SUBSEMIGROUPS OF LIE GROUPS

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von

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Preface

The present article is projected to be the core of chapter IV of a monograph entitled The Lie Theory of Semigroups in which the authors (K.H. Hofmann, J.D. Lawson and myself) want to give an account of the state of the art in this relatively new area, explain important problems and indicate possible directions in further research. The first three chapters will deal with the theory convex cones as needed in this context, a theory of certain classes of convex cones in real Lie algebras and the theory of local semigroups in Lie groups. The present chapter tries to put the available, not yet very systematic, information on global semigroups in Lie groups into an organic form. Whenever material from the ealier chapters is needed the reference will be marked by the respective roman number. References without roman numbers are references within chapter IV. We give a fairly extensive introduction to this chapter in which we try to give credit to those who used the relevant concepts and methods first. This is not always easy: First of all, many ideas were developed in discussions while working on joint papers or following other peoples presentations. But also, some of the sources in the literature need a lot of filling in and even corrections before they can be used, in spite of the substantial amount of information they provide.

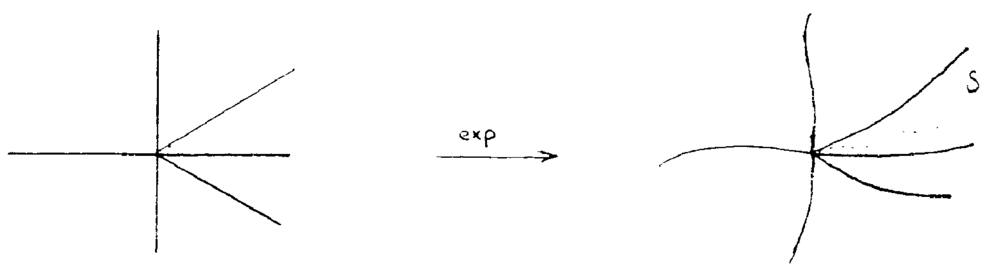
I would like to acknowledge the enormous influence the three years of collaboration with K.H. Hofmann have had on everything that is in this article. Moreover I would like to thank J.D. Lawson, J. Faraut, M. Mizony and L. Rothkrantz for many helpful discussions over the last two years.

Introduction

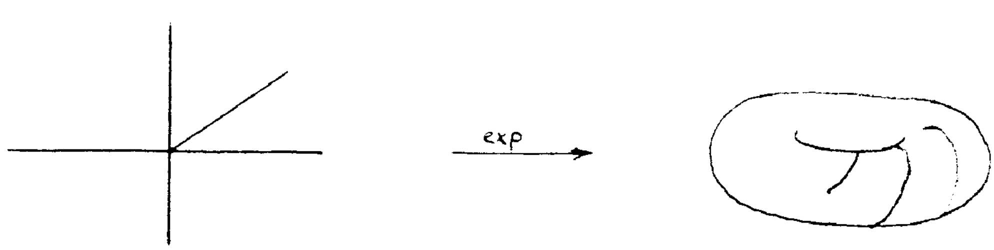
Section 1: Preanalytic semigroups and their tangent objects

In his lecture Topological Semigroups: History ,Theory, Applications , delivered at the 'Jahrestagung der Deutschen Mathematiker-Vereinigung' in 1975 , K.H. Hofmann advocated a programm in order to develop a Lie Theory of Semigroups. He left open what precisely this should be, but it was to be about semigroups with some sort of differentiable structure, a tangent object and an exponential function. It is clear that such a theory should somehow be able to deal with subsemigroups of Lie groups and it is natural to ask how general that situation is. Necessary conditions for a semigroup to be (locally) embeddable in a Lie group were for instance studied in [Hou73] and [Gr79] (cf. also [Gr83] and [Gr84]). The results in this direction are not yet completely satisfactory and there is still work going on, but nevertheless the current research concentrates on the study of (local) semigroups in Lie Groups. We will exclusively deal with this situation.

One of the main features in Lie group theory is the local identification of the Lie group and its tangent object, the Lie algebra, via the exponential function. It is this identification which allows to translate difficult analytical questions into more accessible algebraic questions, solve these and finally translate the solutions back into solutions of the analytical problem. Any reasonable Lie theory of semigroups should contain some variant of this translation mechanism. The starting point has be the definition of a tangent object. We want this tangent object of a given subsemigroup S of a Lie group G to be a subset $\underline{L}(S)$ of the Lie algebra $\underline{L}(G)$ of G. The natural procedure now is to pick a neighborhood U of the identity **1** in G which is a diffeomorphic image of a neighborhood B of zero in $\underline{L}(G)$ under the exponential function and then take as $\underline{L}(S)$ the set of limit directions of sequences s_n in $\exp^{-1}(S \cap U)$:



This is essentially the point of view of Hofmann and Lawson in [HL83a]. Vinberg, for special cases, uses a similar definition in [Vi80]. On account of the limiting process it makes no difference whether one considers S or its closure $\overline{S}$ in G as far as the tangent object is concerned, i.e. $\underline{L}(S) = \underline{L}(\overline{S})$. Actually, it is shown in [HL83a] that $\underline{L}(S) = \{ x \in \underline{L}(G) : \mathbb{R}^+ x \cap B \subset \exp^{-1}(\overline{S} \cap U) \}$. Since S is a semigroup this shows also $\underline{L}(S) = \{x \in \underline{L}(G) : \exp \mathbb{R}^+ x \in \overline{S}\}$. Ol'shanskii uses this as a definition for the tangent object in [Ol82a]. There are no problems with this definition as long as S is closed. But if we consider the semigroup consisting of one half of dense one parameter subgroup of a torus, we see that the above concept does not yield any good information even though there is a natural candidate for a tangent object:



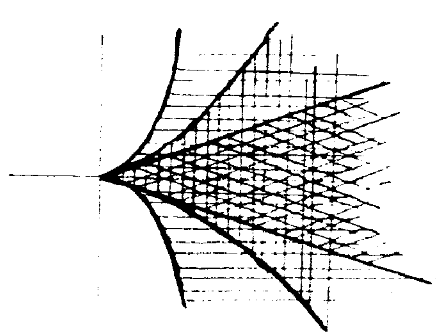
One way around this problem would be to define the tangent object as $\{x \in \underline{L}(G): \exp R^+ x \in S\}$, but doing this one looses more than gains. In fact the tangent object as defined before is a wedge, i.e. a closed convex set which is in addition stable under multiplication by positive scalars. Moreover this tangent wedge satisfies

$$(*) \qquad e^{ad(x)}\underline{L}(S) = \underline{L}(S) \quad \text{for all } x \in \underline{L}(S) \cap -\underline{L}(S)$$

a property which turns out to be crucial for the theory and cannot be dismissed (cf.[HH86b]). But all of this depends on the closure process, so the example of the dense wind suggests to take the closure in a smaller group. This path, suggested in [HL83a], is followed in the present article: We will call a subsemigroup S of a Lie group G preanalytic if the group G(S) generated by S in G is analytic. For a preanalytic semigroup S we define the tangent object as $\underline{L}(S) = \{ x \in \underline{L}(G): \exp R^+ x \subset cl(S)\}$ where cl(S) is the closure of S in G(S). With this definition $\underline{L}(S)$ is a wedge satisfying the property (*). We call $\underline{L}(S)$ the tangent wedge of S.

Section 2: Ray semigroups and infinitesimally generated semigroups

The definition of the tangent wedge is the first necessary device to translate problems from the analytical to the algebraical level (and we see already that in the case of semigroups there will be also a geometrical aspect). It is easy to see that, in contrast to the group situation a preanalytic semigroup is not uniquely determined by is tangent wedge:



Therefore the process of translating the solutions of algebraic and geometric problems into solutions of the original analytic problem can only work satisfactorily if the semigroup S can somehow be recovered from its tangent wedge. The most obvious class of semigroups to look at then are the ray-semigroups . Here we call a semigroup S a ray-semigroup if it is generated by exp(M) where M is in $\underline{L}(G)$. It is this class of semigroups which is of interest in geometric control theory since the problem of reachability and controllability of systems on Lie groups and homogeneous spaces translates into the problem of deciding which semigroup is generated by exp(M) for some subset M of $\underline{L}(G)$. Control theorists realised that they could assume M to be a wedge and also came across property (*)(cf.[Hir83],[JK81],[JS72]). They also observed that the semigroup generated by a set M (and hereby we mean the semigroup generated by exp(M)) contains interior points if and only if M generates $\underline{L}(S)$ as a Lie algebra. Since the existence of interior points is of eminent importance, this gives a hint which classes of semigroups, other than ray-semigroups, are to be considered for our purpose. For, if one looks for instance at an open quadrant in $\mathbb{R}^2$, then this is a ray-semigroup whose tangent wedge is the corresponding closed quadrant. Thus only the closure of this semigroup can be recovered from its tangent wedge. In [HL83a] this kind of pathology is excluded by only considering semigroups which are generated by $\underline{L}(S)$. Such semigroups they call analytic . By what was said above analytic semigroups contain interior points when considered in G(S) and can be fully recovered from their tangent wedges. Some nice results have been proven in [HL83a] for analytic semigroups, but one has to be aware of the fact that the closure of an analytic semigroup is not always analytic, even if the closure is taken only in G(S)(cf.[HL83a]).

This is not a very satisfactory situation, since our defintion of the tangent wedge depends on the closure and we don't want to fall out of our class of semigroups under consideration already in the most basic definitions. Moreover, for technical reasons it is often convenient to consider closures so that we would very soon have to leave our proper domain of interest. Thus we enlarge the class of semigroups under consideration as follows: A preanalytic subsemigroup S of a Lie group G is called *infinitesimally generated* if exp($\underline{L}$(S)) generates G(S) as a group, if S contains exp($\underline{L}$(S)) and if S is contained in the closure of the semigroup generated by $\underline{L}$(S) where the closure is taken in G(S). A subsemigroup is called *strictly infinitesimally generated* if it is analytic in the sense of Hofmann and Lawson. The concepts of ray-semigroups, analytic semigroups and infinitesimally generated semigroups are very close - a statement we shall make precise in this article.

Let us pause here to draw a short resumé of what has been said up to now: The program is to set up a Lie theory of semigroups including the possibility to translate analytic problems into algebraic and geometric problems, solve them and translate the solutions back to solutions of the original problem. It is by no means obvious what the basic objects of our study should be and all the definitions one has come up with until now are subject to immediate criticism drawn from the inability of the concepts to deal with certain natural examples. Thus one has to make a choice and for this article it will be, that object of our study is the class of infinitesimally generated semigroups.

Section 3: Groups associated with infinitesimally generated semigroups

After one has decided which class of semigroups one wants to study and has made the basic definitions, it is natural to study the groups associated with these semigroups. There are two groups which are naturally associated with any subsemigroup S of a group G. On the one hand there is the group G(S) generated by S, and on the other hand there is the biggest group $S \cap S^{-1}$ contained in S. As for the infinitesimally generated subsemigroups of a Lie group the group G(S) poses no particular problems since it is build into the definition of infinitesimal generation. Thus, it turns out that G(S) is the analytic subgroup of G corresponding to the Lie algebra generated by $\underline{L}(S)$ in $\underline{L}(G)$. It is much harder to get hold of $H(S) = S \cap S^{-1}$. Hofmann and Lawson showed in [HL83a] that, for analytic semigroups H(S) is the analytic subgroup of G corresponding to the Lie algebra $\underline{L}(S) \cap -\underline{L}(S)$ in $\underline{L}(G)$. Moreover they show that H(S) is closed. This last fact is an almost immediate consequence of the closure in the definition of the tangent wedge and hence again gives rise to criticism of that definition. It turns out the the same results are true for infinitesimally generated semigroups, but they are considerably harder to prove. The key result that has to be added to the techniques of local sections from [HL83a] is the fact that for infinitesimally generated semigroups we can find arbitrarily small neighborhoods U of H(S) in S such that $S \backslash U$ is a onesided semigroup ideal in S, i.e. $(S)(S\backslash U) \subset S\backslash U$ or $(S\backslash U)(S) \subset (S\backslash U)$ depending on whether we want to talk about left or right ideals. This result turns out to have several nice and useful applications in the theory of infinitesimally generated semigroups.

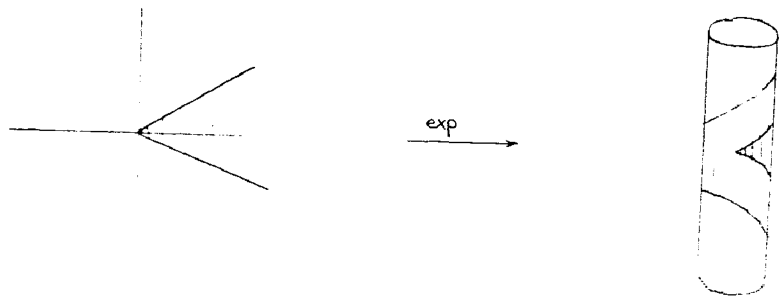
Section 4: Functorial Properties

There is not too much that can be said at this point about functorial properties of infinitesimally generated semigroups, but there are a few remarks in place concerning preimages and semidirect products of such semigroups. These remarks will be useful in study of examples.

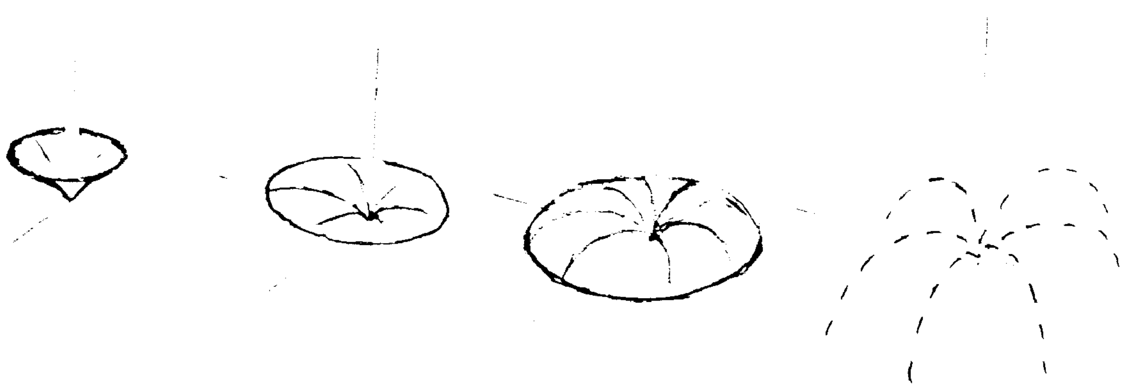
Section 5: Examples

It is obvious that in order to develop a new theory one needs to have a good knowledge of examples in order to develop a good intuition and not to be trapped in too bold conjectures. We start out with examples which are very easy from the point of view of Lie groups and Lie algebras, but which show already that it is by no means sufficient to take a wedge W satisfying the equation (*)

above and consider the semigroup S generated by W in order to get an infinitesimally generated semigroup with tangent wedge W. This strongly contrasts the situation of local semigroups (cf.[HH86b]). In fact it turns out very quickly that there are two different reasons why a Lie wedge ,i.e. a wedge satisfying (*), cannot be obtained as the tangent wedge of a preanalytic, and then of an infinitesimally generated, semigroup. The first is of topological nature, as exemplified in a cylinder contrasting $\mathbf{R}^2$:



One can avoid this kind of obstruction for instance by only considering simply connected Lie groups, but there remains a second type of obstruction which is of algebraic nature and occurs for the first time in the Heisenberg group (cf.[HL81],[HHL85]): Here any wedge containing a central point in its interior generates the whole Heisenberg group as a semigroup. This is all the more surprising since there are local semigroups up to a certain size having such a wedge as a tangent wedge. These local semigroups may be viewed as tilting over when they are made bigger:

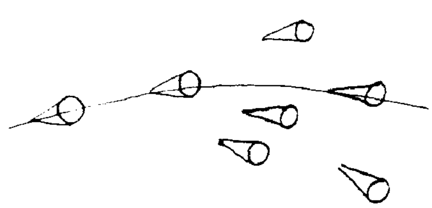


We also include some examples of a more complicated nature such as certain subsemigroups of the oscillator group (cf.[Hi86d]) and a fairly detailed study of subsemigroups in Sl(2,$\mathbf{R}$) (cf.[HH85b]).

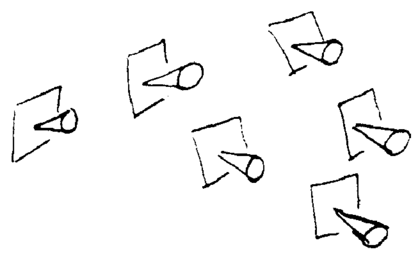
Section 6: Global Lie wedges

It has been stressed above that we need to be able translate properties from the global to the local situation and vice versa. Thus, after having established the fact that tangent wedges of preanalytic semigroups are Lie wedges, the first question in view of the examples will be: Which Lie wedges are tangent wedges of infinitesimally generated semigroups? This question is far from being solved in general and to simplify matters one may at first restrict oneself to the situation of simply connected Lie groups. We call a Lie wedge global if it is the tangent wedge of an infinitesimally generated semigroup in a simply connected Lie group. Again there are some minor technical difficulties depending on whether the wedge generates the Lie algebra or not.

The main concepts used to deal with the question of globality in this section are those of left invariant wedge fields, admissible curves and positive functions. All these concepts have been used before in specialized contexts (cf.[Vi80], [Ol82a,b], [HH86b]), and we expect them to play a very important role in the general theory to be built. The basic idea behind all of this is to associate in a differentiable way to each point of a homogeneous space G/H a wedge in the corresponding tangent space and call piecewise differentiable curve admissible if its derivative at each point where it exists sits in the right wedge:



A positive function is then a C^1- function from G/H into the reals such that the differential at each point is positive on the wedge at this point (we draw the kernel of the differential):



Given this machinery one tries to construct semigroups by considering end points of admissible curves in G/H and take their inverse image in G. At this point one can see why one has to undergo the trouble of dealing with homogeneous spaces: this is needed to construct in a "controlled fashion" semigroups with nontrivial subgroups. It should be noted here that the possibility of constructing semigroups in this way is tied up very closely with the existence of positive functions. In order to get such functions we use the well known ordertheoretical concept of causality and the close relation between order preserving and positive functions (cf.[Ol82b]).What comes out of all of this is the fact that once one has a global Lie wedge W every Lie wedge W' which is properly surrounded by W, i.e. $W'\setminus(W'\cap -W') \subset \mathrm{int}W$ (cf.[HL83a]), is again global provided the analytic subgroup H' corresponding to $W'\cap -W'$ is closed (this last requirement has to be expected after the results of section 3; cf.[Hi86a] for a special case). The condition that W has to surround W' can not be dropped but it is possible to derive similar results from slightly different situations ([Hi86c]).

The results described above suggest to study subsemigroups which have an as big as possible tangent wedge. The biggest possible wedge which is not all of the Lie algebra is halfspace. If a halfspace is a Lie wedge its bounding hyperplane is a Lie algebra. This is a very special situation and the analysis of old results (cf.[Ho65]) allow us to show that any halfspace Lie wedge is global. Using some general reasoning on global wedges and results from [HH85c] one can now see that all semialgebras, i.e. tangent wedges of locally divisible local semigroups (cf.[HH85c] for the exact definitions), which are of dimension less or equal to three, are global. Results of this type are important on the one hand since the low dimensional examples have always played an important role even in proofs (cf.[HH85a,d],[HHL85]) and on the other hand since they provide further examples which may give useful insights in general case. Along these lines we use the methods developed so far to show that any Lorentzian cone which is a semialgebra is global (cf.[Hi86c] for an ordertheoretical interpretation of this fact).

There is another situation where one obtains globality results with relative ease. If W is a wedge in an associative algebra A such that $WW \subset W$ then W, considered as a wedge in the Lie algebra L_A coming from A is global. This observation, due to Hofmann and Lawson, has not really been exploited yet, but it should play an important role in the future, especially when representations are studied.

Section 7: Maximal semigroups in Lie groups

It has been mentioned that one is lead to study semigroups with as big as possible tangent wedge. Semigroups whose tangent wedge is a halfspace can be characterized by the property that their topological boundary is a subgroup distinct from the semigroup itself (cf.[Po77],[Do76]). From this characterization and some results related to the ones in [Ho65], mentioned above, allow a classification of such semigroups up to a maximal normal subgroup contained in the semigroup (cf.[HM68],[Po77]).

It is by no means true that all maximal proper subsemigroups have halfspaces as tangent wedges even if they contain interior points. The first case where this fails to be true is Sl(2,R) as was remarked in [HH85b]. Nevertheless it is true for large classes of Lie groups. Using a general machinery for maximal semigroups in Lie groups as developed by J.D.Lawson it was shown in [HHL85] that the closure of a maximal open proper subsemigroup of a nilpotent Lie group is always strictly infinitesimally generated with a halfspace as tangent wedge. Moreover this tangent wedge contains the whole commutator subalgebra. Using some methods developed in [BJKS82] we generalize this result to groups which are semidirect products of a compact group with a nilpotent normal subgroup and complex solvable groups. Finally we show how this result can be used to solve problems from geometric control theory as described above (cf.[Hi86b]). We note that Lawson has recently been able to generalize our result to semidirect products of compact groups with a solvable normal subgroup (without the statement about the commutator algebra which is wrong in this case).

Section 8: Divisibility and local divisibility

A semigroup S is called divisible if for each $s \in S$ and $n \in \mathbf{N}$ there is a $g \in S$ such that $g^n = s$. We know from the local theory that for a local semigroup (S,U) in a Lie group the analogous statement holds if and only if the tangent wedge is a semialgebra (cf.[HL83a]). The example of Sl(2,R) shows that not every semigroup generated by a semialgebra is divisible. On the other hand no example of a divisible subsemigroup of a Lie group whose tangent wedge is not a semialgebra is known. Hofmann and Lawson have shown in [HL83b] that such an example cannot exist if the semigroup is not allowed to contain a non trivial subgroup . Using the techniques developed in this article we give a proof of this fact which is on the one hand a little shorter than the original proof and on the other hand will hopefully give rise to a proof in the general situation.

Chapter IV: SUBSEMIGROUPS OF LIE GROUPS

In contrast to the basic Lie theory for *local* subsemigroups of Lie groups which was described in Chapter III the Lie theory of *global* subsemigroups of Lie groups still resists to be put into a satisfactory systematic form. The purpose of this chapter is to give an account of results and examples known, to show some connections and to suggest a path to follow in further research.

We restrict our interest to finite dimensional Lie groups even though some of the concepts introduced below make perfect sense also for infinite dimensional Lie groups. As far as the subsemigroups are concerned we will concentrate on certain classes which will allow us to build a Lie theory in the sense that they admit a reasonably big tangent object which reflects important properties of the subsemigroups. These classes are introduced in the first two sections.

Section 1: Preanalytic semigroups and their tangent objects

Recall that a subgroup A of a Lie group G is called *analytic* if and only if there is a connected Lie group A_L and an injective morphism $f: A_L \to G$ of Lie groups with $A = f(A_L)$. We write $\underline{L}(A) = \underline{L}(f)(\underline{L}(A_L))$ where $\underline{L}$ is the functor which associates with a Lie group its Lie algebra. We will usually identify the underlying groups of A_L and A via f and thus write A_L for "A with its Lie group topology"; likewise we identify $\underline{L}(A_L)$ and $\underline{L}(A)$ via $\underline{L}(f)$: in this case the exponential function $\exp_{A_L}: \underline{L}(A_L) \to A_L$ is just the restriction of the exponential function $\exp_G: \underline{L}(G) \to G$ to $\underline{L}(A_L)$. The assignment $A \mapsto \underline{L}(A)$ is a bijection from the set of all analytic subgroups A of G onto the set of all Lie subalgebras $\underline{L}(A)$ of $\underline{L}(G)$. The unavoidable difficulty with this part of Lie theory is that not all analytic subgroups are closed. The simplest example is the subgroup $A = \{(x, x\sqrt{2}) + \mathbb{Z}^2 : x \in \mathbb{R}\}$ in the 2-torus $G = \mathbb{R}^2/\mathbb{Z}^2$. For $f: A_L \to G$ we can take the function $x \mapsto (x, x\sqrt{2}) + \mathbb{Z}^2 : \mathbb{R} \to G$, so that the Lie group topology of A makes A isomorphic to $\mathbb{R}$; but A is dense in G. One calls A "a dense winding subgroup of the torus" or in short a "dense wind".

If this complication arises already on the level of group theory, we certainly will have to take precautions in the case of any Lie theory for subsemigroups of Lie groups that this complication is adequately covered. A fundamental theoretical tool is the following nontrivial fact on analytic subgroups in (finite dimensional) Lie groups due to Yamabe (cf. [Ya 50]).

THEOREM IV.1.1: A subgroup of a (finite dimensional) Lie group G is analytic if and only if it is arcwise connected.

This brings us to the following definition:

DEFINITION IV.1.2: A subsemigroup S of a Lie group G is called *preanalytic* if and only if the subgroup $\langle S\rangle_{Gr}$ generated by S in G is arcwise connected. We will write $G(S)$ for this group with its Lie group topology. In particular $\underline{L}(G(S))$ is a well-defined Lie subalgebra of $\underline{L}(G)$.

We not that the closure $\overline{\langle S\rangle}_{Gr}$ of $\langle S\rangle_{Gr}$ is a closed connected Lie subgroup of G, and as far as S is concerned, for most purposes we can restrict our attention to the case that this group is G. We observe that, in this case, $\underline{L}(G(S))$ is an ideal in $\underline{L}(G)$ and $G(S)$ is normal in G, in fact $G(S)$ contains the commutator group of G (cf. [Bou 72]). In many questions we can assume that we are working in $G(S)$; if we do this, however, we should recall that a refinement of the topology may have intervened.

Theorem 1.1 allows us to make the following simple remark.

REMARK IV.1.3: (i) Every arcwise connected semigroup containing the identity in a Lie group is preanalytic.
(ii) Every semigroup in a Lie group is preanalytic if it has non-empty interior.

EXAMPLE IV.1.4: (i) Let $G = \mathbb{R}^2$ and $S = \{(x,y) \in G: x = y = 0 \text{ or } x,y > 0\}$, then S is a preanalytic semigroup.
(ii) Let $G = \mathbb{R}^2/\mathbb{Z}^2$ be the torus and $S = \{(x,y) + \mathbb{Z}^2 \in G: y = x\sqrt{2},\ x \geq 0\}$, then S is a preanalytic semigroup in the torus such that $G(S)$ is isomorphic to $\mathbb{R}$.
(iii) Let G and S be as in (ii) and set $T = \{(x,y) + \mathbb{Z}^2 \in G: y = x\sqrt{3},\ x \geq 0\}$, then T is another dense winding pre-

analytic one parameter subsemigroup of the torus. Then $S \cap T$ is a dense subsemigroup of the torus which is not preanalytic. In particular it does not determine a unique smallest preanalytic subsemigroup containing it, since the two one-parameter subsemigroups S and T intersect precisely in it.

We now proceed to associate with any preanalytic subsemigroup S of a Lie group G a tangent object. Note that for any BCH-neighborhood B in $\underline{L}(G)$ on which $\exp: \underline{L}(G)\ G$ is injective the set $S_B = \exp^{-1}(S \cap \exp B)$ is a local semigroup with respect to B. Since the tangent object of the semigroup S should be locally determined we want it to be determined by the local semigroup S_B. Moreover if S happens to be an analytic subgroup of G the tangent object should be just the Lie algebra of S. This leads to the following definition:

DEFINITION IV.1.5: Let S be a preanalytic subsemigroup of a Lie group G. We consider the exponential function $\exp_{G(S)}: \underline{L}(G(S)) \to G(S)$ and call $\underline{L}(\exp^{-1}_{G(S)}(S))$ the *tangent wedge* of S (cf. I.3.1). It will be denoted by $\underline{L}(S)$.

