1,0)$ for $v \in ((1,0)+\mathbb{R}^+{\cdot}(1,0)) \setminus \{(1,0)\}$ and $\mathbb{R}{\cdot}(0,1)$ otherwise. Thus if $x \in \mathbf{L}(G)$ is given by the matrix

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $x \in \mathbf{L}(S_f)$ if and only if the following equations hold:

(a) $(a,c) \in \mathbb{R}^+{\cdot}(1,0) \cup \mathbb{R}^+(0,1)$,

(b) $(sa, sc) \in \mathbb{R}{\cdot}(1,0)$ for all $s > 1$,

(c) $(a+rb, c+rd) \in \mathbb{R}{\cdot}(0,1)$ for all $r > 0$.

We conclude that $a = b = c = 0$ and $d \in \mathbb{R}$ arbitrary which means that $\mathbf{L}(S_f)$ is even a vector space.

2.5. Example. (*Nonnegative matrices*) Let S be the semigroup of all invertible $n \times n$-matrices with nonnegative entries (cf. Example 1.2). Note that for $v = (v_1, ..., v_n) \in (\mathbb{R}^+)^n$ the set of subtangent vectors $\mathbf{L}_v((\mathbb{R}^+)^n)$ is equal to $\mathbb{R}^{o_1} \times ... \times \mathbb{R}^{o_n}$ where o_j stands for $+$ if $v_j = 0$ and for 1 if $v_j \neq 0$. Testing (7) against the standard basis for $\mathbb{R}^n$ we find that $\mathbf{L}(S_f) = \{x = (x_{ij}) \in \mathrm{gl}(n): x_{ij} \geq 0 \text{ for } i \neq j\}$.

If $f: V \to \mathbb{R}$ in Theorem 2.1 is differentiable and v is a regular point of f, then $\mathbf{L}_v(f_v \downarrow)$ is just a halfspace bounded by the kernel of $df(v)$ and $x \cdot v \in \mathbf{L}_v(f_v \downarrow)$ reads $df(v)(x \cdot v) \leq 0$. If v is singular these two expressions are no longer equivalent since $\mathbf{L}_v(f_v \downarrow)$ may still be nontrivial, but it turns out that it suffices to consider only regular points:

2.6. Proposition. *Let A be Banach space and $f: V \to A$ continuously differentiable, then we have*

$$\mathbf{L}(S_f) = \{x \in \mathbf{L}(G): df(v)(x \cdot v) \ \ \forall v \in V\}. \tag{8}$$

Proof. For any $x \in \mathbf{L}(G)$ and $v \in V$ we consider the function $\gamma_{x,v}: \mathbb{R} \to A$ defined by $\gamma_{x,v}(t) = f((\exp tx) \cdot v)$. This function is differentiable , and since $f(v) = \gamma_{x,v}(0)$, it suffices to show that $\gamma_{x,v}$ is decreasing (not necessarily strictly) on $[0, \infty[$ for all $v \in V$ if and only if $df(v)(x \cdot v) \leq 0$ for all $v \in V$. To this end we note that $\gamma'_{x,v}(t) = df((\exp tx) \cdot v)(x \cdot ((\exp tx) \cdot v))$. Since we deal with arbitrary $v \in V$ we see that $\gamma'_{x,v}(t) \leq 0$ for all $v \in V$ and all $t \in \mathbb{R}$ if and only if $df(v)(x \cdot v) \leq 0$ for all $v \in V$, which proves our claim. ∎

2.7. Example. (*Quadratic forms*) Let V be a finite dimensional real vector space, $B: V \times V \to \mathbb{R}$ a symmetric bilinear map and $q: V \to \mathbb{R}$ defined by $q(v) = B(v, v)$ the associated quadratic form, then we have

$$\mathbf{L}(S_q) = \{x \in \mathbf{L}(G): B(v, x \cdot v) \leq 0 \ \ \forall v \in V\}. \tag{9}$$

