TUM -MATH-05-82-M08-200/1-FMA

Anforderungen an: Bibliothek der Fakultät
für Mathematik und Informatik
Technische Universität München
Arcisstr. 21 / Postfach 202420

D - 8000 München 2 Germany

Typescript: A. Bußmann
Druck: Institute für Mathematik und Informatik der
Technischen Universität München

FOUNDATIONS OF K-THEORY FOR C*-ALGEBRAS

J. Hilgert

TECHNISCHE UNIVERSITÄT MÜNCHEN

INSTITUT FÜR MATHEMTAIK

TUM-M8208

MAI 1982

TABLE OF CONTENTS

Abstract ....

Chapter
I. K-Theory for Unital C^*-Algebras ....
II. Excision Theorems ....
III. K-Theory for Non Unital C^*-Algebras .....
IV. K-Theory as Homotopy Functor ....
V. Suspensions and Higher K-Groups ....
VI. Bott Periodicity and the Six-Term-Sequence ....
VII. Multiplicative Structures ....

References ....

Abstract

Though K-theory for C^*-algebras has been an object of investigation for several years and various approaches to the theory are known, no rigorous and concise treatment, accessible to any interested functional analyst, seems to exist. This work is designed to fill this gap. It builds the theory from the scratch, following and generalizing Karoubis approach to K-theory for locally compact spaces (cf. [K]).

First we set up K-theory for unital C^*-algebras. We define relative K-groups $K_\alpha(\varphi)$ $\alpha = 0,1$ for unital C^*-morphisms φ and prove two excision theorems, which will allow us to define K-theory for non unital C^*-algebras. Moreover, we show that the K-functors do not distinguish between homotopic C^*-morphisms. This will enable us to define K_n of a C^*-algebra for all $n \in \mathbb{N}$ and to establish a long exact sequence in K-theory associated to a short exact sequence of C^*-algebras. Finally, we describe some multiplicative structures in K-theory for C^*-algebras.

CHAPTER I: K-THEORY FOR UNITAL C*-ALGEBRAS

This chapter is devoted largely to the introduction of notations and terminology which will be used throughout. We also give the definitions and establish some basic properties of the K-groups for unital C*-algebras.

I.1. DEFINITION (cf. [K]: II.2.1). Let $\mathcal{C}$ be an additive category. Let $\mathcal{C}(E,F)$ denote the set of $\mathcal{C}$-morphisms $E \to F$. A *Banach structure* on $\mathcal{C}$ is given by a completely normable topological vector space structure (over $\mathbb{C}$) on all $\mathcal{C}(E,F)$ such that the composition of morphisms $\mathcal{C}(E,F)\times\mathcal{C}(F,G) \to \mathcal{C}(E,G)$ is bilinear and continuous. A Banach category is an additive category provided with a *Banach structure*.

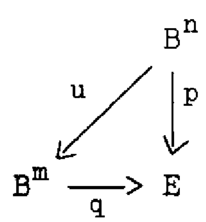
I.2. DEFINITION (cf. [K]: II.2.1). Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories and $\varphi : \mathcal{C} \to \mathcal{C}'$ an additive functor. Then φ is called *quasi surjective* if every object of $\mathcal{C}'$ is a direct factor of an object isomorphic to an object of the form $\varphi(E)$ with $E \in \mathrm{Ob}(\mathcal{C})$.

φ is called *full* if $\mathcal{C}(E,F) \to \mathcal{C}'(\varphi(E),\varphi(F))$ is surjective. If $\mathcal{C}$ and $\mathcal{C}'$ are Banach categories, the functor φ is called a *Banach functor* if $\mathcal{C}(E,F) \to \mathcal{C}'(\varphi(E),\varphi(F))$ is linear and continuous.

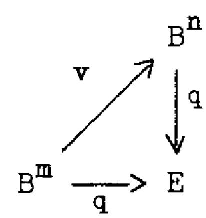
I.3. LEMMA (cf. [K]: II.2.9). Let B be a unital C*-algebra. Let $P(B)$ be the category of finitely generated projective (left) modules over B and module maps. Then $P(B)$ is a Banach category.

Proof. The proof is done in several steps. First we consider an object $E \in \mathrm{Ob}(P(B))$ and endow it with a completely normable topological vector space structure. If

$E \in Ob(P(B))$ is free, we can give it the product norm of B^n. If $E \in Ob(P(B))$ is not free, then there is a projection of B-modules $p: B^n \to E$ onto E. Equip E with the quotient topology. Then E is complete.
Next, we show that this topology does not depend on the particular choice of p. Let $q: B^m \to E$ be another B-module projection onto E, then we get the following commutative diagrams



and



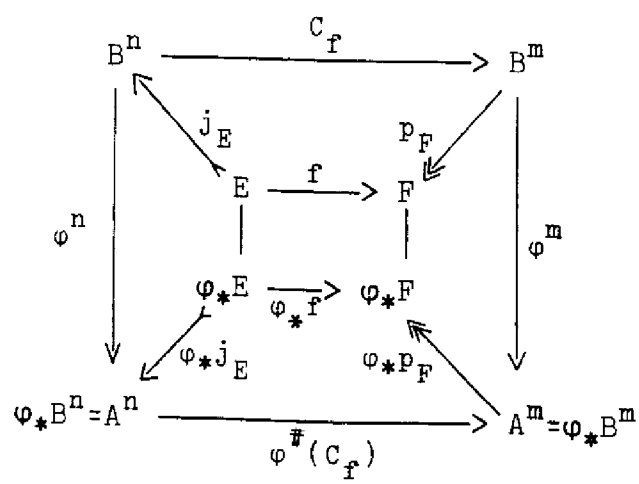
where the existence of the module maps u and v follows from the projectivity of E. Now u and v are automatically continuous, since they are implemented by $m \times n$, respectively, $n \times m$ matrices with entries in B in the usual way. Thus the two quotient topologies agree. We call the topology on E the canonical topology. The canonical topology on E is actually the same as the induced topology given by any injection $j: E \to B^n$ which inverts the projection p on the right, i.e. satisfies $p \circ j = 1_E$. Indeed, consider $f := j \circ p: B^n \to B^n$, then f is a module homomorphism, hence is continuous. Therefore j is continuous, too, since E carries the quotient topology with respect to p. Clearly, p is continuous by definition, and $p \circ j = 1_E$. Define $g: j(E) \to E$ by $g(x) = p(x)$. If $U \subset E$ is open, then $p^{-1}(U)$ is open in B^n and $g^{-1}(U) = p^{-1}(U) \cap j(E)$. Hence $g^{-1}(U)$ is open in $j(E)$ w.r.t. the subspace topology, thus g is continuous. Moreover, if $j': E \to j(E)$ denotes the corestriction of j, we have $g \circ j' = 1_E$ and $j' \circ g = 1_{j(E)}$. This implies that j' is a homeomorphism and thus j is an embedding, which proves our claim.

The next step is to equip $P(B)(E,F) = \mathrm{Hom}_B(E,F)$ with the structure of a completely normable vector space. The topology of uniform convergence on bounded sets turns the vector space $\mathcal{L}(E,F)$ of all continuous linear operators $E \to F$ w.r.t. the canonical topologies of E and F into a complete topological vector space. This topology is compatible with the operator norm $\|f\| = \sup_{\|m\|_E \leq 1} \|f(m)\|_F$ for any pair of norms on E and F compatible with the canonical topologies on E,F . We show that $P(B)(E,F)$ is a closed vector subspace of $\mathcal{L}(E,F)$: It is clear that $\mathrm{Hom}_B(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$. Pick norms on E and F , which are compatible with the canonical topologies. Endow $\mathcal{L}(E,F)$ with the corresponding operator norm. To prove that $\mathrm{Hom}_B(E,F)$ is closed in $\mathcal{L}(E,F)$ it is enough to show that if $f_i \in \mathrm{Hom}_B(E,F)$ converges to f in $\mathcal{L}(E,F)$, f has to be in $\mathrm{Hom}_B(E,F)$, i.e. $f(bm) = bf(m)$ for all $b \in B$, $m \in E$. But $\|f(bm)-bf(m)\|_F \leq \|f(bm)-f_i(bm)\|_F + \|f_i(bm)-bf_i(m)\|_F + \|bf_i(m)-bf(m)\|_F \leq \|f(bm)-f_i(bm)\|_F + \|b\|_B \|f_i(m)-f(m)\|_F$. Now uniform continuity proves that $\|f(bm)-bf(m)\| = 0$, i.e. $f(bm) = bf(m)$. To complete the proof of the lemma we have to show that the composition of morphisms $P(B)(E,F) \times P(B)(F,G) \to P(B)(E,G)$ is bilinear and continuous. But this is clear since the composition of linear operators $\mathcal{L}(E,F) \times \mathcal{L}(F,G) \to \mathcal{L}(E,G)$ is bilinear and continuous w.r.t. the topologies of uniform convergence on bounded sets. ■

Note that there is no canonical norm on an $E \in \mathrm{Ob}(P(B))$ so we don't ask for a Banach space structure on E , as it might seem natural. This problem does not occur in [K] II.2.9, because Karoubi gives no proof.

I.4. LEMMA (cf. [K]: II.2.9). Let B and A be unital C^*-algebras. Let $\varphi\colon B \to A$ be a unital C^*-homomorphism. Consider A as right B-module via $a\cdot b = a\varphi(b)$ and let $A \otimes_B E$ be the algebraic tensor product of the right B-module A and the left B-module E. Then $A \otimes_B E$ is a left A-module with $a\cdot(a'\otimes x) = aa'\otimes x$. Then the assignment $\varphi_*\colon P(B) \to P(A)$ defined by $\varphi_*(E) = A\otimes_B E$ on objects and by $\varphi_*(f) = \mathrm{id}_A \otimes_B f$ on morphisms (cf. [M] §2 for these definitions) is a quasi surjective Banach functor. Further, φ_* is full iff φ is surjective.

Proof. It is easy to see that φ_* is a functor. Moreover is φ_* clearly quasi surjective, since $\varphi_*(B^n) = A^n$. Now let $f\colon E \to F$ be a morphism in $P(B)(E,F)$. We first consider the case $E = B^n$ and $F = B^m$. Then $f\colon B^n \to B^m$ may be identified with that $m{\times}n$-matrix $C = (c_{jk})_{\substack{j=1\ldots m\\ k=1\ldots n}}$ with entries in B for which $pr_j f(x) = \sum_{k=1}^{n} c_{jk}\, pr_k(x)$ for $j = 1,\ldots,m$. Then $\varphi_*(E) = A\otimes_B B^n = (A\otimes_B B)^n$ may be canonically identified with A^n via the isomorphism $a \mapsto a\otimes 1\colon A \to A\otimes_B B$. Likewise we identify $\varphi_* F$ with A^m. It is readily observed that the matrix associated with $\varphi_*(f)\colon \varphi_* E \to \varphi_* F$ after identifying $\varphi_*(E)$ with A^n and $\varphi_*(F)$ with A^m is $\varphi^{\#}(C) := (\varphi(c_{jk}))_{\substack{j=1\ldots m\\ k=1\ldots n}}$. This shows the continuity and linearity of $\varphi_*\colon P(B)(E,F) \to P(A)(\varphi_* E,\varphi_* F)$. Now, let $E,F \in \mathrm{Ob}(P(B))$ be arbitrary. Select projections $p_E\colon B^n \to E$, $p_F\colon B^m \to F$ and corresponding coprojections $j_E\colon E \to B^n$, $j_F\colon F \to B^m$. The commutative diagram



may be rephrased in the commutative diagram

$$\begin{array}{ccc}
P(B)(E,F) & \xrightarrow{\quad \varphi_* \quad} & P(A)(\varphi_* E, \varphi_* F) \\
\Big\downarrow {\scriptstyle P(B)(p_E, j_F)} & & \Big\downarrow {\scriptstyle P(A)(\varphi_* p_E, \varphi_* j_F)} \\
P(B)(B^n, B^m) & \xrightarrow[C \,\to\, \varphi^*(C)]{} & P(A)(A^n, A^m)
\end{array}$$

This proves linearity and contiuity of φ_* , provided one can show that the injections $P(B)(p_E, j_F)$ and $P(A)(\varphi_* p_E, \varphi_* j_F)$ are embeddings. But $P(B)(j_E, p_F) \circ P(B)(p_E, j_F) = P(B)(p_E j_E, p_F j_F) = P(B)(id_E, id_F) = id_{P(B)(E,F)}$. Thus $P(B)(p_E, j_F)$ is a coretraction of completely normable spaces, hence is an embedding.

The same argument works for $P(A)(\varphi_* p_E, \varphi_* j_F)$. This concludes the proof that φ_* is a Banach functor. The assertion that φ_* is full iff φ is surjective is clear if E and F are free, since for φ surjective one can lift any A-matrix to a B-matrix. The general case follows easily from the described embeddings. ∎

We are now ready to define the K_0 and K_1 groups for unital C^*-algebras and pairs of unital algebras. We follow Karoubi's approach.

I.5. DEFINITION (cf. [K] II.1.7). Let B be a unital C^*-algebra. Consider the set Γ_B of isomorphism classes of modules in $\mathcal{P}(B)$. For $E \in \mathrm{Ob}(\mathcal{P}(B))$ denote the class of E by $[E]$. Define an equivalence relation on Γ_B by setting $[E] \sim [F]$ if there exists a $G \in \mathrm{Ob}(\mathcal{P}(B))$ such that $E \oplus G \cong F \oplus G$. Denote the class of $[E]$ in $\Gamma_{B/\sim}$ by $\overline{[E]}$. Then $\Gamma_{B/\sim}$ is a cancellative monoid w.r.t. the addition $\overline{[E]} + \overline{[F]} = \overline{[E \oplus F]}$. Let $K_0(B)$ be the Grothendieck group of $\Gamma_{B/\sim}$, i.e. the group of formal differences of elements of $\Gamma_{B/\sim}$.

Note that this definition is based only on the ring structure of B. The full C^*-algebra structure of B does not enter.

I.6. DEFINITION (cf. [K] 2.13). Let $\varphi: B \to A$ be a unital C^*-homomorphism. Consider the set of triples $\Gamma(\varphi) := \{(E,F,\alpha): E,F \in \mathrm{Ob}(\mathcal{P}(B)), \alpha: \varphi_* E \to \varphi_* F \text{ an isom.}\}$. Two triples (E,F,α) and (E',F',α') are called *isomorphic*, written $(E,F,\alpha) \cong (E',F',\alpha')$, if there exist isomorphisms $f: E \to E'$ and $g: F \to F'$ which make the following square commute.

$$\begin{array}{ccc} \varphi_* E & \xrightarrow{\alpha} & \varphi_* F \\ \downarrow{\scriptstyle \varphi_* f} & & \downarrow{\scriptstyle \varphi_* g} \\ \varphi_* E' & \xrightarrow[\alpha']{} & \varphi_* F' \end{array}$$

A triple (E,F,α) is called *elementary* if $E = F$ and α is homotopic to $1_{\varphi_* E}$ within $\mathrm{Aut}(\varphi_* E)$. Define an addition on $\Gamma(\varphi)$ by setting $(E,F,\alpha) + (E',F',\alpha') := (E \oplus E', F \oplus F', \alpha \oplus \alpha')$. This definition makes $\Gamma(\varphi)$ into a commutative monoid. Define a congruence relation on $\Gamma(\varphi)$ by setting, $\sigma \sim \sigma'$, for $\sigma, \sigma' \in \Gamma(\varphi)$, if there exist elementary triples τ and τ' such that $\sigma + \tau \cong \sigma' + \tau'$. Denote the equivalence class of (E,F,α) by $d(E,F,\alpha)$. Then $K_0(\varphi)$ is defined as the quotient monoid of $\Gamma(\varphi)$ modulo $\sim$. It turns out that $K_0(\varphi)$ is a group.

Note that $A = 0$ is viewed as unital C^*-algebra. Then, for $\varphi: B \to 0$, we can identify $K_0(\varphi)$ and $K_0(B)$ (cf. [K] II.2.13).

The following lemmas are stated and proved in [K] and are stated here only for the sake of completeness. They will enable us to give an alternative description of $K_0(\varphi)$, which will be useful in actual calculations.

I.7. LEMMA (cf. [K] II.2.14). $K_0(\varphi)$ is an abelian group. The inverse of $d(E,F,\alpha)$ is $d(F,E,\alpha^{-1})$. ∎

I.8. LEMMA (cf. [K] II.2.15). For $d(E,F,\alpha)$ and $d(E,F,\alpha')$ in $K_0(\varphi)$ with α homotopic to α' within the space of isomorphisms from $\varphi_* E$ to $\varphi_* F$, we have $d(E,F,\alpha) = d(E,F,\alpha')$. ∎

I.9. LEMMA (cf. [K] II.2.16). For $d(E,F,\alpha)$ and $d(F,G,\beta)$ in $K_0(\varphi)$ we have that $d(E,F,\alpha) + d(F,G,\beta) = d(E,G,\beta\alpha)$. ∎

I.10. LEMMA (cf. [K] II.2.20).
The maps $j^*: K_0(\varphi) \to K_0(B)$, given by $d(E,F,\alpha) \mapsto \overline{[E]} - \overline{[F]}$ and $\varphi^*: K_0(B) \to K_0(A)$, given by $\overline{[E]} - \overline{[F]} \mapsto \overline{[\varphi_* E]} - \overline{[\varphi_* F]}$ are well defined group homomorphisms yielding the exact sequence $K_0(\varphi) \xrightarrow[j^*]{} K_0(B) \xrightarrow[\varphi^*]{} K_0(A)$. Moreover, if there exists a C^*-homomorphism $\psi: A \to B$ such that $\varphi \circ \psi = \mathrm{id}_A$, we get a split exact sequence $0 \to K_0(\varphi) \xrightarrow[j^*]{} K_0(B) \xrightarrow[\varphi^*]{} K_0(A) \to 0$. ■

I.11. LEMMA (cf. [K] II.2.25). Let $\varphi: B \to A$ be a surjective unital C^*-homomorphism. If $\tau = (E,E,\alpha') \in \Gamma(\varphi)$ is an elementary triple, then $\tau \cong (E,E,\mathrm{id}_{\varphi_* E})$. ■

I.12. LEMMA (cf. [K] II.2.26). Let $\varphi: B \to A$ be as in I.11. If we replace elementary triples in the definition of $K_0(\varphi)$ by triples of the form $(E,E,\mathrm{id}_{\varphi_* E})$ and proceed in the same fashion otherwise, we get the same group $K_0(\varphi)$. ■

I.13. THEOREM (cf. [K] II.2.28). Let φ be as in I.11, then $d(E,F,\alpha) = 0$ in $K_0(\varphi)$ iff there is a $G \in \mathrm{Ob}(P(B))$, which can be chosen to be free, and a module isomorphism $\beta: E \oplus G \to F \oplus G$ such that $\varphi_* \beta = \alpha \oplus \mathrm{id}_{\varphi_* G}$. ■

Note that this description of $K_0(\varphi)$ does no longer involve the topological structure of A and B .