Discuss how Definition 1.5 would change if one would replace $\exp_{G(S)}$ by $\exp_G: \underline{L}(G) \to G$ using the example of the "dense wind".

REMARK IV.1.6: Definition IV.1.5 implies that for any BCH-neighborhood B in $\underline{L}(G(S))$ we have $\underline{L}(S) = \underline{L}(\exp^{-1}_{G(S)}(S \cap \exp_{G(S)} B))$ so that $\underline{L}(S)$ is a Lie wedge by chapter III since $\exp^{-1}_{G(S)}(S \cap \exp_{G(S)} B)$ is a local semigroup. □

Next we aim for an alternative description of $\underline{L}(S)$ analogous to Proposition 3.2

PROPOSITION IV.1.7: Let S be a preanalytic subsemigroup S of a Lie group G and $x \in \underline{L}(G)$, then the following statements are equivalent:

(1) $x \in \underline{L}(S)$

(2) $\exp \mathbb{R}^+ x \subseteq \mathrm{cl}_{G(S)} S$ where $\mathrm{cl}_{G(S)} S$ denotes the closure of S in $G(S)$.

Proof: Verbatim the proof of III.3.2 with $B = \underline{L}(G(S))$. □

Again we see that it is important to note that in condition 1.7(2) we use the closure of S in G(S) and not in G; the latter may be bigger (think of the dense wind).

Proposition 1.7 suggests to study $\{x \in \underline{L}(G): \exp \mathbb{R}^+ x \subseteq S\}$ as the, possibly more appropriate, tangent object for S. But it is not even clear whether this is a Lie wedge.

Section 2: Ray semigroups and infinitesimally generated semigroups

It has been pointed out that a Lie theory for subsemigroups of Lie groups should be dealing with semigroups whose properties are determined to some extent by their behaviour in a neighborhood of the identity. One class of such subsemigroups are those which are generated by one-parameter-semigroups.

DEFINITION IV.2.1: A subsemigroup S of a Lie group G is called a *ray semigroup* if S is generated (algebraically) by a family of one-parameter-semigroups, i.e. if there is a subset $K \subseteq \underline{L}(G)$ such that $S = \langle \exp \mathbb{R}^+ K \rangle$.

Note that ray semigroups are arcconnected, hence preanalytic. Therefore we can talk about the tangent wedge of a ray semigroup and see that Proposition 1.7 implies that the closure of a ray semigroup S (in G(S)) is completely determined by its tangent wedge.

Show that the semigroups from Examples 1.4(i) and (ii) are ray semigroups and determine their tangent wedge.

Ray semigroups have an important feature which makes them handy to work with: They have a big interior. There are several ways to prove this. One method comes from differential geometry and is essentially due to E. Cartan. We use a category argument as it is given in [JS 72].

THEOREM IV.2.2: Let G be a Lie group and S be a ray semigroup in G such that G = G(S). Then we have

(i) $(\text{int } S)^- = \overline{S}$

(ii) $(\text{int } \overline{S}) = \text{int } S$.

Proof: We split the proof of Theorem 2.2 into a few lemmas which are of separate interest.

LEMMA IV.2.3: Let S be a ray semigroup in G such that $G(S) = G$ and Σ be a family of one-parameter semigroup generating S. Then some finite subset $\Gamma \subseteq \Sigma$ generates G as a group.

Proof: Let the dimension of G be n. Let Δ be any non-empty finite subset of Σ. Let H be the subgroup generated by Δ, then H is a connected analytic subgroup of G determined by a Lie subalgebra $\underline{L}(H)$ of $\underline{L}(G)$. We write $n(\Delta) = \dim \underline{L}(H)$. Thus the function $\Delta \mapsto n(\Delta)$ is defined on the set of finite subsets of Σ and takes values in $\{1,2,\ldots, \dim G\}$. This function attains its maximum at Δ, say. Suppose $n(\Delta) < \dim G$.

Then H is proper in G, since $\dim \underline{L}(H) < \dim \underline{L}(G)$. Therefore $\Delta \neq \Sigma$, otherwise $S \subseteq H$, whence $H = G$. Now pick $\sigma \in \Sigma \backslash \Delta$ such that $\sigma(\mathbb{R}^+) \nsubseteq H$. Let K be the subgroup generated by $H \cup \sigma(\mathbb{R}^+)$. Again we have that K is a connected analytic group and $\underline{L}(H) + \mathbb{R}x \subseteq \underline{L}(K)$ with $\sigma(t) = \exp tx$ for $t \in \mathbb{R}$. Thus $n(\Delta \cup \{\sigma\}) = \dim \underline{L}(K) > \dim \underline{L}(H) = n(\Delta)$, contradicting maximality of $n(\Delta)$. □

LEMMA IV.2.4: Let S be a ray semigroup in G such that $G(S) = G$ then for any open neighborhood U of $\mathbf{1}$ in G we have $U \cap \operatorname{int} S \neq \emptyset$, where $\operatorname{int} S$ denotes the interior of S.

Proof: Lemma 2.3 shows that we can find a finite set $\{x_1,\ldots,x_k\}$ in $\underline{L}(G)$ such that G is generated by the one parameter groups $\gamma_j(t) = \exp tx_j$ for $j = 1,\ldots,k$. This means that

$$G = G(S) = \bigcup_{n\in\mathbb{N}} (\Gamma \cap U)^n$$

where $\Gamma = \bigcup_{j=1}^{k} \gamma_j(\mathbb{R})$. Now the Category Theorem shows that there is an $n_o \in \mathbb{N}$ such that $(\Gamma \cap U)^{n_o}$ has non-empty interior.

Next consider the map $F: \mathbb{R}^{kn_o} \to G$ defined by

$\vec{t} = (t_{1,1},\ldots,t_{1,k},t_{2,1},\ldots,t_{n_o k}) \mapsto \gamma_1(t_{1,1}) \cdot \ldots \cdot \gamma_k(t_{1,k}) \cdot \ldots \cdot \gamma_k(t_{n_o,k})$.

We then know that F is (real) analytic and its image has non-empty interior. Thus by Sard's Theorem there is a $\vec{t}_o \in \mathbb{R}^{kn_o}$ such that the derivative $dF(\vec{t}_o): \mathbb{R}^{kn_o} \to T_{F(\vec{t}_o)}$, where $T_{F(\vec{t}_o)}$ is the tangent space to G in $F(\vec{t}_o)$, has full rank. But then the analyticity of F implies that the set $\{\vec{t} \in \mathbb{R}^{kn_o} : dF(\vec{t})$ has full rank$\}$ is dense in $\mathbb{R}^{kn_o}$, and hence any neighborhood B_o of zero in $\mathbb{R}^{kn_o}$ contains an element $\vec{t}_1 \in (\mathbb{R}^+)^{kn_o}$ and a neighborhood of zero B_1 such that $B_1 + \vec{t}_1 \subseteq (\mathbb{R}^+)^{kn_o} \cap B_o$ and $F(\vec{t}_1 + B_1)$ contains a neighborhood of $F(\vec{t}_1)$ in G. But since $F((\mathbb{R}^+)^{kn_o}) \subset S$ this proves the lemma. □

LEMMA IV.2.5: Let S be a subsemigroup of a topological group G then $(\text{int } S)S \subset \text{int } S$, i.e. the interior of S is a semigroup ideal of S.

Proof: Let $g \in \text{int } S$ and U be a neighborhood of g which is contained in S, then for any $s \in S$ we have $gs \in Us \subseteq S$. But since G is a topological group Us is a neighborhood of gs so that $gs \in \text{int } S$. □

Now we are ready to prove Theorem 2.2 . Note first that for $s \in S$ and a neighborhood U of the identity $\mathbb{1}$ in G we obtain by Lemma 2.4 that $U \cap \text{int } S \neq \emptyset$ and hence by Lemma 2.5 that $sU \cap \text{int } S \neq \emptyset$. Therefore $s \in (\text{int } S)^-$, i.e. $S \subset (\text{int } S)^-$. But then also $\overline{S} \subset (\text{int } S)^-$ so that Theorem 2.2(i) is proved. In order to show $\text{int } S = \text{int } \overline{S}$ consider $s \in U \in \overline{S}$ where U is open. Then there exists an open set V containing $\mathbb{1}$ such that $sV^{-1} \subseteq U$. Let $W = V \cap \text{int } S$ then $sW^{-1} \subseteq U \subseteq \overline{S}$ (and $W \neq \emptyset$ by 2.2(i)); since sW^{-1} is open, there exist $t \in S$ and $w \in W$ such that $sw^{-1} = t$. But then $s = tw \in tW \subseteq \text{int } S$ which proves the claim. □

We have seen above that the Example 1.4(i) is a ray semigroup. Note further that in this example $\exp(\underline{L}(S))$ is not contained in S. We know from 1.7 that $\exp_{G(S)}(\underline{L}(S)) \subseteq \text{cl}_{G(S)}S$, but the closure of a ray semigroup need not be a ray semigroup as we will see in the example section. In order to avoid such pathologies one considers another class of subsemigroups of Lie groups which still can be dealt with by Lie theoretic methods and which turns out to be close to the class of ray semigroups in a very precise sense.

DEFINITION IV.2.6: A subsemigroup S of a Lie group G is called *infinitesimally generated* if

(i) S is preanalytic.

(ii) $\exp(\underline{L}(S))$ generates $G(S)$ as a group.

(iii) $\exp(\underline{L}(S)) \subseteq S \subseteq \text{cl}_{G(S)} \langle \exp \underline{L}(S) \rangle$ where $\langle \exp \underline{L}(S) \rangle$ again denotes the semigroup generated by $\exp \underline{L}(S)$.

The semigroup S is called *strictly infinitesimally generated* if $S = \langle \exp \underline{L}(S) \rangle$.

Condition 2.6(ii) has been introduced in order to assure that the group generated by S can be recovered from the Lie algebra generated by $\underline{L}(S)$. How this is done will be shown in the next section. We do not know whether condition 2.6(ii) is a consequence of 2.6(i) and (iii).

REMARK IV.2.7: Every infinitesimally generated subsemigroup S of a Lie group G contains a unique smallest strictly infinitesimally generated subsemigroup with the same Lie wedge, namely $\langle \exp \underline{L}(S) \rangle$. Every strictly infinitesimally generated semigroup is a ray semigroup.

The relation between ray semigroups and infinitesimally generated semigroups is described in the following

THEOREM IV.2.8: Let T be an arbitrary subsemigroup in a Lie group G and let T_R be the ray-semigroup generated by all one-parameter subsemigroups of T. Then there exists a unique strictly infinitesimally generated subsemigroup $S = \langle \exp \underline{L}(T_R) \rangle$ such that the following conditions are satisfied:

(i) $T_R \subseteq S \subseteq T_R^* = \mathrm{cl}_{G(S)} T_R \subseteq \overline{T}_R = \mathrm{cl}_G(T_R)$

(ii) $\underline{L}(T_R) = \underline{L}(S) = \underline{L}(T_R^*)$

(iii) $G(T_R) = G(S)$

(iv) $\mathrm{int}_{G(S)}\, T_R = \mathrm{int}_{G(S)} S = \mathrm{int}_{G(S)}\, T_R^*$.

Proof: Note first that T_R is contained in S by the very definition of S. On the other hand $\exp \underline{L}(T_R)$ is contained in $G(T_R)$ by the definition of $\underline{L}(T_R)$ so that $G(T_R) = G(S)$. But now Proposition 1.7 shows that $\exp \underline{L}(T_R) \subset \mathrm{cl}_{G(S)} T_R = T_R^*$ and hence also $S \subseteq T_R^*$. Moreover this proposition shows that $\underline{L}(T_R) = \underline{L}(T_R^*)$ so that we can conclude $\underline{L}(T_R) = \underline{L}(S) = \underline{L}(T_R^*)$. Thus it only remains to show (iv). But this follows directly from Theorem 2.2(ii) since T_R is a ray semigroup. □

If we choose T in Theorem 2.8 to be closed we can say even more:

PROPOSITION IV.2.9: Let T be a closed subsemigroup in a Lie group G which contains the identity. Then the semigroup T_R generated by all one-parameter semigroups in T is strictly infinitesimally generated.

Proof: Since T is closed we have $cl_{G(T_R)}T_R \subset (G(T_R) \cap T)^- \cap G(T_R) = T \cap G(T_R)$. Therefore, if a one-parameter subsemigroup lies entirely in $cl_{G(T_R)}T_R$, then it lies in T and hence in T_R by the definition of T_R. Thus Proposition 1.7 implies that $\exp \underline{L}(T_R) \subset T_R$. But this just means that T_R is generated by $\exp \underline{L}(T_R)$ so that T_R is strictly infinitesimally generated. □

If we start out with an infinitesimally generated semigroup we obtain the same result:

REMARK IV.2.10: *If T is a infinitesimally generated subsemigroup in a Lie group G, then the semigroup T_R generated by all one-parameter semigroups in T is equal to the semigroup generated by $\exp(\underline{L}(T))$.*

Proof: Let S be the semigroup $\langle\exp(\underline{L}(T_R))\rangle$ generated by $\underline{L}(T_R)$. Since $\exp(\underline{L}(T)) \subseteq T$ we have $S = \langle\exp \underline{L}(T_R)\rangle \subset \langle\exp \underline{L}(T)\rangle \subset T_r \subset S$ by Theorem 2.8. □

Section 3: Groups associated with infinitesimally generated semigroups

The study of groups associated with a given semigroup is an important feature in semigroup theory. In particular if S is a subsemigroup of a group G there are two groups which are in a natural way associated with S. There is the subgroup $G(S)$ generated by S on the one hand and the group of units $H(S) = S \cap S^{-1}$ contained in S on the other hand. Since we are dealing with infinitesimally generated semigroups here, we know a priori that $G(S)$ is an analytic subgroup. Therefore it suffices to know $\underline{L}(G(S))$ in order to identify $G(S)$:

PROPOSITION IV.3.1: *Let S be an infinitesimally generated semigroup in a Lie group G, then $\underline{L}(G(S)) = \langle\langle\underline{L}(S)\rangle\rangle$ where $\langle\langle\underline{L}(S)\rangle\rangle$ denotes the Lie algebra generated by $\underline{L}(S)$.*

Proof: Note first that without loss of generality we may assume that $G = G(S)$. Moreover $\underline{L}(S) \subseteq \underline{L}(G(S))$ by definition so that $\langle\langle\underline{L}(S)\rangle\rangle \subseteq \underline{L}(G(S))$ since $\underline{L}(G(S))$ is a Lie algebra. Therefore it suffices to show $\underline{L}(G(S)) \subseteq \langle\langle \underline{L}(S) \rangle\rangle$. Let H be the analytic subgroup of G corresponding to $\langle\langle \underline{L}(S) \rangle\rangle$, then $\exp \underline{L}(S) \subseteq H$ and hence by Condition 2.6(ii) $G = G(S) \subseteq H$. In other words $\underline{L}(G(S)) \subseteq \langle\langle \underline{L}(S) \rangle\rangle$. □

We see from Proposition 3.1 that the definition of infinitesimally generated semigroups S yields almost immediately a characterization of the group generated by S in terms of its tangent wedge $\underline{L}(S)$. The analogous problem for the group of units $H(S) = S \cap S^{-1}$ contained in S is much harder. This is due to the freedom we allowed for S in Definition 2.6(iii) . We start with the simplest case where S is generated as a semigroup by $\exp(\underline{L}(S))$, i.e. the case where no closures come into play.

PROPOSITION IV.3.2: Let S be a strictly infinitesimally generated subsemigroup of a Lie group G. Then its group of units $H(S) = S \cap S^{-1}$ is an analytic subgroup of G and $\underline{L}(H(S)) = \underline{L}(S) \cap -\underline{L}(S)$. Moreover, $H(S)$ is closed in $G(S)$!

Proof: As in 2.8 associate with $H(S)$ the ray semigroup $H(S)_R$ generated by all one-parameter semigroups contained in $H(S)$. Then $H(S)_R$ is an analytic group, so we can speak of $\underline{L}(H(S)_R)$. Since $H(S)_R \subset H(S) \subset S$ we have $\underline{L}(H(S)_R) \subseteq \underline{L}(S)$. Similarly $\underline{L}(H(S)_R) \subseteq \underline{L}(S^{-1}) = -\underline{L}(S)$. Conversely, suppose $x \in \underline{L}(S) \cap -\underline{L}(S)$. Then $tx \in \underline{L}(S)$ for all $t \in \mathbb{R}$. Thus $\exp(tx) \in S$ for all t, which implies $\exp(tx) \in H(S)_R$ for all t. Hence $x \in \underline{L}(H(S)_R)$ and thus $\underline{L}(H(S)_R) = \underline{L}(S) \cap -\underline{L}(S)$.

Since $\exp(\underline{L}(H(S)_R)) \subseteq \exp \underline{L}(S) \subseteq S$, we have that the subgroup generated by $\exp(\underline{L}(H(S)_R))$ is contained in S (note that the subgroup and the subsemigroup generated by $\exp(\underline{L}(H(S)_R))$ coincide). Hence $\exp(L(H(S)_R))$ is contained in $H(S)_R$. To finish the proof we show that $H(S)$ is generated by $\exp(\underline{L}(H(S)_R))$.

Let $g \in H(S)$, then $g = \exp(x_1) \cdot \ldots \cdot \exp(x_n)$ for some $x_1, \ldots, x_n \in \underline{L}(S)$ since $H(S) \subset S$ and S is strictly infinitesimally generated. Here we need to interupt the proof of 3.2 in order to state a simple lemma:

LEMMA IV.3.3: Let S be a subsemigroup of a topological group G and $H(S) = S \cap S^{-1}$ be the group of units in S then $(S \setminus H(S))$ is a semigroup ideal in S.

Proof: Let $s \in S \setminus H(S)$ and $g \in S$. If $sg \in H(S)$ then $g^{-1}s^{-1} \in S$, hence $s^{-1} = gg^{-1}s^{-1} \in S$ so that $s \in H(S)$ contradicting our assumptions.

Since a similar argument works for $gs \in H(S)$ the proof is finished. □

Proof of 3.2 continued. We conclude from Lemma 3.3 that $\exp(x_1)\cdot\ldots\cdot\exp(x_n) \in H(S)$. Moreover, since $x_1,\ldots,x_n \in \underline{L}(S)$ we have $\exp(tx_k) \in H(S)$ for all $0 \le t < 1$ and $k = 1,\ldots,n$ again by Lemma IV.3.3 so that finally $\exp(tx_k) \in H(S)$ for all $t \in \mathbb{R}$ and $k = 1,\ldots,n$ and thus $g \in H(S)_R$. But this shows that $H(S)_R = H(S)$ and $H(S)$ is analytic with $\underline{L}(H(S)) = \underline{L}(S) \cap -\underline{L}(S)$.

To prove the last statement note that Proposition 1.7 implies that $cl_{G(S)}H(S)$ is a closed connected Lie subgroup of $G(S)$ with $\underline{L}(cl_{G(S)}H(S)) \subseteq \underline{L}(cl_{G(S)}S) \cap -\underline{L}(cl_{G(S)}S) = \underline{L}(S) \cap -\underline{L}(S) = \underline{L}(H(S)) \subseteq$ $\subseteq \underline{L}(cl_{G(S)}H(S))$. Thus $H(S) = cl_{G(S)}H(S)$. □

At this point we are ready to improve Proposition 3.2 insofar as we may replace the hypothesis "strictly infinitesimally generated" by "infinitesimally generated". The prize we have to pay here for this is that we no longer can guarantee the connectedness of the group of units.

PROPOSITION IV.3.4: <u>Let</u> S <u>be an infinitesimally generated subsemigroup of a Lie group</u> G. <u>Then its group of units</u> $H(S) = S \cap S^{-1}$ <u>is a Lie subgroup of</u> $G(S)$ <u>with</u> $\underline{L}(H(S)) = \underline{L}(S) \cap -\underline{L}(S)$.

Proof: We may assume without loss of generality that $G = G(S)$. Let S_R be the semigroup generated by all one parameter semigroups in S, then S_R is strictly infinitesimally generated by Remark 2.10 . Thus Proposition 3.2 applies to S_R and we know that $H(S_R) = S_R \cap S_R^{-1}$ is a connected Lie subgroup of G with $\underline{L}(H(S_R)) = \underline{L}(S) \cap -\underline{L}(S)$ since $G = G(S_R)$ and $\underline{L}(S_R) = \underline{L}(S)$. But $H(\overline{S})$ is a closed subgroup of G, hence a Lie subgroup and $\underline{L}(H(\overline{S})) \subset \underline{L}(S) \cap -\underline{L}(S)$. Thus $\underline{L}(H(\overline{S})) = \underline{L}(S) \cap -\underline{L}(S)$ whence $H(S)$ is a subgroup of $H(\overline{S})$ containing $H(S_R)$ which at the same time is the identity component of $H(\overline{S})$. Thus $H(S)$ contains any component of $H(\overline{S})$ it meets. This proves the claim. □

In order to prove also the connectedness of the group of units of an infinitesimally generated semigroup S we need to have an analogue of Lemma 3.3 which is strong enough to comply with the discrepancy between the semigroup generated by $\exp \underline{L}(S)$ and its closure. Thus the lemma we are looking for should replace $H(S)$ by a neighborhood of $H(S)$ in S. The following lemma provides us with a nice way to obtain small neighborhoods of $H(S)$ in S.

LEMMA IV.3.5: Let S be an infinitesimally generated subsemigroup of a Lie group G, and assume that $G(S) = G$. Let F be a vectorsubspace of $\underline{L}(G)$ complementary to $E = \underline{L}(S) \cap -\underline{L}(S)$. Then we can find an open neighborhood C of 0 in $\underline{L}(G)$ with the following properties:

(i) The map $(h,u) \to hu: H(S) \times U_F \to H(S)U_F = U$, where $U_F = \exp(C \cap F)$, is an $H(S)$-equivalent homeomorphism onto an open neighborhood of $H(S)$.

(ii) $S \cap U = H(S)(S \cap U_F)$.

Proof: Let B be a Campbell-Hausdorff-neighborhood of 0 in $\underline{L}(G)$. If $\underline{L}(S)$ is a halfspace then $\exp|_B^{-1}(\exp B \cap S) \subseteq \underline{L}(S)$ since it is then a semialgebra by II.2.7 and we set $C_1 = B$ and $W = \underline{L}(S)$. If $\underline{L}(S)$ is not a halfspace we find a wedge W surrounding $\underline{L}(S)$, i.e. satisfying $\underline{L}(S) \setminus (\underline{L}(S) \cap -\underline{L}(S)) \subset \operatorname{int} W$ and a neighborhood C_1 of 0 in B with $\exp|_{C_1}^{-1}(\exp C_1 \cap S) \subseteq W$ by chap.III since $(S \cap B, B)$ is a local semigroup. There is no loss of generality in assuming that C_1 is of the form $C_E * C_F$ with an open neighborhood C_E of 0 in E and an open neighborhood C_F of 0 in F; indeed the differential of the function $(x,y) \mapsto x * y: (E \cap C) \times (F \cap C) \to \underline{L}(G)$ at zero is the identity so that the function is a local diffeomorphism at 0. Moreover we may assume that $(C_F * C_F) \cap C_E = \{0\}$, making C_F smaller if necessary.

We first note that $H(S) \exp C_1 = H(S) \exp C_E \exp C_F = H(S) \exp C_F$ and that $H(S) \exp C_1$ is an open neighborhood of $H(S)$. Also note that $C_F = (C_E * C_F) \cap F = C_1 \cap F$. Since $H(S)$ is closed by Proposition 3.4 we can find a neighborhood C of 0 in $\underline{L}(G)$ such that $H(S) \cap \exp C \subset \exp C_E$. Thus, making C smaller if necessary we may assume that $C \subset C_1$ is a ball with $H(S) \cap (\exp C)^2 \subset C_E$. This means that the map $(h,u) \mapsto hu: H(S) \times \exp C_F \to H(S) \exp C_F = H(S) \exp C$ is a homeomorphism. In fact $h_1u_1 = h_2u_2$ implies
$h_1^{-1}h_2 = u_2u_1^{-1} \in H(S) \cap (\exp C)^2 \cap (\exp C_F)^2 \subset \exp C_E \cap (\exp C_F)^2 = \{0\}$.
Now let $s \in S \cap U$ where $U = H(S)\exp(C \cap F)$. Then $s = hu$ with some $h \in H(S)$ and $u \in U_F = \exp(C \cap F)$, hence $u = h^{-1}s \in S \cap U$ and so $S \cap U \subseteq H(S)(S \cap U_F)$; the reverse containment is trivial. □

We can now prove the desired analogue of Lemma 3.3 .

LEMMA IV.3.6: Let S be an infinitesimally generated subsemigroup of a Lie group G and assume that $G(S) = G$. Then there exist arbitrarily small neighborhoods $\mathcal{O}$ of $\mathbb{1}$ in G such that $S \setminus (H(S)(\mathcal{O} \cap S))$ is a right semigroup ideal in S.

Proof: Let C, C_E, C_F, U, U_F and W be as in Lemma 3.5 and define a map $p_F: U \to C_F$ by $p_F(g) = \exp^{-1}u$, where $g = hu$ with $h \in H(S)$ and $u \in U_F$. Note that we may define a norm $\| \ \|$ on F such that $\|x\| = \omega(x)$ with $w \in W^*$ for all $x \in F \cap W$. We may assume that $C_F = \{x \in F: \|x\| < 1\}$. Consider the ball $C_F' = \{x \in F: \|x\| < \frac{1}{2}\}$ and set $\mathcal{O} = \exp C_E * C_F'$. Suppose now that we can find $s_1 \in S \setminus H(S)(\mathcal{O} \cap S)$ and $s_2 \in S$ be such that $s_1 s_2 \in H(S)(\mathcal{O} \cap S)$.

Remark 2.10 and Theorem 2.2 imply that S has dense interior so that we may assume that $s_2 \in \operatorname{int} S$. But then $s_2 \in \langle \exp \underline{L}(S) \rangle$ by Theorem 2.8 so that we can find a continuous piecewise differentiable curve $\gamma: [0,1] \to G$ with $\gamma(0) = s_1$ and $\gamma(1) = s_1 s_2$ satisfying $\gamma'(t) \in d\lambda_{\gamma(t)}(\mathbb{1})\, \underline{L}(S)$ where $\lambda_g: G \to G$ is defined by $\lambda_g(g') = gg'$ for $g, g' \in G$ and $\mathbb{1}$ is the identity in G. Moreover we may assume that $s_1 \in H(S)U_F$ replacing s_1 by $\gamma(t)$ for $0 < t < 1$ if necessary. Note that then $0 < \|p_F(\gamma(1))\| < \frac{1}{2} \leq \|p_F(\gamma(0))\| < 1$. Since $\gamma([0,1]) \subseteq \operatorname{int} S \cap U \subseteq H(S)(S \cap U_F) \subseteq H(S)(\exp(C_F \cap W))$ by Lemma 3.5 we know that the map $\phi: t \mapsto \|p_F(\gamma(t))\|$ is piecewise differentiable so there exists a $t_o \in \,]0,1[$ such that $\phi'(t_o) < 0$. We may assume that $\gamma(t_o) \in U_F$ shifting s_1 by an appropriate $h \in H(S)$.

We also may assume that under the identification $C_E * C_F \leftrightarrow \exp(C_E * C_F)$ the wedges $F \cap d\lambda_g(\mathbb{1})\underline{L}(S) \subset W$ for all $g \in C$. Thus by [HH86b 4.2] we obtain that $\phi(t)$ is nondecreasing in a neighborhood of t_o. This is a contradiction to $\phi'(t_o) < 0$ and we have proved our claim. □

Finally we are in the position to show that the group of units of an infinitesimally generated semigroup is connected.

THEOREM IV.3.7: Let S be an infinitesimally generated subsemigroup of a Lie group G. Then its group of units $H(S) = S \cap S^{-1}$ is an analytic subgroup of G and $\underline{L}(H(S)) = \underline{L}(S) \cap -\underline{L}(S)$. Moreover $H(S)$ is closed in $G(S)$.