2.8. Example. Let $V = \mathbb{R}^2$ and $G = \mathrm{Gl}(n)$ with the identity representation. Moreover let $B: V \times V \to \mathbb{R}$ be an arbitrary bilinear map and $q(v) = B(v, v)$. The semigroup S_q is defined by the equation $B(g(v), g(v)) \leq B(v, v)$ which is of the form

$$\alpha r^2 + \beta rs + \gamma s^2 \leq 0 \tag{$*$}$$

for r and s in $\mathbb{R}$ where α, β and γ depend on g. If we set $t = \frac{r}{s}$ the equation reads $\alpha t^2 + \beta t + \gamma$ for all $t \in \mathbb{R}$. Since $(*)$ implies that α and γ are nonpositive it suffices to check that the polynomial is nonpositive at its critical value. This yields the equation

$$4\alpha\gamma \leq \beta^2. \tag{$**$}$$

Note that the equation (9) for $\mathbf{L}(S_q)$ is also of the form $(*)$, so $(**)$ can also be use to calculate $\mathbf{L}(S_q)$. We give two examples where $g \in \mathrm{Gl}(V)$ and $x \in \mathrm{gl}(V)$, respectively, are represented as a 2×2-matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) $B((r,s),(r',s')) = rr' + ss'$.

Then $q(g\cdot(r,s)) \le q(r,s)$ reads $(ar+bs)^2 + (cr+ds)^2 \le r^2 + s^2$ and a simple calculation shows that $\alpha = a^2 + c^2 - 1$, $\beta = 2ab + 2cd$ and $\gamma = b^2 + d^2 - 1$. Therefore (**) yields $4(a^2+c^2-1)(b^2+d^2-1) \le 4(ab+cd)^2$ which can be rewritten as $(\det g)^2 \le a^2 + b^2 + c^2 + d^2 - 1$. Thus $g \in S_q$ if and only if

$$
\begin{aligned}
(\det g)^2 &\le a^2 + b^2 + c^2 + d^2 - 1,\\
a^2 + c^2 - 1 &\le 0,\\
b^2 + d^2 - 1 &\le 0.
\end{aligned}
$$

In order to determine the tangent cone $\mathbf{L}(S_q)$ we note that it consists of all x which satisfy $(ar+bs)r + s(cr+ds) \le 0$, that is we have $\alpha = a$, $\beta = b + c$ and $\gamma = d$. Thus $x \in \mathbf{L}(S_q)$ if and only if

$$
\begin{aligned}
a &\le 0,\\
d &\le 0,\\
(a+d)^2 &\le (b+c)^2 + (a-d)^2.
\end{aligned}
$$

(b) $B((r,s),(r',s')) = rr' - ss'$.

In this case $q(g\cdot(r,s)) \le q(r,s)$ reads $(ar+bs)^2 - (cr+ds)^2 \le r^2 - s^2$ and again a simple calculation shows that $\alpha = a^2 - c^2 - 1$, $\beta = 2ab - 2cd$ and $\gamma = b^2 - d^2 - 1$. Now (**) yields $4(a^2 - c^2 - 1)(b^2 - d^2 - 1) \le 4(ab - cd)^2$ which can be rewritten as $-(\det g)^2 \le a^2 + b^2 - c^2 - d^2 - 1$. Thus $g \in S_q$ if and only if

$$
\begin{aligned}
-(\det g)^2 &\le a^2 + b^2 - c^2 - d^2 - 1,\\
a^2 - c^2 - 1 &\le 0,\\
b^2 - d^2 - 1 &\le 0.
\end{aligned}
$$

This time the tangent cone $\mathbf{L}(S_q)$ consists of all x which satisfy $(ar+bs)r - s(cr+ds) \le 0$, that is, we have $\alpha = a$, $\beta = b - c$ and $\gamma = -d$. Thus $x \in \mathbf{L}(S_q)$ if and only if

$$
\begin{aligned}
a &\le 0,\\
-d &\le 0,\\
(a-d)^2 &\le (b-c)^2 + (a+d)^2.
\end{aligned}
$$