We now turn to the definition of K_1 . Here we can rely only partly on previous work. The notion of relative K_1-groups has to my knowledge, not been used before in the context of C^*-algebras.

I.14. DEFINITION. Let $\varphi: B \to A$ be a unital C^*-algebra homomorphism. Consider the set of pairs $\Gamma_1(\varphi) :=$ $\{(E,\alpha): E \in \mathrm{Ob}(P(B)), \alpha \in \mathrm{Aut}\, E, \varphi_*\alpha = \mathrm{id}_{\varphi_* E}\}$. Two pairs (E,α) and (E',α') are called *isomorphic*, written $(E,\alpha) \cong (E',\alpha')$, if there is an isomorphism $h: E \to E'$ which makes the following square commute:

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \alpha \downarrow & & \downarrow \alpha' \\ E & \xrightarrow[h]{} & E' \end{array}$$

A pair (E,α) is called *elementary* if α is homotopic to id_E in $\mathrm{Aut}\, E$ relative to A , written $\alpha \simeq \mathrm{id}_E \,\mathrm{rel}\, A$. This means that if σ_r is the homotopy between α and id_E , we have $\varphi_*\sigma_t = \mathrm{id}_{\varphi_* E}$ for all $t \in I$. Define an addition on $\Gamma_1(\varphi)$ by $(E,\alpha)+(E',\alpha') := (E \oplus E', \alpha \oplus \alpha')$. For $\sigma, \sigma' \in \Gamma_1(\varphi)$, define a relation $\sim$ by $\sigma \sim \sigma'$ if there exist elementary pairs τ and τ' such that $\sigma+\tau \cong \sigma'+\tau'$. It is easy to check that $\sim$ is a congruence. Denote the equivalence class of (E,α) by $d(E,\alpha)$. Now set $K_1(\varphi) := \Gamma_1(\varphi)/\sim$. For $A = 0, \varphi: B \to 0$, we set $K_1(B) := K_1(\varphi)$.

It is easy to see that $K_1(\varphi)$ is a monoid with zero as neutral element. In the following we shall show that $K_1(\varphi)$ is an abelian group and give an alternative desciption of $K_1(\varphi)$, which will prove useful in calculations.

I.15. LEMMA. With the notation of I.14 we have that $d(E,\alpha)+d(E,\alpha^{-1}) = 0$. Thus $K_1(\varphi)$ is a group.

Proof. It sufficies to show that $\alpha \oplus \alpha^{-1} \simeq \mathrm{id}_{E\oplus E}\ \mathrm{rel}\,A$

Let $\sigma_t := \begin{pmatrix} 1-t^2 & -(2-t^2)t \\ t & 1-t \end{pmatrix}^{-1} \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix}$

then we have $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_E \oplus \mathrm{id}_E$ and

$$\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \alpha \oplus \alpha^{-1}.$$

Moreover, we see from I.4 and $\varphi_*(\alpha) = \mathrm{id}_{\varphi_* E}$ that

$$\varphi_*(\sigma_t) = \begin{pmatrix} 1-t & -(2-t^2)t \\ t & 1-t \end{pmatrix}^{-1} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

now with entries in $\mathbb{C}\cdot 1_A$ instead of $\mathbb{C}\cdot 1_B$. ■

I.16. LEMMA. Let B, A, φ, E be as in I.14. Let $\alpha, \alpha' \in \mathrm{Aut}\, E$ be such that $\varphi_*(\alpha) = \mathrm{id}_{\varphi_* E} = \varphi_*(\alpha')$ and $\alpha \simeq \alpha'\ \mathrm{rel}\, A$. Then $d(E,\alpha) = d(E,\alpha')$.

Proof. $d(E,\alpha') - d(E,\alpha) = d(E,\alpha') + d(E,\alpha^{-1}) = d(E\oplus E, \alpha'\oplus\alpha^{-1})$. The last term is zero, since $\alpha'\oplus\alpha^{-1} \simeq \alpha\oplus\alpha^{-1} \simeq \mathrm{id}_E \oplus \mathrm{id}_E\ \mathrm{rel}\, A$ by I.15. ■

I.17. LEMMA. Let $\varphi: B \to A$ be a unital C^*-morphism. Then

a) $K_1(\varphi)$ is abelian.

b) $d(E,\alpha) + d(E,\beta) = d(E,\alpha\beta) = d(E,\beta\alpha)$ for all $\alpha, \beta \in \mathrm{Aut}\, E$.

Proof. a) We want to show $d(E\oplus F, \alpha\oplus\beta) = d(F\oplus E, \beta\oplus\alpha)$. Let $h: E\oplus F \to F\oplus E$ be the isomorphism which simply interchanges summands. Then the following square commutes:

$$\begin{array}{ccc} E\oplus F & \longrightarrow & F\oplus E \\ {\scriptstyle \alpha\oplus\beta}\downarrow & & \downarrow{\scriptstyle \beta\oplus\alpha} \\ E\oplus F & \longrightarrow & F\oplus E \end{array}$$

Thus $(E\oplus F, \alpha\oplus\beta)$ is isomorphic to $(F\oplus E, \beta\oplus\alpha)$ which proves the first claim.

b) By adding elementary pairs, we get $d(E,\alpha\beta) = d(E\oplus E, \alpha\beta + id_E)$. Thus it suffices to show $\alpha\beta + id_E \simeq \alpha\oplus\beta \simeq \beta\oplus\alpha \simeq \beta\alpha + id_E$ rel A. By I.15 we have $(\alpha\oplus\beta)^{-1}(\alpha\beta + id_E) = \beta\oplus\beta^{-1} \simeq id_E \oplus id_E$ rel A . Multiplying the homotopy from the left with $\alpha\oplus\beta$ we obtain $\alpha\beta\oplus id_E \simeq \alpha\oplus\beta$ rel A . Similarly $(\alpha\beta\oplus id_E)(\beta\oplus\alpha)^{-1} \simeq id_E \oplus id_E$ rel A and $\alpha\beta\oplus id_E \simeq \beta\oplus\alpha$ rel A . Interchanging the roles of α and β now proves the claim. ■

I.18. LEMMA. Let B, A, φ, E, α be as in I.14. Then $d(E,\alpha) = 0$ in $K_1(\varphi)$ iff there is a $G \in Ob(P(B))$ such that $\alpha\oplus id_E \simeq id_{E\oplus G}$ rel A in $Aut(E\oplus G)$. We can choose G to be free.

Proof. If $d(E,\alpha) = 0$ then there exist elementary pairs (G,η) and (G',η') in $K_1(\varphi)$ and an isomorphism $h: E\oplus G \to G'$ which satisfies $h\circ(\alpha\oplus\eta) = \eta'\circ h$. By adding another elementary pair to (G,η) and (G',η') , if necessary, we can choose G to be free. Hence we have that $\alpha\oplus id_G \simeq \alpha\oplus\eta$ rel A and $\alpha\oplus\eta = h^{-1}\circ\eta'\circ h \simeq h^{-1}\circ id_{G'}\circ h$ rel A. Thus $\alpha\oplus id_E \simeq id_{E\oplus G}$ rel A . The converse is clear. ■

I.19. THEOREM. Let $\varphi: B \to A$ be a surjective unital C^*-morphism. Then $d(E,\alpha) = d(F,\beta)$ in $K_1(\varphi)$ iff there is a free $G \in Ob(P(B))$ such that $\alpha\oplus id_F\oplus id_G \simeq id_E\oplus\beta\oplus id_G$ rel A in $Aut(E\oplus F\oplus G)$.

Proof. If $d(E,\alpha) = d(F,\beta)$, then $d(E\oplus F, \alpha\oplus\beta^{-1}) = 0$, thus by I.18 there exists a free $G \in Ob(P(B))$ such that $\alpha\oplus\beta^{-1}\oplus id_G \simeq id_{E\oplus F\oplus G}$ rel A . Multiplying the homotopy by $id_E\oplus\beta\oplus id_G$ we get that $\alpha\oplus id_F\oplus id_G \simeq id_E\oplus\beta\oplus id_G$ rel A . The converse is clear. ■

Note that this description of $K_1(\varphi)$ does still depend on the notion of homotopy.

Before we turn to yet another way to view K_1, let us note that K_0 and K_1 are covariant functors from the category of unital C^*-algebras and unital C^*-morphisms into the category of abelian groups. The proof is routine, so we only describe how K_0 and K_1 act on a C^*-morphism $\varphi\colon B \to A$. Since the notation $K_i(\varphi)$, for $i = 0,1$, is already in use, we denote the image of φ under K_i by φ_i^*. Then φ_i^* is the group homomorphism from $K_i(B)$ to $K_i(A)$ defined by $\varphi_0^*(\overline{[E]}-\overline{[F]}) = \overline{[\varphi_* E]}-\overline{[\varphi_* F]}$ and $\varphi_1^*(d(E,\alpha)) = d(\varphi_* E,\varphi_*\alpha)$ respectively.

We now give another description of $K_1(\varphi)$, which is extremely useful in relating homotopy and K-theory as well as in many calculations. First, we describe $Gl(A)$ for a unital C^*-algebra A. Let $Gl_n(A) \subset M_n(A)$ be the set of $n \times n$ matrices with entries in A. It is well known that $Gl_n(A)$ is a topological group, which is open in $M_n(A)$. Let $Gl_n^o(A)$ be the connected component of 1 in $Gl_n(A)$. Denote the quotient group $Gl_n(A)/Gl_n^o(A)$ by G_n. For each $n \in \mathbb{N}$ we obtain a map from $Gl_n(A)$ to $Gl_{n+1}(A)$ sending $a \in Gl_n(A)$ to $a \oplus 1_A = \begin{pmatrix} a & 0 \\ 0 & 1_A \end{pmatrix}$. Note that this map sends $Gl_n^o(A)$ into $Gl_{n+1}^o(A)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
G_1 & \to & G_2 & \to \dots & G_n & \to \dots \to & \varinjlim G_n & =: & G_\infty \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
Gl_1(A) & \to & Gl_2(A) & \to \dots & Gl_n(A) & \to \dots \to & \varinjlim Gl_n(A) & =: & Gl(A) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
Gl_1^o(A) & \to & Gl_2^o(A) & \to \dots & Gl_n^o(A) & \to \dots \to & \varinjlim Gl_n^o(A) & =: & Gl_\infty^o(A)
\end{array}$$

Here, $\varinjlim$ denotes the direct limit. The diagram clearly commutes, and since the direct limit commutes with exact sequences, we get the following short exact sequence of groups $0 \to Gl^{o}_{\infty}(A) \to Gl(A) \to G_{\infty} \to 0$. Give $Gl(A)$, $Gl^{o}_{\infty}(A)$ and G_{∞} the inductive limit topology, i.e. a set $C \subset Gl(A)$ is closed iff $C \cap Gl_n(A)$ is closed in $Gl_n(A)$ for all $n \in \mathbb{N}$.

I.20. LEMMA. Let X_n be a directed system of Hausdorff spaces such that $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$. Let $X = \varinjlim X_n$ be the inductive limit with the inductive limit topology. Then any compact set $K \subset X$ is contained in an X_n for some $n \in \mathbb{N}$.

Proof. Suppose K is not contained in any X_n . We may assume $X_{n+1} \neq X_n$. Then there exists a sequence $\{k_n\}$ of points $k_n \in (X_{n+1} \setminus X_n) \cap K$. Since K is compact, $\{k_n\}$ has a cluster point $k \in K \subseteq X$. Now $k \in X_l$ for some $l \in \mathbb{N}$. The set $\{k, k_1, k_2, \ldots\}$ is closed since it intersects each X_n in a finite, hence closed set; thus it is compact. On the other hand, $\{k, k_1, k_2 \ldots\}$ is the inductive limit of the discrete subspaces $\{k, k\ , \ldots, k_n\}$, hence it is discrete as subspace of X . This is a contradiction. ■

I.21. PROPOSITION. For a unital C^*-algebra A the group $Gl^{o}_{\infty}(A)$ is the connected component of 1 in $Gl(A)$.

Proof. By the preceding lemma, we see that any path in $Gl(A)$ is actually a path in $Gl_n(A)$ for some $n \in \mathbb{N}$. Thus any $a \in Gl^{o}(A)$ is in $Gl^{o}_n(A)$ for some n . The reverse inclusion is clear. ■

For a topological group G we denote its connected component by G^{o} and the quotient G/G^{o} by $\pi_o(G)$.

I.22. THEOREM. Let B and A be unital C^*-algebras and $\varphi: B \to A$ a unital C^*-morphism. Then φ induces a natural group homomorphism $\varphi^\#: Gl(B) \to Gl(A)$ and we have that $K_1(\varphi) \cong \pi_0(\ker \varphi^\#)$.

Proof. The map $\varphi^\#$ is the map induced by the $\varphi^\#: Gl_n(B) \to Gl_n(A)$ (cf. I.4 for the definition) on the direct limits. Note first that for $A = 0$, $Gl(A) = 0$ and $\varphi^\#$ is the zero map. Thus in particular we prove that $K_1(B) \cong \pi_0(Gl(B))$. Now we define the map $\tau: K_1(\varphi) \to \pi_0(\ker \varphi^\#)$ which will be the desired isomorphism. Let $p_i: B^n \to E$ be a projection onto E and $i: E \to B^n$ a corresponding coprojection. Let E' be the complement of E in B^n w.r.t. (p_i, i). For any $\alpha \in Aut(E)$ with $\varphi_*\alpha = id_{\varphi_* E}$ we define $\alpha_i := \alpha \oplus id_{E'}$. Then $\alpha_i \in Gl_n(B) \subset Gl(B)$. In fact $\alpha_i \in \ker \varphi^\#$, since $\varphi^\#(\alpha_i) = \varphi_*\alpha_i = \varphi_*\alpha \oplus \varphi_* id_{E'} = id_{A^n}$ (cf. I.4). Denote the class of α_i in $\pi_0(\ker \varphi^\#)$ by $[\alpha_i]$. Now, for $d(E,\alpha) \in K_1(\varphi)$ we set $\tau(d(E,\alpha)) = [\alpha_i]$. To show that τ is well defined, we have to show that $[\alpha_i]$ does not depend on the embedding. Suppose $p_j: B^m \to E$, $j: E \to B^m$ is another pair of projection and coprojection. Let E'' be the complement of E in B^m w.r.t. (p_i, j). Then $d(B^n, \alpha_i) = d(E,\alpha) = d(B^m, \alpha_j)$. Thus, by I.19 there exists a $G \in Ob(P(B))$ such that $\alpha_i \oplus id_{B^m} \oplus id_G \cong id_{B^n} \oplus \alpha_j \oplus id_G \ \mathrm{rel}\, A$. We can assume that G is free, say $G = B^k$. This is the same as saying $\alpha_i \oplus 1_{B^m} \oplus 1_{B^k}$ is pathconnected to $1_{B^n} \oplus \alpha_j \oplus 1_{B^k}$ in $\ker \varphi^\#$. Since $1_{B^n} \oplus \alpha_j$ is pathconnected to $\alpha_j \oplus 1_{B^n}$ in $\ker \varphi^\#$ we have $[\alpha_j] = [\alpha_i]$. The same kind of argument shows in general that $\tau(d(E,\alpha)) = \tau(d(F,\beta))$ if $d(E,\alpha) = d(F,\alpha)$. Thus τ is well defined. For any $\alpha \in \ker \varphi^\#$ there is a number n_α such that $\alpha \in Gl_{n_\alpha}(B)$. Define $\tau': \pi_0(\ker \varphi^\#) \to K_1(\varphi)$ by $\tau([\alpha]) = d(B^{n_\alpha}, \alpha)$. Using the same methods as above it is now routine to check τ' is well defined, a group homomorphism and the inverse of τ. This concludes the proof. ■

We now state a lemma which actually has been a key ingredient in the proof of theorem I.13 and which will be used again and again in the sequel.

I.23. LEMMA (cf. [K]: II.2.21). Let A and B be unital Banach algebras and $\varphi: B \to A$ a continuous surjective ring homomorphism. If $\gamma: I \to A$ is a path such that $\gamma(t) \in Gl_1(A)$ for all $t \in I$ and $\gamma(0) = \varphi(b)$ for some $b \in Gl_1(B)$. Then there exists a $b' \in Gl_1(B)$ such that $\varphi(b') = \gamma(1)$ and b' is connected to b in $Gl_1(B)$.