Proof: We may assume that $G = G(S)$. It only remains to show that $H(S)$ is connected by Proposition 3.4 . Thus suppose that $h \in H(S) \setminus H(S)_o$ where $H(S)_o$ is the connected component of $H(S)$. By Lemmas 3.5 and 3.6 we may find a neighborhood $\mathcal{O}$ of $\mathbb{1}$ in G such that $H(S)_o\mathcal{O}$

is the connected component of $H(S)\mathcal{O}$ and $S\setminus H(S)(\mathcal{O}\cap S)$ is a right semigroup ideal of S. Now let $s,s' \in \operatorname{int} S \cap H(S)(\mathcal{O}\cap S)$ be so close to h and h^{-1} respectively that $ss' \in \mathcal{O} \cap \operatorname{int} S$. This is possible since S has dense interior.

But by Theorem 2.8 there exist $x_1,\dots,x_n$ in $\underline{L}(S)$ such that $s = \exp x_1 \cdot\ldots\cdot \exp x_k$ and $s' = \exp x_{k+1} \cdot \ldots \cdot \exp x_m$. If we set $\gamma(t) = \exp x_1 \cdot \ldots \cdot \exp (t-m-1)x_m$ for $t \in [m-1,m]$ then we find a $t_o \in [0,n]$ such that $\gamma(t_o) \in S\setminus H(S)(\mathcal{O}\cap S)$ since s and ss' belong to different connected component of $H(S)(\mathcal{O}\cap S)$. Thus ss' is of the form g_1g_2 with $g_1 \in S\setminus H(S)(\mathcal{O}\cap S)$ and $g_2 \in S$, hence cannot be contained in $\mathcal{O}\cap S$. This contradiction proves the claim. $\square$

Section 4: Functorial properties

This section is devoted to the study of the behaviour of preanalytic semigroups and their tangent objects with respect to the simplest functorial constructions like taking images and preimages under Lie-homomorphisms or forming products.

PROPOSITION IV.4.1: Let $\phi: G \to H$ be a morphism of Lie groups and $\underline{L}(\phi): \underline{L}(G) \to \underline{L}(H)$ be the associated morphism of Lie algebras.

(i) If S is a subsemigroup of G with $G(S) = G$ then $\phi(S)$ is preanalytic and $\underline{L}(\phi)(\underline{L}(S)) \subseteq \underline{L}(\phi(S))$.

(ii) If $T \ni \mathbb{1}$ is a subsemigroup of H with $G(T) = H$ and ϕ is surjective then $G(\phi^{-1}(T)) = G$ and $\underline{L}(\phi^{-1}(T)) = \underline{L}(\phi)^{-1}(\underline{L}(T))$.

Proof: (i) Note first that the group generated by $\phi(S)$ is the image of ϕ, hence an analytic group. Moreover, since $\underline{L}(\phi(S))$ is defined in terms of $G(\phi(S))$ we may assume that ϕ is a quotient map. Finally we may assume that S is closed since $\phi(\overline{S}) \subset \overline{\phi(S)}$. Therefore $\underline{L}(\phi)(\underline{L}(S)) \subseteq \underline{L}(\phi(S))$ by Proposition 1.7 . In fact $\exp_G \mathbb{R}^+ x \subseteq S$ implies $\exp_H \mathbb{R}^+ \underline{L}(\phi)(x) = \phi \exp_G \mathbb{R}^+ x \subseteq \phi(S)$ so that $\underline{L}(\phi)(x) \in \underline{L}(\phi(S))$.
(ii) Choose any $g \in G$, then we find $t_1,\dots,t_n \in T$ such that $t_1^{\varepsilon_1} \cdot \ldots \cdot t_n^{\varepsilon_n} = \phi(g)$ where $\varepsilon_k = \pm 1$. Since ϕ is surjective we can find $g_1,\dots,g_n \in G$ such that $\phi(g_k) = t_k$. Then $g_1^{\varepsilon_1} \cdot \ldots \cdot g_n^{\varepsilon_n} g^{-1} \in \ker \phi \subset \phi^{-1}(T)$ hence $g \in \langle\phi^{-1}(T)\rangle$. Note that

$\phi(\phi^{-1}(T)) = T$ so that the inclusion $\underline{L}(\phi^{-1}(T)) \subset \underline{L}(\phi)^{-1}\underline{L}(T)$ follows from (i). Conversely if $x \in \underline{L}(\phi)^{-1}(\underline{L}(T))$ then $\exp_H \mathbb{R}^+ \underline{L}(\phi)(x) \subseteq \overline{T}$ so that $\exp_G \mathbb{R}^+ x \subseteq \phi^{-1}(\overline{T})$. But H is metrizable so that [Bou 58] (cf. also I.3.9) implies that $\phi^{-1}(\overline{T}) \subset \overline{\phi^{-1}(T)}$ since any Cauchy sequence in $\overline{T}$ can be lifted to a Cauchy sequence in $\phi^{-1}(\overline{T})$. In fact, for any $s \in \phi^{-1}(\overline{T})$ we find a sequence $\{h_n\}$ in T converging to $\phi(s)$ and hence a sequence $\{s_n\}$ in G with $s_n \in \phi^{-1}(h_n) \subseteq \phi^{-1}(T)$ such that s_n converges to s. Therefore $s \in \overline{\phi^{-1}(T)}$ and $\exp_G \mathbb{R}^+ x \subseteq \phi^{-1}(\overline{T}) \subseteq \overline{\phi^{-1}(T)}$. But this just means $x \in \underline{L}(\phi^{-1}(T))$ by Proposition 1.7 . □

Note that Proposition 4.1(i) is sharp in a sense since it is easy to construct examples of semigroups whose quotient semigroups have much bigger tangent wedges than the quotient wedge of the original tangent wedge:

EXAMPLE IV.4.2: Let S be a Lorentzian cone in $\mathbb{R}^3$ and N be the discrete subgroup generated by any nonzero point on the boundary of S. If $\phi: \mathbb{R}^3 \to \mathbb{R}^3/N$ is the quotient map then $\underline{L}(\phi) = \mathbb{R}^3 \to \mathbb{R}^3$ is the identity, but $\phi(S)$ has a halfspace as tangent wedge (cf. Figure 1).

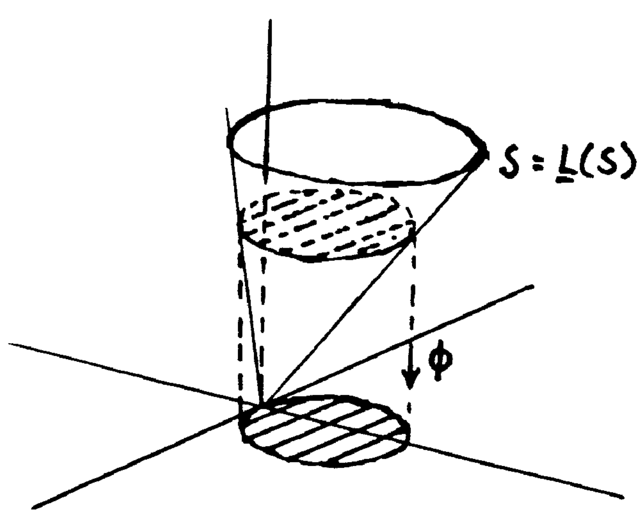


Figure 1

It is fairly obvious that for two infinitesimally generated subsemigroups S_1 and S_2 of the Lie groups G_1 and G_2 we find that $S_1 \oplus S_2 = \{(s_1,s_2) \in G_1 \oplus G_2 : s_1 \in S_1,\ s_2 \in S_2\}$ is an infinitesimally generated semigroup with $\underline{L}(s_1 + s_2) = \underline{L}(S_1) \oplus \underline{L}(S_2)$. We want to present the analogous statement for semidirect products. In order to fix notation we first give a selfcontained description of the semidirect sum of Lie algebras and the semidirect product of Lie groups:

THEOREM IV.4.3: Let $\mathfrak{n}$ and $\mathfrak{h}$ be two Lie algebras and let $\delta: \mathfrak{h} \to \mathrm{Der}\,\mathfrak{n}$ be a morphism of Lie algebras, where $\mathrm{Der}\,\mathfrak{n}$ is the derivation algebra of $\mathfrak{n}$. Then the product space $\mathfrak{n} \times \mathfrak{h}$ is a Lie algebra $\mathfrak{n} \oplus_\delta \mathfrak{h}$, called the semidirect sum, with respect to the bracket

$$[(x,y),\ (x',y')] = (\delta(y)x' - \delta(y')x + [x,x'],\ [y,y'])$$

and there exist Campbell-Hausdorff-neighborhoods $B_\mathfrak{h}$, $B_\mathfrak{n}$ and B of $\mathfrak{h}, \mathfrak{n}$ and $\mathfrak{n} \oplus_\delta \mathfrak{h}$ respectively, such that

(i) $\nu: B_\mathfrak{n} \times B_\mathfrak{h} \to B$ given by $\nu(x,y) = (x,o) * (o,y)$ is a diffeomorphism with inverse function $\mu: B \to B_\mathfrak{n} \times B_\mathfrak{h}$, where $*$ is the Campbell-Hausdorff-multiplication on $\mathfrak{n} \oplus_\delta \mathfrak{h}$.

(ii) $B_\mathfrak{n} \times B_\mathfrak{h}$ is a local group with respect to the multiplication $\#: B \times B \to \mathfrak{n} \oplus_\delta \mathfrak{h}$ defined by $((x,y),\ (x',y')) \mapsto (x *_\mathfrak{n} e^{\delta(y)}x',\ y *_\mathfrak{h} y')$ where $*_\mathfrak{n}$ and $*_\mathfrak{h}$ are the Campbell-Hausdorff-multiplications of N and H respectively.

(iii) The maps $\nu: (B_\mathfrak{n} \times B_\mathfrak{h}, \#) \to (B,*)$ and $\mu: (B,*) \to (B_\mathfrak{n} \times B_\mathfrak{h}, \#)$ are local isomorphisms.

Proof: It is easy to check that $\mathfrak{n} \oplus_\delta \mathfrak{h}$ is a Lie algebra. It is also clear that ν has a local inverse μ since the differential of ν at zero is the identity. Thus it suffices to show that ν transports $\#$ into $*$ to prove the theorem. To do this we show first that

$$(o,y) * (x,o) = (e^{\delta(y)}x,\ o) * (o,y).$$

In fact, we have $(o,y) * (x,o) * (o,-y) = e^{\mathrm{ad}_{\mathfrak{n} \oplus_\delta \mathfrak{h}}(o,y)}(x,o)$. But $\mathrm{ad} = \mathrm{ad}_{\mathfrak{n} \oplus_\delta \mathfrak{h}}$ is given by

$$\mathrm{ad}(o,y)(x',y') = (\delta(y)x',\ \mathrm{ad}_\mathfrak{h}\, y\ (y'))$$

so that $e^{\mathrm{ad}(o,y)} = (e^{\delta(y)}) \times (e^{\mathrm{ad}_\mathfrak{h}\, y})$ and hence

$$e^{\mathrm{ad}(o,y)}(x,o) = (e^{\delta(y)}x,\ o).$$

Now we calculate

$$\begin{aligned}
&\nu(x,y) * \nu(x',y') = (x,o) * (o,y) * (x',o) * (o,y') = \\
&= (x,o) * (e^{\delta(y)}x',\ o) * (o,y) * (o,y') = \\
&= (x *_\mathfrak{n} e^{\delta(y)}x',\ o) * (o,\ y *_\mathfrak{h} y') = \\
&= \nu(x *_\mathfrak{n} e^{\delta(y)}x',\ y *_\mathfrak{h} y') = \nu((x,y) \# (x',y'))
\end{aligned}$$

and the proof is finished. □

From this theorem we can now derive the interaction of semidirect products of groups and semidirect sums of Lie algebras as well as a (local) knowledge of the exponential function of such groups:

THEOREM IV.4.4: *Let* H *and* N *be Lie groups and* $\alpha: H \to \mathrm{Aut}\, N$ *be a morphism. Let* $\lambda: \mathrm{Aut}\, N \to \mathrm{Aut}\, \underline{L}(N)$ *be the natural morphism described by the commutative diagram*

$$\begin{array}{ccc} \underline{L}(N) & \xrightarrow{\lambda(\phi)} & \underline{L}(N) \\ \Big\downarrow \exp & & \Big\downarrow \exp \\ N & \xrightarrow[\phi]{} & N \end{array}$$

Furthermore, let $\delta: \underline{L}(H) \to \mathrm{Der}\, \underline{L}(N)$ *be given by* $\delta = \underline{L}(\lambda \circ \alpha) = \underline{L}(\lambda)\, \underline{L}(\alpha)$, *then we have*

(i) $e^{\delta(y)} = \lambda(\alpha \exp_H y)$

(ii) *The cartesian product* $N \times H$ *is a topological group* $N \rtimes_\alpha H$ *with respect to the multiplication*

$$(n,h)(n',h') = (n\alpha(h)(n'), hh')$$

(iii) $N \rtimes_\alpha H$ *is a Lie group with Lie algebra* $\underline{L}(N) \oplus_\delta \underline{L}(H)$ *and the exponential function* $\exp_{N \rtimes_\alpha H} = \exp$ *is locally defined on a Campbell-Hausdorff-neighborhood* B *by* $\exp|_B = (\exp_N \times \exp_H) \circ \mu$, *where* μ *is the map given in Theorem* IV.4.3.

Proof: Note first that we have the following commutative diagram:

$$\begin{array}{ccccc} \underline{L}(H) & \xrightarrow{L(\alpha)} & \underline{L}(\mathrm{Aut}\, N) & \xrightarrow{L(\lambda)} & \mathrm{Der}\, \underline{L}(N) \\ \Big\downarrow \exp_H & & \Big\downarrow & & \Big\downarrow D \mapsto e^D \\ H & \xrightarrow{\alpha} & \mathrm{Aut}\, N & \xrightarrow{\lambda} & \mathrm{Aut}\, \underline{L}(N) \end{array}$$

This proves (i). Statement (ii) is easy to check. Note that in order to prove (iii) it suffices to show that $\exp$ is a local diffeomorphism satisfying $\exp z \exp z' = \exp z * z'$ for z, z' in a Campbell-Hausdorff-neighborhood. But part (i) and Theorem 4.4 show that we may calculate for small x, x', y and y

$$(\exp_N x, \exp_H y)(\exp_N X', \exp_H Y') =$$

$$= ((\exp_N x)\,\alpha(\exp_H y)(\exp_N x'), (\exp_H y)(\exp_H y')) =$$

$$= (\exp_N(x)\,\exp_N(\underline{L}(\alpha(\exp_H y))\,x'), (\exp_H y)(\exp_H y')) =$$

$$= (\exp_N(x)\,\exp_N(e^{\delta(y)}x'), (\exp_H y)(\exp_H y')) =$$

$$= (\exp_N(x * e^{\delta(y)} x'), \exp_H(y * y')).$$

If we now set $x = \mu_N(\xi,\eta)$, $x' = \mu_N(\xi',\eta')$, $y = \mu_H(\xi,\eta)$ and $y' = \mu_H(\xi',\eta')$ where μ_N and μ_H are the map μ followed by the respective projection N and H, this calculation shows that

$$(\exp(\xi,\eta))(\exp(\xi',\eta')) = \exp_N \times \exp_H(\mu(\xi,\eta) * \mu(\xi',\eta')) =$$

$$= \exp_N \times \exp_H(\mu((\xi,\eta) * (\xi',\eta'))) = \exp((\xi,\eta) * (\xi',\eta')). \quad \square$$

Now we are ready to state and prove the desired result on semidirect products of infinitesimally generated semigroups.

THEOREM IV.4.5: <u>Let N and H be Lie groups and $\alpha: H \to \mathrm{Aut}\, N$ be a morphism of Lie groups. Assume that S_N and S_H are closed infinitesimally generated subsemigroups of N and H respectively such that $\alpha(h)S_N \subset S_N$ for all $h \in S_H$, then the set $S_N \rtimes_\alpha S_H$ defined as $S_N \rtimes_\alpha S_H = \{(n,h) \in N \rtimes_\alpha H,\ n \in S_N,\ h \in S_H\}$ is an infinitesimally generated subsemigroup of $N \rtimes_\alpha H$ with tangent wedge $\underline{L}(S_N \rtimes_\alpha S_H) = \underline{L}(S_N) + \underline{L}(S_H)$.</u>

Proof: Note first that $S_N \rtimes_\alpha S_H$ is a closed subsemigroup of $N \rtimes_\alpha H$ since $\alpha(S_H)S_N \subset S_N$. Moreover $S_N \rtimes_\alpha S_H$ is preanalytic, since it generates the group $G(S_N) \rtimes_\alpha G(S_H)$ which is analytic, so that $\underline{L}(S_N \rtimes_\alpha S_H)$ makes sense. Recall that $\exp: \underline{L}(N) \oplus_\delta \underline{L}(H) \to N \rtimes_\alpha H$ restricted to $\underline{L}(N) = \underline{L}(N) \oplus_\delta (o)$ and $\underline{L}(H) = (o) \oplus_\delta \underline{L}(H)$ gives the exponential functions $\exp_N: \underline{L}(N) \to N$ and $\exp_H: \underline{L}(H) \to H$ respectively. Thus $\exp(\underline{L}(S_N) \oplus_\delta (o)) \subset S_N \rtimes_\alpha \{\mathbf{1}_H\} \subset S_N \rtimes_\alpha S_H$ and $\exp((o) \oplus_\delta \underline{L}(S_H)) \subset \{\mathbf{1}_N\} \rtimes_\alpha S_H \subset S_N \rtimes_\alpha S_H$. This shows that $\underline{L}(S_N)$ and $\underline{L}(S_H)$ are contained in $\underline{L}(S_N \rtimes_\alpha S_H)$. But $\underline{L}(S_N \rtimes_\alpha S_H)$ is a wedge and hence contains $\underline{L}(S_N) + \underline{L}(S_H)$. Since $S_N \rtimes_\alpha S_H$ is closed this also implies that $\exp(\underline{L}(S_N) + \underline{L}(S_H)) \subset S_N \rtimes_\alpha S_H$. Note that the infinitesimally generated semigroups S_N and S_H generate $S_N \rtimes_\alpha S_H$ so that $S_N \rtimes_\alpha S_H$

is as well generated by $\exp(\underline{L}(S_N) + \underline{L}(S_H))$. Thus $S_N \rtimes_\alpha S_H$ is infinitesimally generated and it only remains to show that $\underline{L}(S_N \rtimes S_H) \subset \underline{L}(S_N) + \underline{L}(S_H)$. In order to do this it suffices to find a neighborhood B of zero in $\underline{L}(N) \oplus_\delta \underline{L}(H)$ such that the local semigroup $\exp^{-1}(S_N \rtimes_\alpha S_H) \cap B$ has tangent wedge $\underline{L}(S_N) + \underline{L}(S_H)$. Now choose B and μ as in Theorem 4.4(iii) , then we get a commutative square

$$\begin{array}{ccc} B_{\mathfrak{n}} \times B_{\mathfrak{h}} & \xrightarrow{\exp_N \times \exp_H} & N \rtimes_\alpha H \\ \uparrow \mu & & \| \\ B & \xrightarrow[\exp]{} & N \rtimes_\alpha H \end{array}$$

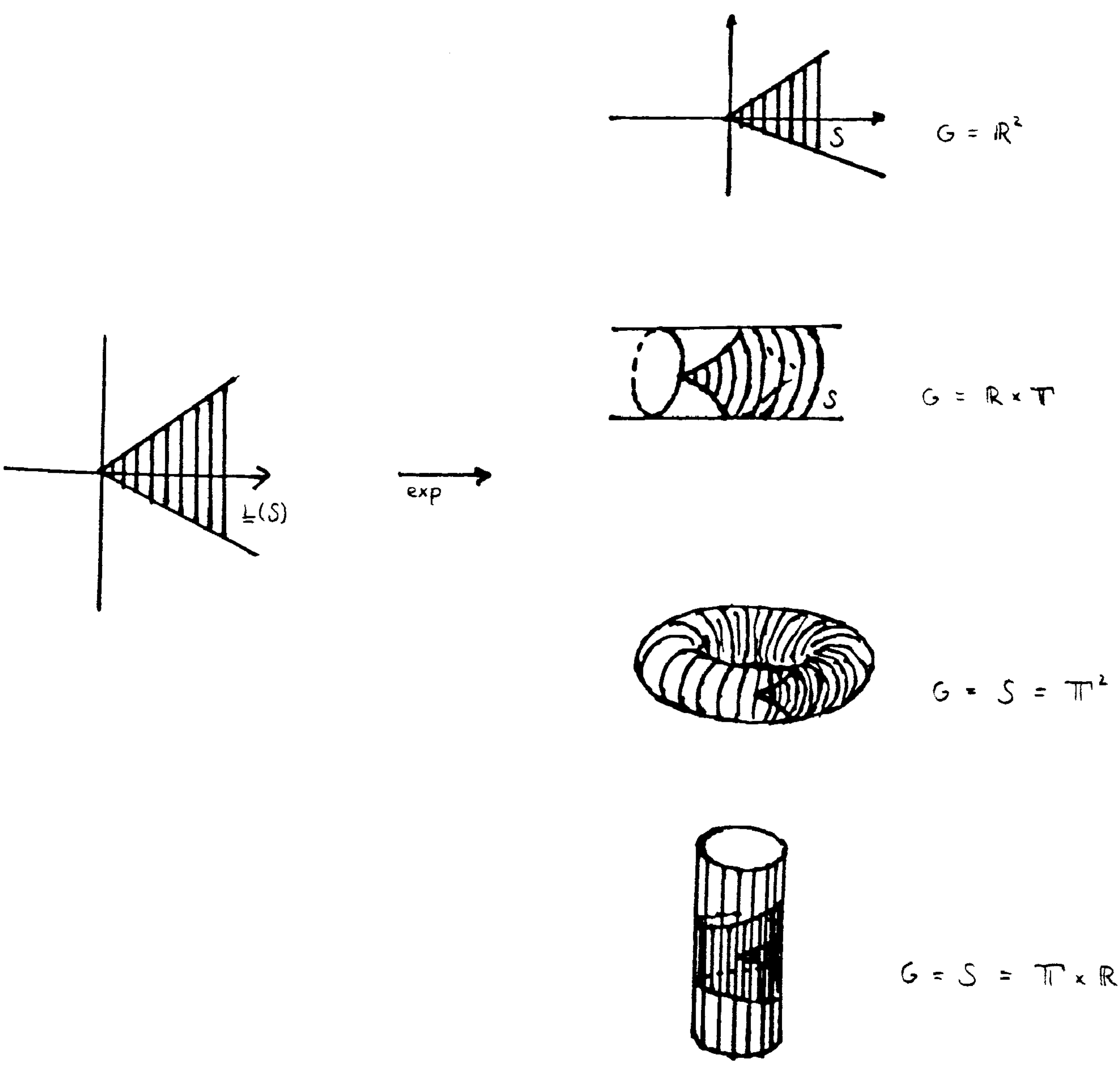
since μ is a diffeomorphism with differential identity at zero, it follows from chap.III that the local semigroups $(\exp^{-1}(S_N \rtimes_\alpha S_H) \cap B, *)$, $\Sigma = ((\exp_N \times \exp_H)^{-1} (S_N \rtimes_\alpha S_H \cap \exp B), \#)$ have the same tangent wedge. But clearly the tangent wedge of Σ is $\underline{L}(S_N) + \underline{L}(S_H)$. □

Section 5: Examples

In this section we provide a number of examples in order to develop some intuition and also illustrate some problems we will have to deal with later on. We start with some abelian examples which will show how crucial the topology of the Lie group enters in the setting of global subsemigroups. The Heisenberg group will supply examples for some algebraic phenomena which one couldn't detect locally as well as examples for the fact that the exponential image of the tangent wedge of an infinitesimally generated semigroup may very well be quite thin in the semigroup. An example for a class of groups in which there are almost no obstructions for global semigroups is the class of almost abelian groups (cf. II.2.13). This is due to the fact that there are so many codimension one groups in these groups. Moreover we will study the non-exponential three dimensional solvable Lie group, in which we will find an example for an infinitesimally generated semigroup which is not strictly infinitesimally generated, and the oscillator group, which played an important role in the classification of the Lorentzian semialgebras. Finally we present an extensive study of the special linear group in two dimensions, which may be viewed as a prototype of a noncompact semisimple Lie group.

EXAMPLE IV.5.1: The simplest, but nevertheless instructive, type of examples is the case where the Lie group G under consideration is abelian. If G is a vectorgroup then the infinitesimally generated semigroups are identical with their tangent wedges under the identification L(G) = G. If G is a torus any closed infinitesimally generated semigroup is a group (cf. Example 1.4). If G is a cylinder then there are wedges which are tangent wedges of infinitesimally generated semigroup as well as wedges which are not (cf. Figure 1).

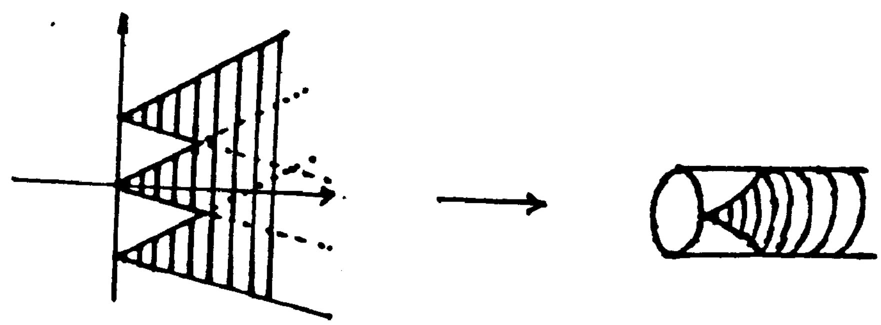
Figure 1



These examples show that the global topology of the group and the position of the wedge in the algebra will play an important role when we are dealing with the question for which cones and wedges W in $\underline{L}(G)$ we can actually find preanalytic semigroups S in G with $W = \underline{L}(S)$.

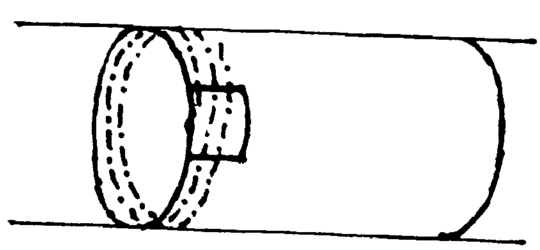
EXAMPLE IV.5.2: Of course there are many subsemigroups S in G which are not infinitesimally generated. This may even happen if the quotient of S by a discrete subgroup is infinitesimally generated by the same tangent wedge (cf. Figure 2).

Figure 2



EXAMPLE IV.5.3: Another question which exhibits some of its difficulties already in the abelian case is that of global and local divisibility. Given a divisible subsemigroup S of G it may be necessary choose a very small neighborhood U of **1** in order to have that $U \cap S$ is locally divisible (cf. chap.III and Figure 3).

Figure 3



EXAMPLE IV.5.4: Let G be the Heisenberg group, i.e. the group of all real 3×3-matrices of the form

$$(a,b,c) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

This group is a good example for the fact that not only the topology but also the algebraic structure of a Lie group can make it impossible to find a semigroup with some prescribed tangent wedge. In fact for any wedge W in $\underline{L}(G)$ containing the center of $\underline{L}(G)$ in its interior the semigroup generated by $\exp W$ is all of G. We prove this by showing the following slightly more general lemma:

LEMMA IV.5.5: _Let S be a subsemigroup of Heisenberg group G containing central elements in its interior, then_ $S = G$.