3. Infinitesimally generated semigroups

According to the general philosophy of Lie theory – treating analytic problems by translating them into algebraic ones and translating the algebraic solution back to the analytic situation – one of the basic problems in the Lie theory of semigroups is the question, which wedges can occur as tangent objects of semigroups. On the *local* level this question is completely answered by a generalisation of Lie's Fundamental Theorem (cf. [HH86]): A wedge in a Lie algebra is the tangent object of a *local* semigroup if and only if it is a Lie wedge. On the global level, even though characterisations of tangent wedges of semigroups exist, a satisfactory solution (in terms of verifiable conditions) is still missing. We do *not* address this problem in this article since we start from given (global) semigroups. But for the general program mentioned above there still remains the problem to which extent the given semigroup is determined by its tangent wedge. In contrast to the Lie theory of groups there may be many semigroups with the same tangent wedge. It is therefore important to know if a (closed) semigroup S is *infinitesimally generated.* By this we mean that S is the smallest closed semigroup containing $\exp \mathbf{L}(S)$. We note here that the concept of infinitesimally generated semigroups is quite a bit more delicate if the assumption of closedness is dropped.

We will treat here the example of contractions with respect to a quadratic form on a finite dimensional real vector space. Thus we let V be a real vector space, $q: V \to \mathbb{R}$ a quadratic form and $S = S_q = \{g \in \mathrm{Gl}(V): q(g\cdot v) \leq q(v)\ \forall v \in V\}$. Moreover we let $B: V \times V \to \mathbb{R}$ be the symmetric bilinear form associated to q and note that for positive definite B, using the polar decomposition of matrices, it is fairly elementary to show that the smallest closed subsemigroup of $\mathrm{Gl}(V)$ containing $\exp \mathbf{L}(S)$ is $\exp(\mathbf{L}(S) \cap \mathbf{p}_B)H(S)$ where $\mathbf{p}_B$ is the space of all matrices which are symmetric with respect to B (cf. [La86]). In fact the same is true for arbitrary nondegenerate B, but the existence of a polar decomposition in the indefinite case is quite a bit more complicated to show (cf. [BK79]). We present an argument here which goes back to Ol'shanskiĭ(cf. [Ol81]) and is in some sense typical in the study of semigroups generated by a given wedge. The idea is to show that the map $\Phi: (\mathbf{L}(S) \cap \mathbf{p}_B) \times H(S) \to \exp(\mathbf{L}(S) \cap \mathbf{p}_B)H(S)$ defined by $\Phi(x,h) = (\exp x)h$ is a homeomorphism, then transport the curve $t \mapsto (\exp x \exp tx')$ for $x', x \in (\mathbf{L}(S) \cap \mathbf{p}_B)$ back to $(\mathbf{L}(S) \cap \mathbf{p}_B) \times H(S)$ via Φ and to show, using Lemma 2.2, that it cannot leave that set.

In order to show that Φ is indeed a homeomorphism, we have to make a detour in the complexification V_C of V. First we recall that we may define a hermitean form $B_C: V_C \times V_C \to \mathbb{C}$ via $B_C(v+iw, v'+iw') = B(v,v') + B(w,w') + iB(w,v') - iB(v,w')$. The associated quadratic form $q_C: V_C \to \mathbb{R}$ defined by $q_C(v+iw) = B_C(v+iw, v+iw)$ is given by $q_C(v+iw) = q(v) + q(w)$. This shows that $S \subseteq S_C$ where $S_C = \{g \in \mathrm{Gl}(V_C): q(g\cdot v_C) \leq q_C(v_C)\ \forall v_C \in V_C\}$. Now note that B_C is nongenerate if B is, so that we have adjoint operations ${}^{t_B}: \mathrm{gl}(V) \to \mathrm{gl}(V)$ and ${}^{*_B}: \mathrm{gl}(V_C) \to \mathrm{gl}(V_C)$ defined by $B(x\cdot v, w) = B(v, x^{t_B}\cdot w)$ for all $v, w \in V$ and $B_C(x\cdot v_C, w_C) = B_C(v_C, x^{*_B}\cdot w_C)$ for all $v_C, w_C \in V_C$ respectively. The adjoint operations are convenient for the description of $H(S)$ and $H(S_C)$ and their Lie algebras:

3.1. Remark.

(a) $H(S) = \{g \in \mathrm{Gl}(V): g^{t_B} = g^{-1}\}$.