Proof. Let $V := \{\varphi \in A: \|y-1_A\| < 1\}$, then we can define a logarithm on V. Find a partition $0 = t_0 \dots t_n = 1$ of I such that $\gamma(t_i)^{-1}\gamma(t_{i+1}) \in V$ for $i = 0 \dots n-1$. Define $a_i := \log(\gamma(t_i)^{-1}\gamma(t_{i+1}))$. We get $\gamma(1) = \gamma(0)\cdot\exp(a_1)\cdot\ldots\cdot\exp(a_{n-1})$. Choose $b_i \in B$ such that $\varphi(b_i) = a_i$ and define b' by $b' := b\cdot\exp(b_1)\cdot\ldots\cdot\exp(b_{n-1})$. But $\varphi(b') = \varphi(b)\cdot\varphi(\exp b_1)\cdot\ldots\cdot\varphi(\exp b_{n-1})$ and since φ is continuous, $\varphi(\exp b_i) = \exp(\varphi(b_i))$. Thus $\varphi(b') = \gamma(1)$. Moreover b' is connected to b in $Gl_1(B)$ via the path $t \mapsto b\cdot\exp(tb_1)\cdot\ldots\cdot\exp(tb_{n-1})$ with $t \in [0,1]$. ■

I.24. LEMMA. Let B and A be unital C^*-algebras and φ a surjective C^*-morphism. For $E \in Ob(P(B))$ and $\alpha \in Aut(E)$ such that $\varphi_*(\alpha) \simeq_{\sigma t} id_{\varphi_* E}$, there exists a $\beta \in Aut\ E$ with $\beta \simeq \alpha$ in $Aut(E)$ and $\varphi_*(\beta) = id_{\varphi_* E}$.

Proof. By I.3 the set $End\ E$ can be given a Banach space structure and $\varphi_*: End\ E \to End\ \varphi_* E$ is continuous. By I.4, the map φ_* is surjective. Clearly, φ_* is a ring homomorphism. We apply I.23 to obtain a $\beta \in \mathrm{Aut}(E)$ with $\varphi_*(\beta) = 1_{\mathrm{End}(\varphi_* E)} = \mathrm{id}_{\varphi_* E}$ and β connected to α in $\mathrm{Aut}(E)$, i.e., $\beta \simeq \alpha$ in $\mathrm{Aut}(E)$. ∎

Next, we prove the analogue of I.10 for K_1. Note the important role lemma I.24 plays in the proof. First, to simplify language, we introduce the notion of a retract.

I.25. DEFINITION. A C^*-algebra A is called a retract of the C^*-algebra B if there exists a C^*-surjection $\varphi: B \to A$ and a C^*-morphism $\psi: A \to B$ such that $\varphi \circ \psi = \mathrm{id}_A$. The map ψ is required to be unital if B, A and φ are.

I.26. PROPOSITION. Let B and A be unital C^*-algebras and $\varphi: B \to A$ a unital C^*-surjection. Then we get an exact sequence $K_1(\varphi) \xrightarrow[\pi_1^*]{} K_1(B) \xrightarrow[\varphi_1^*]{} K_1(A)$. Moreover, if A is a retract of B, we get a split exact sequence

$$0 \longrightarrow K_1(\varphi) \xrightarrow[\pi_1^*]{} K_1(B) \underset{\varphi_1^*}{\overset{\psi_1^*}{\rightleftarrows}} K_1(A) \longrightarrow 0 .$$

Proof. The maps φ_1^* and ψ_1^* are the images of φ and ψ under the functor K_1. The map $\pi_1^*: K_1(\varphi) \to K_1(B)$ is defined by $\pi_1^*(d(E,\alpha)) = d(E,\alpha)$. Note that the right hand $d(E,\alpha)$ denotes the class of (E,α) in $K_1(B)$. It is routine to verify that π^* is a well defined group homomorphism. From the definition of $K_1(\varphi)$, it follows that $\varphi_1^* \circ \pi_1^* = 0$. Now, let $d(E,\alpha) \in \ker \varphi_1^*$. Then $d(\varphi_* E, \varphi_* \alpha) = 0$. By I.19, there exists a free $G \in \mathrm{Ob}(P(B))$, say $G = B^n$, such that $\varphi_*\alpha \oplus \mathrm{id}_G \simeq \mathrm{id}_{\varphi_* E \oplus G}$ in $\mathrm{Aut}(\varphi_* E \oplus G) = \mathrm{Aut}(\varphi_*(E \oplus B^n))$. Lemma I.24 now shows the existence of $\beta \in \mathrm{Aut}(E \oplus B^n)$ such that $\alpha \oplus \mathrm{id}_{B^n} \simeq \beta$ in $\mathrm{Aut}(E \oplus B^n)$ and $\varphi_* \beta = \mathrm{id}_{\varphi_*(E \oplus B^n)}$. Thus $d(E,\alpha) = d(E \oplus B^n, \alpha \oplus \mathrm{id}_{B^n}) = d(E \oplus B^n, \beta)$. But $(E \oplus B^n, \beta)$ defines an element in $K_1(\varphi)$. Thus $d(E,\alpha)$ is in the image of π_1^*.

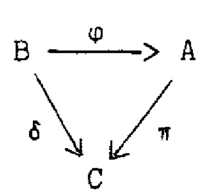
If A is a retract of B, the retraction ψ induces a splitting for φ_1^*. Thus φ_1^* is surjective and it only remains to show that π_1^* is an injection. To this end, view $K_1(\varphi)$ as $\pi_0(\ker \varphi^\#)$ and $K_1(B)$ as $\pi_0(\mathrm{Gl}\ B)$. Then π_1^* maps the class of $a \in \ker \varphi^\#$ to its class in $\mathrm{Gl}(B)$. Suppose now that $\pi_1^*([a]) = 0$, i.e. that there is a path $\gamma\colon I \to \mathrm{Gl}(B)$ connecting a and $1_{\mathrm{Gl}(B)}$. Consider $\gamma'(t) := \gamma(t)\cdot\psi^\#(\varphi^\#(\gamma(t)^{-1}))$, then $\varphi^\#(\gamma'(t)) = \varphi^\#(\gamma(t))\cdot\varphi^\#\psi^\#\varphi^\#(\gamma(t)^{-1}) = \varphi^\#(\gamma(t))\cdot\varphi^\#(\gamma(t)^{-1}) = 1_{\mathrm{Gl}(A)}$. Thus $\gamma'(t)$ is a path in $\ker \varphi^\#$. We have $\gamma'(1) = 1_{\mathrm{Gl}(B)}$ and $\gamma'(0) = \gamma(0)\cdot\psi^\#\varphi^\#\gamma(0)^{-1} = a\psi^\#\varphi^\#(a^{-1}) = a\psi^\# 1 = a$. Thus a is actually connected to $1_{\mathrm{Gl}(B)}$ inside $\ker \varphi^\#$, hence $a \in (\ker \varphi^\#)^\circ$ and $[a] = 0$. ■

CHAPTER II: EXCISION THEOREMS

The purpose of this chapter is to prove two theorems, called the excision theorems, which will allow us to define a K-theory for non unital C^*-algebras. Alain Connes proved a result which is analogous to our excision theorem for K_0 , using the notion of classes of stably homotopic quasiisomorphisms. This notion is essentially the same as our $K_0(\varphi)$. In his proof he uses, however, analytic as well as algebraic techniques, whereas the present proof shows that the excision theorem for K_0 is a purely algebraic theorem once I.13 is achieved.

Let B and A be unital rings and $\varphi\colon B \to A$ a unital ring homomorphism. Then A becomes a right B-B-module with respect to $a \cdot b := a\varphi(B)$. For $E \in \mathrm{Ob}(P(B))$ we can form the tensor product $A \otimes_B E$. Then $A \otimes_B E$ is a finitely generated projective left A-module, i.e. $A \otimes_B E \in \mathrm{Ob}(P(A))$. For any $f \in P(B)(E,F)$ we have $\mathrm{id}_A \otimes f \in P(B)(A \otimes_B E, A \otimes_B F)$. The assignment $\varphi_*\colon P(B) \to P(A)$ defined by $\varphi_*(E) = A \otimes_B E$ and $\varphi_*(f) = \mathrm{id}_A \otimes f$ is precisely the functor we used already in I.4. There is a canonical map $\varphi_E\colon E \to \varphi_* E$ given by $\varphi_E(m) = 1_A \otimes m$. This φ_E is a generalized module map, i.e. $\varphi_E(bm) = \varphi(b)\varphi_E(m)$ (cf. [M] §2).

II.1. PROPOSITION. Consider the commutative triangle of unital rings and unital ring homomorphisms



Then for any $E \in \mathrm{Ob}(P(B))$ the modules $\pi_*(\varphi_* E)$ and $\delta_* E$ are canonically isomorphic and this isomorphism makes the following square commute.

$$\begin{array}{ccc} E & \xrightarrow{\varphi_E} & \varphi_* E = A \otimes E \\ {\scriptstyle \delta_E}\downarrow & & \downarrow{\scriptstyle \pi_{\varphi_* E}} \\ C \otimes E = \delta_* E & \xrightarrow{\cong} & \pi_*(\varphi_* E) = C \otimes (A \otimes E) \end{array}$$

Proof. First consider the case where E is free, say B^n. Then $\pi_*(\varphi_* E) = C \otimes_A (A \otimes_B E) \cong C \otimes_B E = \delta_* E$ via the map that sends $c \otimes_A (a \otimes_B m)$ to $c\pi(a) \otimes_B m$. For E, E' such that $E \oplus E' = B^n$ the distributive law for tensor products and direct sums shows that the map $C \otimes_A (A \otimes_B E) \to C \otimes_B E$ given by restricting and corestricting the canonical isomorphism between $\pi_*(\varphi_* B^n)$ and $\delta_* B^n$ is a well defined module isomorphism. It is easy to check that the square commutes. ∎

Next we introduce the map which the excision theorem will show to be an isomorphism.

II.2. PROPOSITION. Consider the commutative square of unital C^*-algebras

$$\begin{array}{ccc} D & \xrightarrow{\rho} & C \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ B & \xrightarrow[\varphi]{} & A \end{array}$$

Moreover assume ρ and φ to be surjective. Then there exists a natural group homomorphism $j^*: K_0(\rho) \to K_0(\varphi)$ given by $j^*(d(E,F,\alpha)) = d(j_{1*}E, j_{1*}F, j_{2*}\alpha) = d(B \otimes E, B \otimes F, id_A \otimes \alpha)$.

Proof. By II.1 we know that $A \otimes_B (B \otimes_D E)$ is canonically isomorphic to $A \otimes_C (C \otimes_D E)$ for all $E \in Ob(P(D))$. Under this identification $id_A \otimes_C \alpha$ is an isomorphism from $A \otimes_B (B \otimes_D E)$ to $A \otimes_B (B \otimes_D F)$ so $d(B \otimes_D E, B \otimes_D F, id_A \otimes \alpha)$ defines an element in $K_0(\varphi)$. It is clear that j^* is additive, so in order to show that it is well defined it suffices to show that $d(E,F,\alpha) = 0$ implies that $d(B \otimes_D E, B \otimes_D F, id_A \otimes \alpha) = 0$. Suppose that $d(E,F,\alpha) = 0$ in $K_0(\rho)$ then by I.13 there exists $G \in Ob(P(B))$ and an isomorphism $h: E \oplus G \to F \oplus G$ such that the following square commutes

$$\begin{array}{ccc} C \otimes_D (E \oplus G) & \xrightarrow{id_C \otimes h} & C \otimes_D (F \oplus G) \\ \downarrow{\scriptstyle \alpha \oplus id} & & \downarrow{\scriptstyle id} \\ C \otimes_D (E \oplus G) & \xrightarrow[id_C \otimes h]{} & C \otimes_D (F \oplus G) \end{array}$$

We apply the functor j_2^* to this square to obtain, with the obvious identifications, a commutative square

$$\begin{array}{ccc} A \otimes_B (B \otimes_D (E \oplus G)) & \xrightarrow{id_A \otimes id_B \otimes h} & A \otimes_B (B \otimes_B (F \oplus G)) \\ \downarrow{\scriptstyle (id_A \otimes \alpha) \oplus id} & & \downarrow{\scriptstyle id} \\ A \otimes_B (B \otimes_D (E \oplus G)) & \xrightarrow[id_A \otimes id_B \otimes h]{} & A \otimes_B (B \otimes_D (F \oplus G)) \end{array}$$

Since $id_B \otimes_D h: B \otimes_D (E \oplus G) \to B \otimes_D (F \oplus G)$ is an isomorphism this proves that $d(B \otimes_D E, B \otimes_D F, id_A \otimes_C \alpha) = 0$. ∎

Now we describe a method of constructing projective modules over a pullback, which will be essential in what follows. Let

$$\begin{array}{ccc} D & \xrightarrow{\rho} & C \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ B & \xrightarrow[\varphi]{} & A \end{array}$$

be a pullback square of unital rings. Further let $\hat{E} \in Ob(P(B))$ and $M \in Ob(P(C))$ be such that there exists an isomorphism $\beta: A \otimes_C M \to A \otimes_B \hat{E}$. We define a module E over D as the pullback of the following diagram

$$\begin{array}{ccc} E & \overset{p_C}{\dashrightarrow} & M \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle \beta\circ j_{2M}} \\ \hat{E} & \underset{\varphi_{\hat{E}}}{\dashrightarrow} & A\otimes_B \hat{E} \end{array}$$

Then $E = \{(\hat{e},m) \in \hat{E} \oplus M : \beta\circ j_{2M}(m) = \varphi_{\hat{E}}(\hat{e})\}$. The module structure is given by $d\cdot(\hat{e},m) = (j_1(d)\cdot\hat{e},\rho(d)m)$.

II.3. THEOREM (cf. [M] §2). Assume that in addition to these circumstances φ is surjective. Then $E \in Ob(P(D))$. Moreover $B \otimes_D E$ is naturally isomorphic to $\hat{E}$ and $C \otimes_D E$ is naturally isomorphic to M.

Proof. We only give the natural maps which the theorem proves to be isomorphisms. After identifying $B \otimes_B \hat{E}$ with $\hat{E}$ and $C \otimes_C M$ with M, we note that they are given by $id_B \otimes p_B : B \otimes_D E \to \hat{E}$ and $id_C \otimes p_C : C \otimes_D E \to M$. ■

II.4. THEOREM (Excision for K_0). Given is a pullback square

$$\begin{array}{ccc} D & \xrightarrow{\rho} & C \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ B & \xrightarrow[\varphi]{} & A \end{array}$$

of unital C^*-algebras with a surjective φ. Then the map $j^*: K_0(\rho) \to K_0(\varphi)$, defined in II.1 is an isomorphism.

Proof. We split the proof in two lemmas.

II.5. LEMMA. The map $j^*: K_0(\rho) \to K_0(\varphi)$ is surjective.

Proof. Let $d(\hat{E},\hat{F},\hat{\alpha}) \in K_0(\varphi)$, i.e. $\hat{E},\hat{F} \in \mathrm{Ob}(P(B))$ and $\hat{\alpha}: A \otimes_B \hat{E} \to A \otimes_B \hat{F}$ is an isomorphism. By adding an elementary triple if necessary, we can assume w.l.o.g. that $\hat{F}$ is free, say, $\hat{F} = B^n$. Then $A \otimes_B \hat{F} = A^n$. Define a D-module E via the pullback

$$\begin{array}{ccc} E & \overset{p_C}{\dashrightarrow} & C^n \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle j_2 C^n} \\ \hat{E} & \xrightarrow[\hat{\alpha}\circ\varphi_{\hat{E}}]{} & A^n \end{array}$$

Now Theorem II.3. applies and thus $E \in \mathrm{Ob}(P(D))$. Moreover, $\alpha := \mathrm{id}_C \otimes p_C: C \otimes_D E \to C^n$ is an isomorphism. Thus (E,D^n,α) defines an element of $K_0(\rho)$. We want to show that $(B \otimes_D E, B \otimes_D D^n, \mathrm{id}_A \otimes \alpha) \cong (\hat{E},B^n,\hat{\alpha})$. For $h := \mathrm{id}_B \otimes p_B: j_{1*}E \to \hat{E}$, the natural isomorphism from II.3, we consider the diagram:

$$\begin{array}{ccc}
A \otimes_C (C\otimes_D E) = A \otimes_B (B\otimes_D E) & \xrightarrow[\cong]{id_A \otimes h} & A \otimes_B \hat{E} \\
\Big\downarrow {\scriptstyle id_A \otimes \alpha} & & \Big\downarrow {\scriptstyle \hat{\alpha}} \\
A \otimes_C (C\otimes_D D^n) = A \otimes_B (B\otimes_D D^n) & \xrightarrow[id_{A^n}]{} & A \otimes_B B^n
\end{array}$$

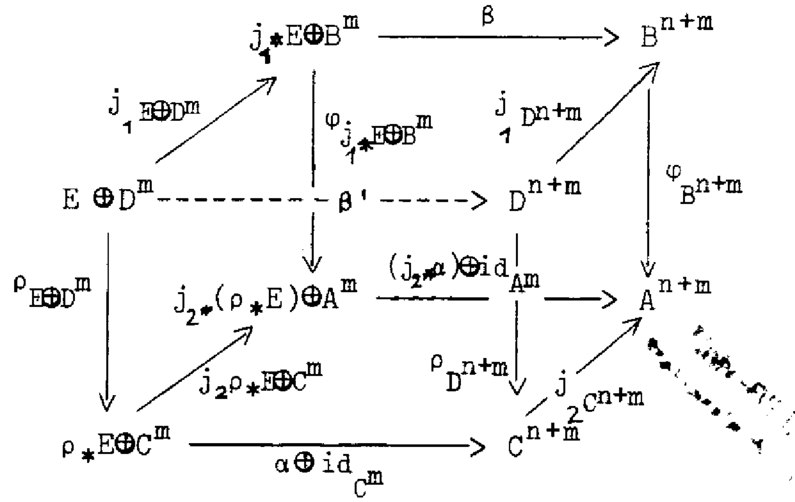
We check the commutativity of the square on elementary tensors in $A \otimes_B (B\otimes_D E)$, which we can w.l.o.g. assume to be of the form $a \otimes_B (1_B \otimes_D e)$. Then $\hat{\alpha}(\varphi_* h(a \otimes_B (1_B \otimes_D e))) = \hat{\alpha}(a \otimes_B p_B(e)) = a\cdot\hat{\alpha}(\varphi_A \circ p_B(e)) = a\cdot(j_{2C^n} \circ p_C(e)) = a \otimes_C p_C(e) = j_{2*}\alpha(a \otimes_C (1_C \otimes_D^E e))$. The commutativity of the square implies that $(B \otimes_D E, B \otimes_D D^n, id_A \otimes \alpha) \cong (\hat{E}, B^n, \hat{\alpha})$. This concludes the proof. ■

II.6. LEMMA. The map $j^*\colon K_0(\varphi) \to K_0(\varphi)$ is injective.