Proof: Note first that we may identify G with $\underline{L}(G)$ if endow $\underline{L}(G)$ with the Campbell-Hausdorff-multiplication

$$x * y = x + y + \frac{1}{2}[x,y] \quad \text{for all} \quad x,y \in \underline{L}(G). \tag{5.1}$$

The Lie algebra $\underline{L}(G)$ can be represented as the real 3×3-matrices of the form

$$[\alpha,\beta,\gamma] = \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}$$

If we set $x = [1,0,0]$, $y = [0,1,0]$ and $z = [0,0,1]$ then $\mathbb{R}z$ is the center of G. For $w \in \xi x + \eta y$ and $w' = x - \xi y$ we calculate for $z_o \in \mathbb{R}z$ and $n \in \mathbb{N}$

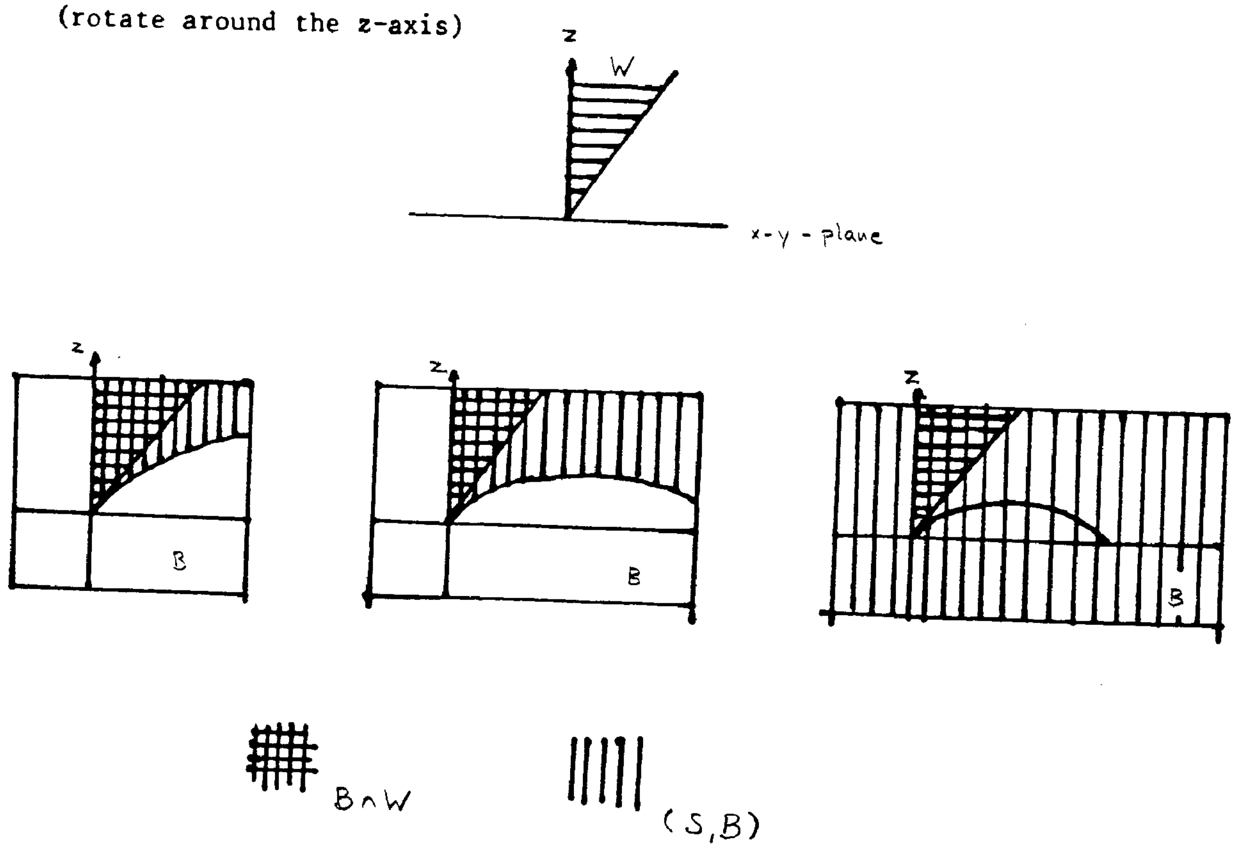
$$(n(z_o + w)) * (n(z_o + w')) = 2nz + \xi n(x-y) + \eta n(y+x) - \frac{1}{2} n^2(\xi^2 - \eta^2)z.$$

If S is a semigroup containing a whole neighborhood of z_o this calculation shows that there exists an element $u \in \text{int } S$ in the x-y-plane. Rotating w and w' around the z-axis we also find $-u \in \text{int } S$. But then $\mathbf{1} = u * -u \in \text{int } S$ so that $S = G$. □

Note that Example 5.4 contrasts the assertion of the local theory that for any cone K in $\underline{L}(G)$ there exists a *local* semigroup (S,B) having K as tangent object. The crux here is that the neighborhood B must be chosen small enough (cf. Figure 4).

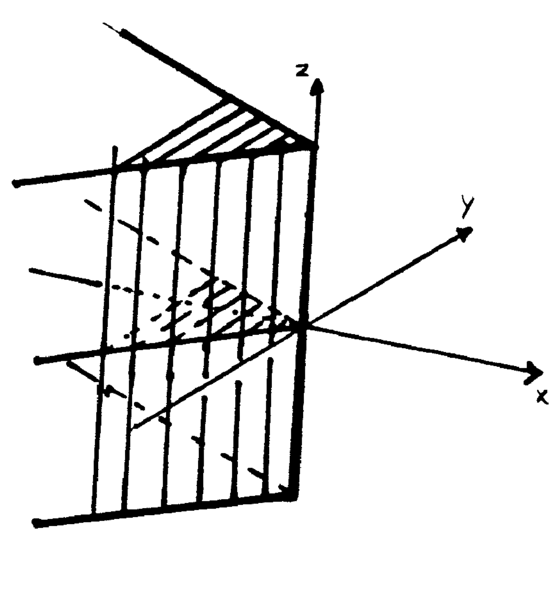
Figure 4

(rotate around the z-axis)



EXAMPLE IV.5.6: In spite of Lemma 5.4 the Heisenberg group is full of infinitesimally generated semigroups. If we still identify G with $\underline{L}(G)$ under the Campbell Hausdorff multiplication then we see that one particularly simple class are all wedges containing the center (cf. Figure 5).

Figure 5

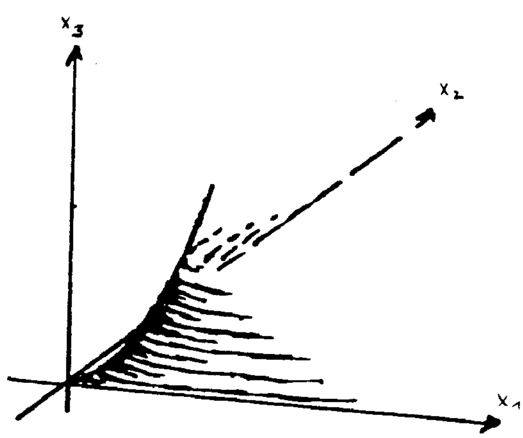


There are also strictly infinitesimally generated subsemigroups in the Heisenberg group which do not contain a nontrivial group:

EXAMPLE IV.5.7: Let G be the Heisenberg group and $S = \{(a,b,c) \in G: 0 \le a,b, \ 0 \le c \le ab\}$ then S is a strictly infinitesimally generated semigroup with $\underline{L}(S) = \{(\alpha,\beta,\gamma) \in \underline{L}(G): \gamma = 0, \ 0 \le \alpha,\beta\}$.

Proof: It is straight forward to check that S is a closed semigroup. Identifying G with a three dimensional vectorspace we may visualize S as the region in the first octant bounded by the surface $x_3 = x_1 x_2$ and the x_1-x_2-plane (cf. Figure 6).

Figure 6



The one-parameter semigroup $\sigma(t) = \exp t\,[1,0,0] = (t,0,0)$ and $\tau(t) = \exp t\,[0,1,0] = (0,t,0)$ generate S. In fact, if $(a,b,c) \in S$ and $b > 0$ then

$$(5.2) \qquad (a,b,c) = \sigma(\tfrac{c}{b})\tau(b)\sigma(a - \tfrac{c}{b}).$$

If $b = 0$ then $c = 0$, hence $(a,b,c) = \sigma(a)$. It remains to show that $\underline{L}(S) \subseteq \{(\alpha,\beta,\gamma) \in \underline{L}(G): \gamma = 0, \ 0 \le \alpha,\beta\}$. Since S is a closed ray semigroup by what we have just seen, Proposition 1.7 shows that $\underline{L}(S) = \{x \in \underline{L}(G): \exp \mathbb{R}^+ x \subseteq S\}$. But $\exp t[\alpha,\beta,\gamma] = (t\alpha, t\beta, t\gamma + t^2\alpha\beta)$ so that $\underline{L}(S) \subseteq \{[\alpha,\beta,\gamma] \in \underline{L}(G): \gamma = 0, \ \alpha,\beta \ge 0\}$. □

We have seen in chap.II that almost abelian algebras were full of semialgebras because of the abundance of hyperplane subalgebras. Similarly we find a log of infinitesimally generated subsemigroups of the corresponding groups:

EXAMPLE IV.5.8: Let G be the group of real $(n+1)\times(n+1)$-matrices of the form

$$\begin{bmatrix} r E_n & v \\ 0 & 1 \end{bmatrix}$$

where $r \in \mathbb{R}$, $v \in \mathbb{R}^n$ and $E_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity. Such a group we call an *almost abelian* group. The Lie algebra $\underline{L}(G)$ is an almost abelian algebra and can be represented as the matrices of the form

$$\begin{bmatrix} r E_n & v \\ 0 & 0 \end{bmatrix} \quad r \in \mathbb{R}, \quad v \in \mathbb{R}^n.$$

Then the exponential map $\exp = \underline{L}(G) \to G$ is a diffeomorphism and maps every wedge W in $\underline{L}(G)$ homeomorphically onto a subsemigroup of G. In oarticular there exists a subsemigroup S of G with $\underline{L}(S) = W$ for any wedge W in $\underline{L}(G)$.

To show this statement just note that any wedge in $\underline{L}(G)$ is a semialgebra by II.2.1.3 and hence we can apply Corollary II.1.30. □

EXAMPLE IV.5.9: Let G be the semidirect product of $\mathbb{C}$ by $\mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{C}$ by rotation then the set $S = \{(c,r) \in G : |c| \le r\}$ is a closed infinitesimally, but not strictly infinitesimally generated semigroup with $\underline{L}(S) = \{(\gamma,\rho) \in \underline{L}(G) : |\gamma| \le \rho,\ \gamma \in \mathbb{C};\ \rho \in \mathbb{R}\}$ where $\underline{L}(G)$ is the corresponding Lie algebra semidirect sum of $\mathbb{C}$ and $\mathbb{R}$.

Proof: Note first that we may represent G as the set of 3×3-matrices of the form:

$$(c,r) = \begin{bmatrix} e^{ir} & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^r \end{bmatrix}$$

Then the Lie algebra $\underline{L}(G)$ is given by

$$(\gamma,\rho) = \begin{bmatrix} i\rho & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}$$

and the exponential function is the usual matrix exponential function. It is not surjective.

It is easy to check that S is a closed subsemigroup of G.

Since S also contains inner points it is preanalytic and $\underline{L}(S) = \{(\gamma,\rho) \in \underline{L}(G)\colon \exp \mathbb{R}^+((\gamma,\rho)) \subseteq S\}$. A one parameter group in G is given by

$$(5.3)\qquad \exp t(\gamma,\rho) = \left(\frac{\gamma(e^{it\rho}-1)}{i\rho},\ t\rho\right) \quad \text{for } \rho \neq 0$$

hence $\exp \mathbb{R}^+(\gamma,1) \subseteq S$ if and only if

$$|\gamma(e^{it}-1)| \le t \quad \text{for all } t \in \mathbb{R}^+.$$

But this is equivalent to each of the following statements

$$2|\gamma|^2 (1-\cos t) \le t^2 \quad \text{for all } t \in \mathbb{R}^+$$

$$4|\gamma|^2 \sin^2 \frac{t}{2} \le t^2 \quad \text{for all } t \in \mathbb{R}^+$$

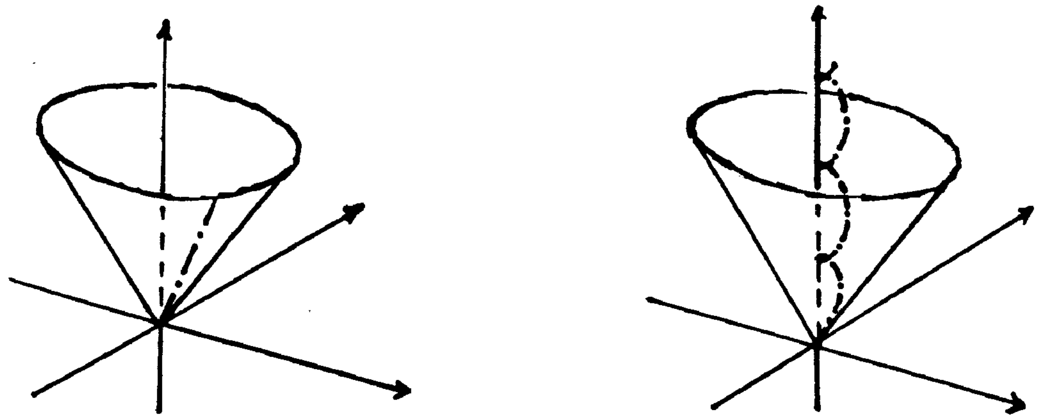
$$|\gamma|\ \left|\sin \frac{t}{2}\right| \le \frac{t}{2} \quad \text{for all } t \in \mathbb{R}^+$$

$$|\gamma|\ \frac{|\sin t|}{t} \le 1 \quad \text{for all } t \in \mathbb{R}^+$$

$$|\gamma| \le 1.$$

Since $\exp(\gamma,0) = (\gamma,0)$ is in S if and only if $\gamma = 0$ this shows that $\underline{L}(S) = \{(\gamma,\rho) \in \underline{L}(G)\colon |\gamma| \le \rho\}$ (cf. Figure 7).

Figure 7



In order to show that S is infinitesimally but not strictly infinitesimally generated we consider the semigroup S_R generated by $\underline{L}(S)$. Recall that

$$\exp \underline{L}(S) = \{(\frac{\gamma(e^{i\rho t}-1)}{i\rho}, \rho t): \rho > 0, |\gamma| \leq \rho, t \geq 0\}.$$

Thus $\exp \underline{L}(S) \subseteq T = \{(c,r) \in G: |c| < r\} \cup \{(0,0)\}$. But it is easy to check that T is a semigroup so that $S_R \subset T$.

Conversely, let $(c,r) \in T$. If we set

$$c_k = de^{ia_k}(e^{i\,r/n}-1) \quad \text{where} \quad |d| \leq 1, \quad a_k \in \mathbb{R}$$

we obtain

$$(c_1, \frac{r}{n}) \cdots (c_n, \frac{r}{n}) = (e(e^{i\,r/n}-1) \sum_{k=1}^{n} e^{i(a_k + \frac{(k-1)}{n})}, r)$$

Thus if we set $a_k = \frac{-(k-1)}{n}$ we get

$$(c_1, \frac{r}{n}) \cdots (c_n, \frac{r}{n}) = (dn(e^{i\,r/n}-1), r).$$

Now we choose n so that

$$\frac{|c|}{r} \leq |e^{i\,r/n}-1| \frac{n}{r} < 1.$$

Then we can find $d \in \mathbb{C}$ with $|d| < 1$ such that $c = d(e^{i\,r/n}-1)n$. Since $(c_k, \frac{r}{n}) = \exp(\frac{ir}{n} de^{i\alpha_k}, \frac{r}{n})$ this shows that $T = S_R$. Thus the claim follows since T is dense in S. □

EXAMPLE IV.5.10: Let H be the Heisenberg group, represented as pairs $(v,z) \in \mathbb{R}^2 \times \mathbb{R}$ endowed with the multiplication

$$(5.4) \qquad (v,z) \cdot (v',z') = (v+v', z+z' + \frac{1}{2}\langle dv,v'\rangle),$$

where $\langle\,,\,\rangle: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the scalar product and $d: \mathbb{R}^2 \to \mathbb{R}^2$ is given by the matrix $\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$. Note that this is just the Campbell-Hausdorff-multiplication on the Heisenberg algebra $\underline{L}(H)$ represented as pairs $(\zeta,\xi) \in \mathbb{R}^2 \times \mathbb{R}$ with bracket

$$[(\zeta,\xi), (\zeta',\xi')] = (0, \langle d\zeta,\zeta'\rangle).$$

For $t \in \mathbb{R}$ set

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

and let $\mathbb{R}$ act on H by $r \cdot (v,z) = (R(r)v, z)$. The semidirect product $H \rtimes \mathbb{R}$ with respect to this action we call the *oscillator group* $\mathcal{O}$. The product on $\mathcal{O}$ is given by

(5.5) $$(v,z,r)(v',z',r') = (v + R(r)v',\ z + z' + \frac{1}{2} \langle dv,\ R(r)v'\rangle,\ r + r').$$

Then we can show that for any invariant generating cone W in $\underline{L}(\mathcal{O})$ the closed subsemigroup S of $\mathcal{O}$, generated by $\exp W$ has tangent cone W and there is a neighborhood U of $\mathbf{1}$ in $\mathcal{O}$ such that $S \cap U = \exp W \cap U$.

In order to prove this statement we need to have a good description of the exponential function. We start by calculating the one-parameter subgroups of $\mathcal{O}$:

Note first that Theorem 4.4 shows that we may identify the underlying spaces of $\mathcal{O}$ and $\underline{L}(\mathcal{O})$ so that the generator x of a one-parameter subgroup $\gamma(t)$ is just $x = \gamma'(0)$, since in our representation of $\mathcal{O}$ the multiplication is globally given by the multiplication $\#$ and the differential of ν at zero is the identity. Now let $\Phi(t) = (v(t), z(t), r(t))$ be a one parameter subgroup of $\mathcal{O}$. Then $r(t) = tr_o$ and for $t,s \in \mathbb{R}$

$$(v(s) + R(sr_o)v(t),\ z(s) + z(t) + \frac{1}{2} \langle dv(s),\ R(sr_o)\ (t)\rangle) = (v(s+t),\ z(s+t).$$

Fixing s and letting t tend to zero we obtain

$$(\dot{v}(s),\ \dot{z}(s)) = (R(sr_o)\dot{v}(0),\ \dot{z}(0) + \frac{1}{2} \langle dv(s),\ R(sr_o)\dot{v}(0)\rangle)$$

since $v(0) = 0$ and $z(0) = 0$. But since $R(sr_o) = e^{sr_o d}$ we conclude $v(s) = (e^{sr_o d} - \mathbf{1})v_o$ where $r_o dv_o = \dot{v}(0)$ for $r_o \neq 0$. Thus we have in this case

$$\begin{aligned}
\dot z(s) &= \dot z(0) + \tfrac{1}{2} \langle d(e^{sr_o d} - \mathbb{I})v_o,\ r_o e^{sr_o d} d(v_o)\rangle = \\
&= \dot z(0) + \tfrac{r_o}{2} \langle e^{sr_o d}\, dv_o,\ e^{sr_o d}\, dv_o\rangle - \tfrac{r_o}{2} \langle dv_o,\ e^{sr_o d}\, dv_o\rangle = \\
&= \dot z(0) + \tfrac{r_o}{2} \langle dv_o,\ dv_o\rangle - \tfrac{r_o}{2} \langle dv_o,\ de^{sr_o d}\, v_o\rangle = \\
&= \dot z(0) + \tfrac{r_o}{2} \|v_o\|^2 - \tfrac{r_o}{2} \langle v_o,\ e^{sr_o d}\, v_o\rangle
\end{aligned}$$

since d and $e^{sr_o d}$ are orthogonal. Integration now yields

$$z(s) = s(\dot z(0) + \tfrac{r_o}{2} \|v_o\|^2) - \tfrac{1}{2} \langle dv_o,\ e^{sr_o d}\, v_o\rangle$$

since d is skewsymmetric. Thus the exponential function $\exp\colon \underline{L}(\mathcal{O}) \to \mathcal{O}$ is given by

$$(5.6)\qquad \exp(v,z,r) = \begin{cases} (\frac{1}{r}(\mathbb{I} - e^{rd})dv,\ z + \frac{1}{2r}\|v\|^2 - \frac{1}{2r^2}\langle dv, e^{rd}v\rangle, r) & \text{for } r \neq 0 \\ (v,z,o) & \text{for } r = 0 \end{cases}$$

In fact, we only need to note that $\frac{d}{dt} \exp t(v,z,r)\Big|_{t=0} = (v,z,r)$ and use the fact that $d^{-1} = -d$ is orthogonal in the above calculations.

From this we calculate easily that $\exp|_B\colon B \to B$ is a diffeomorphism, where $B = \mathbb{R}^2 \times \mathbb{R} \times]-2\pi, 2\pi[$. Therefore the set

$C = \{((v,z,r),\ (v',z',r')) \in \underline{L}(\mathcal{O}) \times \underline{L}(\mathcal{O})\colon -2\pi < r + r' < 2\pi\}$ is contained in the set

$\{((v,z,r),(v',z',r')) \in \underline{L}(\mathcal{O}) \times \underline{L}(\mathcal{O})\colon \exp(v,z,r) \exp t(v',z',r') \in \exp B$ for all $t \in [0,1]\}$ and hence we can apply Theorem II.1.31 to obtain that

$\exp((v,z,r))\exp((v',z',r')) \in \exp W$ for all $((v,z,r),(v',z',r')) \in C \cap (W \times$

where W is any generating semialgebra in W.

Note that $\mathbb{R}^2 \times \mathbb{R} \times [2\pi, \infty[$ is a semigroup ideal in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$. Therefore for any semialgebra W in $\underline{L}(\mathcal{O})$ which is contained in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$ the set $S = \exp W \cup (\mathbb{R}^2 \times \mathbb{R} \times [2\pi, \infty[)$ is a subsemigroup of $\mathcal{O}$. But clearly $\underline{L}(S) = W$. Finally we recall (chap.II) that any invariant cone in $\underline{L}(\mathcal{O})$ is isomorphic to the one given by the invariant form

$$q((v,z,r),\ (v',z',r')) = rz' + r'z + \langle v,v'\rangle$$

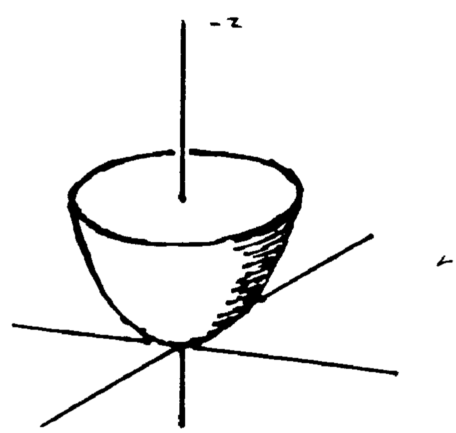
and the restriction $r \geq 0$. Thus the argument above applies and proves the statement on page 27.

We want to give a geometric description of the semigroup generated by $\exp W$ where W is a generating invariant cone in $\underline{L}(O)$. By what we have just seen we may assume that $W = \{(v,z,r) \in \underline{L}(O)\colon 2rz + \|v\|^2 \le 0,\ r \ge 0\}$.

Let $r \in]o,2\pi[$ and consider $(\exp W) \cap (\mathbb{R}^2,\mathbb{R},r)$. Note first that for $\exp(v,z,r) = (v',z',r')$ we have $\exp(e^{td}v,z,r) = (e^{td}v',z',r')$, i.e. the set $(\exp W) \cap (\mathbb{R}^2,\mathbb{R},r)$ is invariant under rotations in the v-plane. If now $v = (x,o)$ then $dv = (o,x)$ and $e^{rd}v = (x \cos r,\ x \sin r)$. Therefore $\langle dv,\ e^{rd}v\rangle = x^2 \sin r$. Moreover $\|(\mathbb{1} - e^{rd})dv\|^2 = \|(\mathbb{1} - e^{rd})v\|^2 = 2\|v\|^2 - 2\langle v,\ e^{rd}v\rangle = 2\|v\|^2(1 - \cos r)$ since $\langle v,\ e^{rd}v\rangle = x^2 \cos r$. Since $2rz + \|v\|^2 \le 0$ just means $z + \frac{1}{2r}\|v\|^2 \le 0$ this shows that $\exp W \cap (\mathbb{R}^2,\mathbb{R},r)$ is the region below the paraboloid given by

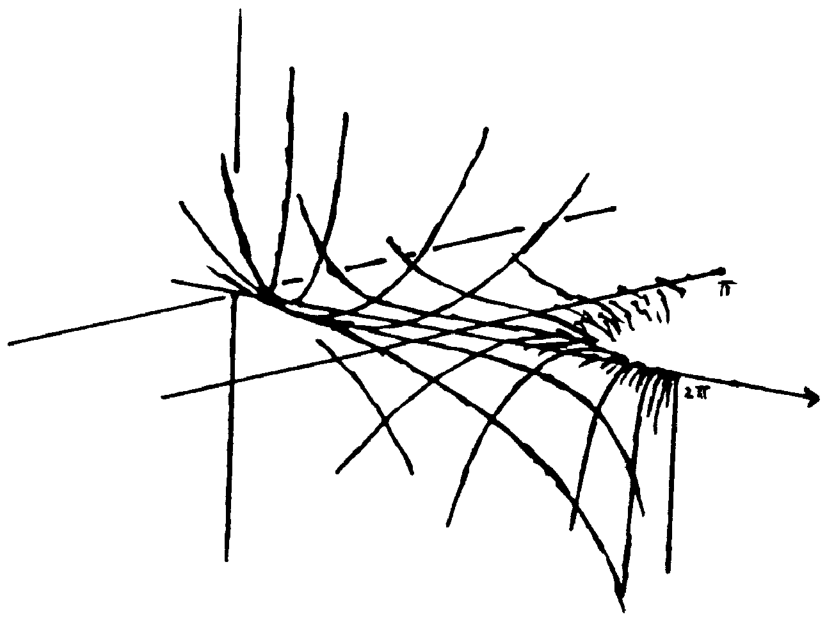
$$(5.7)\qquad \left(v,\ \|v\|^2 \frac{-\sin r}{4(1-\cos r)},\ r\right),\quad v \in \mathbb{R}^2 \qquad \text{(cf. Figure 8)}$$

<u>Figure 8</u>



Note that $\lim\limits_{r\to o} \frac{1-\cos r}{\sin r} = \lim\limits_{r\to o} \frac{\sin r}{\cos r} = 0$ so that $\frac{\sin r}{2(1-\cos r)}$ approaches $\pm\infty$ as r approaches $2\pi n$ with $n \in \mathbb{N}$ depending on whether on approaches from the left or on the right (cf. Figure 9).

Figure 9



EXAMPLE IV.5.11: It is not possible in Example 5.10 to replace the oscillatorgroup by an other group with the same Lie algebra:
Let $\mathcal{O}$ be the oscillator group and $0 \neq z \in Z(\underline{L}(\mathcal{O}))$ where $Z(\underline{L}(\mathcal{O}))$ is the center of the oscillator algebra. Let S be the closed subsemigroup of $\mathcal{O}$ generated by $\exp W$ where W is a generating invariant cone in $\underline{L}(\mathcal{O})$. Then $N = \exp \mathbb{Z}z)$ is a discrete Lie subgroup of $\mathcal{O}$ and the subsemigroup SN/N of $\mathcal{O}/N$ has a halfspace bounded by the hyperplane ideal of $\underline{L}(\mathcal{O})$ as tangent wedge.