(a′) $H(S_C) = \{g \in \mathrm{Gl}(V_C): g^{*_B} = g^{-1}\}$.

(b) $\mathbf{L}(H(S)) = \{x \in \mathrm{gl}(V): x^{t_B} = -x\}$.

(b′) $\mathbf{L}(H(S_C)) = \{x \in \mathrm{gl}(V_C): x^{*_B} = -x\}$.

Proof. (a) and (a′) follow from polarisation of B and B_C. Formulae (b) and (b′) now follow by differentiation. ∎

We note in passing that, given a B-orthogonal basis $\{e_1, ..., e_n\}$ for V such that $q(e_j) = 1$ for $j = 1, \ldots, p$ and $q(e_j) = -1$ for $j = p+1, \ldots, n$, we can express the adjoint operations in terms of the usual transpose ^{t} and complex conjugate transpose *. In fact, if $x \in \mathrm{gl}(V)$ is given as a blockmatrix

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then x^{t_B} is given as the blockmatrix

$$x^{t_B} = \begin{pmatrix} A^t & -C^t \\ -B^t & D^t \end{pmatrix}.$$

Similarly for $x \in \mathrm{gl}(V_C)$ we have

$$x^{*_B} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}.$$

The adjoint operations are even more useful for our purposes, since they allow us to show that there is a vector space complement of $\mathbf{L}(H(S))$ which at the same time is an $\mathbf{L}(H(S))$-module.

3.2. Remark. We set $\mathbf{k}_B = \{x \in \mathrm{gl}(V): x^{t_B} = -x\}$ and $\mathbf{p}_B = \{x \in \mathrm{gl}(V): x^{t_B} = x\}$. Similarly let $\mathbf{k}_{B_C} = \{x \in \mathrm{gl}(V_C): x^{*_B} = -x\}$ and $\mathbf{p}_{B_C} = \{x \in \mathrm{gl}(V_C): x^{*_B} = x\}$. Then

(a) $\sigma_B: \mathrm{gl}(V) \to \mathrm{gl}(V)$ defined by $\sigma_B(x) = -x^{t_B}$ is a Lie algebra automorphism.

(a') $\sigma_{B_C}: \mathrm{gl}(V_C) \to \mathrm{gl}(V_C)$ defined by $\sigma_B(x) = -x^{*_B}$ is a Lie algebra automorphism.

(b) $\mathrm{gl}(V) = \mathbf{k}_B \oplus \mathbf{p}_B$ (vector space direct sum).

(b') $\mathrm{gl}(V_C) = \mathbf{k}_{B_C} \oplus \mathbf{p}_{B_C}$ (vector space direct sum).

(c) $[\mathbf{k}_B, \mathbf{k}_B] \subseteq \mathbf{k}_B$, $[\mathbf{k}_B, \mathbf{p}_B] \subseteq \mathbf{p}_B$, $[\mathbf{p}_B, \mathbf{p}_B] \subseteq \mathbf{k}_B$.

(c') $[\mathbf{k}_{B_C}, \mathbf{k}_{B_C}] \subseteq \mathbf{k}_{B_C}$, $[\mathbf{k}_{B_C}, \mathbf{p}_{B_C}] \subseteq \mathbf{p}_{B_C}$, $[\mathbf{p}_{B_C}, \mathbf{p}_{B_C}] \subseteq \mathbf{k}_{B_C}$.

(d) $i\mathbf{k}_{B_C} = \mathbf{p}_{B_C}$.

(e) $\mathbf{k}_B = \mathbf{k}_{B_C} \cap \mathrm{gl}(V)$ and $\mathbf{p}_B = \mathbf{p}_{B_C} \cap \mathrm{gl}(V)$.