Proof. Suppose $j^*(d(E,F,\alpha)) = d(B \otimes_D E, B \otimes_D F, id_A \otimes \alpha) = 0$. As before, we can assume F to be free, say $F = D^n$. By I.13 we can find a $T \in Ob(\mathcal{P}(B))$, which also can be assumed free, say $T = B^m$, and an isomorphism $\beta\colon B \otimes_D E \oplus B^m \to B^{n+m}$ such that the following square commutes:

$$\begin{array}{ccc}
B \otimes_D E \oplus B^m & \xrightarrow{\beta} & B^n \oplus B^m \\
\Big\downarrow {\scriptstyle \varphi_{B\otimes E} \oplus \varphi_{B^m}} & & \\
A \otimes_B (B\otimes_D E) \oplus A^m & & \Big\downarrow {\scriptstyle \varphi_{B^{n+m}}} \\
\parallel & & \\
A \otimes_C (C\otimes_D E) \oplus A^m & \xrightarrow[(id_A \otimes \alpha)\oplus id]{} & A^n \oplus A^m
\end{array}$$

Consider the following commutative diagram:



Note that both, the right and the left square are pull-back squares. This implies the existence of the map β', induced by $\beta\circ j_{1\,E\oplus D^n}$ and $(\alpha\oplus id_{C^m})\circ\rho_{E\oplus D^m}$ and of β'^{-1}, induced by $\beta\circ j_{1\,D^{n+m}}$ and $(\alpha^{-1}\oplus id_{C^m})\circ\rho_{D^{n+m}}$. Now with β' being an isomorphism, the commutativity of the front square proves, again by I.13, that $d(E,D^n,\alpha)=0$. Thus j^* is injective.

This concludes the proof of the lemma and thereby the proof of Theorem II.4. ■

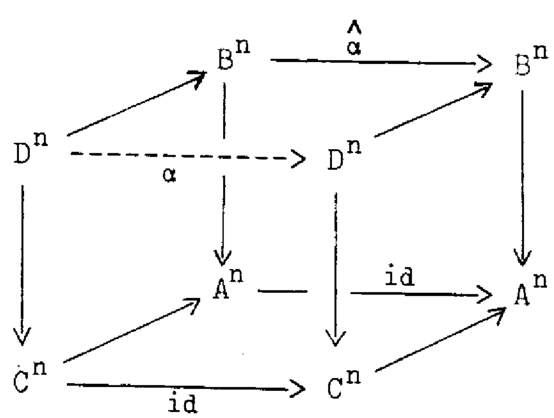
Now, we turn to K_1. We are going to prove a completely analogous result as for K_0. We shall, however, have to use an argument which is not purely algebraic. This was to be expected since we lack an analogue of I.13 for K_1.

II.7. THEOREM (Excision for K_1). Consider a pullback square of unital C^*-algebras.

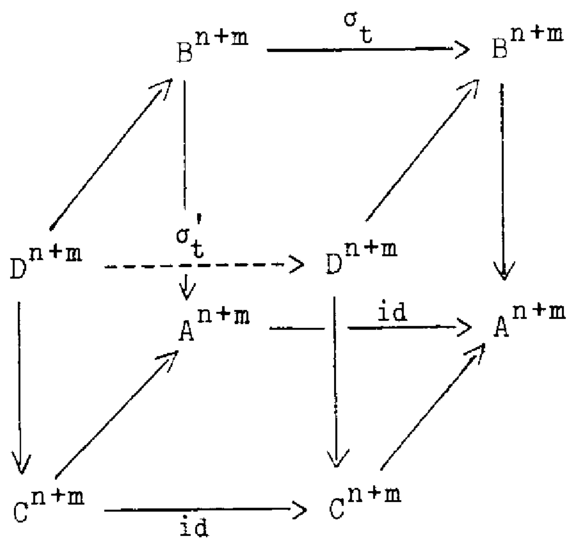
$$\begin{array}{ccc} D & \xrightarrow{\rho} & C \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ B & \xrightarrow[\varphi]{} & A \end{array}$$

with surjective φ. Then there is a natural group isomorphism $j^*: K_1(\varphi) \to K_1(\varphi)$, given by $j^*(d(E,\alpha)) = (j_{1*}E, j_{1*}\alpha) = (B \otimes_D E, id_B \otimes_D \alpha)$.

Proof. Note first that if $\sigma_t: E \to F$ is a homotopy in End E, then $id_B \otimes_D \sigma_t: B \otimes_D E \to B \otimes_D E$ is a homotopy in End $B \otimes_D E$. First we show that j^* is surjective. Let $d(\hat{E},\hat{\alpha}) \in K_1(\varphi)$. By adding an elementary pair if necessary, we can assume w.l.o.g. that $\hat{E}$ is free, say, $\hat{E} = B^n$. The fact that D is a pullback allows us to establish a map $\alpha: D^n \to D^n$ via the commutive diagram



The map α is induced by $\hat{\alpha} \circ j_{1 D^n}$ and ρ_{D^n}. As in II.6 we see that α is invertible. The commutativity of the front square now implies that (D^n, α) defines an element of $K_1(\rho)$. Moreover, the commutativity of the top square implies that $j_{1*}\alpha = \hat{\alpha}$ which shows that $d(j_{1*}D^n, j_{1*}\alpha) = d(\hat{E},\hat{\alpha})$. Thus j^* is surjective. Now suppose that $j^*(d(E,\alpha)) = 0$ for some $d(E,\alpha) \in K_1(\rho)$. As before we can assume that E is free, say $E = B^n$. By I.19 we can find a $G \in Ob(P(B))$, w.l.o.g. G free, say $G = B^m$, and a homotopy σ_t in $Aut(B^{n+m})$ such that $j_{1*}\alpha \oplus id_{B^m} \simeq_{\sigma_t} id_{B^{n+m}}$ rel A. From the following commutative diagram we derive as before the existence of a familiy of automorphism $\sigma_t': D^{n+m} \to D^{n+m}$:



The family of maps σ'_t is a homotopy as follows directly from the fact that the family σ_t is one. Thus $\alpha \oplus id_{D^m} \simeq_{\sigma'_t} id_{D^{n+m}}$ rel A and therefore $d(D^n,\alpha) = 0$ by I.19. This concludes the proof. ■

CHAPTER III: K-THEORY FOR NON UNITAL C*-ALGEBRAS

In this chapter we define $K_0(L)$ and $K_1(L)$ for non-unital C*-algebras. We shall also examine the functorial properties of K_0 and K_1. Moreover, for a short exact sequence of C*-algebras $0 \to L \to B \to A \to 0$, we define a connecting homomorphism $K_1(A) \to K_0(L)$ which will allow us to put the sequences from I.10 and I.26 together.

II.1. LEMMA (cf. [K] II.3.22). Let B and A be unital C*-algebras and $\varphi: B \to A$ a unital C*-morphism. Then there is a natural group homomorphism $\partial_\varphi: K_1(A) \to K_0(\varphi)$ which makes the following sequence exact:

$$K_1(B) \xrightarrow{\varphi_1^*} K_1(A) \xrightarrow{\partial_\varphi} K_0(\varphi) \xrightarrow{\pi_0^*} K_0(B) \xrightarrow{\varphi_0^*} K_0(A)$$

Proof. The maps φ_1^*, π_0^* and φ_0^* have been defined in I.10 and I.26. We give the definition of ∂_φ : Let $d(E',\alpha')$ be in $K_1(A)$. Then there exists an $F' \in Ob(P(A))$ such that $E' \oplus F'$ is free over A, say $E' \oplus F' = A^n$. Then $\partial_\varphi(d(E',\alpha')) := d(B^n, B^n, \alpha' \oplus id_{F'})$. This makes sense because $\varphi_* B^n = A^n = E' \oplus F'$. The proof that ∂_φ is well defined and satisfies the desired properties can be found in [K] II.3.22.

III.2. LEMMA. Consider a commutative square of unital C*-algebras.

$$\begin{array}{ccc} D & \overset{\rho}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ B & \underset{\varphi}{\twoheadrightarrow} & A \end{array}$$

Then for $j^{*(1)}: K_1(\rho) \to K_1(\varphi)$ and $j^{*(0)}: K_0(\rho) \to K_0(\varphi)$, the maps defined in II.7 and II.2, the following diagram is commutative:

$$\begin{array}{ccccccccccc}
K_1(\rho) & \xrightarrow{{}_{\rho}\pi_1^*} & K_1(D) & \xrightarrow{\rho_1^*} & K_1(C) & \xrightarrow{\partial_\rho} & K_0(\rho) & \xrightarrow{{}_{\rho}\pi_0^*} & K_0(D) & \xrightarrow{\rho_0^*} & K_0(C) \\
\downarrow j^{*(1)} & & \downarrow (j_1)_1^* & & \downarrow (j_2)_1^* & & \downarrow j^{*(0)} & & \downarrow (j_1)_0^* & & \downarrow (j_2)_0^* \\
K_1(\varphi) & \xrightarrow[{}_{\varphi}\pi_1^*]{} & K_1(B) & \xrightarrow[\varphi_1^*]{} & K_1(A) & \xrightarrow[\partial_\varphi]{} & K_0(\varphi) & \xrightarrow[{}_{\varphi}\pi_0^*]{} & K_0(B) & \xrightarrow[\varphi_0^*]{} & K_0(A)
\end{array}$$

Proof. All the maps have been defined before. Subscripts ρ and φ only indicate for which morphism we construct the natural maps. The commutativity of the second and the fifth square follows from the functoriality of K_1 and K_0. Let $d(E,\alpha) \in K_1(\rho)$. Then $(j_1)_1^* \circ {}_{\rho}\pi_1^*(d(E,\alpha)) = (j_1)_1^*(d(E,\alpha)) = d(j_{1*}E, j_{1*}\alpha) = {}_{\varphi}\pi_1^* \circ j^{*(1)}(d(E,\alpha))$, where the middle terms mean equivalence classes in $K_1(D)$ and $K_1(B)$, respectively. Let $d(E',\alpha') \in K_1(C)$ and $E' \oplus F' = \mathbb{C}^n$. Then $j_{2*}E' \oplus j_{2*}F' = A^n$. Thus $j^{*(0)} \circ \partial_\rho(d(E',\alpha')) = j^{*(0)}(d(D^n, D^n, \alpha' \oplus id_{F'})) = d(B^n, B^n, j_{2*\alpha'} \oplus id_{j_{2*}F'}) = \partial_\varphi d(j_{2*}E', j_{2*}\alpha') = \partial_\varphi \circ (j_2)_1^*(d(E',\alpha'))$. Finally, let $d(E,F,\alpha) \in K_0(\rho)$. Then $(j_1)_0^* \circ {}_{\rho}\pi_0^*(d(E,F,\alpha)) = (j_1)_0^*(\overline{[E]} - \overline{[F]}) = \overline{[j_{1*}E]} - \overline{[j_{1*}F]} = {}_{\varphi}\pi_0^*(d(j_{1*}E, j_{1*}F, j_{2*}x)) = {}_{\varphi}\pi_0^* \circ j^{*(0)}(d(E,F,\alpha))$. ■

III.3. LEMMA. Let $0 \longrightarrow L \xrightarrow{\hat{j}} B \xrightarrow{\varphi} A \longrightarrow 0$ be a short exact sequence of C^*-algebras such that B, A and φ are unital. Let $\tilde{L}$ be the C^*-algebra we obtain from L by adjoining an identity. Let $j_1 : \tilde{L} \to B$ be the unital C^*-morphism induced by $\hat{j}$. Then we get a commutative diagram of C^*-algebras. Moreover, the right square is a pullback square.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \tilde{L} & \xrightarrow{\rho} & \mathbb{C} & \longrightarrow & 0 \\
& & \Vert & & \downarrow j_1 & & \downarrow j_2 & & \\
0 & \longrightarrow & L & \xrightarrow{\hat{j}} & B & \xrightarrow{\varphi} & A & \longrightarrow & 0
\end{array}$$

Proof. The map ρ is the canonical surjection $\tilde{L} \to \tilde{L}/L = \mathbb{C}$. The map j_2 is the canonical injection which sends $\lambda \in \mathbb{C}$ to $\lambda \cdot 1_A$. With these maps the right square is clearly commutative. Let $\hat{L} := \{(b,\lambda) \in B\oplus\mathbb{C}: \varphi(b) = \lambda \cdot 1_A\}$ be the pullback of φ and j_2, then it is easy to check that the map $h: \tilde{L} \to \hat{L}$, defined by $h(l+\lambda 1) := (\hat{j}(l)+\lambda 1, \lambda)$ is an isomorphism of C^*-algebras. ■

Now we are ready to define K_0 and K_1 for arbitrary C^*-algebras. This definition makes also sense for unital C^*-algebras and we shall show that the two definitions coincide for those. Note first that for any C^*-algebra L there is a short exact sequence $0 \to L \to \tilde{L} \overset{\rho}{\to} \mathbb{C} \to 0$. By the preceding lemma and the excision theorems we see that, for any short exact sequence $0 \to L \to B \overset{\rho}{\to} A \to 0$ with A,B and φ unital, $K_i(\varphi) \cong K_i(\rho)$. If L is also unital we get a pullback square of unital C^*-algebras

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\rho} & \mathbb{C} \\ \downarrow \cong & & \downarrow \mathrm{id} \\ L\oplus\mathbb{C} & \xrightarrow{pr_2} & \mathbb{C} \end{array}$$

The isomorphism is given by $(a,c) \longmapsto a+c1_L \oplus c$. Thus $K_i(\rho) \cong K_i(pr_2)$. But we can view pr_2 as $0\oplus id_{\mathbb{C}}$: $L\oplus\mathbb{C} \to 0\oplus\mathbb{C}$ and relative K-theory preserves direct sums as the reader can show easily. Thus $K_i(pr_2) = K_i(0) \oplus K_i(id_{\mathbb{C}}) = K_i(L) \oplus 0$.

III.4. DEFINITION. Let L be any C^*-algebra such that we have a short exact sequence of C^*-algebras $0 \to L \to B \overset{\varphi}{\to} A \to 0$ with B,A and φ unital. Then define $K_i(L) := K_i(\varphi)$ for $i = 0,1$.

The above remarks make sure that the definition III.4 makes sense and does not create ambiguity for unital C^*-algebras.

For any C^*-morphism $\varphi: B \to A$ we can define a unital C^*-morphism $\tilde{\varphi}: \tilde{B} \to \tilde{A}$ that sends an $(b,\lambda) \in \tilde{B}$ to $(\varphi(b),\lambda) \in \tilde{A}$. If B, A and φ are already unital, then the composition of $\tilde{\varphi}: \tilde{B} \to \tilde{A}$ with the isomorphism $B \oplus \mathbb{C} \to \tilde{B}$ and $\tilde{A} \to A \oplus \mathbb{C}$ described above, is the isomorphism $\varphi \oplus id_{\mathbb{C}}: B \oplus \mathbb{C} \to A \oplus \mathbb{C}$. Thus we see as before that in this case $K_i(\tilde{\varphi}) = K_i(\varphi) \oplus K_i(id_{\mathbb{C}}) = K_i(\varphi)$.

III.5. DEFINITION. Let B and A be C^*-algebras and $\varphi: B \to A$ be a C^*-morphism. If $\tilde{\varphi}: \tilde{B} \to \tilde{A}$ is the unital C^*-morphism induced by φ, we define $K_i(\varphi) := K_i(\tilde{\varphi})$ for $i = 0,1$.

Now we can assign to each C^*-morphism $\varphi: B \to A$ group homomorphisms $\varphi_i^*: K_i(B) \to K_i(A)$ where $\varphi_i^* = (\tilde{\varphi})_i^*|_{K_i(B)}$ and $(\tilde{\varphi})_i^*: K_i(\tilde{B}) \to K_i(\tilde{A})$ is the map defined in I.10 and I.26 respectively. The fact that φ_i^* maps $K_i(B)$ actually into $K_i(A) \subset K_i(\tilde{A})$ follows from the following commutative diagram, III.2, I.10 and I.26

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{B} & \longrightarrow & \tilde{B} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\tilde{\varphi}} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & \tilde{A} & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

III.6. THEOREM. The assignment K_i which sends a C^*-algebra A to $K_i(A)$ and a C^*-morphism $\varphi: B \to A$ to $\varphi_i^*: K_i(B) \to K_i(A)$ is a covariant functor from the category of C^*-algebras and C^*-morphisms into the category of abelian groups.