Proof: Note first that SN/N clearly is contained in $S_oN/N = S_o/N$, hence $\underline{L}(SN/N) \subseteq \underline{L}(S_o/N) = \underline{L}(S_o)$ which is a halfspace bounded by the Heisenberg algebra. Conversely we have that $\exp$ is a diffeomorphism from a tube around the center of $L(\mathcal{O})$ onto the image of this tube. But the projection of W along $Z(\underline{L}(\mathcal{O}))$ onto $\{\mathbb{R}^2 \times \{0\} \times \mathbb{R}\}$ is $\{\mathbb{R}^2 \times \{0\} \times \mathbb{R}^+\}$ so that for any $x \in \{\mathbb{R}^2 \times \{0\} \times \mathbb{R}^+ \setminus \{0\}\}$ we can find a $y \in W \cap \{\mathbb{R}^2 \times \mathbb{Z}z \times \mathbb{R}^+\}$ projecting down to x. Hence $(\exp x)N = (\exp y)N \in SN/N$ and thus, choosing x close to the origin we derive $\underline{L}(SN/N) = \{\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+\}$. □

The remaining part of this section is devoted to the study of the special linear group $Sl(2,\mathbb{R})$. In a sense this group is the epitome of a (semi-) simple non-compact Lie group and its Lie algebra $sl(2,\mathbb{R})$ is one of the basic building blocks in the theory and classification of semisimple Lie algebras. Thus $SL(2,\mathbb{R})$ is the canonical starting point for the study of infinitesimally generated semigroups in semisimple Lie groups. Since the information available on semigroups in general semisimple Lie groups is rather sparse at the moment, we present a fairly extensive study of the situation in $SL(2,\mathbb{R})$ and its simply connected covering group $\widetilde{SL}(2,\mathbb{R})$.

Recall that $SL(2,\mathbb{R})$ is the set of real 2×2-matrices with determinant 1. Its Lie algebra $sl(2,\mathbb{R})$ is the set of real 2×2-matrices of zero trace. The exponential function $\exp: sl(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is given by the usual matrix exponential function. We will simply write $sl(2)$ and $SL(2)$ instead of $sl(2,\mathbb{R})$ and $SL(2,\mathbb{R})$.

For computational convience we introduce the nomalized trace $\tau: sl(2) \to \mathbb{R}$ defining $\tau(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \frac{1}{2}(a+d)$ and a normalized Cartan-Killing-form $k: sl(2) \times sl(2) \to \mathbb{R}$ defined by $k(X,Y) = \tau(XY)$. Then $k(X,Y) = \frac{1}{8}\chi(X,Y)$ with $\chi: sl(2) \times sl(2) \to \mathbb{R}$, defined by $\chi(X,Y) = \text{trace}(\text{ad } X \text{ ad } Y)$, where the trace is taken on the space of endomorphisms of the vectorspace $sl(2)$, i.e. χ is the Cartan-Killing-form. By abuse of notation we will write $k(X)$ for $k(X,X)$.

For g in $SL(2)$ we obtain a Lie algebra automorphism I_g of $sl(2)$ via $I_g(X) = gXg^{-1}$. The map $g \to I_g$ from $SL(2)$ to $Aut(sl(2))$ is a Lie group morphism whose kernel is the two element group $\{\mathbb{1},-\mathbb{1}\}$. Its image is the connected component $Aut_o(sl(2))$ at the identity, and this group is generated by all automorphisms of the form e^{adX} with X in $sl(2)$. This means that $Aut_o(sl(2))$ is the adjoint group of $SL(2)$.

We will write $g\cdot X = gXg^{-1}$ for g in $Sl(2)$ and $X \in A$ where A is the Banachalgebra of linear operators on $\mathbb{R}^2$. In this fashion $SL(2)$ acts linearly and automorphically on A and $sl(2)$, and automorphically on $SL(2)$. The exponential function is equivariant relative to these actions, i.e. $\exp: sl(2) \to SL(2)$ satisfies $g\cdot \exp X = \exp(g\cdot X)$. By a slight abuse of language we call the action of $SL(2)$ on $sl(2)$ the adjoint action. This action needs to be understood very well in the following.

The form k is bilinear, symmetric, non-degenerate and *invariant* in the sense that $k(g\cdot X, g\cdot Y) = k(X,Y)$ for g in $SL(2)$ and that $k([X,Y],Z) = k(X,[Y,Z])$. In fact if ϕ is any automorphism of $sl(2)$, then $k(\phi X,\phi Y) = \frac{1}{8}\text{trace}(\text{ad }\phi X \text{ ad }\phi Y) = \frac{1}{8}\text{trace}(\phi \text{ ad } X \text{ ad } Y\phi^{-1}) = \frac{1}{8}$ $\text{tr}(\text{ad } X \text{ ad } Y) = k(X,Y)$. There is, up to scalar multiplication, only one invariant form.

The points X in $sl(2)$ for which $k(X) \leq 0$ will play an important role in our further discussion. They form a double cone, called *the standard double cone*. In obvious ways this cone is reminiscent of the light cone in special relativity. We want to distinguish one of the two cones. For this purpose, we need to introduce a basis for $sl(2)$; there will be involutive automorphisms of $sl(2)$ that interchange the two cones.

We identify A with the algebra of real 2 by 2 matrices and set

(5.8) $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $T = P+Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $U = p-Q = [$

Call $\{H,P,Q\}$ *the first basis* and $\{H,T,U\}$ *the second basis* for $sl(2)$. The multiplication is then given by

(5.9a) $[H,P] = 2P$, $[H,Q] = -2Q$, $[P,Q] = H$, and

(5.9b) $[H,T] = 2U$, $[H,U] = 2T$, $[U,T] = 2H$.

We observe that

(5.9c) $k(h \cdot H + p \cdot P + q \cdot Q) = h^2 + pq$; $k(h \cdot H + t \cdot T + u \cdot U) = h^2 + t^2 - u^2$.

In particular, $k(H) = k(T) = 1$, $k(P) = k(Q) = 0$, and $k(U) = -1$. The first basis is adapted to the general theory of semisimple algebras, but for the purpose of geometric representation, we prefer the second basis and denote the plane $\mathbb{R} \cdot H + \mathbb{R} \cdot T$ with the letter E, and call it *the horizontal* plane, while the line $\mathbb{R} \cdot U$ will be called *the vertical line*. Once and for all, we will write $X = X' + x \cdot U$ with X' horizontal. Moreover we introduce a non-canonical Hilbert space structure on $sl(2)$ through the inner product $\langle X_1, X_2 \rangle = h_1 h_2 + t_1 t_2 + x_1 x_2$ for $X_j = h_j H + t_j T + x_j U$ and $j = 1,2$. We will write $|X| = \langle X,X \rangle^{1/2}$, and observe that $k(X) = |X'|^2 - x^2$.

With respect to these definitions, the standard double cone is given by the set $\{X: |X'| \leq |x|\}$. We distinguish one part of the double cone by setting $K = \{X: |X'| \leq x\}$. The boundary of K given by $\{X: k(X) = 0, x \geq 0\}$ is denoted by N. We observe that K as well as N is invariant as a consequence of the connectivity of the adjoint group. Note here that $Aut_0(sl(2))$ has index two in the full automorphism group and that one convenient representative $\alpha: sl(2) \to sl(2)$ of the second coset is defined by $\alpha X = TXT$, i.e., by

(5.10) $\alpha(H) = -H$, $\alpha(P) = Q$, $\alpha(Q) = P$, $\alpha(U) = -U$, $\alpha(T) = T$.

Note that α exchanges K and $-K$ and N and $-N$.

Recall (cf. [HH 85b]) that we obtain the two dimensional subalgebras of sl(2) as follows.

PROPOSITION IV.5.12: *For a plane* B *in* sl(2) *the following statements are equivalent:*

a) B *is a subalgebra.*
b) $B = X^{\perp}$ *for some* $X \neq 0$ *with* $k(X) = 0$.
c) $B = X^{\perp}$ *for some* $X \in B$.
d) $B^{\perp} \subset B$.
e) B *is tangent to* $N \cup -N$.

All such B *are conjugate under* $\exp \mathbb{R}U$.

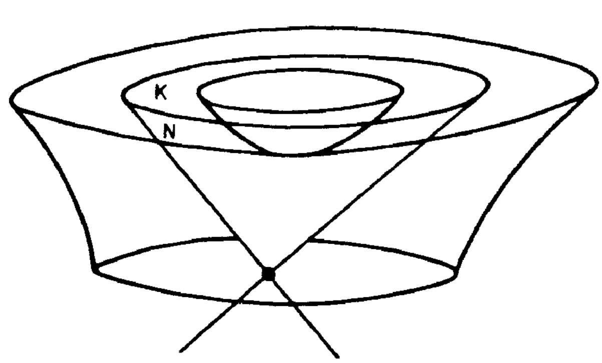
Note also that we can completely describe the orbits in sl(2) under the adjoint action (cf. [HH 85b]).

PROPOSITION IV.5.13: (cf. Figure 10) *The orbits in* sl(2) *under the adjoint action are:*

a) *In the interior of the standard double cone*
$SL(2) \cdot (uU) = \{X = X' + xU: x^2 - |X'|^2 = u^2, \; xu > 0\}$.
b) *In the exterior of the standard double cone*
$SL(2) \cdot (tT) = \{X = X' + xU: |X'|^2 - x^2 = t^2\}$.
c) *On the boundary of the standard double cone*
$SL(2) \cdot 0 = \{0\}$; $SL(2) \cdot P = N \setminus \{0\}$; $SL(2) \cdot Q = -N \setminus \{0\}$.

The hyperboloids of a) *and* b) *constitute the level sets of* k *for non-zero values.*

Figure 10



Next we will develop an explicit description of the exponential function $\exp: sl(2) \to SL(2)$. For any X in $sl(2)$, we have $X^2 = k(X)\mathbb{1}$, whence all even powers of X are scalar multiplex of X. We define the power series

(5.11) $C(z) = 1 + \frac{z}{2!} + \frac{z^2}{4!} + \ldots$ and $S(z) = 1 + \frac{z}{3!} + \frac{z^2}{5!} + \ldots,$

and note the formulae

(5.12a) $C(z^2) + zS(z^2) = e^z.$

(5.12b) $C(x) = \begin{cases} \cosh\sqrt{x} & \text{for } 0 \le x, \\ \cos\sqrt{-x} & \text{for } 0 > x, \end{cases} \quad \sqrt{|x|}\, S(x) = \begin{cases} \sinh\sqrt{x} & \text{for } 0 \le x \\ \sin\sqrt{-x} & \text{for } 0 > x. \end{cases}$

(5.12c) $C(z)^2 - zS(z)^2 = 1.$

(5.12d) $C'(z) = \frac{1}{2}S(z).$

Formulae 5.12a) and 5.12b) are obvious, the last two identities can easily be shown by considering just positive z and using the analyticity of S and C.

We have *the fundamental formula for the exponential function*

(5.13) $\exp X = C(k(X))\mathbb{1} + S(k(X))X.$

In particular, we find the element $\exp X$ inside A always in the plane spanned by $\mathbb{1}$ and X, and, since $\mathrm{tr}X = 0$, we have $\tau(\exp X) = C(k(X))$. Moreover we note that $k(X) = 0$ implies $\exp X = \mathbb{1} + X$ so that on $N \cup -N$ the exponential function is affine.

To discuss the singularities of the exponential function, consider the function C which is holomorphic on the whole complex plane. From 5.12 follows that for $x > -\pi^2$ we have $C'(x) > 0$. Hence $C: [-\pi^2, \infty[\to [-1, \infty[$ is a homeomorphism, and thus has an inverse $c: [-1,\infty[\to [-\pi^2, \infty[$ which must be real analytic on $]-1,\infty[$.

Now let us introduce a half space in A given by $A^* = \tau^{-1}]-1,\infty[$. We define a function which we provisionally call logarithm $\mathrm{Log}: A^* \to sl(2)$ by

(5.14) $\mathrm{Log}\, g = \frac{1}{S(c(\tau(g)))}(g - \tau(g)\mathbb{1})$ for $g \in A^*$.

By (1.1) we have $k(g-\tau(g)\mathbf{1}) = \tau((g-\tau(g)\mathbf{1})^2) = \tau(g^2) - 2(\tau(g))^2 + (\tau(g))^2 = \tau(g^2) - (\tau(g))^2 = -\det g + (\tau(g))^2$. Thus $k(\mathrm{Log}\, g) = S(c(\tau(g)))^{-2}\,(\tau(g)^2 - \det g)$.

Now we specialize to $g \in SL(2) \cap A^*$, i.e. to $\det g = 1$, and find $k(\mathrm{Log}\, g) = S(c(\tau(g)))^{-2}\,(\tau(g)^2 - 1)$. If for the moment we say $y = \tau(g)$ and $x = c(y)$, then $y = C(x)$ and from (5.12c) we find $c(y) = x = S(x)^{-2}(1-y^2) = S(c(y))^{-2}(1-y^2)$. Thus for $g \in SL(2) \cap A^*$ we have $k(\mathrm{Log}\, g) = c(\tau(g)) > \pi^2$ and then

$\exp \mathrm{Log}\, g = C(k(\log g))\mathbf{1} + S(k(\log g))\mathrm{Log}\, g = y\,\mathbf{1} + S(x) \cdot \frac{1}{S(x)}(g - y\cdot\mathbf{1}) = g.$

So we have

$$\exp \mathrm{Log}\, g = g \quad \text{for} \quad g \in SL(2) \cap A^*.$$

Now $\tau(\exp X) > -1$ iff $C(K(X)) > -1$ by (5.13), hence $\exp X \in A^*$ iff $k(X) > -\pi^2$ iff $|X'|^2 - x^2 > -\pi^2$. This gives an invariant open domain $\mathcal{D} = \{X \in sl(2): k(X) > -\pi^2\} = \{X = X' + xU: x^2 < |X'|^2 + \pi^2\}$. For $x \in \mathcal{D}$ we have $\exp X \in A^*$ so that we may consider the analytical function $X \to \mathrm{Log}\exp X: \mathcal{D} \to sl(2)$. Since exp is a local diffeomorphism around zero, every X near zero may be represented in the form $X = \mathrm{Log}\, g$ for some g near $\mathbf{1}$ by (5.15). For these X we have $\mathrm{Log}\exp X = \mathrm{Log}\exp\mathrm{Log}\, X = \mathrm{Log}\, g = X$. The analytical function $X \to \mathrm{Log}\exp X$ thus agrees on a neighborhood of zero with the identity function hence is equal to the identity function. We thus have the first part of

THEOREM IV.5.14: a) <u>The exponential function</u> $\exp: sl(2) \to SL(2)$ <u>induces an isomorphism of real analytical manifolds from</u> $\mathcal{D}$ <u>onto</u> $SL(2) \cap A^*$ <u>whose inverse is given by</u> $\mathrm{Log}: SL(2) \cap A^* \to \mathcal{D}$,
$\mathrm{Log}\, g = S(C^{-1}(\tau(g)))^{-1}(g - \tau(g)\mathbf{1})$.
b) $SL(2) \cap \tau^{-1}(]-\infty,-1]) \cap \exp(sl(2)) = \{-\mathbf{1}\}$.
c) <u>The set of singular points of</u> exp <u>is</u> $\exp^{-1}(-\mathbf{1}) = \{SL(2)\cdot(\pi + 2\pi\mathbb{Z})U\}$.

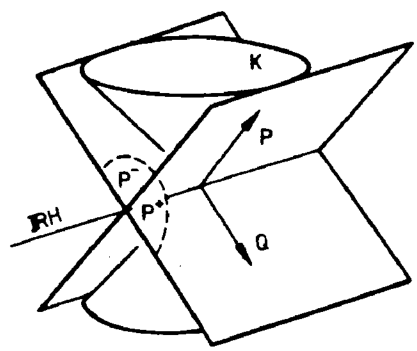
Proof: To show b) and c) simply note that from 5.12.b) and 5.13 it follows that $\exp X \in \exp(sl(2)) \cap \tau^{-1}(]-\infty,-1])$ is equivalent to $\sqrt{-k(X)} \in \pi + 2\pi\mathbb{Z}$. □

Let us finally remark that, as a consequence of Theorem 5.14(a) such as $\mathcal{B} = \mathbb{R}H + \mathbb{R}P$ is mapped diffeomorphically under exp since $\mathcal{D}$ is an open neighborhood of $\mathcal{B}$.

We need to recall some facts from the local theory of semigroups in SL(2). We start by fixing the notation for the relevant sets.

DEFINITION IV.5.15: For $X \neq 0$ and $k(X) = 0$ we set
$X^+ = \{Y \in sl(2): k(X,Y) \le 0\}$, $X^- = -X^+$;
$\mathcal{P}^+ = P^- \cap Q^- = \{hH + pP + qQ: (h,p,q) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+\}$;
$\mathcal{P}^- = P^+ \cap (-Q)^+ = \{hH + pP + qQ: (h,p,q) \in \mathbb{R} \times \mathbb{R}^+ \times -\mathbb{R}^+\}$.
(The reader should keep the distinction between $\mathcal{P}^+$ and P^+ etc. in mind

Figure 11



With these definitions we have (cf. II.1.18)

PROPOSITION IV.5.16: Any generating semialgebra on $sl(2)$ is the intersection of half space semi-algebras each of which is of the form X^+ for some $X \neq 0$ with $k(X) = 0$.

THEOREM IV.5.17: If W is a wedge in $sl(2)$, and $\dim H(W) = 1$, then W is a Lie semialgebra iff it is conjugate to one of the following.

a) $\mathcal{P}^+$, $\mathcal{P}^-$ or $-\mathcal{P}^-$, if $\dim(W-W) = 3$,
b) the half-planes in P bounded by $\mathbb{R}H$ if $\dim(W-W) = 2$ and $k(X) > 0$ for some $X \in H(W)$,
c) the half-planes in $P^\perp$ bounded by $\mathbb{R}P$ if $\dim(W-W) = 2$ and $k(X) = 0$ for all $X \in H(W)$,
d) $\mathbb{R}H$, $\mathbb{R}U$ or $\mathbb{R}P$ if $\dim(W-W) = 1$.

THEOREM IV.5.18: A wedge W in $sl(2)$ is a Lie wedge iff it is either a cone or else a semialgebra.

Proof of 5.17 and 5.18 : Exercise (use Proposition 5.13 and 4.12). □

We have now laid the ground to study infinitesimally generated semigroups in $Sl(2)$. There is one, that will turn out to play a special role. If we identify A with the set of two by two matrices as we did before, this semigroup can be described as the set of matrices in $SL(2)$ with non-negative entries. *We denote this semigroup by* $SL(2)^+$.

PROPOSITION IV.5.19: <u>The exponential function induces on isomorphism</u> $\exp: P^+ \to SL(2)^+$ <u>of analytic manifolds with boundary</u> (cf. 5.15).

Proof: After Theorem 5.14 , it suffices to show (i) $\exp P^+ \subseteq SL(2)^+$ and (ii) $\operatorname{Log} SL(2)^+ \subseteq P^+$ with log given in (5.14). To show (i), let $H = hH + pP + qQ$ with $p,q \geq 0$ and set $\exp(X) = \binom{a\ b}{c\ d}$. We have $k(X) = h^2 + pq$ and set $t = k(X)$. If $k(X) = 0$, then $X = P$ or $X = Q$ and $\exp X = \mathbf{1} + P$ (resp. $\mathbf{1} + Q$) which is contained in $SL(2)^+$. So assume $t > 0$, and conclude from (5.13) that $a = (\cosh t) + \frac{(\sinh t)}{t} h$, $b = \frac{\sinh t}{t} p$, $c = \frac{\sinh t}{t} q$, and $d = \cosh t - \frac{\sinh t}{t} h$. Since $t > 0$ we have $\sinh t > 0$, so that $a,b,c \geq 0$. But from $\frac{h}{t} = \frac{h}{\sqrt{h^2 + pq}} \leq 1$ it follows that $d \geq 0$, too.

To prove (ii), let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2)^+$. Then $\det g = 1$ implies $ad = 1 + cb \geq 1$, whence $1 \leq \sqrt{ad} \leq \frac{a+d}{2}$ which means $\tau(g) \geq 1$. Thus $C^{-1}(\tau(g)) = (\operatorname{ar\,cosh} \tau(g))^2 \geq 0$ and $S(C^{-1}(\tau(g))) \geq 0$ (with equality precisely for $g = \mathbf{1}$). By (5.14) we have $\operatorname{Log} g \in P^+$ if $b \geq 0$ and $c \geq 0$, which is then the case. □

As a consequence of this proposition, we know that $SL(2)^+$ is a uniquely divisible semigroup whose tangent object $L(SL(2)^+)$ is P^+. Here we mean by uniquely divisible that for any $s \in SL(2)^+$ and any $n \in \mathbb{N}$ there exists a unique $s_1 \in SL(2)^+$ with $(s_1)^n = s$. Moreover, the Campbell-Hausdorff multiplication $(X,Y) \to X * Y$ allows an analytic extension to a semigroup multiplication $*: P^+ \times P^+ \to P^+$.

The preceding calculations permit us to demonstrate that the Lie wedges of quite reasonable semigroups in $SL(2)^+$ are not semialgebras. Indeed, let S be the set of all matrices $g = \binom{a\ b}{c\ d}$ in $SL(2)^+$ with $a \geq 1$. Except for $g = \mathbf{1}$ we have $\tau(g) > 1$ and then $t = S(\operatorname{arcosh} \tau(g)) > 0$. Thus $\operatorname{Log} g = (2t)^{-1} \begin{bmatrix} a-d & b \\ c & d-a \end{bmatrix}$ with $d = (1 + bc)a^{-1}$ whence $\operatorname{Log} g = gH + pP + qQ$ with $h = \frac{1}{2ta}(a^2 - 1 - bc) \geq -\frac{bc}{2ta}$. One checks that

$L(S) = \{hH + pP + qQ: (h,p,q) \in (\mathbb{R}^+)^3\}$: Indeed, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \exp r(hH + pP + qQ) \in S$ for $r > 0$, then $a = a(r) = C(r^2k) + S(r^2k)rh \geq 0$, whence $h = \lim_{r \to 0} \frac{a(r)}{r} \geq 0$.

Whether S is in fact the smallest closed subsemigroup $\overline{S}_R$ (cf. Theorem 2.8) containing $\exp L(S)$ we do not know, but our calculation shows at any rate that $\exp(L(S)) \subseteq \overline{S}_R \subseteq S$, and since $L(S)$ is no semialgebra, $\exp L(S)$ is not a neighborhood of $\mathbf{1}$ in $\overline{S}_R$, let alone S.

PROPOSITION IV.5.20: Let S be a preanalytic semigroup in $SL(2)$ and $\overline{S}_R$ is smallest closed semigroup in $SL(2)$ containing $\exp(\underline{L}(S))$. Then we have the following possibilities:

a) $\overline{S}_R$ is a circle group $\exp \mathbb{R}X$ with $k(X) < 0$,

b) $\overline{S}_R = G$,

c) a conjugate of $\overline{S}_R$ is contained in $SL(2)^+$.

Proof: W·log we set $S = \overline{S}_R$. Then S is infinitesimally generated, and is therefore completely characterized by its tangent Lie wedge $L(S)$. If $L(S)$ contains an element X with $k(X) < 0$, then S contains $\exp \mathbb{R}X$, a circle group, whence $\mathbb{R}X \subseteq L(S)$. Thus $L(S)$ is not a cone and hence, by Theorem 5.18 a semialgebra. This means that $L(S) = sl(2)$ or $\mathbb{R}X$ since no other semialgebras contain elements of negative k-length and have $\mathbb{R}X$ in the edge. Now assume that $L(S) \cap \text{interior } (K \cup -K) = \emptyset$. Find a half space X^+ with $k(X) = 0$ and $K \subseteq X^+$ as well as $L(S) \subseteq -X^+$, and a half space Y^+ with $k(Y) = 0$ and $-K \subseteq Y^+$ as well as $L(X) \subseteq -Y^+$. Then $-X^+ \cap -Y^+$ is a semialgebra containing $L(S)$, and by Theorem 5.17 it is conjugate to P^+. Thus a conjugate of S is contained in $SL(2)^+$.

In order to sharpen this result we will show that $SL(2)^+$ is a maximal proper connected subsemigroup of $SL(2)$, i.e. that any connected subsemigroup T of $SL(2)$ containing $SL(2)^+$ is either $SL(2)^+$ or $SL(2)$. For this purpose we need a Lemma.

LEMMA IV.5.21: For any element $X \in sl(2)$ there are the following mutually exclusive possibilities for the element $\exp X = g$:

(i) $X \in P^+ \cup -P^+$ (i.e. $g \in Sl(2)^+ \cup (Sl(2)^+)^{-1}$).

(ii) There is a positive number s such that $k(sH * X) < 0$ (i.e. $(\exp sH)g$ lies on a circle group in $Sl(2)$).

Remark: For X, Y and $Z \in \mathcal{D}$ with $\exp X \exp Y = \exp Z$ we write $Z = X * Y$.

Proof: We set $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$; then (5.14) says that for a suitable number $\alpha \in \mathbb{R}$ we have $X = \log g = \alpha(g - \tau(g) \cdot \mathbb{1}) = \frac{\alpha}{2} \begin{pmatrix} x-w & 2y \\ 2z & w-y \end{pmatrix} = \alpha(\frac{1}{2}(x-w)H + yP + zQ)$.
Thus case (i) occurs precisely when $yz \geq 0$. We now assume $yz < 0$, i.e. $xw - 1 = yz < 0$, i.e., $xw < 1$.

Now $\exp(sH * X) = (\exp sH)g = \begin{bmatrix} tx & ty \\ \frac{z}{t} & \frac{w}{t} \end{bmatrix}$ with $t = e^s$.

Thus with a suitable scalar β we then have

$$sH * X = \beta(\frac{1}{2}(tx - \frac{w}{t})H + ty + \frac{z}{t} Q).$$

Then $k(sh * X) < 0$ iff $\frac{1}{4}(tx - \frac{w}{t})^2 + xw - 1 = \frac{1}{4}(tx - \frac{w}{t})^2 + yz < 0$ iff $(tx + \frac{w}{t})^2 < 4$.

There are two cases to consider: If $xw \leq 0$, then it is easy to find a $t > 0$ with $(tx + \frac{w}{t})^2 < 4$. If $xw > 0$, then the function $u \to (ux + \frac{w}{u})^2$ attains a minimum for $t = (w/x)^{1/2}$, and this minimum is equal to $xw < 4$. In either case if we take $s = \log t$ we have $k(sH * X) < 0$. □

LEMMA IV.5.22: <u>Let</u> S <u>be a subsemigroup of</u> Sl(2) <u>containing</u> $Sl(2)^+$. <u>If</u> S <u>meets the interior of</u> $\exp(K \cup -K) \cup (Sl(2)^+)^{-1}$, <u>then</u> $S = Sl(2)$.

Proof: Let $s \in S$, then each neighborhood U of s contains inner points of S: Indeed the identity neighborhood $s^{-1}U$ contains inner points of $Sl(2)^+$ hence of S, and so $U = s(s^{-1}U)$ contains inner points of S. Therefore, if S meets the interior of $\exp(K \cup -K)$, then an open subset of some circle group is in S, and it then follows that this whole circle group and $\exp \mathbb{R}H$. Thus $S = Sl(2)$ in this case. Now assume that S contains a point s in the interior of $(Sl(2)^+)^{-1}$. Then $(Sl(2)^+)^{-1}$ contains an open subset V of S. But then $V^{-1} \subset Sl(2)^+$, whence the identity neighborhood VV^{-1} is contained in S. But since Sl(2) is connected, the semigroup generated by any symmetric identity neighborhood is Sl(2). □

LEMMA IV.5.23: <u>Let</u> S <u>be a subsemigroup of</u> Sl(2) <u>containing</u> $Sl(2)^+$. <u>Suppose that</u> S <u>contains a boundary point</u> s <u>of</u> $(Sl(2)^+)^{-1}$ <u>which is not contained in</u> $Sl(2)^+$. <u>Then</u> $S = Sl(2)$.