Proof. (a),(a') and (d) are straightforward calculations. Items (b), (b'), (c) and (c') follow immediately if one notes that the decompositions in (b) and (b') are eigenspace decompositions. Finally (e) is a consequence of the fact that ^{t_B} is the restriction of ^{*_B} to $\mathrm{gl}(V)$. ∎

With the decomposition from Remark 3.2 it is possible to essentially reduce the study of the semigroup generated by $\exp \mathbf{L}(S)$ to the study of the semigroup generated by $\exp W_p$ where $W_p = \mathbf{L}(S) \cap \mathbf{p}_B$.

3.3. Remark. Let S_{min} be the smallest closed subsemigroup of $\mathrm{Gl}(V)$ which contains $\exp \mathbf{L}(S)$ and S_p the smallest closed subsemigroup of $\mathrm{Gl}(V)$ containing $\exp W_p$, then

(a) $hW_ph^{-1} \subseteq W_p$ for all $h \in H(S)$.

(b) $S_{min} = S_p H(S)_o$, where $H(S)_o$ is the connected component of $H(S)$.

Proof. We have $B(hxh^{-1}\cdot v, w) = B(v,(h^{-1})^{t_B}x^{t_B}h^{t_B}\cdot w) = B(v, hxh^{-1}\cdot w)$ for any $x \in W_p$, which shows that $hxh^{-1} \in \mathbf{p}_B$. Moreover $B(hxh^{-1}\cdot v, v) = B(xh^{-1}\cdot v, h^{-1}\cdot v) \leq 0$ for all $v \in V$ shows that $hxh^{-1} \in \mathbf{L}(S)$. This proves (a), and (b) follows since any product $(\exp x)h(\exp x')h'$ may be written in the form $(\exp x)(h \exp x')h^{-1}hh' = (\exp x)(\exp hxh^{-1})hh'$. ∎

In order to prove the anounced result we now only have to show that S_p is contained in $(\exp W_p)H(S)_o$. But we still have to record some results from our complexification procedure in the form it will be used later.

3.4. Lemma. *Let $W_C = \mathbf{L}(S_C) \cap \mathbf{p}_{B_C}$, then we have*

(a) $\mathbf{L}(S_C) = \mathbf{k}_{B_C} + W_C$.

(b) $e^{\mathrm{ad}\, x} W_C = W_C$ *for all* $x \in \mathbf{k}_{B_C}$.

(c) $iW_C \subseteq \mathbf{k}_{B_C}$.

(d) $e^{\mathrm{ad}\, x} iW_C = iW_C$ *for all* $x \in \mathbf{k}_{B_C}$.

(e) $iW_C - iW_C = \mathbf{k}_{B_C}$.

Proof. (a) is clear with Remarks 3.1 and 3.2. Formula (b) is a consequence of Remark 3.2 in view of $e^{\mathrm{ad}\, x} y = (\exp x) y (\exp x)^{-1}$ (cf. Remark 3.3). The inclusion (c) follows from Remark 3.2 and (d) just reflects the fact that $e^{\mathrm{ad}\, x}$ is a complex linear map. Finally, in order to prove (e) we note that $iW_p - iW_p$ has to be an ideal in $\mathbf{k}_{B_C}$ because of (d). Therefore $(iW_p - iW_p)_C$ has to be an ideal of $\mathrm{gl}(V_C)$. But one easily calculates that $x \in \mathrm{gl}(V)$ given by $xe_j = 0$ for $j = 1, \ldots, p-1, p+2, \ldots, n$, $xe_p = ae_p$ and $xe_{p+1} = -de_{p+1}$ with $a, d \geq 0$ is contained in $W_p \subseteq W_C$ (cf. Example 2.8b). Thus W_C is neither contained in $\mathbb{C}\mathbf{1}$ nor in $[\mathrm{gl}(V_C), \mathrm{gl}(V_C)]$, hence in no nontrivial ideal of $\mathrm{gl}(V_C)$. This proves (e). ∎

The point of Lemma 3.4e is that, using the general theory of invariant cones (cf. [HHL87]), it shows that any $\mathrm{ad}\, ix$ with ix in the interior $\mathrm{int}\, iW_C$ of iW_C in $\mathbf{k}_{B_C}$ is semisimple with purely imaginary spectrum. In particular any $\mathrm{ad}\, ix$ with $x \in W_p \subseteq W_C$ has a real spectrum.