Proof. The proof is routine and left to the reader as an easy exercise. ■

For any short exact sequence of C^*-algebras $0 \to L \overset{\pi}{\to} B \overset{\varphi}{\to} A \to 0$ we get a short exact sequence $0 \to L \overset{\pi}{\to} \tilde{B} \overset{\tilde{\varphi}}{\to} \tilde{A} \to 0$. By III.1 we obtain an exact sequence of abelian groups $K_1(\tilde{\varphi}) \longrightarrow K_1(\tilde{B}) \longrightarrow K_1(\tilde{A}) \xrightarrow{\partial\varphi} K_0(\tilde{\varphi}) \longrightarrow K_0(\tilde{B}) \longrightarrow K_0(\tilde{A})$. Note that $Gl(\mathbb{C})$ is path-connected and thus $K_1(\mathbb{C}) = 0$. Hence we have that $K_1(A) = K_1(\tilde{A})$. Moreover, by definition $K_i(\tilde{\varphi}) = K_i(\varphi)$. So we get a commutative diagram

$$\begin{array}{ccccccccccc}
K_1(\tilde{\varphi}) & \longrightarrow & K_1(\tilde{B}) & \longrightarrow & K_1(\tilde{A}) & \xrightarrow{\partial_\varphi} & K_0(\tilde{\varphi}) & \xrightarrow{\pi_0^*} & K_0(\tilde{B}) & \xrightarrow{(\tilde{\varphi})_0^*} & K_0(\tilde{A}) \\
\| & & \| & & \| & & \| & & \uparrow & & \uparrow \\
K_1(\varphi) & \longrightarrow & K_1(B) & \longrightarrow & K_1(A) & \xrightarrow{\partial_\varphi} & K_0(\varphi) & \xrightarrow{\pi_0^*} & K_0(B) & \xrightarrow{\varphi_0^*} & K_0(A)
\end{array}$$

Since φ_0^* is just the restriction of $(\tilde{\varphi})_0^*$ we obtain:

III.7. PROPOSITION. Let $0 \to L \overset{\pi}{\to} B \overset{\varphi}{\to} A \to 0$ be a short exact sequence of C^*-algebras. Then the following sequence is exact:

$$K_1(L) \xrightarrow{\pi_1^*} K_1(B) \xrightarrow{\varphi_1^*} K_1(A) \xrightarrow{\partial_\varphi} K_0(L) \xrightarrow{\pi_0^*} K_0(B) \xrightarrow{\varphi_0^*} K_0(A).$$

■

III.8. PROPOSITION. Consider a commutative diagram of C^*-algebras:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\pi} & B & \xrightarrow{\varphi} & A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \xrightarrow{\tau} & D & \xrightarrow{\rho} & C & \longrightarrow & 0
\end{array}$$

Let the rows be exact. Then the diagram

$$\begin{array}{ccccccccccc}
K_1(L) & \xrightarrow{\pi_1^*} & K_1(B) & \xrightarrow{\varphi_1^*} & K_1(A) & \xrightarrow{\partial_\varphi} & K_0(L) & \xrightarrow{\pi_0^*} & K_0(B) & \xrightarrow{\varphi_0^*} & K_0(A) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
K_1(K) & \xrightarrow{\tau_1^*} & K_1(D) & \xrightarrow{\rho_1^*} & K_1(C) & \xrightarrow{\partial_\rho} & K_0(K) & \xrightarrow{\tau_0^*} & K_0(D) & \xrightarrow{\rho_0^*} & K_0(C)
\end{array}$$

obtained from III.7, commutes.

Proof. First consider the case where all algebras and maps in the right square are unital. Then we are in the situation of Lemma III.2 which gives the commutativity of the diagram. In the general case we replace the right square of the algebra-diagram by

$$\begin{array}{ccc}
\tilde{B} & \xrightarrow{\tilde{\varphi}} & \tilde{A} \\
\uparrow & & \uparrow \\
\tilde{D} & \xrightarrow[\tilde{\rho}]{} & \tilde{C}
\end{array}$$

Since the diagram for the general case is gotten from the diagram for the unital case by just restricting some maps, it is clear that it commutes. ■

CHAPTER IV: K-THEORY AS HOMOTOPY FUNCTOR

In this chapter we introduce the notion of homotopic C^*-morphisms and various other concepts arising naturally from homotopy theory of topological spaces. These ideas have been used more or less implicitly by many authors. It seems, however, that nobody ever bothered to write up the powerful consequences in K-theory in a concise form. The main result is that the K-functors do not distinguish between homotopic C^*-morphisms.

IV.1. DEFINITION. Let A and B be C^*-algebras and $\varphi_i: A \to B$ for $i = 0,1$ be two C^*-morphisms. The maps φ_0 and φ_1 are called homotopic, written $\varphi_0 \simeq \varphi_1$, if there exists a family $\phi_t: A \to B$ of C^*-morphisms for $t \in I$ such that $\phi: I \times A \to B$ defined by $\phi(t,a) = \phi_t(a)$ is jointly continuous and $\phi_0 = \varphi_0$ as well as $\phi_1 = \varphi_1$.

IV.2. DEFINITION. Let A and B be C^*-algebras. A C^*-morphism $\varphi: A \to B$ is called a homotopy equivalence if there exists a C^*-morphism $\psi: B \to A$ such that $\varphi \circ \psi \simeq \mathrm{id}_B$ and $\psi \circ \varphi \simeq \mathrm{id}_A$.

IV.3. DEFINITION. A C^*-algebra C is called contractible if $\mathrm{id}_C \simeq 0$. Here 0 denotes the map $C \to C$ that sends everything to zero.

IV.4. LEMMA. Let B be a unital C^*-algebra. Let $0 \neq E,F \in \mathrm{Ob}(P(B))$ and $p_E: B^n \to E$, $j_E: E \to B^n$, $p_F: B^n \to F$, $j_F: F \to B^n$ be pairs of projections and coprojections for the modules E and F. Endow B^n with the product norm and E and F with the subspace norm with respect to j_E and j_F. Then if $\|j_E \circ p_E - j_F \circ p_F\| < (\max\{\|p_E\|, \|p_F\|\})^{-1}$, the map $p_E \circ j_F: F \to E$ is an isomorphism of topological vector spaces.

Proof. First we show that $p_F \circ j_E \circ p_E \circ j_F$ is an automorphism of F. The space $\mathrm{End}(F)$ of endomorphisms of F is a unital Banach algebra w.r.t. the operator norm. Then $\|id_F - p_F \circ j_E \circ p_E \circ j_F\| = \|p_F \circ j_F \circ p_F \circ j_F - p_F \circ j_E \circ p_E \circ j_F\| = \|p_F \circ (j_F \circ p_F - j_E \circ p_E) \circ j_F\| \le \|p_F\| \|j_F \circ p_F - j_E \circ p_E\| \|j_F\| = \|p_F\| \|j_F \circ p_F - j_E \circ p_E\| < 1$. Thus $p_F \circ j_E \circ p_E \circ j_F$ is invertible in $\mathrm{End}(F)$. But $(p_F \circ j_E \circ p_E \circ j_F)^{-1} \circ p_F \circ j_E$ is a left inverse for $p_E \circ j_F$, thus $p_E \circ j_F$ is injective. Similarly we now show that $p_E \circ j_F \circ p_F \circ j_E \in \mathrm{Aut}(E)$ and thus $p_E \circ j_F$ has a right inverse $p_F \circ j_E \circ (p_E \circ j_F \circ p_F \circ j_E)^{-1}$. Hence $p_E \circ j_F$ is surjective. ∎

IV.5. THEOREM. Let A and B be C^*-algebras and $\phi_t : B \to A$ and a homotopy between the C^*-morphisms $\varphi : B \to A$ and $\psi : B \to A$. Then the induced map $\varphi_o^* : K_o(B) \to K_o(A)$ and $\psi_o^* : K_o(B) \to K_o(A)$ are equal.

Proof. For a given free module B^n over a unital C^*-algebra B we can identify projective retracts E of B^n, given by a pair of projection and coprojection (p_E, j_E), with a projection P_E in $M_n(B)$, namely the matrix associated with $j_E \circ p_E$. If A is another unital C^*-algebra and $\varphi : B \to A$ is a unital C^*-morphism, then the module $\varphi_* E = A \otimes_B E$ is given by the projection $\varphi^*(P_E)$. Thus, if $\varphi : B \to A$ is homotopic to $\psi : B \to A$ via an unital homotopy ϕ_t, for any $t \in I = [0,1]$, there exists an open neighbourhood U_t of t such that $\|\phi_t^*(P_E) - \phi_s^*(P_E)\| < (1 + \|\phi_t(P_E)\|)^{-1} \le 1$ for all $s \in U_t$. If $J = \{s \in I : \varphi_*(E) \cong \phi_{s*}(E)\}$ this shows by IV.4 that J is nonempty and open. But if $t \in \bar{J}$, we find an $s \in U_t \cap J$, hence, again by IV.4, $t \in J$. Since I is connected this implies $J = I$. We see that in the case where all the algebras and morphisms are unital the map $(\phi_t)_o^* : K_o(B) \to K_o(B)$, given by $(\phi_t)_o^*(\overline{[E]} - \overline{[F]}) =$

$\overline{[\phi_{t*}(E)]} - \overline{[\phi_{t*}(F)]}$, does not depend on t . In the general case we replace A and B by $\tilde{A}$ and $\tilde{B}$, and φ, ψ, ϕ_t by $\tilde{\varphi}, \tilde{\psi}, \tilde{\phi}_t$. Thus we get that the maps $\tilde{\varphi}_0^*: K_0(\tilde{B}) \to K_0(\tilde{A})$ and $\tilde{\psi}_0^*: K_0(\tilde{B}) \to K_0(\tilde{A})$ are equal. Therefore also their restrictions-corestrictions $\varphi_0^*: K_0(B) \to K_0(A)$ and $\psi_0^*: K_0(B) \to K_0(A)$ are equal. ■

IV.6. THEOREM. If $\varphi, \psi: B \to A$ are homotopic C^*-morphisms, the maps $\varphi_1^*: K_1(B) \to K_1(A)$ and $\psi_1^*: K_1(B) \to K_1(A)$ are equal.

Proof. First we assume that all algebras and maps are unital. View $K_1(B)$ as $\pi_0(Gl(B))$ and $K_1(A)$ as $\pi_0(Gl(A))$. For any $b \in Gl(B)$ and $[b]$ its class in $\pi_0(Gl(B))$, the map $(\phi_t)_1^*: K_1(B) \to K_1(A)$ is given by $(\phi_t^{*})_1^*([b]) = [\phi_t^{*} b]$. But $\phi_t^{*}(b)$ and $\phi_s^{*}(b)$ are path-connected in $Gl(A)$ via $\gamma: I \to Gl(A)$ defined by $\gamma(r) = \phi^{*}_{rt+(1-r)s}(b)$. Thus $(\phi_t)_1^*([b]) = (\phi_s)_1^*([b])$ for all $s,t \in I$. In particular $\varphi_1^* = \psi_1^*$. The general case follows easily from replacing A and B by $\tilde{A}$ and $\tilde{B}$, and all the maps by the corresponding unital maps, just as in IV.5 . ■

IV.7. COROLLARY. Let A and B be C^*-algebras and $\varphi: A \to B$ a homotopy equivalence. Then the induced map $\varphi_i^*: K_i(A) \to K_i(B)$ is an isomorphism.

Proof. Note that the identity on A,B induces the identity on $K_i(A)$ and $K_i(B)$, respectively. Now the claim follows directly from the preceding theorems and the definition of a homotopy equivalence via the usual argument. ■

IV.8. COROLLARY. Let B be a contractible C^*-algebra. Then $K_0(B)$ and $K_1(B)$ are zero.

Proof. Note that the zero map on B induces the zero map on $K_0(B)$. Thus the identity map on $K_0(B)$ is equal to the zero map. ■

CHAPTER V: SUSPENSIONS AND HIGHER K-GROUPS

In this chapter we give the (well known) definition of the suspension of a C^*-algebra. We use it to define $K_n(A)$ for any $n \in \mathbb{N}$. The main result will be a long exact sequence in K-theory associated with a short exact sequence of C^*-algebras.

V.1. DEFINITION. Let B be a C^*-algebra. The *cone* CB over B is defined by $CB := \{f: I \to B : f$ continuous and $f(1) = 0\}$. The *suspension* SB of B is defined by $SB = \{f \in CB : f(0) = 0\}$.

It is easy to check that CB and SB are C^*-algebras. In fact cone and suspension can be viewed as functors from the category of C^*-algebras into itself. The image of a morphism $\varphi: B \to A$ under these functors are given by $C_\varphi: CB \to CA$ aith $C\varphi(f) = \varphi \circ f$ and $S\varphi: SB \to SA$ with $S\varphi(f) = \varphi \circ f$, respectively. Note that if $ev: CB \to B$ denotes the evaluation at 0, we get a short exact sequence $0 \to SB \hookrightarrow CB \to B \to 0$.

V.2. LEMMA. The cone CB is contractible for any C^*-algebra B.

Proof. Consider the family of C^*-morphisms $\phi_t: CB \to CB$ for $t \in I$, defined by $\phi_t(f)(s) = f(1-(1-t)(1-s))$. Then $\phi_0 = id_{CB}$ and $\phi_1 = 0$. It is clear that ϕ_t is a homotopy. ■

V.3. PROPOSITION. Let B be a C^*-algebra. Then we have a natural isomorphism $K_1(B) \cong K_0(SB)$.

Proof. The above remarks and theorem III.7 show that we have an exact sequence in K-theory $K_1(CB) \to K_1(B) \xrightarrow{\partial} K_0(SB) \to K_0(CB)$. But CB is contractible by V.2, hence $K_i(CB) = 0$ by IV.8 . Thus ∂ is an isomorphism.

If A is another C^*-algebra and $\varphi: B \to A$ is a C^*-morphism, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SA & \longrightarrow & CA & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle S_\varphi} & & \uparrow{\scriptstyle C_\varphi} & & \uparrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & SB & \longrightarrow & CB & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

By III.8 we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 = K_1(CA) & \longrightarrow & K_1(A) & \xrightarrow{\partial_A} & K_0(SA) & \longrightarrow & K_0(CA) = 0 \\ & & \uparrow{\scriptstyle \varphi_1^*} & & \uparrow{\scriptstyle (S\varphi)_0^*} & & \\ 0 = K_1(CB) & \longrightarrow & K_1(B) & \xrightarrow{\partial_B} & K_0(SB) & \longrightarrow & K_0(CB) = 0 \end{array}$$

■

V.4. DEFINITION. Let B be a C^*-algebra. Define the n-th K-group of B by $K_n(B) = K_{n-1}(SB) = \ldots = K_0(S^nB)$. Here S^n means the n-fold application of the functor S to B .

Proposition V.4 shows that there is no ambiguity in this definition. Note that the functor S is exact, i.e. it sends exact sequences to exact sequences. In particular, for a short exact sequence of C^*-algebras $0 \to L \xrightarrow{\pi} B \xrightarrow{\varphi} A \to 0$, we get a short exact sequence $0 \to SL \xrightarrow{S\pi} SB \xrightarrow{S\varphi} SA \to 0$. By III.7 this induces an exact sequence $K_1(SL) \xrightarrow{(S\pi)_1^*} K_1(SB) \xrightarrow{(S\varphi)_1^*} K_1(SA) \xrightarrow{\partial_{S\varphi}}$

$K_0(SL) \xrightarrow{(S\pi)_0^*} K_0(SB) \xrightarrow{(S\varphi)_0^*} K_0(SA)$, which we rephrase in the following manner $K_2(L) \xrightarrow{\pi_2^*} K_2(B) \xrightarrow{\varphi_2^*}$ $K_2(A) \xrightarrow{\partial_2} K_0(SL) \xrightarrow{(S\pi)_0^*} K_0(S) \xrightarrow{(S\varphi)_0^*} K_0(SA)$. The naturality of the isomorphism from V.3 shows that the following diagram commutes.

$$\begin{array}{ccccc} K_0(SL) & \xrightarrow{(S\pi)_0^*} & K_0(SB) & \xrightarrow{(S\varphi)_0^*} & K_0(SA) \\ \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\ K_1(L) & \xrightarrow{\pi_1^*} & K_1(B) & \xrightarrow{\varphi_1^*} & K_1(A) \end{array}$$

Thus we can put together the above sequence and
$K_1(L) \longrightarrow K_1(B) \longrightarrow K_1(A) \xrightarrow{\partial} K_0(L) \longrightarrow K_0(B) \longrightarrow K_0(A)$.
We obtain the following theorem.

V.5. THEOREM. Let $0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\varphi} A \longrightarrow 0$ be a short exact sequence of C^*-algebras. Then we have a long exact sequence in K-theory as follows: for $n \geq 1$

$$\longrightarrow K_n(L) \xrightarrow{(S^n\pi)_0^*} K_n(B) \xrightarrow{(S^n\varphi)_0^*} K_n(A) \xrightarrow{\partial_{S^n\varphi}} K_{n-1}(L)$$
$$\xrightarrow{(S_\pi^{n-1})_0^*} K_{n-1}(B) \xrightarrow{(S_\varphi^{n-1})_0^*} K_{n-1}(A) \text{ . } \blacksquare$$

We denote $(S^n\pi)_0^*$ by π_n^* and $(S^n\varphi)_0^*$ by φ_n^* . Moreover we denote $\partial_{S^n\varphi}$ by ∂_n if the map φ is clear from the context.

CHAPTER VI: BOTT PERIODICITY AND THE SIX-TERM-SEQUENCE

In this chapter we shall describe the famous Bott periodicity theorem, which is of great importance. It will, among other things, enable us to install the so called six term sequence, which is a different form of expressing the long exact sequence.

VI.1. DEFINITION. Let G be a topological group. Define $\pi_1(G)$ to be the first homotopy group of G with respect to homotopies and loops based at the identity.

It is well known that in this case the multiplication in $\pi_1(G)$ can be described by pointwise multiplication of loops just as well as by composition of loops.