Proof: We may assume that $s \in \exp(\mathbb{R}H + \mathbb{R}P)$; the case $s \in \exp(\mathbb{R}H + \mathbb{R}Q)$ is treated analogously. If $B = \exp(\mathbb{R}H + \mathbb{R}P)$, then the semigroup $S \cap B$ contains the half space semigroup $S = \exp(\mathbb{R}H + \mathbb{R}^+P)$ in B and the element s outside S'. But then sS' is a neighborhood of the identity in B, and since B is generated as a semigroup by any neighborhood of the identity, we have $B \subseteq S$.

Thus S contains the semigroup generated by $B \cup (Sl(2)^+$ which, by Proposition 5.20 is dense in $Sl(2)$. But it also contains inner points, namely the ones of $Sl(2)^+$.

The assertion then is a consequence of the following Lemma:

LEMMA IV.5.24: <u>If</u> S <u>is a dense subsemigroup of a topological group</u> G <u>and if the interior of</u> S <u>is not empty, then</u> $S = G$.

Proof: Let U be the interior of S. Since $U \neq \emptyset$, there is an $s \in S \cap U^{-1}$. Then sU is an open identity neighborhood which is contained in S. Thus the subgroup $H = S \cap S^{-1}$ is open in G and contained in S. If $g \in G$, then the neighborhood gH of g contains a semigroup element $t \in S$, whence $g \in sH^{-1} = sH \subseteq sS \subseteq S$. □

Now we have the following Proposition:

PROPOSITION IV.5.25: <u>Let</u> S <u>be any proper subsemigroup of</u> $Sl(2)$ <u>containing</u> $Sl(2)^+$. <u>Then</u> $S \cap \operatorname{im}\exp = Sl(2)^+$ <u>and</u> $S \subseteq Sl(2)^+ \cup -Sl(2)^+ = \{\mathbf{1},-\mathbf{1}\} \cdot Sl(2)^+$.

Proof: Since $S \cup -S = \{\mathbf{1},-\mathbf{1}\}\cdot S$ is a semigroup containing S, we may assume without losing generality that $S = -S$. By Lemmas 5.21 , 22 and 23 we know $S \cap \operatorname{im}\exp \subseteq Sl(2)^+$, which proves the first assertion. Now suppose $s \in S$. Since $\operatorname{im}\exp \cup -\operatorname{im}\exp = Sl(2)$ (cf. (5.14)), we know that s or $-s$ in $\operatorname{im}\exp$, hence s or $-s$ is in $Sl(2)^+$. Thus $s \in Sl(2)^+ \cup -Sl(2)^+$. □

The sets $Sl(2)^+$ and $-Sl(2)^+$ are obviously disjoint, hence:

EXAMPLE IV.5.26: a) The subsemigroup $Sl(2)^+$ is a maximal connected proper subsemigroup of $Sl(2)$.
b) The image $PSl(2)^+$ of $Sl(2)^+$ in $PSl(2)$ is a maximal proper subsemigroup of $PSl(2)$. □

The study of the universal covering group $\widetilde{SL(2)}$ of SL(2) will also allow us to show that any proper subsemigroup of SL(2) containing a circle group coincides with that circle group. Thus we can sharpen Proposition 5.20 as follows.

EXAMPLE IV.5.27: Let S be closed preanalytic subsemigroup of SL(2). Then we have the following possibilities:

a) S is a circle group.

b) S = SL(2).

c) There exists a $g \in SL(2)$ such that $g\overline{S}_R g^{-1} \subset SL(2)^+$. If $g\overline{S}_R g^{-1} = SL(2)^+$ then S is either $SL(2)^+$ or $\{\mathbf{1},-\mathbf{1}\}SL(2)^+$. □

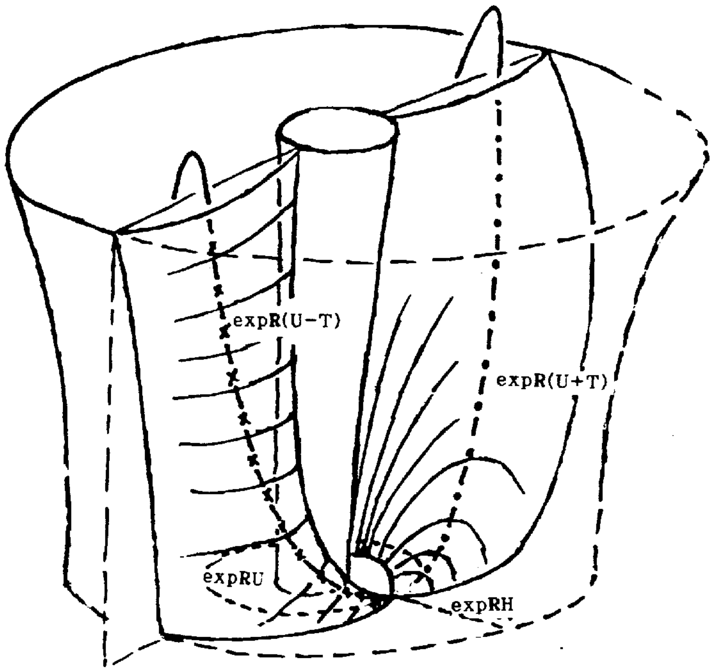
We do not know whether there are closed connected proper semigroups S with $\overline{S}_R \subset SL(2)^+$ which are not contained in $SL(2)^+$.

Proposition 5.20 shows that no semialgebra intersecting the interior of the standard double cone manifests itself as a global semigroup in Sl(2) - unless it is a subalgebra; on the other hand, according to the basic theorem of the local Lie theory of semigroups, all of them define local semigroups.

This situation becomes radically different if we ascend to the universal covering group $\widetilde{SL}(2)$ of SL(2).

The polar decomposition of each element of SL(2) into a product of an element of SO(2) and a triangular matrix shows quickly that SL(2) is topologically the product of a one sphere and plane. Thus the universal covering space is $\mathbb{R}^3$. Therefore, in order to present the universal covering group all that is required is the fixing of a covering map $f: \mathbb{R}^3 \to SL(2)$, presumably one which respects the polar decomposition. The general theory of simple connectivity and universal covering spaces then gives a unique Lie group structure on $\mathbb{R}^3$ for any fixed identity element such that f becomes a covering morphism, and the lifting of the exponential function $\exp: sl(2) \to SL(2)$ to a function $\mathrm{Exp}: sl(2) \to \mathbb{R}^3$ gives the exponential function of the universal covering. Thus, theoretically, there is nothing left to do. (cf. Figure 12)

Figure 12



Except, that for calculations and even for the formation of a geometric intuition of the structure of the covering group, a lot depends on an explicit choice of the "parametrization" f. We propose here a particular one which we find to have many good features. Notably, as convenient domain for f we will take sl(2) itself and respect as much as we can the symmetries defined by the adjoint action of the circle group $\exp \mathbb{R}U$.

Of course, we retain the notation and concepts introduced above.

LEMMA IV.5.28: <u>For</u> $X = X' + xU$ <u>we have the following identities:</u>

$$(\exp xU)\exp(-xU + e^{-x/2\,\mathrm{ad}\,U}X) = \exp xU \exp(e^{-x/2\,\mathrm{ad}\,U}X') =$$
$$= C(|X'|^2)\exp(xU) + S(|X'|^2)X' = C(k(X)\exp(xU) + S\&k(X))X' =$$
$$= \cosh(|X'|)\exp(xU) + \sinh(|X'|)(|X'|^{-1}X').$$

Proof: The first equality is immediate from $X = X' + xU$, whence $e^{t\,\mathrm{ad}\,U}X = e^{t\,\mathrm{ad}\,U}X' + xU$. By the invariance of k we have

$k(e^{-x/2\,\mathrm{ad}\,U}X') = k(X') = |X'|^2$. From (5.13) it then follows that $\exp(e^{-x/2\,\mathrm{ad}\,U}X') = C(|X'|^2)\mathbb{1} + S(|X'|^2)(e^{-x/2\,\mathrm{ad}\,U}X')$. Now we multiply through with $\exp tU$ and note from

$$(5.15)\qquad e^{r\,\mathrm{ad}\,U} = \begin{bmatrix} \cos 2r & \sin 2r & 0 \\ -\sin 2r & \cos 2r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

that $e^{t\,\mathrm{ad}\,U}(X') = \exp(2tU)\cdot X'$ since

$$\exp tU = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \in A.$$

Thus in view of (5.12) all identities are proved. □

We are now ready to give the core definition

DEFINITION IV.5.29: We define a function $f: sl(2) \to SL(2)$ by

$$f(X) = C(k(X'))\exp xU + S(k(X'))X' = (\exp xU)\exp(e^{-x/2\,\mathrm{ad}\,U}X').$$

Note that $f(X) = \cosh|X'|\exp xU + \sinh|X'|\frac{X'}{|X'|}$ for $x' \neq 0$ and $f(xU) = \exp xU$. We will show that this function is a covering map, thus use it to introduce on $sl(2)$ the structure of the universal covering group of $SL(2)$. First we observe that f is analytic and that $f(X)$ is contained in the plane spanned in A by $\exp xU$ and X'. Recall that $\exp xU = (\cos x)\mathbb{1} + (\sin x)U$, and hence

$$(5.16)\qquad f(X) = (\cos h\,|X'|\sin x)U + (\cos h\,|X'|\cos x)\mathbb{1} + \frac{\sin h\,|X'|}{|X'|}X'.$$

From $\tau(U) = \tau(X') = 0$, we immediately obtain

$$(*)\qquad \tau(f(X)) = \cos h\,|X'|\cos x.$$

If we denote by p_u the projection of A onto $\mathbb{R}U$ with kernel spanned by $\mathbb{1}$, H and T, then

$$(**)\qquad p_u(f(X)) = \cos h\,|X'|\sin x.$$

For any element $a \in A$, we define a complex number $z(a) = \tau(a) + ip_u(a)$ and call it *the characteristic number* of $a \in A$. In this way we can write (*) and (**) as

$$(5.17)\qquad z(f(X)) = \cos h\,|X'|\,e^{ix}.$$

We extend $X \to X'$ to a projection $a \to a': A \to \mathbb{R}H + \mathbb{R}T$ with kernel $\mathbb{R}\mathbf{1} + \mathbb{R}U$.

LEMMA IV.5.30: <u>If</u> $a \in A$, <u>then</u> $a = \tau(a) \cdot \mathbf{1} + p_u(a)U + a'$ <u>and</u> $\det a = |z(a)|^2 - |a'|^2$. <u>In particular</u>, $g \in SL(2)$ <u>implies</u> $|z(g)|^2 = 1 + |g'|^2 \geq 1$.

Proof: The claim follows immediately from the definition of $z(a)$ and the fact that with $a = hH + tT$ we have

$$a = \begin{bmatrix} \tau(a) + h & p_u(a) + t \\ -(p_u(a) - t) & (a) - h \end{bmatrix}. \quad \square$$

We apply the Lemma to $f(X)$ and obtain for $X' \neq 0$

$$(5.18) \qquad \frac{X'}{|X'|} = (|z(f(X))|^2 - 1)^{-1/2} f(X)'.$$

Formulae (5.17) and (5.18) show that $f(X)$ determines X' completely and we see from (5.17) that $f(X_1) = f(X_2)$ iff $x_2 - x_1 \in 2\pi\mathbb{Z}$. Moreover if u is a complex number of modulus greater than one and E a horizontal unit vector in $sl(2)$, then there is an X in $sl(2)$ such that $u = z(f(X))$ and $X' = |X'|E$. Note that f *is surjective*. In fact, let $g \in SL(2)$, then $|z(g)| \geq 1$, and by the preceding remarks, we find an X in $sl(2)$ such that $z(f(X)) = z(g)$ and $f(X)' = g'$, since $|g'| = (|z(g)|^2 - 1)^{1/2}$ by Lemma IV.5.30. From this we conclude $f(X) = g$ by (5.17).

We now decompose f canonically into the quotient map $sl(2) \to sl(2)/2\pi\mathbb{Z}U$ and the induced continuous bijection $f^*: sl(2)/2\pi\mathbb{Z} \to SL(2)$. Since f has no singular points, as is readily verified from the definition, f^* is also open. Hence f^* is a homeomorphism and thus f is a covering map. Since f is analytic, we know that there is a Lie group multiplication $(X,Y) \to X \circ Y: sl(2) \times sl(2) \to sl(2)$ satisfying

$$(5.19) \qquad f(X \circ Y) = f(X)f(Y).$$

We denote $(sl(2), \circ)$ by G. In order to establish some basic properties of the multiplication, we observe a number of equivariance properties of f.

First we define an action of the additive group $\mathbb{R}$ on $sl(2)$ as a combination of a rotation around the vertical and a vertical translation: For $r \in \mathbb{R}$ and $X \in sl(2)$ we set

(5.20) $r\cdot X = e^{r/2\,\mathrm{ad}\,U}X' + (r+x)U = (\exp rU)X' + (r+x)U.$

Now we can establish the following Lemma.

LEMMA IV.5.31: a) $f(r\cdot X) = (\exp rU)f(X)$

b) $f(e^{r\,\mathrm{ad}\,U}X) = (\exp rU)f(X)(\exp - rU) = e^{r\,\mathrm{ad}\,U}f(X)$

c) $f(\ X) = T\ f(X)T$ (cf. (1.4)).

Proof:

a) $f(r\cdot X) = f(e^{r/2\,\mathrm{ad}\,U}X' + (r+x)U) =$

$= C(k(X'))\exp(r+x)U + S(k(X'))\,e^{r/2\,\mathrm{ad}\,U}X'$

$= \exp rU\,(C(k(X'))\exp xU + S(k(X'))X') = (\exp rU)f(X).$

b) $f(e^{r\,\mathrm{ad}\,U}X) = f(e^{r\,\mathrm{ad}\,U}X' + xU) = C(k(X'))\exp xU + S(k(X'))e^{r\,\mathrm{ad}\,U}X'$

$= e^{r\,\mathrm{ad}\,U}(C(k(X'))\exp xU + S(k(X'))X') = e^{r\,\mathrm{ad}\,U}f(X)$

$= (\exp rU)f(X)\exp(-rU),$

since $e^{r\,\mathrm{ad}\,U}Y = (\exp rU)Y(\exp -rU)$.

c) Left to the reader as an easy exercise. □

From the discussion above we derive the following

THEOREM IV.5.32: There is a Lie group multiplication $(X,Y)\to X\circ Y$ on $sl(2)$ such that $f\colon G\to SL(2)$ is the universal covering morphism, where G denotes the group $(sl(2),\circ)$ and that the following properties are satisfied:

a) $(rU)\circ(sU) = (r+s)U.$

b) If $E\in\mathcal{E}$, then $rE\circ sE = (r+s)E.$

c) $rU\circ X\circ(-rU) = e^{r\,\mathrm{ad}\,U}X$. Thus the decomposition $\mathcal{E}\oplus\mathbb{R}U$ is invariant under inner automorphisms induced by rU.

d) $rU\circ X = e^{r/2\,\mathrm{ad}\,U}X' + (r+x)U$ and $rU\circ(\mathcal{E}+uU) = \mathcal{E}+(r+u)U.$

e) $X\circ(rU) = e^{-r/2\,\mathrm{ad}\,U}X' + (r+x)U$ and $(\mathcal{E}+uU)\circ rU = \mathcal{E}+(r+u)U.$

f) $X\circ(-X) = (-X)\circ X = 0.$

g) $\pi\mathbb{Z}U$ is the center of G, and we have $X\circ 2n\pi U = X + 2n\pi U$, $X\circ(2n+1)\pi U = -X + (2n+1)\pi U$ for $n\in\mathbb{Z}$.

Proof: a) We have

$f(rU\circ sU) = f(rU)f(sU) = \exp rU\exp sU = \exp(r+s)U = f((r+s)U).$

For $r=s=0$ we note $rU\circ sU = 0 = (r+s)U$. Since liftings are unique, we conclude $rU\circ sU = (r+s)U$.

b) Let $E\in\mathcal{E}$ be of norm one, i.e., $E = hH + tT$ with $h^2+t^2 = 1$. Then

$E^2 = k(E)\mathbb{1} = \mathbb{1}$ and $f(rE \circ sE) = ((\cosh r)\mathbb{1} + (\sinh r)E)((\cosh s)\mathbb{1} + (\sinh s)E) = \cosh(r+s)\mathbb{1} + \sinh(r+s)E = f((r+s)E)$. As before, we conclude $rE \circ sE = (r+s)E$.

c), d) and e) follows from calculations of the following type:

$$f(rU \circ X \circ (-rU)) = (\exp rU)f(X)(\exp(-rU)) = e^{r \operatorname{ad} U} f(X) =$$
$$= e^{r \operatorname{ad} U} (C(k(X')) \exp xU + S(K(X'))X') =$$
$$= C(k(e^{r \operatorname{ad} U} X')) e^{r \operatorname{ad} U} \exp xU + S(k(e^{r \operatorname{ad} U} X'))e^{r \operatorname{ad} U} X' =$$
$$= f(e^{r \operatorname{ad} U} X) \quad \text{since} \quad e^{r \operatorname{ad} U} X' = (e^{r \operatorname{ad} U} X)'.$$

f) We have $f(X \circ -X) = (\exp xU) \exp(e^{-x/2 \operatorname{ad} U} X') \exp(-xU) \cdot \exp(-e^{x/2 \operatorname{ad} U} X')$
$= \exp(e^{x \operatorname{ad} U} e^{-x/2 \operatorname{ad} U} X') \exp(-e^{x/2 \operatorname{ad} U} X') = \mathbb{1}$.

Thus $f(X \circ (-X)) = f(0)$ and as before $X \circ (-X) = 0$.

g) If X is central in G, then $f(X) = \pm\mathbb{1}$, but this implies $X \in \pi\mathbb{Z}U$. Moreover $X \circ 2n\pi U = e^{-n\pi \operatorname{ad} U} X' + (x + 2\pi n)U = X' + xU + 2\pi nU$ by (5.15). The last inequality is shown similarly. □

Theorem IV.5.32 implies that all horizontal and vertical lines through 0 are one parameter groups in G, and that the inverse agrees with the additive inverse. We will show presently, that *each* one parameter group of G lies in a plane containing the vertical line $\mathbb{R}U$. The group of rotations around the vertical is an automorphism group, and thus we can reduce our structural description to one plane containing the vertical, say the plane spanned by H and U, and derive the general information by rotation. This is an important advantage of our parametrisation.

We apply this strategy now to determine the exponential function $\operatorname{Exp}: sl(2) \to G$. From Theorem 5.32a,b) it follows that Exp agrees with the identity function on $\mathbb{R}U$ and E. Moreover Exp is uniquely determined through $\operatorname{Exp} 0 = 0$ and $f(\operatorname{Exp} X) = \exp X$. We write $\operatorname{Exp} X = \overline{X} + \overline{x}U$ with $\overline{X} = (\operatorname{Exp} X)'$. Then the defining equation $f(\operatorname{Exp} X) = \exp X$ reads

$$(5.21) \qquad C(k(\overline{X})) \exp \overline{x}U + S(k(\overline{X}))\overline{X} = C(k(X))\mathbb{1} + S(k(X))X.$$

From this we derive the following

LEMMA IV.5.33: <u>For</u> X <u>in</u> $sl(2)$ <u>we have</u>

a) $S(|\overline{X}|^2)\overline{X} = S(k(X))X'$

b) $C(|\overline{X}|^2) \cos \overline{x} = C(k(X))$; $C(|\overline{X}|^2) \sin \overline{x} = S(k(X))x$

c) $z(\exp X) = \cosh(|\overline{X}|)e^{i\overline{x}} = C(k(X)) + ix\, S(k(X))$.

Proof: The first three equations follow directly from (5.21) and the fact that $|\overline{X}|^2 = k(\overline{X})$. The last identity is just a reformulation of (5.17) in view of (5.21). □

The equations of Lemma 5.33 allow us to compute the exponential function. In fact they tell us that $\overline{X}$ is a scalar multiple of X'. Thus if we know the functions $\rho = \rho(X) = |\overline{X}|$ and $\operatorname{sgn} S(k(X))$ then we know $\overline{X}$. In particular, as we already announced, $\operatorname{Exp} X \in \operatorname{span}\{X',U\}$. But the complete information on ρ and $\overline{x}$ and thus on Exp is contained in the last equation of Lemma 5.33 which we call the *characteristic equation*. By a slight abuse of language we call the complex number $z(\operatorname{Exp} X)$ *the characteristic number* of X.

We want to determine the shape of the one parameter groups of G from the characteristic equation. Since Exp X is contained in the plane spanned by X' and U, it suffices up to sign to present ρ as a function of $\overline{x}$, or vice versa. The special form of the function C and S forces us to treat vectors with positive, negative and zero k-length separately.

The characteristic equation reads

(5.22)

a) $\cosh\rho \cdot e^{i\overline{x}} = 1 + ir$ for $X = 2rP$,

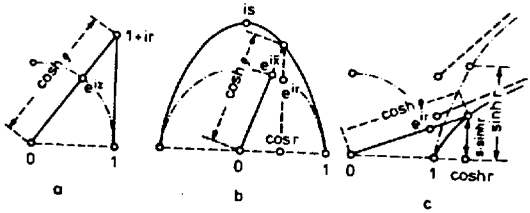
b) $\cosh\rho \cdot e^{i\overline{x}} = \cos r + i\sin r$ for $X = r((s^2-1)^{1/2}H + sU)$, $s \le 1$,

c) $\cosh\rho \cdot e^{i\overline{x}} = \cosh r + i\sinh r$ for $X = r((s^2+1)^{1/2}H + sU)$, $1 \ge 0$.

Note here that it suffices to consider those X listed in (5.22) since they are their rotations around U by π fill all of $\mathbb{R}H + \mathbb{R}U$, so that all other one parameter groups are just the result of rotating one of those described by (5.22).

To get a rough intuition of what these one parameter groups look like we consider the following figure that depicts graphically how (5.22) determines the pair $(\rho,\overline{x})$ from the given data r,s (cf. Figure 13).

Figure 13



An analytic description of the point sets $\operatorname{Exp} \mathbb{R} \cdot X$ in $\operatorname{span}\{H,U\}$ is given by

PROPOSITION IV.5.34: Let X be in $\mathbb{R}^+H + \mathbb{R}^+U$. Then

a) $\operatorname{Exp}(\mathbb{R}^+X) = \{\rho H + \zeta U: \cosh \rho \cos \zeta = 1,\ \rho \geq 0,$ where $0 \leq \zeta \leq \frac{\pi}{2}\}$ for $k(X) = 0$,

b) $\operatorname{Exp}(\mathbb{R}^+X) = \{\rho \operatorname{sgn}(\sin \zeta)H + \zeta U: \cosh^2 \rho((a-1)\cos^2\zeta + 1) = a,$ where $\rho, \zeta \geq 0$ and $a = x^2 k(X)^{-1}\}$ for $k(X) < 0$,

c) $\operatorname{Exp}(\mathbb{R}^+X) = \{\rho H + \zeta U: \cosh^2 \rho((a+1)\cos^2\zeta - 1) = a,$ where $\rho \geq 0$, $0 \leq \zeta \leq \frac{\pi}{2}$ and $a = x^2 k(X)^{-1}\}$ for $k(X) > 0$,

d) $\operatorname{Exp}(\mathbb{R}^-X) = -\operatorname{Exp}(\mathbb{R}^+X)$.

We recall once more that we get all one parameter groups in G by rotating those described by Proposition 5.34 .

Proof: a) It suffices to consider $X = 2rP$ so that by (5.22a)) we get $\cosh \rho\ |\cos \bar{x}| = \sqrt{1+r^2}\ (1+\tan^2\bar{x})^{-1/2} = 1$ and $\cos \bar{x} > 0$.
b) Without loosing generality, we consider $X = r((s^2-1)^{1/2}H + sU)$, where $s \geq 1$ and obtain $a = s^2$. Thus by (5.22b)), we calculate $\cosh^2\rho = \cos^2 r + a \sin^2 r = (\cot^2 r + a)(1+\cot^2 r)^{-1} = a(1+\cot^2\bar{x})(1+a\cot^2\bar{x})$ $= a(\sin^2\bar{x} + a\cos^2\bar{x})^{-1} = a(1+(a-1)\cos^2\bar{x})^{-1}$.
c) We consider $X = r((s^2+1)^{1/2}H + sU)$, where $s \geq 0$ and obtain $a = s^2$ so that by (5.22c)), $\cos \bar{x} = \cosh r(\cosh \rho)^{-1}$ and $\cosh^2\rho = a \sinh^2 r + \cosh^2 r = (a+1)\cos^2 r - a$. Thus $\cos^2 x \cosh^2\rho = \dfrac{\cosh^2\rho + a}{a+1}$, and the claim follows.
d) Clear with Theorem 5.32 . □

Using the identity $\operatorname{arcosh} S = \log(S - (S^2-1)^{1/2})$ it is now a matter of elementary calculation to derive

(5.23)

a) $|\bar{x}| = \operatorname{arc\,cos}(\frac{1}{\cosh \rho})$ for $k(X) = 0$,

b) $\rho = \log(s + \sqrt{s^2 - 1}\ |\sin \bar{x}|) - \frac{1}{2}\log(1 + (s^2 - 1)\cos^2 \bar{x})$ for $k(X) < 0$ and $s = \frac{|x|}{\sqrt{-k(X)}}$,

c) $|\bar{x}| = \operatorname{arc\,cos}\ ((\frac{s^2}{s^2+1}\frac{1}{\cosh^2\rho} + \frac{1}{s^2+1})^{1/2})$ for $k(X) > 0$ and $s = \frac{|x|}{\sqrt{k(X)}}$.

We observe that the point sets described by (5.23b)) and c), "converge" to the set described by (5.23a)) if s tends to infinity.
It is important to develop an intuitive idea of these results. Figure 14 should help in this regard.

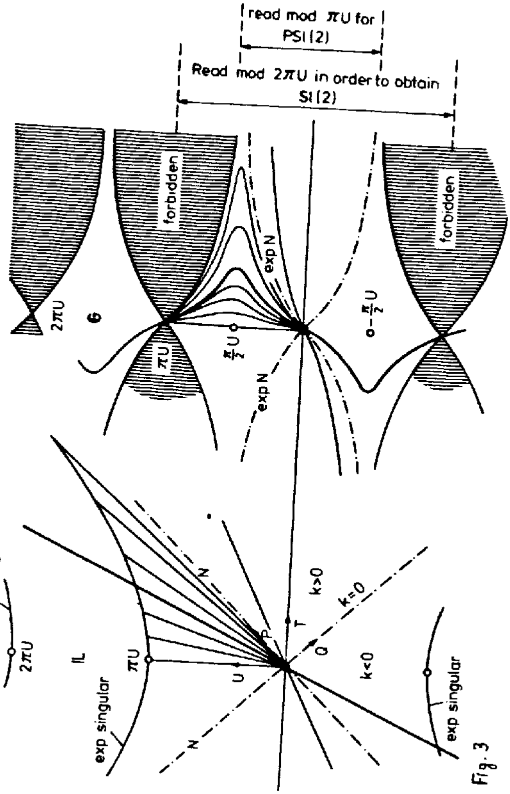
The dark area in Figure 14 is the complement of the image of the exponential function. Reading Figure 14 "modulo 2 ", i.e., considering the plank between level $-\pi$ and π and identifying opposite boundary points we obtain a picture of SL(2). Proceeding in the same way with levels $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ we get a representation of PSL(2). In particular, PSL(2) is exponential, i.e., has a surjective exponential function.