We are now ready to prove that Φ is a homeomorphism.

3.5. Lemma. *Let $\Psi: \mathbf{p}_B \times H(S) \to \mathrm{Gl}(V)$ be defined by $\Psi(x,h) = (\exp x)h$. Then $\Phi = \Psi|_{W_p \times H(S)}$ is a homeomorphism onto its image. Moreover Ψ is differentiable and regular for all $(x,h) \in W_p \times H(S)$.*

Proof. We note first that Ψ is a differentiable map, and regular in any point (x,h) such that exp is regular in x (cf. [He78, Chap.II, Exercise A1]). Since the spectrum of $\mathrm{ad}\, x$ is real for all $x \in W_p$, it follows from [He78, Chap. II,§1, Thm.1.7] that there is a neighborhood U of W_p in $\mathbf{p}_B$ such that Ψ is regular for all $(x,h) \in U \times H(S)$. Therefore it only remains to show that Φ is injective.

Claim 1: $\exp x = \exp x'$ for $x, x' \in W_p$ implies $x = x'$.

In fact, it follows that x and x' commute (cf. [HHL87, Lemma V.6.7]) since exp is regular in x. Therefore we have $\exp(x - x') = \mathbf{1}$. But we have that $\mathrm{ad}\, x$ and $\mathrm{ad}\, x'$ can be put *simultaneously* into Jordan canonical form with real eigenvalues, which means that also $\mathrm{ad}(x - x')$ has a real Jordan canonical form, so that $e^{\mathrm{ad}(x-x')} = \mathrm{Ad}(\exp(x - x')) = \mathbf{1}$ cannot hold unless $\mathrm{ad}\, x = \mathrm{ad}\, x'$, that is

$\operatorname{ad}(x-x')=0$. But then $x-x'=r\mathbf{1}$ for some $r\in\mathbb{R}$ which implies $x=x'$ since $\exp(x-x')=\mathbf{1}$.

Claim 2: $(\exp x)h=(\exp x')h'$ for $x,x'\in W_p$ and $h,h'\in H(S)$ implies $x=x'$ and $h=h'$.

To show Claim 2 we may assume that $h'=\mathbf{1}$. We set $p=\exp x$ and calculate $ph=(ph)^{t_B}=h^{t_B}p^{t_B}=h^{-1}p$ so that $\exp h^{-1}2xh=h^{-1}(\exp 2x)h=h^{-1}p^2h=p^2=\exp 2x$. But we know that $h^{-1}xh\in W_p$ so that Claim 1 shows $2h^{-1}xh=2x$, hence $hp=ph$ which in turn implies $h=\mathbf{1}$. Now we conclude again from Claim 1 that $x=x'$. ∎

Now we want to transport the curve $t\mapsto \exp x\exp tx'$ for $x,x'\in W_p$ back to $\mathbf{p}_B\times H(S)$ via Φ and use Lemma 2.2 to show that the resulting curve cannot leave $W_p\times H(S)$. Thus we fix x and x' in W_p and define $\gamma:\mathbb{R}\to \mathrm{Gl}(V)$ by $\gamma(t)=\exp x\exp tx'$. Since $x\in W_p$ and Φ is regular for all $(x,h)\in W_p\times H(S)$ we find an $\varepsilon\geq 0$ and a differentiable curve $\gamma_o:[0,\varepsilon]\to \mathbf{p}_B\times H(S)$ such that $\gamma_o(0)=(x,\mathbf{1})$ and $\Phi\circ\gamma_o(t)=\gamma(t)$ for all $t\in[0,\varepsilon]$. In order to show that the semigroup, and hence also the closed semigroup, generated by $\exp W_p$ is contained in $(\exp W_p)H(S)$, it is enough to show that