For a unital C^*-algebra A, let $L_n(A)$ be the group of loops in $Gl_n(A)$, based at 1, under pointwise multiplication. Let $N_n(A)$ be the subgroup of loops which are homotopic to a constant loop. $N_n(A)$ is normal in $L_n(A)$. There is a canonical injection $L_{n-1}(A) \to L_n(A)$ which maps $f \in L_{n-1}(A)$ to $f \oplus 1 = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \in L_n(A)$. This map sends $N_{n-1}(A)$ into $N_n(A)$. Thus we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\pi_1(Gl_1(A)) & \longrightarrow & \dots & \longrightarrow & \pi_1(Gl_n(A)) & \longrightarrow & \dots & \longrightarrow & \varinjlim \pi_1(Gl_n(A)) \\
\uparrow & & & & \uparrow & & & & \uparrow \\
L_1(A) & \longrightarrow & \dots & \longrightarrow & L_n(A) & \longrightarrow & \dots & \longrightarrow & \varinjlim L_n(A) \\
\uparrow & & & & \uparrow & & & & \uparrow \\
N_1(A) & \longrightarrow & \dots & \longrightarrow & N_n(A) & \longrightarrow & \dots & \longrightarrow & \varinjlim N_n(A)
\end{array}$$

If we let $L(Gl(A))$ be the group of loops in $Gl(A)$ based at 1 and $N(Gl(A))$ the subgroup of $L(Gl(A))$ consisting of the contractible loops, Lemma I.20 shows that $L(Gl(A)) = \varinjlim L_n(A)$ and $N(Gl(A)) = \varinjlim N_n(A)$. Thus we get the following Proposition.

VI.2. PROPOSITION. Let A be a unital C^*-algebra. Then $\pi_1(Gl(A)) = \varinjlim \pi_1(Gl_n(A))$. ■

VI.3. PROPOSITION. Let A be a unital C^*-algebra. Then $\pi_0(Gl(\widetilde{SA})) \cong \pi_1(Gl(A))$.

Proof. Let G be any topological group and H be a closed subgroup of H. Denote the group of continuous functions from the one-sphere S^1 into G, which send the base point of S^1 into H, by $C(S^1,G,H)$. Then we know that $\pi_0(C(S^1,G,H)) = \pi_1(G,H)$, the relative homotopy group. Moreover, if H is connected we have $\pi_1(G,H) = \pi_1(G,1)$. Now note that $\widetilde{SA} = \{f: I \xrightarrow[\text{cont.}]{} A : f(1) = f(0) \in \mathbb{C}\cdot 1_A\}$. We identify $Gl(\widetilde{SA})$ with $C(S^1,Gl(A),Gl(\mathbb{C}\cdot 1_A))$ in the obvious way. Then, since $Gl(\mathbb{C}\cdot 1_A)$ is pathconnected we get $\pi_0(Gl(\widetilde{SA})) \cong \pi_1(Gl(A),Gl(\mathbb{C}\cdot 1_A)) = \pi_1(Gl(A),1_{Gl(A)})$ and since we defined $\pi_1(Gl(A))$ as $\pi_1(Gl(A),1_{Gl(A)})$ this proves the claim. ■

VI.4. THEOREM. (Bott Periodicity, cf. [K] III.1.11). Let A be a unital Banach algebra. Then the map $\gamma_A: K_0(A) \longrightarrow \pi_1(Gl(A))$ induced by the assignment that sends the isomorphy class $[E]$ of a finitely generated projective A-module to the homotopy class of the loop $t \longmapsto z(t)P_E + 1 - P_E$, where the projection $P_E \in M_\infty(A)$ is as in IV.4 and $z(t) = e^{2\pi i t}$, is an isomorphism, called the Bott isomorphism. ■

Note that $z(t)P_E + 1 - P_E = \exp(t \cdot P_E)$. Note also that the Bott isomorphism is natural in the sense that for A and B unital C^*-algebras and $\varphi\colon B \to A$ a unital C^*-morphism the map $\varphi_1^*\colon \pi_1(Gl(B)) \longrightarrow \pi_1(Gl(A))$ induced by the map $\varphi^*\colon Gl(B) \longrightarrow Gl(A)$ makes the following diagram commute:

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\varphi_0^*} & K_0(A) \\ \Big\downarrow \gamma_B & & \Big\downarrow \gamma_A \\ \pi_1(Gl(B)) & \xrightarrow{\varphi_1^*} & \pi_1(Gl(A)) \end{array}$$

VI.5. COROLLARY. Let A be a unital C^*-algebra. Then $K_0(A) \cong K_1(SA)$ via the Bott map.

Proof. We have seen in the proof of III.7 that $K_1(B) = K_1(\widetilde{B})$ for any C^*-algebra B . Thus $K_1(SA) = K_1(\widetilde{SA})$ and the claim follows from VI.4. ■

VI.6. COROLLARY. Let A be a C^*-algebra. Then we have that $K_0(S^2A) \cong K_0(A)$.

Proof. If A is unital we have $K_0(S^2A) \cong K_1(SA)$ by V.3 and $K_1(SA) \cong K_0(A)$ by VI.5. In the general case we have a split exact sequence $0 \to S^2A \to S^2\widetilde{A} \to S^2\mathbb{C} \to 0$. If $\rho\colon K_0(\widetilde{A}) \to K_0(S^2\widetilde{A})$ denotes the composition of the Bott map $\gamma_{\widetilde{A}}$ and the map from V.3 and $\rho'\colon K_0(\mathbb{C}) \to K_0(S^2\mathbb{C})$ the corresponding map for $\mathbb{C}$ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(S^2A) & \longrightarrow & K_0(S^2\widetilde{A}) & \longrightarrow & K_0(S^2\mathbb{C}) & \longrightarrow & 0 \\ & & & & \cong \Big\uparrow \rho & & \cong \Big\uparrow \rho' & & \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(\widetilde{A}) & \longrightarrow & K_0(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

Since the rows are exact we see that $\rho|_{K_0(A)}$ is an isomorphism between $K_0(A)$ and $K_0(S^2A)$. ■

For a short exact sequence of C^*-algebras
$0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\varphi} A \longrightarrow 0$ we can now write down the socalled sixterm sequence

$$\begin{array}{ccccccc}
K_2(A) & \xrightarrow{\partial_2} & K_1(L) & \xrightarrow{\pi_1^*} & K_1(B) & \xrightarrow{\varphi_1^*} & K_1(A) \\
& \nwarrow{\scriptstyle \rho\ \cong} & \uparrow{\scriptstyle \partial_2 \circ \rho} & & & & \downarrow{\scriptstyle \partial_1} \\
& & K_0(A) & \xleftarrow[\varphi_0^*]{} & K_0(B) & \xleftarrow[\pi_0^*]{} & K_0(A)
\end{array}$$

The map $\partial_2 \circ \rho$ is often referred to as the exponential map because of the structure of the Bott map, which is an essential part in ρ .

VI.7. THEOREM. The six term sequence is exact.

Proof. We only have to show exactness at $K_0(A)$. First consider the case where B, A and φ are unital. Then, by writing down all the maps whose composition $\partial_2 \circ \rho$ is, we get the following commutative diagram.

$$\begin{array}{ccccccccccc}
K_0(A) & \xrightarrow{\gamma_A} & \pi_1(Gl(A)) & \xrightarrow{\cong} & \pi_0(Gl(\widetilde{SA})) & \xrightarrow{\cong} & K_1(SA) & \xrightarrow{\cong} & K_0(S^2A) & \xrightarrow{\partial_1 \varphi} & K_1(L) \\
\uparrow{\scriptstyle \varphi_0^*} & & \uparrow{\scriptstyle \varphi_1^*} & & \uparrow{\scriptstyle (S\varphi)_0^*} & & \uparrow{\scriptstyle \varphi_1^*} & & \uparrow{\scriptstyle \varphi_2^*} & & \\
K_0(B) & \xrightarrow[\gamma_B]{} & \pi_1(Gl(B)) & \xrightarrow{\cong} & \pi_0(Gl(\widetilde{SB})) & \xrightarrow{\cong} & K_1(SB) & \xrightarrow{\cong} & K_0(S^2B) & &
\end{array}$$

We condense this to the commutative diagram

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow[\cong]{\rho_A} & K_0(S^2A) & \xrightarrow{\partial_2 \varphi} & K_1(L) \\
\Big|{\scriptstyle \varphi_0^*} & & \Big|{\scriptstyle \varphi_2^*} & & \\
K_0(B) & \xrightarrow[\rho_B]{\cong} & K_0(S^2B) & &
\end{array}$$

Since $K_0(S^2B) \xrightarrow{\varphi_2^*} K_0(S^2A) \xrightarrow{\partial_2} K_1(L)$ is part of the long exact sequence this proves that the sixterm sequence is exact at $K_0(A)$. In the general case we replace A and B by $\tilde{A}$ and $\tilde{B}$ respectively to get a diagram

$$\begin{array}{ccccc} K_0(\tilde{A}) & \xrightarrow{\rho_{\tilde{A}}} & K_0(S^2\tilde{A}) & \xrightarrow{\partial_{2\tilde{\varphi}}} & K_1(L) \\ \uparrow (\tilde{\varphi})_0^* & & \uparrow (\tilde{\varphi})_2^* & & \\ K_0(\tilde{B}) & \xrightarrow[\rho_{\tilde{B}}]{} & K_0(S^2\tilde{B}) & & \end{array}$$

We saw in VI.6 that the restrictions $\rho_A|_{K_0(A)}$ and $\rho_{\tilde{B}}|_{K_0(B)}$ are isomorphisms $K_0(A) \longrightarrow K_0(S^2A)$ and $K_0(B) \longrightarrow K_0(S^2B)$ respectively. Moreover, from III.7 we know that $\varphi_0^* = (\tilde{\varphi})_0^*|_{K_0(B)}$. If we can show that $\varphi_2^* = (\tilde{\varphi})_2^*|_{K_0(S^2B)}$ and $\partial_{2\varphi} = \partial_{2\tilde{\varphi}}|_{K_0(S^2\tilde{A})}$ we get a diagram as in the unital case, which proves the exactness of the sexterm sequence at $K_0(A)$. But we see as in III.7, since $0 \longrightarrow S^2A \longrightarrow S^2\tilde{A} \longrightarrow S^2\mathbb{C} \longrightarrow 0$ and $0 \longrightarrow S^2B \longrightarrow S^2\tilde{B} \longrightarrow S^2\mathbb{C} \longrightarrow 0$ are split exact, that $\varphi_2^* = (S^2\varphi)_0^* = (S^2\tilde{\varphi})_0^*|_{K_0(S^2B)}$. Moreover, out of a similar reasoning $\partial_{2\varphi} = \partial_{2\tilde{\varphi}}|_{K_0(S^2A)}$. ■

CHAPTER VII: MULTIPLICATIVE STRUCTURES

In this chapter we give a few canonical multiplicative structures relating the K-theory of two nuclear C^*-algebras B_1 and B_2 to the K-theory of their tensor product. Karoubi described those for unital algebras in a fairly abstract manner in [K2]. We give a more concrete description, also for nonunital algebras. Moreover, we describe a way of providing $K_n(B)$ with a module structure.

For two C^*-algebras B_1 and B_2 we can form the algebraic tensor product $B_1 \otimes_{\mathbb{C}} B_2$. We can provide $B_1 \otimes_{\mathbb{C}} B_2$ with C^*-crossnorms. If the algebras are nuclear all possible C^*-crossnorms agree. Denote the completion of $B_1 \otimes_{\mathbb{C}} B_2$ w.r.t. this norm by $B_1 \bar{\otimes} B_2$. The tensor product $\bar{\otimes}$ is natural w.r.t. morphisms $B_1 \to A_1$ and $B_2 \to A_2$. Moreover, tensoring with a fixed nuclear algebra is an exact functor. From now on all C^*-algebras are assumed to be nuclear. Let $E_i \in Ob(P(B_i))$ for $i = 1,2$. If the E_i are free, say $E_i = B_i^{ni}$ then we have a canonical isomorphism between $E_1 \otimes_{\mathbb{C}} E_2$ and $(B_1 \otimes_{\mathbb{C}} B_2)^{n_1 \cdot n_2}$. Define $E_1 \otimes_{\mathbb{C}} E_2$ to be the closure of $E_1 \otimes_{\mathbb{C}} E_2$ in $(B_1 \bar{\otimes} B_2)^{n_1 \cdot n_2}$ with the product norm. Now suppose that E_i is embedded in B_i^{ni} as a retract. Let $j_i: E_i \longrightarrow B_i^{ni}$ be the embeddings and $p_i: B_i^{ni} \longrightarrow E_i$ the retractions. We topologize $E_1 \otimes_{\mathbb{C}} E_2$ with quotient topology of the map $p_1 \otimes p_2$ which is clearly surjective. As in I.3 we see that $j_1 \otimes j_2$ is an embedding w.r.t. to this topology. Thus we define $E_1 \bar{\otimes} E_2$ as the closure of $E_1 \otimes_{\mathbb{C}} E_2$ in $B_1^{n_1} \bar{\otimes} B_2^{n_2} = (B_1 \bar{\otimes} B_2)^{n_1 \cdot n_2}$. We have to show that this definition does not depend on the particular embeddings. In fact, since $E_1 \otimes_{\mathbb{C}} E_2$ is finitely generated projective over $B_1 \otimes_{\mathbb{C}} B_2$, we see as in I.3 that the topology on $E_1 \otimes_{\mathbb{C}} E_2$ does not depend on the choice of embeddings and projections. But the closure in $B_1^{n_1} \bar{\otimes} B_2^{n_2}$ is just the completion of $E_1 \otimes_{\mathbb{C}} E_2$ w.r.t. that topology, since $j_1 \otimes j_2$ is an embedding. Next, we show that $E_1 \bar{\otimes} E_2 \in \mathrm{Ob}(P(B_1 \otimes B_2))$. The map $p_1 \otimes p_2 : B_1^{n_1} \otimes B_2^{n_2} \longrightarrow E_1 \otimes E_2$ is continuous, so there exists a unique extension to the completions $p_1 \bar{\otimes} p_2 : B_1^{n_1} \bar{\otimes} B_2^{n_2} \longrightarrow E_1 \bar{\otimes} E_2$. Similarly we get a unique map $j_1 \bar{\otimes} j_2 : E_1 \bar{\otimes} E_2 \longrightarrow B_1^{n_1} \bar{\otimes} B_2^{n_2}$. But we have $(p_1 \otimes p_2) \circ (j_1 \otimes j_2) = \mathrm{id}_{E_1 \otimes_{\mathbb{C}} E_2}$ so the uniqueness of the extension shows that $(p_1 \bar{\otimes} p_2) \circ (j_1 \bar{\otimes} j_2) = \mathrm{id}_{E_1 \bar{\otimes} E_2}$. Thus $p_1 \bar{\otimes} p_2$ is onto and $j_1 \bar{\otimes} j_2$ is one to one. Uniqueness of the extension also ensures that $(j_1 \bar{\otimes} j_2) \circ (p_1 \bar{\otimes} p_2)$ is an idempotent.

VII.1. PROPOSITION. Let B_1 and B_2 be unital nuclear C^*-algebras and $E_2 \in \mathrm{Ob}(P(B_2))$. Then tensoring with E_2 is an additive, exact functor from $P(B_1)$ to $P(B_1 \bar{\otimes} B_2)$ which is natural w.r.t. C^*-morphisms $A_i \to A_i$ for $i = 1,2$.

Proof. In view of the above remarks it is easy to check that it is a functor. It is enough to show that the functor transforms short exact sequences into short exact sequences. Let $0 \to E \overset{\alpha}{\to} F \overset{\beta}{\to} H \to 0$ be short exact in $P(B_1)$. The algebraic tensor product with E_2 is an exact functor, so we have an exact sequence $0 \longrightarrow E \otimes_{\mathbb{C}} E_2 \xrightarrow{\alpha \otimes \mathrm{id}} F \otimes_{\mathbb{C}} E_2 \xrightarrow{\beta \otimes \mathrm{id}} H \otimes_{\mathbb{C}} E_2 \longrightarrow 0$. Moreover, since H is projective, we have a splitting $\gamma : H \to F$ which induces a splitting $j \otimes \mathrm{id} : H \otimes_{\mathbb{C}} E_2 \longrightarrow F \otimes_{\mathbb{C}} E_2$. We get a sequence $0 \longrightarrow E \bar{\otimes} E_2 \xrightarrow{\alpha \bar{\otimes} \mathrm{id}} F \bar{\otimes} E_2 \xrightarrow{\beta \bar{\otimes} \mathrm{id}} H \bar{\otimes} E_2 \longrightarrow 0$ and a map $\gamma \bar{\otimes} \mathrm{id} : H \bar{\otimes} E_2 \longrightarrow F \bar{\otimes} E_2$. Uniqueness of the completion shows that $(\beta \bar{\otimes} \mathrm{id}) \circ (\gamma \bar{\otimes} \mathrm{id}) = \mathrm{id}_{H \bar{\otimes} E_2}$, thus $\beta \bar{\otimes} \mathrm{id}$ is surjective and the sequence is exact at $H \bar{\otimes} E_2$. But the splitting $\gamma \otimes \mathrm{id}$ induces a splitting $\delta \bar{\otimes}$ id: $F \bar{\otimes} E_2 \longrightarrow E \bar{\otimes} E_2$. So we see similarly as above that the sequence is exact at $E \bar{\otimes} E_2$. Again by the uniqueness of the completion of a map we see that $(\beta \bar{\otimes} \mathrm{id}) \circ (\alpha \bar{\otimes} \mathrm{id}) = 0$. Moreover if $\mathfrak{a} \in F \bar{\otimes} E_2$ is in the kernel of $\beta \bar{\otimes} \mathrm{id}$ and $\mathfrak{a}_k \in F \otimes E_2$ tend to $\mathfrak{a}$ then $\beta \otimes \mathrm{id}(\mathfrak{a}_k) = \beta \bar{\otimes} \mathrm{id}(\mathfrak{a}_k)$ tends to zero in $H \otimes_{\mathbb{C}} E_2$. Replacing $\mathfrak{a}_k$ by $[\mathfrak{a}_k - (\gamma \otimes \mathrm{id}) \circ (\beta \otimes \mathrm{id})(\mathfrak{a}_k)]$ we can assume that $\mathfrak{a}_k \in \ker(\beta \otimes \mathrm{id}) = \mathrm{im}(\alpha \otimes \mathrm{id})$. Then the sequence $\mathfrak{b}_k := \delta \otimes \mathrm{id}(\mathfrak{a}_k)$ converges and $(\alpha \otimes \mathrm{id})(\mathfrak{b}_k) = \mathfrak{a}_k$, since $\delta \otimes \mathrm{id}$ is a topological isomorphism from $\mathrm{im}(\alpha \otimes \mathrm{id})$ to $E \otimes E_2$ with inverse $\alpha \otimes \mathrm{id}$. Thus $\mathfrak{a} = \alpha \bar{\otimes} \mathrm{id}(\lim_k \mathfrak{b}_k)$ and the sequence is exact also at $F \bar{\otimes} E_2$. It is an easy consequence of this that the tensorproduct $\bar{\otimes}$ distributes over direct sums.