THEOREM IV.5.35: a) Exp <u>induces an isomorphism of real analytical manifolds from</u> $\mathcal{D} = \{X: k(X) > -\pi^2\}$ <u>onto the open area between the surfaces</u> $\operatorname{Exp}(-\mathcal{N}) + \pi u$ <u>and</u> $\operatorname{Exp} \mathcal{N} - \pi u$, <u>i.e. the open set</u>
$\{X = \rho E + \zeta U: E \in \mathcal{E}, |E| = 1, \zeta \in]-\pi,\pi[$ and $\cosh \rho \cos \zeta > -1\}$.
b) <u>The exterior of the standard double cone gets mapped onto</u>
$\{X = \rho E + \zeta U: E \in \mathcal{E}, |E| = 1, \zeta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\cosh \rho \cos \zeta > 1\}$
c) <u>The singular points of</u> Exp <u>are</u> $\operatorname{Exp}^{-1}\{n\pi U: n \in \mathbb{Z} \setminus \{0\}\}$. <u>This set arises from the following upon rotation about the</u> U-<u>axis</u>:

$$\{X = \pi n((s^2 - 1)^{1/2}H + sU): s \geq 1, n \neq 0\}.$$

Proof: a) By Theorem 5.14 , $\exp = f \circ \operatorname{Exp}$ induces an analytic isomorphism from $\mathcal{D}$ onto $SL(2) \cap A^*$. Thus exp induces an analytic isomorphism from $\mathcal{D}$ onto the component of zero in $f^{-1}(SL(2) \cap A^*) = \{X \in \mathbf{L}: \tau(f(X)) > -1\}$. Since $\tau(f(X)) = \cosh |X'| \cos x$ by (2.1), we are looking for the zero component of the set of all $X = \rho E + \zeta U$ with $E \in \mathcal{E}$ and $\cosh \rho \cos \zeta > -1$. But the horizontal planes through the ζU with $\cos \zeta = -1$ separate the set $f^{-1}(SL(2) \cap A^*)$ into components, so the claim follows.

Figure 14



b) Note first that by Proposition 5.34 , the inequality $k(X) < 0$ implies $\cosh^2\rho\,(1 + (a-1)\cos^2\bar{x}) = a$ for some $a \in\,]1,\infty[$, hence $\cosh^2\rho\,\cos^2\bar{x} < 1$. Conversely, $k(X) > 0$ implies $\cosh^2\rho\,((a+1)\cos^2\bar{x} - 1) = a$ for some $a \in\,]0,\infty[$, hence $\cosh^2\rho\,\cos^2\bar{x} > 1$ and the claim follows.

c) The set of singular points of Exp is invariant under the adjoint action, hence, after a), is the union of the orbits of the singular points of the form $X = tU$ (cf. Proposition 6.13). The derivative of Exp in the point X has the kernel $\oplus\{\ker(\mathrm{ad}X - 2\pi in\,\mathbf{I}): n = \pm 1, \pm 2, \ldots\}$ (after extension of the scalars to $\mathbb{C}$) by [Bou72] . Now

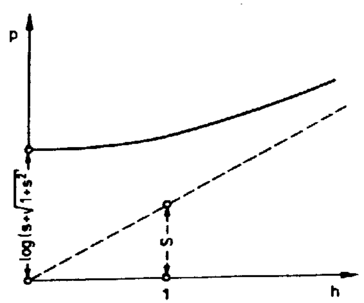
$$\mathrm{ad}\,\pi tU - 2\pi in\,\mathbf{I} = 2\pi \begin{bmatrix} -in & t & 0 \\ -t & -in & 0 \\ 0 & 0 & -in \end{bmatrix}$$

relative to the second basis, and the determinant of this vector space endomorphism of sl(2) is $2\pi ni(n^2 - t^2)$. Thus the singular points on the vertical axis are precisely $n\pi U$, $n = \pm 1, \pm 2, \ldots$ □

To conclude the general description of $\widetilde{SL(2)}$ we describe the two dimensional subgroup Exp $\mathcal{B}$, $\mathcal{B} = \mathbb{R}\cdot H + \mathbb{R}P$ by giving the level lines in sl(2). This means that for a given fixed $\bar{x} \in [0, \frac{\pi}{2}]$ we wish to determine the set $\{\bar{X}: \operatorname{Exp} X = \bar{X} + \bar{x}U;\ X \in \mathcal{B}\}$. But $X = hH + 2pP = (hH + pT) + pU$ gives rise to the characteristic equation $\cosh \rho e^{i\bar{x}} = \cosh h + i\,\frac{p}{h} \sinh h$. If we set $s = \tan \bar{x} = \frac{p}{h} \tanh h$ we get $p = sh \coth h$ and the characteristic equation becomes $\cosh \rho e^{i\bar{x}} = \cosh h(1 + is)$. Now

$\bar{X} = \frac{\rho}{\sqrt{h^2 + p^2}} (h\cdot H + p\cdot T) = h^* H + p^* P$ and $\frac{h^*}{p^*} = \frac{h}{p} = s \coth h$ tends to s for large h. Since for $h = h^* = 0$ we find
$p^* = \rho = \operatorname{arsinh} s = \log(s + (1 + s^2)^{1/2})$ we obtain

Figure 15



Note in particular that the level lines are invariant under reflection at the plane $\mathbb{R}U + \mathbb{R}T$ and that the level line of $-\bar{x}$ is just the negative of the level line of $\bar{x}$. Thus, if $\operatorname{Exp} \mathcal{B} = \{(X', \beta(X')) \in sl(2); X' \in E\}$, where β is the appropriate analytic function, we have $\beta(hH + tT) = \beta(-hH + tT) = -\beta(-hH - tT) = \beta(hH - tT)$.

At this point we are ready to complete the proof of Example 5.27 , i.e. to show that any proper subsemigroup of $SL(2)$ containing a circle group coincides with that circle group. This will be a corollary of the following

THEOREM IV.5.36: Let S be a subsemigroup of $G = sl(2)$ which contains a conjugate K of $\mathbb{R}\cdot U$ (i.e. the lifting of a circle group in $sl(2)$). Then $S = K$ or $S = G$.

Proof: Without loosing generality we take $K = \mathbb{R}\cdot U$ and suppose that there is an $X \in S \setminus \mathbb{R}\cdot U$.
We recall $X = X' + x\cdot U$ according to (5.16) and note that by IV.5.32d) we have $(-x)U \circ X \in E$. But then $0 \neq (-x)U \circ X \in S$. We may therefore assume that $x \in E$. By 5.32c and (5.15), we conclude $-X = e^{(\pi/2)U} X = (\pi/2)U \circ X \circ (-\pi/2)U \in S$. Hence by 5.32f , we note that $re^{\operatorname{ad} X} U = X \circ (rU) \circ (-X) \in S$ for all $r \in \mathbb{R}$. Thus S contains the analytic subgroup generated by $\mathbb{R}U$ and $\mathbb{R}(e^{\operatorname{ad} X} U)$, whose Lie algebra is generated by U and $e^{\operatorname{ad} X} U$, and hence agrees with $sl(2)$. Thus $S = G$. □

COROLLARY IV.5.37: A proper subsemigroup of $SL(2)$ containing a circle group coincides with it.

COROLLARY IV.5.38: Let $G^{\#}$ be a quotient group of G modulo a non-degenerate central subgroup Z and let $S^{\#}$ be a subsemigroup of $G^{\#}$ containing a circle group K. Then $S^{\#} = K^{\#}$ or $S^{\#} = G^{\#}$.

Proof: Let $p: G \to G/Z = G^{\#}$ by the quotient morphism and consider $S = p^{-1}(S^{\#})$. Then S is a subsemigroup of G containing $K = p^{-1}(K^{\#})$. Since all one-parameter groups of G whose image in $G^{\#}$ is a circle group contain the center of G, we conclude that K is connected and thus is a one parameter group. Now K is of the form $\operatorname{Exp} \mathbb{R}\cdot W$ with a conjugate W of U by Proposition 5.13 . Hence Theorem 5.36 applies and shows $S = K$ or $S = G$. But this implies that either $S^{\#} = p(S) = p(K) = K^{\#}$ or $p(S^{\#}) = p(G) = G^{\#}$. □

Note that Proposition 4.1 yields a lot of examples of infinitesimally generated subsemigroups of G by pulling back semigroups from $SL(2)$ and then considering infinitesimally generated subsemigroups of these pull backs (cf. 2.8). Here we will concentrate on semigroups which do not arise in this way.

Recall from Theorem 5.35 that $\mathcal{B} = P^{\perp}$ is mapped diffeomorphically under Exp onto a surface which we may describe by $\operatorname{Exp} \mathcal{B} = \{X \in sl(2): x = \beta(X')\}$ with a suitable analytical function β from the horizontal plane E in $\mathbb{R}$. We set $\Omega^+ = \{X \in sl(2): x \geq \beta(X')\}$ and $\Omega^- = \{X \in sl(2): x \leq \beta(X')\}$. Note that $\Omega^- = \alpha(e^{\pi/2 \operatorname{ad} U}\,\Omega^+)$, since $\beta(hH + tT) = -\beta(hH - tT)$. In contrast to the situation for $SL(2)$ we find that Ω^+ is a semigroup. In order to show this we need the following lemma, which is of separate interest.

LEMMA IV.5.39: <u>Let</u> G <u>be a connected locally compact group and let</u> H <u>be a closed subgroup of</u> G. <u>If</u> A <u>is a closed subgroup of</u> G, <u>isomorphic to</u> $\mathbb{R}$ <u>such that the multiplication</u> $A \times H \to G$ <u>is a homeomorphism, then</u> $G \setminus H$ <u>has two connected components which are both subsemigroups of</u> G <u>and whose boundary is</u> H.

Proof: Using the inversion $g \mapsto g^{-1}: G \to G$ we see that the multiplication $H \times A \to G$ is also a homeomorphism. Let C be one of the connected components of $A \setminus \{1\}$. Define $S = CH$. Then S is one of the connected components of $G \setminus H$. But HC is also a component of $G \setminus H$. Since HC and S intersect at least in C they are equal. From $HC = CH$ and the fact that C and H are semigroups it follows that S is a semigroup. Moreover it follows that $S^{-1} \cap S = \emptyset$. Thus $G = S \uplus H \uplus S^{-1}$ and the other claims follow. (Here $\uplus$ means disjoint union). □

EXAMPLE IV.5.40: We have (interior Ω^+) $= (\mathbb{R}^+ \setminus \{0\})U \circ \operatorname{Exp} \mathcal{B}$ and Ω^+ is a closed semigroup bounded by a two dimensional subgroup. Analogous statements hold for Ω^-.

Proof: Consider the function $F: sl(2) \to SL(2)$ defined by $F(hH + pP + uU) = (\exp uU)(\exp(hH + pP))$. From (5.13) we see that

$$F(hH + pP + uU) = \begin{pmatrix} \cos u & \sin u \\ \sin -u & \cos u \end{pmatrix} \begin{pmatrix} e^h & \frac{p}{h} \sin h \\ 0 & e^{-h} \end{pmatrix}$$

Note that F maps $\mathcal{B}$ diffeomorphically onto the subgroup of upper triangular matrices in $SL(2)$. Moreover the restriction of F to $\mathbb{R}U$ is a covering map of the subgroup $SO(2)$ sitting in $SL(2)$. Thus F is a covering map. If we now consider $sl(2)$ together with the group multiplication $\circ$ provided by F we see that $\mathbb{R}U$ is a closed one parameter group and $\mathcal{B}$ is codimension one connected Lie subgroup such that the multiplication $\mathbb{R}U \times \mathcal{B} \to \widetilde{SL}(2)$ $(uU, hH + pP) \mapsto (uU) \circ (hH + pP)$ is a homeomorphism. Now Lemma 5.39 applies and shows that $(\mathbb{R}^+ \setminus \{0\})U \circ \mathcal{B}$ is an open subsemigroup with boundary $\mathcal{B}$. The uniqueness of the simply connected covering group shows that we find an isomorphic $\phi\colon (sl(2),\circ) \to SL(2)$ of Lie groups such that the following diagram commutes

$$\begin{array}{ccc} (sl(2),\circ) & \xrightarrow{\ \phi\ } & SL(2) \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle f} & \\ & SL(2) & \end{array}$$

Thus $\phi(\mathcal{B}) = \operatorname{Exp} \mathcal{B}$ and $\phi(uU) = \operatorname{Exp}(uU)$ and the claim follows. □

Example 5.40 shows that $e^{r\,\mathrm{ad}\,U}\,\Omega^+$ and $e^{r\,\mathrm{ad}\,U}\,\Omega^-$ are halfspace semigroups in G. In fact, since every two dimensional connected subgroup of G is conjugate to $\operatorname{Exp} \mathcal{B}$ under a rotation, these are the only half space semigroups in G. This allows us to prove the following theorem:

THEOREM IV.5.41: a) *Let* S *be an infinitesimally generated closed subsemigroup in* G. *If* $L(S)$ *is a semialgebra, then* S *is contained in the intersection* $\widetilde{S}$ *of a family of half-space semigroups in* G, *each of which is conjugate either to* Ω^+ *or to* Ω^-, *such that* $L(\widetilde{S}_R) = L(S)$, *i.e.* $\widetilde{S}_R = S$.

b) *For each semialgebra* W *in* $sl(2)$ *there exists exactly one infinitesimally generated closed subsemigroup* S *with* $L(S) = W$.

Proof: By Proposition 5.16, every semialgebra W is the intersection of a family of half-space semialgebras W_j, $j \in I$, each of which determines a unique half space semigroup S_j in G with $L(S_j) = W_j$. In view of the definition of $L(S_j)$ we have $L(\cap S_j) = \cap W_j = W$, thus we set $\widetilde{S} = \cap S_j$. If S is infinitesimally generated by $W = L(S)$ and since $\exp W = \exp(\cap W_j) \le \cap \exp W_j \le \cap S_j = \widetilde{S}$, then S is contained in $\widetilde{S}$, as $\widetilde{S}$ is closed. Now the rest follows. □

Note that Theorem 5.41 leaves the open problem whether $\tilde{S}$ is equal to S or not.

We now generalise the notion of invariance to an arbitrary group G. We say a subsemigroup S in G is *invariant* iff $gSg^{-1} = S$ for all $g \in G$. Clearly a subgroup is invariant iff it is normal. It is also clear that for an invariant subsemigroup S containing the identity in a Lie group G the Lie wedge $L(S)$ is an invariant wedge. If $G = G$ we have only two invariant wedges and we will show that they give rise to invariant subsemigroups. This is, among other things, remarkable, since G is a simple group. We set

(5.24) $\Sigma^+ = \{X \in G = sl(2)\colon X$ is on or above the surface $\exp N\}$ (see Figure 14).

Note that

(5.25) $$\bigcap_{r\in\mathbb{R}} \operatorname{Exp} rU \circ \Omega^+ \operatorname{Exp} -rU = \bigcap_{r\in\mathbb{R}} e^{r\,\mathrm{ad}\,U}\,\Omega^+ =$$
$$= \{X\colon \cos x \cos h\,|X'| \le 1,\ \ x \ge 0\} = \Sigma^+.$$

In fact, the first equality follows from Theorem 5.32c , the last equality $\Sigma^+ = \bigcap_{r\in\mathbb{R}} e^{r\,\mathrm{ad}\,U}\,\Omega^+$ is straightforward, since $e^{r\,\mathrm{ad}\,U}$ is just a rotation under which the ray $\mathbb{R}\cdot P$ sweeps out the surface N while its image $\exp \mathbb{R}\cdot P$ sweeps out the surface $\exp N$. □

THEOREM IV.5.42: a) Σ^+ <u>is a closed invariant non-divisible semigroup with</u> $L(\Sigma^+) = K$.
b) $\Sigma^+ = [0,\pi]U \circ \operatorname{Exp} K$. <u>In particular every element in</u> Σ^+ <u>is the product of two exponentials.</u>
c) Σ^+ <u>is generated by each of its identity neighborhoods.</u>
d) $(-NU) \circ \Sigma^+ = G$.
e) <u>If we set</u> $\Sigma^- = -\Sigma^+$, <u>then analogous statements hold for</u> Σ^-.

Proof: Since $\Sigma^- = \alpha(\Sigma^+)$ it suffices to treat Σ^+.
a) Σ^+ is the intersection of closed halfspace semigroups, hence is a closed semigroup. Obviously, $L(\Sigma^+) = K$ since all conjugates of B are tangent to K. The semigroup Σ^+ cannot be divisible, since there are open sets in $\Sigma^+ \setminus \operatorname{Exp}(sl(2))$ (see Figure 14).
By b) below, $\Sigma^+ \subset (\operatorname{Exp} K) \subset (\operatorname{Exp} K)\,\Sigma^+$. Since K is invariant, so is $\operatorname{Exp} K$ and hence Σ^+.

b) It is clear from Theorem 5.35 and the invariance of $\mathrm{Exp}\,K$ that $\Sigma^+ \supset [0,\pi]U \circ \mathrm{Exp}\,K$. Conversely, $\Sigma^+ \subset [0,\pi]U \circ \mathrm{Exp}\,K$, as follows from Theorem 5.32d and Figure 14.

c) Since $\mathrm{Exp}(K)$ is contained in the semigroup generated by each of the sets $\exp(K \cap V)$ where V is a zero-neighborhood in $\mathrm{sl}(2)$, and since $rU \circ sU = (r+s)U$, claim c) follows from b) above.

d) follows from $(-nU) \circ \Sigma^+ = \Sigma^+ - \{nU\}$ according to Theorem 5.32d . □

THEOREM IV.5.43: <u>If S is a non-zero invariant closed semigroup in G, then $\overline{S}_R = \Sigma^+$ or Σ^-. In particular $S = \Sigma^+$ or Σ^-, if S is infinitesimally generated.</u>

Proof: $L(S)$ is invariant, hence equal to K or $-K$. The closed infinitesimally generated semigroup $\overline{S}_R$ is therefore equal to Σ^+ or Σ^-. The rest is clear.

The existence of the semigroup Σ^+ secures on the Lie group G a partial order compatible with the group structure
Let us summarize what this means for G:

EXAMPLE IV.5.44: a) The group G allows a partial order which is defined by $g \le h$ iff $(-g) \circ h \in \Sigma^+$ iff $h \in g \circ \Sigma^+ = \Sigma^+ \circ g$, and which is compatible with the group structure (i.e. satisfies $f \circ g \le f \circ h$ and $g \circ f \le h \circ f$ for all f, g, and h with $g \le h$).

b) For each $g \in G$ there is a natural number n such that $g \le nU$ $(= U \circ \ldots \circ U$ (n times)). In particular, $(G,\le)$ is directed, i.e. for any pair $g,h \in G$ there is an $f \in G$ with $g \le f$ and $h \le f$.

c) The partial order is compatible with the topology in the sense that the graph of $\le$ is closed in $G \times G$.

Proof: Reflexivity, transitivity and antisymmetry are shown as usual, and the monotonicity laws follow in the standard fashion from the invariance of Σ^+ under inner automorphism. By Theorem 5.42d we have $G = (-\mathbb{N}U) \circ \Sigma^+ = \Sigma^+ \circ (-\mathbb{N}U)$. Since $g \le nU$ is equivalent to $g \in \Sigma^+ \circ (-nU)$, this proves b). The remainder of b) is clear. The graph $\{(g,h) \in G \times G\colon g \le h\} = \{(g,h) \in G \times G\colon (-g) - h \in \Sigma^+\}$ is closed since Σ^+ is closed. This shows c). □

An partially ordered group is called *archimedean* iff $a^n \le b$ for all $n \in \mathbb{Z}$ implies $a = 1$. In the partially ordered group $(G,\le)$ we have

$$E \circ tU \leq \Sigma^+ \quad \text{for} \quad t \geq \pi/2$$

by the definition of Σ^+. Hence whenever $E \in \mathcal{E}$, then by 5.32b we have $tU \in nE \circ \Sigma^+$ for $n \in \mathbb{Z}$: Thus $nE \leq tU$ for all horizontal vectors E and all $t \geq \pi/2$. In view of the fact that every vector X with $k(X) > 0$ is conjugate to a scalar multiple of H by 5.13 we can say that for any X with $k(X) > 0$ there is a vector Y with $k(Y) < 0$ such that all powers of X are dominated by Y. Thus $(G,\leq)$ *is not archimedean*, even though 5.44b) gives a weak archimedean type property. We could have indirectly verified this observation by recalling a theorem on partially ordered groups which says that a directed, partially ordered archimedean group must be commutative. (See [Bl73] p. 317, or [Fu63] p. 95.)

This theorem is based on a theorem of Iwasawa (see loc.cit.) according to which every complete lattice ordered group is commutative, and on the theorem, that every directed archimedean group can be embedded in a complete lattice ordered group (see [Fu], p. 95). Thus the group $(G,\leq)$ is pretty far from being lattice ordered, and we have seen the reason clearly: The set of non-negative elements Σ^+ contains the translates of many subgroups. In fact it is easy to verify that it contains translates of every two dimensional subgroup of G.

Section 6: Global Lie wedges

In Lie group theory we know that there is a bijection between the set of subalgebras of the Lie algebra of a Lie group and its analytic subgroups. The situation appears to be much more complicated in the case of Lie wedges on one hand and global infinitesimally generated subsemigroups on the other. We have of course seen in Section 1 that every infinitesimally generated subsemigroup of a Lie group determines a Lie wedge as its tangent wedge. From Section 5 we know that not every Lie wedge in the Lie algebra even of a simply connected Lie group is the tangent wedge of a global infinitesimally generated subsemigroup - while it always is the tangent wedge of a local infinitesimally generated semigroup. The existence of global subsemigroups for a given Lie wedge in a Lie algebra is a subtle question, and the research in this area moves into largely uncharted territory.

We have seen in Chapter III that a very useful concept in the construction of local semigroups was that of an invariant wedge field. It turns out that this concept is also succesful in the global situation, but it requires more careful preparation to use it properly if one wants to study semigroups with non-trivial subgroups.

In the following let G be a Lie group and H a closed subgroup. Then G/H is an analytic manifold. Let $\lambda_g : G \to G$ be the left multiplication by $g \in G$, then λ_g induces a diffeomorphism $\underline{\lambda}_g : G/H \to G/H$ via $\underline{\lambda}_g(g'H) = gg'H$. The tangent space $T_{\mathbf{1}}(G/H)$ of G/H at $H = \underline{\mathbf{1}}$ can be identified with $\mathfrak{g}/\mathfrak{h}$ where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively. Let $\pi : G \to G/H$ be the canonical quotient map, then a wedge $\underline{W} \subset \mathfrak{g}/\mathfrak{h}$ is called *H-admissible* if $W = d\pi^{-1}(\mathbf{1})(\underline{W})$ is invariant under $\mathrm{Ad}(H)$. If we abreviate gH by $\underline{g}$ we obtain:

LEMMA IV.6.1. Let $\underline{W}$ be H-admissible, then for $\underline{g} = \underline{g}'$ we have

$$d\underline{\lambda}_g(\underline{\mathbf{1}})(\underline{W}) = d\underline{\lambda}_{g'}(\underline{\mathbf{1}})(\underline{W})$$

where $d\underline{\lambda}_g(\underline{\mathbf{1}})$ is the differential of $\underline{\lambda}_g$ at $\underline{\mathbf{1}}$.

Proof: Note first that for $g = g'h$ with $h \in H$ we get $\underline{\lambda}_g = \underline{\lambda}_{g'} \circ \underline{\lambda}_h$ so that it suffices to show that $d\underline{\lambda}_h(\underline{\mathbf{1}})(\underline{W}) = \underline{W}$ for all $h \in H$.

Now let $I_g : G \to G$ be the inner automorphism given by $g' \to g^{-1}g'g$ then we have $\pi \circ I_{h^{-1}} = \pi \circ \underline{\lambda}_h$ for all $h \in H$. Taking derivatives we see that the following diagramm commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(h^{-1})} & \mathfrak{g} \\ {\scriptstyle d\pi(\mathbf{1})}\downarrow & & \downarrow{\scriptstyle d\pi(\mathbf{1})} \\ \mathfrak{g}/\mathfrak{h} & \xrightarrow{d\underline{\lambda}_h(\mathbf{1})} & \mathfrak{g}/\mathfrak{h} \end{array}$$

But since $d\pi(\mathbf{1})W = \underline{W}$ and $\mathrm{Ad}(h^{-1})W = W$ we obtain $d\underline{\lambda}_h(\underline{\mathbf{1}})(\underline{W}) = d\underline{\lambda}_h(\mathbf{1})d\pi(\mathbf{1})(W) =$ $d\pi(\mathbf{1})\mathrm{Ad}(h^{-1})(W) = d\pi(\mathbf{1})(W) = \underline{W}$. □

Lemma 6.1 allows us to make the following definitions.

DEFINITION IV.6.2. Let $\underline{W}$ be H-admissible.

(i) The family $\{\underline{W}(\underline{g}) \subseteq T_{\underline{g}}(G/H) : \underline{W}(\underline{g}) = d\underline{\lambda}_g(\underline{\mathbf{1}})(\underline{W}),\ \pi(g) = \underline{g} \in G/H\}$ is called the *left invariant wedge field* associated with $\underline{W}$. The dual field $\underline{g} \to (\underline{W}(\underline{g}))^* \subseteq T^*_{\underline{g}}(G/H)$ is denoted by $\underline{W}^*(\underline{g})$.

(ii) A piecewise differentiable curve $\underline{\gamma} : [a,b] \to G/H$ is called $\underline{W}$-*admissible* if $d\underline{\gamma}(t) \in \underline{W}(\underline{\gamma}(t))$ whenever $d\underline{\gamma}(t)$ exists.

Note that every wedge W in $\mathfrak{g}$ is $\{\mathbf{1}\}$-admissible and all the definitions in 6.2 make sense for W taking $H = \{\mathbf{1}\}$.

PROPOSITION IV.6.3. Let $\underline{W}$ be H-admissible and $W = d\pi(\mathbf{1})^{-1}\underline{W}$. Then

(i) $W(g) = d\pi(g)^{-1}\underline{W}(\underline{g})$

(ii) If $\gamma : [a,b] \to G$ is piecewise differentiable, then: γ is W-admissible if and only if $\underline{\gamma} = \pi \circ \gamma$ is $\underline{W}$-admissible.

Proof: (i) Let $x \in T_g(G)$ then $x = d\lambda_g(\mathbf{1})y$ with $y \in \mathfrak{g}$ uniquely determined. We have

$$\begin{array}{ll} & d\pi(g)\,x \in \underline{W}(\underline{g}) \\ \text{if and only if} & d\pi(g)d\lambda_g(\mathbf{1})y \in \underline{W}(\underline{g}) \\ \text{if and only if} & d\underline{\lambda}_g(\underline{\mathbf{1}})d\pi(\mathbf{1})y \in \underline{W}(\underline{g}) \\ \text{if and only if} & d\pi(\underline{\mathbf{1}})y \in \underline{W} \\ \text{if und only if} & y \in W \\ \text{if and only if} & x = d\lambda_g(\mathbf{1})y \in W(g). \end{array}$$

(ii) $d\underline{\gamma}(t) = d\pi(\gamma(t)) \circ d\gamma(t)$, so by (i) $d\gamma(t) \in W(\gamma(t))$ if and only if $d\underline{\gamma}(t) \in \underline{W}(\underline{\gamma}(t))$. □

Recall that for any vectorspace complement $\mathfrak{n}$ of $\mathfrak{h}$ in $\mathfrak{g}$ we find an open neighborhood $B_{\mathfrak{n}}$ of zero in $\mathfrak{n}$ such that $(\exp B_{\mathfrak{n}})H$ is an open neighborhood of H in G and

$$\phi : B_{\mathfrak{n}} \times H \longrightarrow (\exp B_{\mathfrak{n}})H$$
$$(x,h) \longmapsto (\exp x)h$$

is a diffeomorphism (this is true since H is closed).
More generally for $\underline{g} \in G/H$ there exists an open neighborhood $\underline{U}$ of $\underline{g}$ and an analytic map $\sigma_{\underline{U}} : \underline{U} \to G$ such that $\pi \circ \sigma_{\underline{U}} = \mathrm{Id}_{\underline{U}}$. This will allow us to lift $\underline{W}$-admissible curves into W-admissible curves.