$$\gamma_o([0,\varepsilon])\subseteq W_p\times H(S) \tag{10}$$

since x and x' were arbitrary elements of W_p. In view of Lemma 2.2 it suffices to show that $\eta'(0)\in \mathbf{L}_{\eta(0)}(W_p)=\mathbf{L}_x(W_p)$ where $\eta(t)$ for $t\in[0,\varepsilon]$ is the $\mathbf{p}_B$-component of $\gamma_o(t)$. Thus we have to calculate the derivatives of the various functions involved.

At this point we have to use some concepts from differential geometry. The curve γ lives on the differentiable manifold $\mathrm{Gl}(V)=G$. Therefore the derivative $\gamma'(t)$ is an element of the tangent space $T_{\gamma(t)}G$. Recall that the tangent space $T_{\mathbf{1}}G$ can be, and will be, identified with the Lie algebra $\mathbf{L}(G)=\mathrm{gl}(V)$. Moreover it is possible to identify the various tangent spaces via the left translations $g'\mapsto gg'=\lambda_g(g')$. For $g\in G$ we have $T_gG=d\lambda_g(\mathbf{1})(T_{\mathbf{1}}G)=d\lambda_g(\mathbf{1})(\mathrm{gl}(V))$. Since $d\exp(0):\mathrm{gl}(V)\to T_{\mathbf{1}}G=\mathrm{gl}(V)$ is the identity we find

$$\gamma'(t)=d\lambda_{\exp x}(\mathbf{1})(x')$$

using the chain rule. We recall from (cf. [He78,Ch.II,§1,Thm.1.7]) that the differential for the exponential function is given by

$$d\exp(x)=d\lambda_{\exp x}(\mathbf{1})\circ\frac{\mathbf{1}-e^{-\operatorname{ad}x}}{\operatorname{ad}x},$$

where the quotient stands for the respective powerseries . If we use the formula for the differential of the product in a Lie group (cf. [He78,Ch.II,Exercise 1(iii)]) we obtain

$$d\Phi(x,\mathbf{1})(y,z)=d\lambda_{\exp x}(\mathbf{1})(z)+d\lambda_{\exp x}(\mathbf{1})\circ\frac{\mathbf{1}-e^{-\operatorname{ad}x}}{\operatorname{ad}x}(y)$$

for $y\in\mathbf{p}_B$ and $z\in\mathbf{L}(H(S))=\mathbf{k}_B$. Now the chain rule yields $d\lambda_{\exp x}(\mathbf{1})(x')=d\lambda_{\exp x}(\mathbf{1})(z)+d\lambda_{\exp x}(\mathbf{1})\circ\frac{\mathbf{1}-e^{-\operatorname{ad}x}}{\operatorname{ad}x}(\eta'(0))$ and hence

$$x'=z(x')+\frac{\mathbf{1}-e^{-\operatorname{ad}x}}{\operatorname{ad}x}(\eta'(0)), \tag{11}$$

where $z(x')$ is an element of $\mathbf{k}_B$ which is uniquely determined by x'. Recall that $d\Phi(x,1)$ is invertible, so the map $\frac{1-e^{-\operatorname{ad} x}}{\operatorname{ad} x}$ is, too. But then the inverse is given by the powerseries $\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}}$. Thus it suffices to show that

$$\text{(12)} \qquad \frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}}(\mathbf{L}(S)) \subseteq \mathbf{L}_x(\mathbf{L}(S)).$$