Now let $\varphi_i : B_i \longrightarrow A_i$ be unital C*-morphisms between nuclear C*-algebras. We went to show that for $E_i \in \mathrm{Ob}(P(B_i))$ the modules $(E_1 \bar{\otimes} A_2) \otimes_{B_1 \bar{\otimes} B_2} (E_1 \bar{\otimes} E_2)$ and $(A_1 \otimes_{B_1} E_1) \bar{\otimes} (A_2 \otimes_{B_2} E_2)$ are equal. If E_1 and E_2 are free this is clear. But since the modules are projective and all the involved tensor products distribute over direct sums the general case follows easily. ■

The isomorphism classes of objects in $P(B_i)$ form commutative monoids S_i w.r.t. taking direct sums as addition. The assignment $\hat{\mu}: S_1 \times S_2 \longrightarrow T$, where T is the monoid of isomorphism classes of objects in $P(B_1 \bar{\otimes} B_2)$, is bilinear. If **Absem** is the category of abelian monoids, **Ab** the category of abelian groups and G: **Absem** $\longrightarrow$ **Ab** the Grothendieck functor then $G(S_i) = K_0(B_i)$ and $G(T) = K_0(B_1 \bar{\otimes} B_2)$. Moreover we have the following isomorphisms:

$$\begin{array}{ccc} \mathrm{Bil}(S_1 \times S_2, T) \longrightarrow \mathrm{Absem}(S_1, \mathrm{Absem}(S_2 T)) & \longrightarrow & \mathrm{Absem}(S_1, \mathrm{Ab}(G(S_2), G(T))) \\ & & \swarrow \\ \mathrm{Bil}(G(S_1) \times G(S_2), G(T)) & \longleftarrow & \mathrm{Ab}(G(S_1), \mathrm{Ab}(G(S_2), G(T))) \end{array}$$

Thus $\hat{\mu}$ induces a bilinear map $\mu\colon K_0(B_1) \times K_0(B_2) \longrightarrow K_0(B_1 \bar{\otimes} B_2)$. The formula is $\mu(\overline{[E_1]}-\overline{[F_1]}, \overline{[E_2]}-\overline{[F_2]}) = \overline{[E_1 \bar{\otimes} E_2]}-\overline{[E_1 \bar{\otimes} F_2]}-\overline{[F_1 \bar{\otimes} E_2]}+\overline{[F_1 \bar{\otimes} F_2]}$. Thus we get the following lemma.

VII.2. LEMMA. Let B_i be unital C^*-algebras. Then the tensor product $\bar{\otimes}\colon \mathrm{Ob}(P(B_1)) \times \mathrm{Ob}(P(B_2)) \longrightarrow \mathrm{Ob}(P(B_1 \bar{\otimes} B_2))$ induces a bilinear map $\mu\colon K_0(B_1) \times K_0(B_2) \longrightarrow K_0(B_1 \bar{\otimes} B_2)$ which is natural with respect to unital C^*-morphisms $\varphi : B_i \longrightarrow A_i$. ■

Let $\varphi_i\colon B_i \longrightarrow A_i$ be unital C^*-morphisms. We define a unital C^*-algebra $P(\varphi_1, \varphi_2)$ as the following pullback

$$\begin{array}{ccccc} & & B_1 \bar{\otimes} A_2 & & \\ & \nearrow & & \searrow {\scriptstyle \varphi_1 \bar{\otimes} \mathrm{id}} & \\ P(\varphi_1,\varphi_2) & & & & A_1 \bar{\otimes} A_2 \\ & \searrow & & \nearrow {\scriptstyle \mathrm{id} \bar{\otimes} \varphi_2} & \\ & & A_1 \bar{\otimes} B_2 & & \end{array}$$

The maps $\mathrm{id}_{B_1} \bar{\otimes} \varphi_2\colon B_1 \bar{\otimes} B_2 \longrightarrow B_1 \bar{\otimes} A_2$ and $\varphi_1 \bar{\otimes} \mathrm{id}_{B_2}$: $B_1 \bar{\otimes} B_2 \longrightarrow A_1 \bar{\otimes} B_2$ induce a map $\chi\colon B_1 \bar{\otimes} B_2 \longrightarrow P(\varphi_1, \varphi_2)$.

VII.3. LEMMA. For $\varphi_i\colon B_i \longrightarrow A_i$ surjective unital C^*-morphisms the following sequence is exact:
$0 \longrightarrow \ker \varphi_1 \bar{\otimes} \ker \varphi_2 \longrightarrow B_1 \bar{\otimes} B_2 \xrightarrow{\chi} P(\varphi_1, \varphi_2) \longrightarrow 0$.

Proof. Let $L_i := \ker \varphi_i$. Tensor the exact sequence $0 \longrightarrow L_2 \longrightarrow B_2 \longrightarrow A_2 \longrightarrow 0$ with L_1 to get the exact sequence $0 \longrightarrow L_1 \bar{\otimes} L_2 \longrightarrow L_1 \bar{\otimes} B_2 \longrightarrow L_1 \bar{\otimes} A_2 \longrightarrow 0$.

Similarly we get $0 \longrightarrow L_1 \bar{\otimes} B_2 \longrightarrow B_1 \bar{\otimes} B_2 \longrightarrow A_1 \otimes B_2 \longrightarrow 0$. So we get an exact sequence $0 \longrightarrow \frac{L_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2} \longrightarrow \frac{B_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2} \longrightarrow \left(\frac{B_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2}\right) \Big/ \left(\frac{L_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2}\right) \longrightarrow 0$. Using the second isomorphism theorem we can rephrase this to $0 \longrightarrow L_1 \bar{\otimes} A_2 \longrightarrow \frac{B_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2} \longrightarrow A_1 \bar{\otimes} B_2 \longrightarrow 0$. The map χ induces a map $\bar{\chi}: \frac{B_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2} \longrightarrow P(\varphi_1,\varphi_2)$. Moreover, there is a map $\tau: L_1 \bar{\otimes} A_2 \longrightarrow P(\varphi_1,\varphi_2)$ induced by $L_1 \bar{\otimes} A_2 \hookrightarrow B_1 \bar{\otimes} A_2$ and $L_1 \bar{\otimes} A_2 \xrightarrow{0} A_1 \bar{\otimes} B_2$. It is easy to see that $A_1 \bar{\otimes} B_2$ is the cokernel of τ, using the fact that $P(\varphi_1,\varphi_2)$ is the pullback of $A_1 \bar{\otimes} B_2$ and $B_1 \bar{\otimes} A_2$. Thus we obtain the following diagram, which is easily checked to be commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 \bar{\otimes} A_2 & \longrightarrow & \frac{B_1\bar{\otimes}B_2}{L_1\bar{\otimes}L_2} & \longrightarrow & A_1 \bar{\otimes} B_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow \bar{\chi} & & \| & & \\
0 & \longrightarrow & L_1 \bar{\otimes} A_2 & \xrightarrow{\tau} & P(\varphi_1,\varphi_2) & \longrightarrow & A_1 \bar{\otimes} B_2 & \longrightarrow & 0
\end{array}$$

This proves that $\bar{\chi}$ is an isomorphism, whence the claim. ∎

We now turn to the case where $\varphi_i: \tilde{B}_i \longrightarrow \frac{\tilde{B}_i}{B_i} = \mathbb{C}$ is the canonical surjection.

VII.4. LEMMA. Let B_i be C^*-algebras and $\varphi_i: \tilde{B}_i \longrightarrow \mathbb{C}$ be the canonical surjections. For $P := P(\varphi_1,\varphi_2)$ and $\chi: \tilde{B}_1 \bar{\otimes} \tilde{B}_2 \longrightarrow P$ the natural map, the induced map $\chi_1^*: K_1(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) \longrightarrow K_1(P)$ is surjective.

Proof. The algebra P is given by the following pull-back

$$\begin{array}{ccccc} & & \widetilde{B}_1 & & \\ & \nearrow & & \searrow^{\varphi_1} & \\ P & & & & \mathbb{C} \\ & \searrow & & \nearrow_{\varphi_2} & \\ & & \widetilde{B}_2 & & \end{array}$$

An element in $K_1(P)$ is therefore given by invertible

$$\Delta_i = \left(b_i^{(\alpha\beta)}\right)_{\alpha,\beta=1\ldots n} \in Gl_n(\widetilde{B}_i) \quad \text{for some} \quad n \in \mathbb{N}, \quad \text{such that}$$

$$\Lambda := \left(\varphi_1\left(b_1^{(\alpha\beta)}\right)\right)_{\alpha,\beta=1\ldots n} = \left(\varphi_2\left(b_2^{(\alpha\beta)}\right)\right)_{\alpha,\beta=1\ldots n} \in Gl_n(\mathbb{C}) .$$

Define matrices in $Gl_n(\widetilde{B}_1 \bar{\otimes} \widetilde{B}_2)$ by

$$\Lambda_1 := \left(b_1^{(\alpha\beta)} \bar{\otimes} 1_{B_2}\right)_{\alpha,\beta=1\ldots n} \quad \text{and} \quad \Lambda_2 := \left(1_{B_1} \bar{\otimes} b_2^{(\alpha\beta)}\right)_{\alpha,\beta=1\ldots n} .$$

It is now easy to check that $\chi^{\#}: Gl_n(\widetilde{B}_1 \bar{\otimes} \widetilde{B}_2) \to Gl_n(P)$ maps $\Lambda_1 \cdot \Lambda^{-1} \cdot \Lambda_2$ to the pair (Δ_1, Δ_2), which proves the claim. ■

We now extend our definition of the cup product to non-unital algebras. The lemma and the long exact sequence yield an exact sequence $0 \longrightarrow K_0(B_1 \bar{\otimes} B_2) \longrightarrow K_0(B_1 \bar{\otimes} B_2) \xrightarrow{\chi_0^*} K_0(P)$. Moreover, the Mayer Vietoris sequence for P (cf. [H]) shows that the map $\gamma_0^*: K_0(P) \longrightarrow K_0(\widetilde{B}_1 \bar{\otimes} \mathbb{C}) \oplus K_0(\mathbb{C} \bar{\otimes} \widetilde{B}_2)$, induced by the natural map $\gamma: P \longrightarrow (\widetilde{B}_1 \bar{\otimes} \mathbb{C}) \bar{\oplus} (\mathbb{C} \bar{\otimes} \widetilde{B}_2)$, is injective. We get the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(B_1 \bar{\otimes} B_2) & \longrightarrow & K_0(\widetilde{B}_1 \bar{\otimes} \widetilde{B}_2) & \xrightarrow{\chi_0^*} & K_0(P) & \xrightarrow{\gamma_0^*} & K_0(\widetilde{B}_1 \bar{\otimes} \mathbb{C}) \oplus K_0(\mathbb{C} \bar{\otimes} \widetilde{B}_2) \\ & & & & \uparrow{\scriptstyle \mu} & & & & \\ 0 & \longrightarrow & K_0(B_1) \oplus K_0(B_2) & \xrightarrow{j} & K_0(\widetilde{B}_1) \oplus K_0(\widetilde{B}_2) & \longrightarrow & K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) & & \end{array}$$

In order to define the cup product $\mu\colon K_0(B_1)\oplus K_0(B_2) \longrightarrow K_0(B_1 \bar{\otimes} B_2)$ as the restriction of the product on $K_0(\tilde{B}_1)\oplus K_0(\tilde{B}_2)$ we have shown that $\chi_0 \circ \mu \circ j = 0$. But the maps $pr_1 \circ \gamma_0^* \circ \chi_0^*$ and $pr_2 \circ \gamma_0^* \circ \chi_0^*$ are given by $(id_{\tilde{B}_1} \bar{\otimes} \varphi_2)_0^*$ and $(\varphi_1 \bar{\otimes} id_{\tilde{B}_2})_0^*$ respectively, if $pr_i\colon K_0(\tilde{B}_1)\oplus K_0(\tilde{B}_2) \longrightarrow K_0(\tilde{B}_2)$ is the projection on the i-th summand. Moreover, the cup product is natural, i.e. the following diagram commutes

$$\begin{array}{ccc} K_0(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) & \longrightarrow & K_0(\tilde{B}_1 \bar{\otimes} \mathbb{C}) \\ \uparrow \mu & & \uparrow \mu \\ K_0(\tilde{B}_1)\oplus K_0(\tilde{B}_2) & \longrightarrow & K_0(\tilde{B}_1)\oplus K_0(\mathbb{C}) \end{array}$$

Thus $pr_1 \circ \gamma_0^* \chi_0^* \circ \mu \circ j = 0$. Similarly $pr_2 \circ \gamma_0^* \circ \chi_0^* \circ \mu \circ j = 0$ and hence $\gamma_0^* \circ \chi_0^* \circ \mu \circ j = 0$ and since γ_0^* is injective, we have $\chi_0 \circ \mu \circ j = 0$.

Recall that $K_0(B_i)$ is defined as $K_0(\varphi_i)$ and $K_0(B_1 \bar{\otimes} B_2)$ is equal to $K_0(\chi)$. Thus we have a cup product on the relative K_0-groups $\mu\colon K_0(\varphi_1)\times K_0(\varphi_2) \longrightarrow K_0(\chi)$. We want to define such a cup product for arbitrary unital C^*-surjections $\varphi_i\colon B_i \longrightarrow A_i$ and the induced map $\chi_B\colon B_1 \bar{\otimes} B_2 \longrightarrow P(\varphi_1,\varphi_2)$. If $L_i := \ker \varphi_i$ and $\rho_i\colon \tilde{L}_i \longrightarrow \tilde{L}_i/L_i = \mathbb{C}$ are the natural surjections, our construction applies and we get a cup product $\mu\colon K_0(\rho_1)\times K_0(\rho_2) \longrightarrow K_0(\chi_L)$ where $\chi_L\colon \tilde{L}_1 \bar{\otimes} \tilde{L}_2 \longrightarrow P(\rho_1,\rho_2)$ is the induced map. The following lemma, together with the excision theorem will establish a natural isomorphism $j\colon K_0(\chi_L) \longrightarrow K_0(\chi_B)$. Thus we can define a cup product using the following diagram.

$$\begin{array}{ccc}
K_0(\rho_1) \times K_0(\rho_2) & \xrightarrow{\ \mu\ } & K_0(\chi_L) \\
\Big\downarrow \cong & & \Big\downarrow j \\
K_0(\varphi_1) \times K_0(\varphi_2) & \dashrightarrow & K_0(\chi_B)
\end{array}$$

VII.5. LEMMA. For $i = 1,2$, let the following diagram be a pullback square of unital C^*-algebras:

$$\begin{array}{ccc}
D_i & \xrightarrow{\ \rho_i\ } & C_i \\
\Big\downarrow \delta_i & & \Big\downarrow \gamma_i \\
B_i & \xrightarrow[\ \varphi_i\ ]{} & A_i
\end{array}$$

Moreover, let ρ_i and φ_i be surjective and δ_i and γ_i be injective. Then the following square of unital C^*-algebras is a pullback square:

$$\begin{array}{ccc}
D_1 \bar{\otimes} D_2 & \xrightarrow{\ \chi_D\ } & P(\rho_1,\rho_2) \\
\Big\downarrow \delta_1\bar{\otimes}\delta_2 & & \Big\downarrow (\delta_1\bar{\otimes}\gamma_2, \gamma_1\bar{\otimes}\delta_2) \\
B_1 \bar{\otimes} B_2 & \xrightarrow[\ \chi_B\ ]{} & P(\varphi_1,\varphi_2)
\end{array}$$

Proof. The proof is achieved in two steps. First we show that for any C^*-algebra S the following square is a pullback:

$$\begin{array}{ccc}
S\bar{\otimes} D_i & \xrightarrow{\ id\bar{\otimes}\rho_i\ } & S\bar{\otimes} C_i \\
id\bar{\otimes}\delta_i \Big\downarrow & & \Big\downarrow id\bar{\otimes}\gamma_i \\
S\bar{\otimes} B_i & \xrightarrow[\ id\bar{\otimes}\varphi_i\ ]{} & S\bar{\otimes} A_i
\end{array}$$

To see this, note first that $J_i := \ker \rho_i$ is naturally isomorphic to $\ker \varphi_i$. Moreover tensoring over $\mathbb{C}$ with a fixed C^*-algebra is an exact functor. If Q is the pullback of $id_s \bar{\otimes} \gamma_i$ and $id_s \bar{\otimes} \varphi_i$, we get the following diagram:

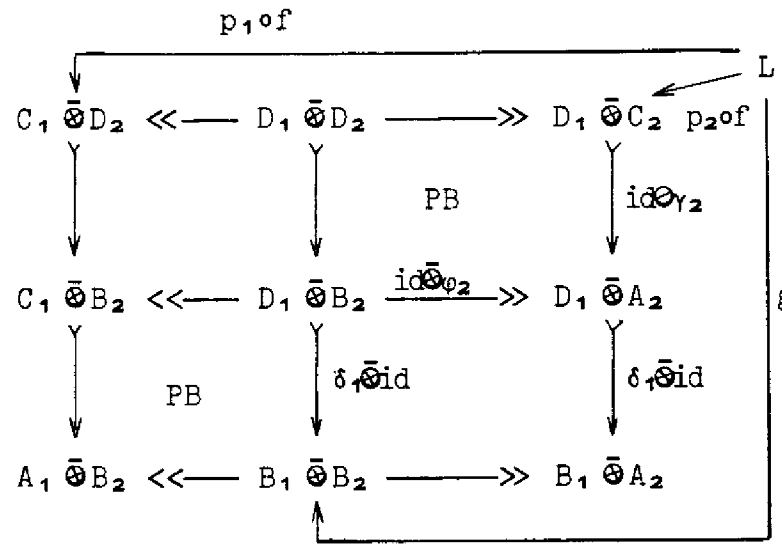
$$\begin{array}{ccccccccc}
0 & \longrightarrow & S \bar{\otimes} J & \overset{f}{\dashrightarrow} & Q & \longrightarrow & S \bar{\otimes} C_i & \longrightarrow & 0 \\
 & & \| & & \uparrow g & & \| & & \\
0 & \longrightarrow & S \bar{\otimes} J & \longrightarrow & S \bar{\otimes} D_i & \longrightarrow & S \bar{\otimes} C_i & \longrightarrow & 0
\end{array}$$

The map $f\colon S \bar{\otimes} J_i \longrightarrow Q$ is induced by the inclusion $S \bar{\otimes} J_i \longrightarrow S \bar{\otimes} B_i$ and the zero map $S \bar{\otimes} J_i \longrightarrow S \bar{\otimes} C_i$. The map $g\colon S \bar{\otimes} D_i \longrightarrow Q$ is induced by $id \bar{\otimes} \delta_i$ and $id \bar{\otimes} \rho_i$. Thus the diagram commutes and g is an isomorphism, which proves the claim.

Now let L be a unital C^*-algebra such that the following is a commutative square of unital C^*-algebras.

$$\begin{array}{ccc}
L & \overset{f}{\longrightarrow} & P(\rho_1, \rho_2) \\
g \downarrow & & \downarrow (\delta_1 \bar{\otimes} \gamma_2, \gamma_2 \bar{\otimes} \delta_2) \\
B_1 \bar{\otimes} B_2 & \underset{\chi_B}{\longrightarrow} & P(\varphi_1, \varphi_2)
\end{array}$$

We have to show that this square induces a unique map σ from L to $D_1 \bar{\otimes} D_2$ such that $f = (id_s \bar{\otimes} \rho_i) \circ \sigma$ and $g = (id_s \bar{\otimes} \delta_i) \circ \sigma$. Consider the following commutative diagram



The preceding shows that the lower left and the upper right squares are pullbacks. From the lower one we get a unique map $h: L \to D_1 \bar{\otimes} B_2$ fitting in the commutative diagram. The fact that $\delta_1 \bar{\otimes} \mathrm{id}_{B_2}$ and $\delta_1 \bar{\otimes} \mathrm{id}_{A_2}$ are monic now implies that $(\mathrm{id}_{D_1} \bar{\otimes} \varphi_2) \circ h = (\mathrm{id}_{D_1} \bar{\otimes} \gamma_2) \circ (p_2 \circ f)$. Using the upper pullback square we not get the unique map $\sigma: L \longrightarrow D_1 \bar{\otimes} D_2$ with the desired properties. ■

The cup product induces naturally several other multiplications. To define those we need the following well known fact.

VII. 6. LEMMA. Let A and B be C^*-algebras. Then we have a natural isomorphism $l_{np}: S^n(A) \bar{\otimes} S^p(P) \longrightarrow S^{n+p}(A \bar{\otimes} B)$.

Proof. Note that $S^n(A) = C_0(S^n) \bar{\otimes} A$ and $S^p(B) = C_0(S^p) \bar{\otimes} B$. Here S^n denotes the n-dimensional sphere with base point and $C_0(S^n)$. Moreover there is a natural isomorphism the complex valued continuous functions on S^n vanishing at the base point.

$\lambda_{n,p}: C_0(S^n) \bar{\otimes} C_0(S^p) \longrightarrow C_0(S^n \wedge S^p) = C_0(S^{n+p})$ where $\wedge$

denotes the wedge product. Now the map 1_{np} is given by the following composition of maps:
$C_0(S^n)\bar{\otimes}A\bar{\otimes}C_0(S^p)\bar{\otimes}B \xrightarrow{g} C_0(S^n)\bar{\otimes}C_0(S^p)\bar{\otimes}(A\bar{\otimes}B) \xrightarrow{\lambda_{np}\bar{\otimes}id_{A\bar{\otimes}B}}$
$C_0(S^{n+p})\bar{\otimes}(A\bar{\otimes}B)$. ■

Now we can define a cup product $\mu_{np}: K_n(A)\times K_p(B) \longrightarrow K_{n+p}(A\bar{\otimes}B)$ by $\mu_{np} = (1_{np})_0^* \circ \mu$ where we identify $K_n(A)$ with $K_0(S^nA)$, $K_p(B)$ with $K_0(S^pB)$ and $K_{n+p}(A\bar{\otimes}B)$ with $K_0(S^{n+p}(A\bar{\otimes}B))$.

$$\begin{array}{ccc} K_0(S^nA)\times K_0(S^pB) & \xrightarrow{\mu} & K_0(S^nA\bar{\otimes}S^pB) \\ & \searrow^{\mu_{np}} & \downarrow (1_{np})_0^* \\ & & K_0(S^{n+p}(A\bar{\otimes}B)) \end{array}$$

We know that $A\bar{\otimes}B$ is isomorphic to $B\bar{\otimes}A$, thus the question arises how it will effect the product μ_{np} if we switch the factors $K_n(A)$ and $K_p(B)$. To answer this question we have to study a group action of the symmetric group of order k on $K_k(B)$ for any $k\in\mathbb{N}$ and an arbitrary C^*-algebra B .

VII.7. LEMMA. Let B be a C^*-algebra. Define a map $T: SB \longrightarrow SB$ by $T(f)(t) = f(1-t)$. Then the induced map in K-theory $T_n^*: K_n(SB) \longrightarrow K_n(SB)$ is given by $T_n^*(u) = -u$.

Proof. Define $\check{B} := \{f: I\to B \text{ , continuous}\}$ as the C^*-algebra of paths in B . Then we get a short exact sequence $0 \longrightarrow SB \xrightarrow{j} \check{B} \xrightarrow{ev} B\bar{\oplus}B \longrightarrow 0$ where ev denotes evaluation at the endpoints. Thus we have $K_n(SB)\cong K_n(ev)$. It is easy to see that $\check{B}$ is homotopy equivalent to B and that after identifying $K_n(B)$

with $K_n(\check{B})$ the map $ev_n^*: K_n(B) \longrightarrow K_n(B \oplus B)$ is given by the diagonal map. From the long exact sequence we get a commutative diagram

$$\begin{array}{ccc} K_{n+1}(B \oplus B) & \xrightarrow{\partial_n} & K_n(SB) \\ \downarrow s & & \downarrow T_n^* \\ K_{n+1}(B \oplus B) & \xrightarrow{\partial_n} & K_n(SB) \end{array}$$

Here s denotes the map that switches summands. Moreover, we have $K_{n+1}(B) \xrightarrow{ev_{n+1}^*} K_{n+1}(B \ominus B) \xrightarrow{\partial_n} K_n(SB) \xrightarrow{j_n^*} K_n(B) \xrightarrow{ev_n^*} K_n(B \oplus B)$ exact. From the above we see that ev_n^* is injective, thus j_n^* is the zero map and hence ∂_n is surjective. For $u \in K_n(SB)$ we find a $v \in K_{n+1}(B \oplus B)$ such that $u = \partial_n(v)$. Let v be $v_1 \oplus v_2$ with $v_1, v_2 \in K_{n+1}(B)$, then $u + T_n^*(u) = \partial_n v + \partial_n(s(v)) = \partial_n(v + s(v)) = \partial_n(v_1 \oplus v_2 + v_2 \oplus v_1) = \partial_n((v_1+v_2) \oplus (v_1+v_2)) = \partial_n \circ ev_{n+1}^*(v_1+v_2) = 0$. Thus $T_n^*(u) = -u$. ■

Let Σ_n be the symmetric group of order n. It acts on $S^n = S^1 \wedge \ldots \wedge S^1$ by sending $[x_1 \ldots x_n] \in S^1 \wedge \ldots \wedge S^1$ to $[x_{\sigma^{-1}(1)} \ldots x_{\sigma^{-1}(n)}] \in S^1 \wedge \ldots \wedge S^1$. This action induces a group homomorphism $\tau_n: \Sigma_n \longrightarrow \mathrm{Aut}(C_0(S^n))$, given by $\tau_n(\sigma)(g[t_1 \ldots t_n]) = g[t_{\sigma(1)} \ldots t_{\sigma(n)}]$ for $\sigma \in \Sigma_n$ and $g \in C_0(S^n)$. For any C^*-algebra B we get a group homomorphism $\wedge: \Sigma_n \longrightarrow \mathrm{Aut}(S^n B) = \mathrm{Aut}(C_0(S^n) \bar{\otimes} B)$ by sending $\sigma \in \Sigma_n$ to $\hat{\sigma} = \tau(\sigma) \bar{\otimes} id_B$. This in turn induces a group homomorphism $\wedge^*: \Sigma_n \longrightarrow \mathrm{Aut}(K_0(S^n B))$.

VII.8. PROPOSITION. Let B be a C^*-algebra. Then the group homomorphism $\wedge^*: \Sigma_n \longrightarrow \mathrm{Aut}(K_0(S^n B))$ is given by $\hat{\sigma}^* = \mathrm{sign}(\sigma) \cdot id_{K_0(S^n B)}$.

Proof. It clearly suffices to show that $\hat{\sigma}^* = -\mathrm{id}$ for any transposition σ. So let σ_{ij} be the transposition that interchanges the i-th and j-th coordinates. We can assume that $i = 1$ and $j = 2$ since for $\alpha = (i1)(j2) \in \Sigma_n$, $\sigma_{ij} = \alpha\sigma_{12}\alpha^{-1}$, and if $\hat{\sigma}^*_{12} = -\mathrm{id}$, then $\hat{\sigma}^*_{ij} = \hat{\alpha}^*(-\mathrm{id})\hat{\alpha}^* = -\mathrm{id}$. The automorphism $\tau_n(\sigma_{ij}) : C_0(S^n) = C_0(S^2) \otimes C_0(S^{n-2}) \longrightarrow C_0(S^2) \bar{\otimes} C_0(S^{n-2})$ is given by $\tau_2(\sigma_{ij}) \bar{\otimes} \mathrm{id}_{C_0(S^{n-2})}$. Thus, replacing B by $S^{n-2}(B)$, we see that it suffices to consider the case $n = 2$. Now we view $S^2(B)$ as the continuous functions from $I^2 \longrightarrow B$ vanishing on the boundary. Then the action of the transposition σ on $S^2(B)$ is given by interchanging the arguments, i.e.
$\hat{\sigma}(g)(x,y) = g(y,x)$ for any $g \in S^2(B)$. Consider the homomorphism $f: \mathbb{R} \longrightarrow \mathrm{int}\, I$ given by
$f(x) = \frac{1}{2} + \frac{x}{2(1+|x|)}$. It satisfies $f(-x) = 1-f(x)$. Define two endomorphisms α_1 and α_2 of $S^2(B)$ by setting $\alpha_1(g)(x,y) = g(f(f^{-1}(x)+f^{-1}(y)), f(f^{-1}(y)-f^{-1}(x)))$ and $\alpha_2(g)(x,y) = g(f(f^{-1}(y)+f^{-1}(x)), f(f^{-1}(x)-f^{-1}(y)))$ for $(x,y) \in \mathrm{int}\, I^2$ and $\alpha_i(g)(x,y) = 0$ if $(x,y) \in \partial I^2$. It is routine to check that this definition makes sense. Note that α_1 is homotopic to the identity via
$\Phi_t(g)(x,y) = g(f(f^{-1}(x)+tf^{-1}(y)), f(f^{-1}(x)+tf^{-1}(y)))$.
Similarly we see that α_2 is homotopic to $\hat{\sigma}$. If we define $T: S^2(B) = S(SB) \longrightarrow S(SB) = S^2(B)$ by
$T(g)(x,y) = g(x,1-y)$ then $\alpha_2 = T \circ \alpha_1$ and Lemma VII.7 combined with the homotopy invariance of the functor K_0 show that $\hat{\sigma}^* = -\mathrm{id}_{K_0(S^2B)}$. ■

VII.9. PROPOSITION. Let A and B be C^*-algebras and $S: A \bar{\otimes} B \longrightarrow B \bar{\otimes} A$ the canonical isomorphism, then the following diagram commutes.

$$\begin{array}{ccc}
K_0(S^nA) \times K_0(S^pB) & \xrightarrow{\mu_{np}} & K_0(S^{n+p}(A \bar{\otimes} B)) \\
 & & \Big\downarrow (-1)^{up} \\
\Big\downarrow \text{switch} & & K_0(S^{n+p}(A \bar{\otimes} B)) \\
 & & \Big\downarrow S^{n+p*}_{(s)_0} \\
K_0(S^pB) \times K_0(S^nA) & \xrightarrow{\mu_{pn}} & K_0(S^{n+p}(B \bar{\otimes} A))
\end{array}$$

Proof. The diagram is induced by the following diagram

$$\begin{array}{ccccc}
(C_0(S^n)\bar{\otimes}A)\times(C_0(S^p)\bar{\otimes}B) & \longrightarrow & C_0(S^n)\bar{\otimes}A\bar{\otimes}C_0(S^p)\bar{\otimes}B & \longrightarrow & C_0(S^n\wedge S^p)\bar{\otimes}(A\bar{\otimes}B) \\
 & & & & \Big\downarrow \sigma\bar{\otimes}\mathrm{id}_{A\bar{\otimes}B} \\
\Big\downarrow \text{switch} & & & & C_0(S^p\wedge S^n)\bar{\otimes}(A\bar{\otimes}B) \\
 & & & & \Big\downarrow \mathrm{id}\bar{\otimes}s \\
(C_0(S^p)\bar{\otimes}B)\times(C_0(S^n)\bar{\otimes}A) & \longrightarrow & C_0(S^p)\bar{\otimes}B\bar{\otimes}C_0(S^n)\bar{\otimes}A & \longrightarrow & C_0(S^p\wedge S^n)\bar{\otimes}(B\bar{\otimes}A)
\end{array}$$

which is commutative if σ is the permutation that sends $[x_1 \ldots x_n, y_1 \ldots y_p]$ to $[y_1 \ldots y_p, x_1 \ldots x_n]$. The claim follows because sign $\sigma = (-1)^{np}$. ∎

The cup product can be used to provide the K-groups of an algebra with multiplicative structures. In fact, if, for two C^*-algebras A and B, there exists a C^*-morphism $m: A \bar{\otimes} B \longrightarrow B$, then $m \circ \mu: K_0(A)\times K_0(B) \longrightarrow K_0(B)$ is a bilinear map. If $A = B$ it is in fact a ring multiplication, if m is associative.

Now identify $K_{2n}(B)$ with $K_0(B)$ and $K_{2n+1}(B)$ with $K_1(B)$. We obtain a $\mathbb{Z}_2$-graded multiplication μ on $K_*(A) = \bigoplus_{i=0,1} K_i(A)$ cross $K_*(B) = \bigoplus_{i=0,1} K_i(B)$ as follows: for elements $a = (a_0 \oplus a_1) \in K_*(A)$ and $b = (b_0 \oplus b_1) \in K_*(B)$

define $\mu(a,b) = (\mu_{00}(a_0,b_0)+\mu_{11}(a_1,b_1)) \oplus (\mu_{10}(a_1,b_0)+\mu_{01}(a_0,b_1)) \in K_*(A\bar{\otimes}B)$. This multiplication is a bilinear map as follows from the bilinearity of the μ_{ij}.

VII.9. PROPOSITION. Let B be a C^*-algebra and A a subalgebra of the center $z(B)$ of B . Then $K_*(A)$ is a $\mathbb{Z}_2$-graded ring and $K_*(B)$ is a $\mathbb{Z}_2$-graded $K_*(A)$ module.

Proof. After the preceding remarks it suffices to show that there is a C^*-morphism $m: A\bar{\otimes}B \longrightarrow B$ such that the image of the restriction of m to $A\bar{\otimes}A$ is contained in A . Since $A \subset Z(B)$ we have a ring homomorphism $A\otimes B \xrightarrow{m} B$ given on elementary tensors by $m(a\otimes b) = ab$. By the universal property of the maximal cross norm on $A\otimes B$ this map extends to the maximal tensor product $A\otimes_\gamma B$ which is equal to $A\bar{\otimes}B$ since the algebras are nuclear. The rest is clear. ■

Finally note that for a C^*-morphism $\varphi: B \longrightarrow D$ and A a subalgebra of $Z(B)$ and $\varphi(A) \subset Z(D)$, the induced map $\varphi^*: K_*(B) \longrightarrow K_*(D)$ is a module map w.r.t. the rings $K_*(A)$ and $K_*(\varphi A)$. This follows from the commutativity of the following diagram:

$$\begin{array}{ccccc}
K_*(A)\times K_*(B) & \xrightarrow{\mu} & K_*(A\bar{\otimes}B) & \longrightarrow & K_*(B) \\
\downarrow{\scriptstyle \varphi^*\times\varphi^*} & & \downarrow{\scriptstyle (\varphi\otimes\varphi)_*} & & \downarrow{\scriptstyle \varphi^*} \\
K_*(\varphi A)\times K_*(D) & \xrightarrow{\mu} & K_*(\varphi A\otimes D) & \longrightarrow & K_*(D)
\end{array}$$

REFERENCES

[K]: Karoubi, M. *K-Theory, An Introduction* Springer, Berlin, 1978

[K_2]: Karoubi, M. *Foncteurs dérivés et K-théorie.* Springer Lecture Notes in Mathematics No. 136, Berlin 1970

[M]: Milnor, J. *Introduction To Algebraic K-Theory* Princeton University Press, Princeton 1971

[H]: Hilgert, J. *Foundations of K-Theory For C^*-Algebras,* Tulane Dissertation, 1982