In order to prove (12) we have to recall two facts from the general theory of Lie wedges (cf. [HHL87]). Firstly we remark that for any Lie wedge W the condition (LIE) is equivalent to

$$\text{(LIE}'\text{)} \qquad [y, T_x(W)] \subseteq T_x(W) \quad \forall y \in W \cap -W,\ x \in W,$$

where $T_x(W) = \mathbf{L}_x(W) \cap -\mathbf{L}_x(W)$ is the *tangent space* of W at x. Moreover we note that for any wedge W in a finite dimensional vector space L and any $x \in W$ we have

$$\text{(13)} \qquad \mathbf{L}_x(W) = \left(x^\perp \cap W^*\right)^*,$$

where for any set $M \subseteq L$ we set $M^\perp = \{\omega \in \widehat{L}: \langle\omega|y\rangle = 0\ \forall y \in M\}$ and $M^* = \{\omega \in \widehat{L}: \langle\omega|y\rangle \geq 0\ \forall y \in M\}$ if $\langle\ |\ \rangle: \widehat{L} \times L \to \mathbb{R}$ denotes the pairing of L with its dual space $\widehat{L}$. Now we calculate $\mathbf{L}_x(\mathbf{L}(S)) = \left(x^\perp \cap \mathbf{L}(S)^*\right)^* = \left(x^\perp \cap W_p{}^* \cap (\mathbf{k}_B)^*\right)^* = \left(x^\perp \cap W_p{}^* \cap (\mathbf{k}_B)^\perp\right)^*$ since $\mathbf{L}(S) = \mathbf{k}_B + W_p$ and $\mathbf{k}_p$ is a vector space. For $\omega \in \left(x^\perp \cap W_p{}^* \cap (\mathbf{k}_B)^\perp\right)$ we claim that $\widehat{\operatorname{ad}}x(\omega) = 0$, where $\widehat{\operatorname{ad}}x$ is the dual operator of $\operatorname{ad} x$. In fact we know $\langle\widehat{\operatorname{ad}}x(\omega)|y\rangle = \langle\omega|-[y,x]\rangle = 0$ for all $y \in \mathbf{k}_B$, since $-[y,x] \in T_x = \mathbf{L}_x(\mathbf{L}(S)) \cap -\mathbf{L}_x(\mathbf{L}(S)) = (x^\perp \cap \mathbf{k}_B \cap W^*)^\perp$ for $y \in \mathbf{k}_B$. On the other hand we have $-[y,x] \in \mathbf{k}_B$ for all $y \in \mathbf{p}_B$ so that $\langle\widehat{\operatorname{ad}}x(\omega)|y\rangle = \langle\omega| - [y,x]\rangle = 0$ for these y as well. Finally we get $\langle\omega|\frac{\operatorname{ad} x}{1-e^{-\operatorname{ad} x}}(y)\rangle = \langle\frac{\widehat{\operatorname{ad}} x}{1-e^{-\widehat{\operatorname{ad}} x}}(\omega)|y\rangle = \langle\omega|y\rangle \geq 0$ for all $y \in \mathbf{L}(S)$. Thus we have proved the following

3.6. Theorem. *Let V be a finite dimensional real vector space and $B: V \times V \to \mathbb{R}$ a symmetric bilinear form. Set $\mathbf{p}_B = \{x \in \mathrm{gl}(V): B(x\cdot v, w) = B(v, x\cdot w)\ \forall v, w \in V\}$, $W_p = \{x \in \mathbf{p}_B: B(x\cdot v, v) \leq 0\ \forall v \in V\}$ and $H = \{g \in \mathrm{Gl}(V): B(g\cdot v, g\cdot v) = B(v,v)\ \forall v \in V\}$, then*

(a) *$\Phi: W_p \times H \to \mathrm{Gl}(V)$ defined by $(x,h) \mapsto (\exp x)h$ is a homeomorphism onto a closed subset of $\mathrm{Gl}(V)$.*

(b) *$(\exp W_p)H$ is a closed subsemigroup of $\mathrm{Gl}(V)$.* ∎

We finally note that the results of [BK79] show that the semigroup given in Theorem 3.6 is actually *equal* to the semigroup of all contractions with respect to B.

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J. Hilgert
Mathematisches Institut
Universität Erlangen
Bismarckstr. $1\frac{1}{2}$
D-8520 Erlangen
The Federal Republic of Germany