LEMMA IV.6.4. Let $\underline{U}$ be an open set in G/H. If $\sigma_{\underline{U}} : \underline{U} \to G$ is differentiable and satisfies $\pi \circ \sigma_{\underline{U}} = \mathrm{Id}_{\underline{U}}$, then we have $d\sigma_{\underline{U}}(g')(\underline{W}(\underline{g}) \subseteq W(\sigma_{\underline{U}}(\underline{g}))$ for all $\underline{g} \in \underline{U}$.

Proof: $d\pi(\sigma_{\underline{U}}(\underline{g})) \circ d\sigma_{\underline{U}}(\underline{g})(\underline{W}(\underline{g}) = \underline{W}(\underline{g})$. Hence $d\sigma_{\underline{U}}(\underline{g})(\underline{W}(\underline{g})) \subseteq d\pi(\sigma_{\underline{U}}(\underline{g}))^{-1}(\underline{W}(\underline{g})) = W(\sigma_{\underline{U}}(\underline{g}))$ by 6.3.

□

LEMMA IV.6.5. Let $W = \mathrm{Ad}(H)W$ be a wedge containing $\mathfrak{h}$. If $\gamma : [a.b] \to G$ be W-admissible and $\eta : [a,b] \to H$ is piecewise differentiable, then $\gamma_1 : [a,b] \to G$ defined by $\gamma_1(t) = \gamma(t)\eta(t)$ is W-admissible.

Proof: Let $t_\circ \in]a,b[$ and consider the curves $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\eta}$ defined by $\hat{\gamma}_1(t) = \gamma_1(t_\circ)^{-1}\gamma_1(t)$, $\hat{\gamma}_2(t) = \eta(t_\circ)^{-1}\gamma(t_\circ)^{-1}\gamma(t)\eta(t_\circ)$ and $\hat{\eta}(t) = \eta(t_\circ)^{-1}\eta(t)$. Then we have $\hat{\gamma}_1(t) = \hat{\gamma}_2(t)\hat{\eta}(t)$. But now $\hat{\gamma}_1(t_\circ) = \mathbf{1} = \hat{\gamma}_2(t_\circ) = \hat{\eta}(t_\circ)$ so that $d\hat{\gamma}_1(t_\circ) = d\hat{\gamma}_2(t_\circ) + d\hat{\eta}(t_\circ)$. Therefore we calculate $d\gamma_1(t_\circ) = d\lambda_{\gamma_1(t_\circ)}(\mathbf{1}) \circ d\hat{\gamma}_1(t_\circ) \in W(\gamma_1(t_\circ))$ since $d\hat{\eta}(t_\circ) \in \mathfrak{h} \subseteq W$ and $d\hat{\gamma}(t_\circ) = \mathrm{Ad}(\eta(t_\circ)) \circ d\lambda_{\gamma(t_\circ)^{-1}}(\gamma(t_\circ))(d\gamma(t_\circ))$ $\in \mathrm{Ad}(\eta(t_\circ))(d\lambda_{\gamma(t_\circ)^{-1}}(\gamma(t_\circ))(W(\gamma(t_\circ))) = \mathrm{Ad}(\eta(t_\circ))W \quad W$.

□

PROPOSITION IV.6.6. For any W-admissible curve $\underline{\gamma} : [a,b] \to G/H$ exists a W-admissible curve $\gamma : [a,b] \to G$ such that $\pi \circ \gamma = \underline{\gamma}$.

• Proof: Since $\underline{\gamma}([a,b])$ is compact we can find a partition $a = a_o < \ldots < a_n = b$ of $[a,b]$ and open neighborhoods $\underline{U}_k$ of $\underline{\gamma}(a_n)$ such that $\bigcup_{k=o}^{n} \underline{U}_k$ covers $\underline{\gamma}([a,b])$ and $a_k \in \underline{U}_{k-1} \cap \underline{U}_k$. Moreover we can assume that we have differentiable maps $\sigma_k : \underline{U}_k \to G$ such that $\pi \circ \sigma_k = Id_{\underline{U}_k}$. If $g_k = \sigma_k(\underline{\gamma}(a_k))$ and $g_k' = \sigma_k(\underline{\gamma}(a_{k+1}))$ for $k = 0\ldots n-1$ we get $(g_k')^{-1}g_{k+1} \in H$. Multiplying the σ_k with suitable elements in H from the right we may assume that $(g_k')^{-1}g_{k+1}$ is in the identity component H_o of H. Thus we can find piecewise differentiable curves $\eta_k : [a_k,a_{k+1}] \to H$ such that $\eta_k(a_k) = \mathbf{1}$ and $\eta_k(a_{k+1}) = (g_k')^{-1}g_{k+1}$ for $k = 0\ldots n-1$. Now define $\gamma : [a,b] \to G$ by $\gamma(t) = \sigma_k(\underline{\gamma}(t))\eta_k(t)$ for $t \in [a_k,a_{k+1}[$ then γ is continuous since $\sigma_k(\underline{\gamma}(a_{k+1}))\eta_k(a_{k+1}) = g_k'(g_k')^{-1}g_{k+1} = \sigma_{k+1}(\underline{\gamma}(a_{k+1}))\eta_{k+1}(a_{k+1})$ for $k = 0\ldots n-1$. By Lemma 6.5. γ is W-admissible and obviously $\pi \circ \gamma(t) = \pi\sigma_k(\underline{\gamma}(t)) = \underline{\gamma}(t)$.

□

With this machinery of admissible wedges and curves we are now ready to introduce the concept of positive functions which has been used in special cases by Vinberg and Ol'shanskii. Here it will play a role for the global semigroups somewhat like Liapunoff-functionals in stability theory. The existence of positive functions will ensure that admissible curves cannot "come back" to the origin.

DEFINITION IV.6.7. Let $\underline{W}$ be H-admissible. A C^1-map $\underline{\varphi} : G/H \to \mathbb{R}$ is called *$\underline{W}$-positive* if $d\underline{\varphi}(\underline{g}) \in \underline{W}^*(\underline{g})$ for all $\underline{g} \in G/H$. A $\underline{W}$-positive map $\underline{\varphi} : G/H \to \mathbb{R}$ is called *strictly $\underline{W}$-positive* if $d\varphi(\underline{\mathbf{1}}) \in \operatorname{int} \underline{W}^*(\underline{g})$, where int denotes the interior. We denote the set of $\underline{W}$-positive functions by $P(\underline{W})$ and the set of strictly $\underline{W}$-positive functions by $P_s(\underline{W})$.

Note that $\operatorname{int} \underline{W}^*$ and hence also $\operatorname{int} \underline{W}^*(\underline{g})$ is empty as soon as $\underline{W}$ contains a nontrivial vectorspace. Thus $P_s(\underline{W})$ can only be nonempty if $\underline{W}$ is a proper cone.

REMARK IV.6.8. Let $\underline{W}$ be admissible and $W = d\pi(\mathbf{1})^{-1}(\underline{W})$. If $\underline{\varphi} : G/H \to \mathbb{R}$ is a C^1-map and $\varphi = \underline{\varphi} \circ \pi$ then φ is W-positive if and only if $\underline{\varphi}$ is $\underline{W}$-positive.

Proof: Let $x \in W(g)$ then $d\varphi(g)(x) = d\underline{\varphi}(\underline{g})(d\pi(g)(x))$. If $\underline{\varphi}$ is $\underline{W}$-positive we get $d\varphi(g)(x) \geq 0$ by 6.3i and if φ is W-positive we get $d\underline{\varphi}(g)(d\pi(g)(x)) \geq 0$ so that $d\underline{\varphi}(g) \in \underline{W}^*(\underline{g})$ since $d\pi(g)W(g) = \underline{W}(g)$.

□

REMARK IV.6.9. Let $\underline{W}$ be H-admissible, $\underline{\varphi}$ be $\underline{W}$-positive and $\underline{\gamma} : [a,b] \to G/H$ be $\underline{W}$-admissible, then $\underline{\varphi} \circ \underline{\gamma}$ is nondecreasing.

Proof: $\underline{\varphi} \cdot \underline{\gamma}$ is piecewise differentiable and $d(\underline{\varphi}\cdot\underline{\gamma})(t) = d\underline{\varphi}(\underline{\gamma}(t)) \cdot d\underline{\gamma}(t) \in d\varphi(\underline{\gamma}(t))(\underline{W}(\underline{\gamma}(t))) \subseteq \mathbb{R}^+$

□

THEOREM IV.6.10. Let $\underline{W}$ be H-admissible and generating such that $P_s(\underline{W}) \neq \emptyset$. Then there exists a subsemigroup S of G such that $\underline{L}(S) = W = d\pi(\mathbf{1})^{-1}\underline{W}$.

Proof: By the remarks made before we know that $\underline{W}$ is a proper cone. Thus we find a vectorspace complement $\mathfrak{n}$ for $\mathfrak{h}$ in $\mathfrak{g}$ such that $W = W \cap \mathfrak{n} + \mathfrak{h}$ and $W \cap \mathfrak{n}$ is a proper cone satisfying $d\pi(\mathbf{1})(W \cap \mathfrak{n}) = \underline{W}$. Moreover there exists an open neighborhood $B_{\mathfrak{n}}$ of zero in $\mathfrak{n}$ such that $\Phi : B_{\mathfrak{n}} \times H \to (\exp B_{\mathfrak{n}})H$ given by $(x,h) \mapsto (\exp x)H$ is a diffeomorphism and $(\exp B_{\mathfrak{n}})H$ is an open neighborhood of H in G. Thus $U_{\mathfrak{n}} = \pi(\exp B_{\mathfrak{n}})$ is an open neighborhood of $\underline{\mathbf{1}}$ in G/H . Note that $d\pi(\mathbf{1})_{|\mathfrak{n}}$ is invertible so that, by making the neighborhoods smaller if necessary, we may assume that $\rho = \pi\circ\exp_{|B}$ is a diffeomorphism. Now let $\underline{\varphi} \in P_s(\underline{W})$ and consider $\underline{\varphi} \circ \rho : B_{\mathfrak{n}} \to \mathbb{R}$ then $d(\underline{\varphi}\circ\rho)(0) = d\underline{\varphi}(\underline{\mathbf{1}}) \circ d\pi(\mathbf{1})$. If $C \subseteq B_{\mathfrak{n}}$ is a (compact) base of $W \cap \mathfrak{n}$ then $d\pi(\mathbf{1})(C)$ is a compact base of $\underline{W}$ and $d\varphi(\underline{\mathbf{1}}) \in \operatorname{int} \underline{W}^*$ implies that there exists an $\varepsilon > 0$ such that $d(\underline{\varphi}\circ\rho)(0)(C) \subseteq [\varepsilon,\infty[$. By continuity of $d(\underline{\varphi}\circ\rho)$ and compactness of C we can now find a neighborhood $B_1 \subset B_{\mathfrak{n}}$ of zero and a compact convex set $C_1 \subset B_{\mathfrak{n}}$ such that $C \subseteq \operatorname{int}_{\mathfrak{n}} C_1$ and $d(\underline{\varphi}\circ\rho)(\underline{x})(\underline{c}) \geq \frac{\varepsilon}{2}$ for all $\underline{x} \in B_1$ and $\underline{c} \in C_1$. Let $r > 0$ be such that $r\, C_1 \subset B_1$ then, by considering the function $t \mapsto \underline{\varphi} \circ\rho(tr\underline{c})$ for $\underline{c} \in C_1$, we see that $\underline{\varphi}(\rho(r\underline{c})) \geq \varphi(\mathbf{1}) + \int_0^1 \frac{\varepsilon}{2}\, r t dt = \underline{\varphi}(\mathbf{1}) + r\,\frac{\varepsilon}{4}$. We set $\varepsilon_1 = r\,\frac{\varepsilon}{4}$ and $K = \rho(rC_1)$ and thus know that $\underline{\varphi}(K) \subset [\underline{\varphi}(\underline{\mathbf{1}}) + \varepsilon_1,\infty[$. Let $\underline{W}_1 = d\pi(\mathbf{1})(\mathbb{R}^+C_1)$ then $\underline{W} \setminus \underline{o} \subset \operatorname{int}_{\mathfrak{g}/\mathfrak{h}} \underline{W}_1$. If $W_1 = d\pi(\mathbf{1})^{-1}\underline{W}_1 = \mathbb{R}^+C_1 + \mathfrak{h}$ then $W \setminus \mathfrak{h} = W \setminus (W - W) \subset \operatorname{int} W_1$ and [HL83a] implies that we can find a neighborhood B of 0 in $B_{\mathfrak{n}} \times \mathfrak{h}$ and a subset Σ of B such that

(i) $\Sigma * \Sigma \cap B \subset \Sigma \subset W_1$

(ii) $W = \{x \in \mathfrak{g} : \mathbb{R}^+x \cap B \subset \Sigma\}$.

(iii) $\exp_{|B} : B \to \exp B$ is diffeomorphism

(iv) $\gamma : [a,b] \to \exp B$ W-admissible implies $\gamma([a,b]) \subseteq \exp \Sigma$

Since the estimate for $\underline{\varphi}(\rho(r\underline{c}))$ was linear in r we may assume that $rC_1 \subset B$. Let $\underline{U}$ be a neigborhood of $\underline{\mathbf{1}}$ in G/H such that

$$\underline{\varphi}(\underline{U}) \subset [\underline{\varphi}(\underline{\mathbf{1}}) - \frac{\varepsilon_1}{2}, \underline{\varphi}(\underline{\mathbf{1}}) + \frac{\varepsilon_1}{2}] .$$

<u>Claim</u>: Let $\gamma : [a,b] \to G$ be a W-admissible curve with $\gamma(a) = \mathbf{1}$ and $\pi \circ \gamma(b) \in \underline{U}$, then $\pi(\gamma([a,b])) \subset \pi(\exp B)$.

Proof of the claim: Suppose there exists a $t \in]a,b[$ such that $\pi(\gamma(t)) \in (G/H) \setminus \pi(\exp B)$ then we consider $\hat{\gamma}: [a,t_\circ] \to \pi^{-1}(\exp B)$ where $t_\circ$ is the infimum of all such t. Recall that $\phi : B_{\mathfrak{h}} \times H \to (\exp B_{\mathfrak{h}})H$ is a diffeomorphism and $\pi^{-1}(\exp B) \subset (\exp B_{\mathfrak{h}})H$. Thus we find piecewise differentiable curves $\gamma_1 : [a,t_\circ] \to \exp B_{\mathfrak{h}}$ and $\eta : [a,t_\circ] \to H$ such that $\hat{\gamma}(t) = \gamma_1(t)\eta(t)$ for all $t \in [a,t_\circ]$. Since then $\gamma_1(t) = \hat{\gamma}(t)\eta^{-1}(t)$ Lemma 6.5. shows that γ_1 is W-admissible. Thus by (iv) we get $\gamma_1(t) \in \exp \Sigma$ and hence by (i) also $\gamma_1(t) \in \exp \Sigma \cap \exp B_{\mathfrak{h}} = \exp(\Sigma \cap B_{\mathfrak{h}}) \subseteq \exp(W_1 \cap B_{\mathfrak{h}})$ so that $\pi\gamma_1(t) \in \rho(\mathbb{R}^+C_1 \cap B_{\mathfrak{h}})$. Therefore there exists a $t_1 \in [a,t_\circ]$ such that $\pi \circ \gamma(t_1) = \pi \circ \gamma_1(t_1) \in K$. But $\underline{\gamma} = \pi \circ \gamma$ is $\underline{W}$-admissible and hence by Remark 6.9. $\underline{\varphi} \circ \underline{\gamma}$ is nondecreasing. Thus $\underline{\varphi}(\pi\circ\gamma)(b) \geqq \underline{\varphi}(\underline{\mathbf{1}}) + \varepsilon_1$ and hence $\pi\gamma(b)$ cannot be in $\underline{U}$. This contradiction proves the claim.

Let U be a symmetric neighborhood of $\mathbf{1}$ in G such that $U^2 \subset \exp B$ and $\underline{\varphi} \circ \pi(U) \subseteq [\underline{\varphi}(\underline{\mathbf{1}}) - \frac{\varepsilon_1}{2}, \underline{\varphi}(\underline{\mathbf{1}}) + \frac{\varepsilon_1}{2}]$. If $\gamma : [a,b] \to G$ is W-admissible such that $\gamma(b) \in U$ and $\gamma(a) = \mathbf{1}$ then by the claim $\gamma([a,b]) \subset (\exp B_{\mathfrak{h}})H$ so that $\gamma(t) = \gamma_1(t)\eta(t)$ with $\gamma_1(t) \in \exp B_{\mathfrak{h}}$ and $\eta(t) \in H$ and γ_1 is W-admissible. Now let $p_{\mathfrak{h}} : B_{\mathfrak{h}} \times H \to B_{\mathfrak{h}}$ be the canonical projection. By continuity we find a neighborhood U_1 of $\mathbf{1}$ in G such that $U_1(\exp \circ p_{\mathfrak{h}} \circ \phi^{-1}(U_1))^{-1} \subset U$. If then $\gamma(b) \in U_1$ we obtain $\gamma_1(b) \in \exp \circ p_{\mathfrak{h}} \circ \phi^{-1}(U_1)$ so that $\eta(b) = \gamma(b)\gamma_1(b)^{-1} \in U$. Thus $\gamma(b) = \gamma_1(b)\eta(b) \in U^2 \subset \exp B$. Thus $\gamma(b) \in (\exp \Sigma)(\exp \Sigma) \cap \exp B \subset \exp \Sigma$.

Therefore, if $S = \{g \in G : \gamma : [a,b] \to G, \gamma(a) = \mathbf{1}, \gamma(b) = g\}$ is the semigroup of endpoints of W-admissible curves starting at the identity, we get $S \cap U_1 \subset \exp \Sigma \subset S$. Thus $\underline{L}(S) = W$.

□

DEFINITION IV.6.11. Let $\underline{W}$ be H-admissible, then we call $S_{\underline{W}} = \{g \in G : \exists \gamma$ W-admissible from $\mathbf{1}$ to g, $W = d\pi(\mathbf{1})^{-1}\underline{W}\}$ the *semigroup generated by* $\underline{W}$.

Theorem 6.9. tells us that the existence of strictly $\underline{W}$-positive functions guaranties the existence of a global semigroup with tangent cone W. Now suppose that $\underline{W}_1 \subset \underline{W}$ then any $\underline{W}$-positive function is a fortiori $\underline{W}_1$-positive. If we know that the existence of global semigroups entails the existence of strictly positive functions then we could conclude that, given a global semigroup S wit $\underline{L}(S) = W$, any wedge W_1 contained in W such that $W_1 \cap W_1 = W \cap -W$ is the tangent wedge of a global semigroup. We do indeed get conclusions of this type, but we shall have to make some hypotheses on W_1.

First we introduce some basic concepts and facts from the theory of ordered sets, since this is the proper framework for the things to come.

DEFINITION IV.6.12. Let G be a group and H be a subgroup of G. A partial order $\leq$ on the coset space G/H is called *left invariant* if it satisfies

(6.1) $g_1H \leq g_2H$ implies $gg_1H \leq gg_2H$ for all $g, g_1, g_2 \in G$.

The set $P_{\leq} = \{gH \in G/H : H \leq gH\}$ is called the *domain of positivity of* $\leq$.
Note that (6.1) implies that $\leq$ is completely determined by $P_{\leq}$.

REMARK IV.6.13. Let $\pi : G \to G/H$ be the canonical projection and $\leq$ be a left invariant order on G/H. Then $S_{\leq} = \pi^{-1}(P_{\leq})$ is a semigroup with $S_{\leq} \cap S_{\leq}^{-1} = H$.

Proof: $g_1, g_2 \in S_{\leq}$ implies $H \leq g_1H$, $H \leq g_2H$ as that $H < g_1H \leq g_1g_2H$ and $g_1g_2 \in S_{\leq}$ by (6.1) Moreover we have the following equivalences
$\Leftrightarrow$ $g \in S_{\leq} \cap S_{\leq}^{-1} \Leftrightarrow H \leq gH,\ H \leq g^{-1}H \Leftrightarrow H \leq gH \leq gg^{-1}H = H$
$\Leftrightarrow$ $g \in H$.

□

REMARK IV.6.14. Let S be a subsemigroup of G with $S \quad S^{-1} = H$. Then $g_1H \leq_s g_2H :\Leftrightarrow g_1^{-1}g_2 \in S$ defines a left invariant order on G/H.

Proof: Note first that $\leq_s$ is welldefined: Let $g_1H = g_1'H$ and $g_2H = g_2'H$ then $g_2^{-1}(g_2')$, $(g_1')^{-1}g_1 \in H$ so that $g_1^{-1}g_2 \in S$ if and only if $((g_1)^{-1}g_1)g_1^{-1}g_2(g_2^{-1}g_2') \in S$ if and only if $(g_1')^{-1}g_2' \in S$.
Since $\mathbf{1} \in H \subset S$ we have the reflexivity of $\leq_s$. The transitivity follows from $SS \subset S$ and the antisymmetry from $S \cap S^{-1} = H$. Assume finally that $g_1H \leq g_2H$, i.e. that $g_1^{-1}g_2 \in S$. Then $(gg_1)^{-1}(gg_2) = g_1^{-1}g_2 \in S$ so that $gg_1H \leq_s gg_2H$.

□

REMARK IV.6.15. With the notation from 6.8 and 6.9 we have

(i) $S_{(\leq_S)} = S$ for any subsemigroup S of G with $S \cap S^{-1} = H$.

(ii) $\leq_{(S_\leq)} = \leq$ for any left invariant order $\leq$ on G/H.

Proof:

(i) $g \in S_{(\leq_S)}$ if and only if $H \leq_S gH$ if and only if $g \in S$.

(ii) $g_1H \leq_{(S_\leq)} g_2H$ if and only if $g_1^{-1}g_2 \in S_\leq$ if and only if

$H \leq g_1^{-1}g_2H$ if and only if $g_1H \leq g_2H$ □

We collect this information in:

PROPOSITION IV.6.16. Let G be a group and H a subgroup of G. The map $S \to \leq_S$ from the set of subsemigroups of G with $S \cap S^{-1} = H$ to the set of left invariant orders on G/H is a bijection with inverse $\leq \to S_\leq$.

The existence of an order allows us to speak about monotonic functions.

DEFINITION IV.6.17. Let $\leq$ be a partial order on G/H then we call a function $\varphi : G/H \to \mathbb{R}$ *monotonic* if $g_1H \leq g_2H$ implies $\varphi(g_1H) \leq \varphi(g_2H)$. The set of all monotonic functions is denoted by $M_\leq(G/H)$.

Now we return to the context of Lie groups.

DEFINITION IV:6.18. Let G be a Lie group and H a closed subgroup of G. An H-admissible cone $W \subset \mathfrak{g}/\mathfrak{h}$ is called *causal* if there are no closed W-admissible curves in G/H.

REMARK IV.6.19. If W is causal then we can define a left invariant order $\leq_W$ on G/H by setting $g_1H \leq_W g_2H$ if there exists a W-admissible curve in G/H from g_1H to g_2H. This order is called the *causal order on* G/H *induced by* W.

Proof: Reflexivity and transitivity are clear, the antisymmetry follows from the definition of causality. Thus it only remains to show the left invariance. But that is clear since $\gamma_1 = \lambda_g \circ \gamma$ is W-admissible for any W-admissible γ. □

If $P_{\underline{W}}$ is the domain of positivity of $\leq_{\underline{W}}$ we know from 6.8. that $\hat{S}_{\underline{W}} = \pi^{-1}(P_{\underline{W}})$ is a subsemigroup of G such that $\hat{S}_{\underline{W}} \cap (\hat{S}_{\underline{W}})^{-1} = H$ and $g_1H \leq_{\underline{W}} g_2H$ if and only if $g_1^{-1}g_2 \in \hat{S}_{\underline{W}}$ by 6.15.

REMARK IV.6.20. Let $\underline{W}$ be a causal cone then $S_{\underline{W}}H = \hat{S}_{\underline{W}}$.

Proof: Recall that $S_{\underline{W}}$ is the subsemigroup of G which consists of all elements which can be connected to $\mathbf{1}$ by a W-admissible curve. By 6.3 and 6.6 we get $\pi(S_{\underline{W}}) = P_{\underline{W}} = \pi(\hat{S}_{\underline{W}})$ so that $S_{\underline{W}}H = \hat{S}_{\underline{W}}H = \hat{S}_{\underline{W}}$. □

REMARK IV.6.21. Let $\underline{W}$ be H-admissible, then $\underline{W}$ is causal if and only if $S_{\underline{W}} \cap (S_{\underline{W}})^{-1} \subset H$.

Proof: Let $g \in (S_{\underline{W}} \cap S_{\underline{W}}^{-1})\backslash H$ then there exist W-admissible curves $\gamma_1, \gamma_2 : [0,1] \to G$ such that $\gamma_1(0) = \gamma_2(0) = \mathbf{1}$ and $\gamma_1(1) = g = \gamma_2(1)^{-1}$, where $W = d\pi(\mathbf{1})(\underline{W})$. Consider $\gamma_3 : [0,2] \to G$ given by

$$\gamma_3(t) = \begin{cases} \gamma_1(t) & t \in [0,1] \\ g\,\gamma_2(t-1) & t \in \,]1,2] \end{cases}$$

Then γ_3 is a closed, W-admissible curve with $\gamma_3(1) = g \notin H$ hence by 6.3 $\pi \circ \gamma_3$ is a closed, nontrivial, $\underline{W}$-admissible curve.
Conversely, suppose that $\underline{\gamma} : [a,b] \to G/H$ is a closed nontrivial $\underline{W}$-admissible curve, then by 6.6. there exists a W-admissible curve $\gamma : [a,b] \to G$ such that $\pi \circ \gamma = \underline{\gamma}$. Without loss of generality we may assume that $\underline{\gamma}(a) = \underline{\gamma}(b) = \underline{1}$ so that $\gamma(b) \in H$. As in 6.6 we may assume that $\gamma(a) = \mathbf{1}$ and $\gamma(b)$ is in the identity component of H so that piecing together with a path in H yields a W-admissible closed curve $\gamma_1 : [a,b'] \to G$. Since $\underline{\gamma}$ was nontrivial, we find a $t_\circ \in \,]a,b'[$ such that $\gamma_1(t_\circ) \in G\backslash H$. Now consider the curves $\gamma_2 : [a,t_\circ] \to G$ and $\gamma_3 : [t_\circ,b'] \to G$ where $\gamma_2(t) = \gamma_1(t)$ and $\gamma_3(t) = \gamma_1(t_\circ)^{-1}\gamma_1(t)$. Then γ_2 and γ_3 are W-admissible curves with $\gamma_2(a) = \gamma_3(t_\circ) = \mathbf{1}$ and $\gamma_2(t_\circ) = \gamma_1(t_\circ) = \gamma_3(b')^{-1}$. Thus $\gamma_1(t_\circ) \in S_{\underline{W}} \cap S_{\underline{W}}^{-1} \backslash H$.
□

Now suppose $\underline{W}$ is a causal cone, then we have the concept of monotonic functions as well as the concept of positive functions. For $G = \mathbb{R}$, $H = \{0\}$, $W = \mathbb{R}^+$ these would be the monotonously increasing functions and the differentiable functions with positive derivative respectively, i.e. the positive functions are exactly the differentiable monotonic functions. This is true in general.

PROPOSITION IV.6.22. Let $\underline{W}$ be a causal cone in $\mathfrak{g}/\mathfrak{h}$ then

$$P(\underline{W}) = C^1(G/H, \mathbb{R}) \cap M_{\leqq_{\underline{W}}}(G/H) .$$

Proof: Let $\varphi \in C^1(G/H, \mathbb{R}) \cap M_{\leqq_{\underline{W}}}(G/H)$ then we calculate the differential of φ by taking directional derivatives

$$d\varphi(gH)(d\lambda_g(\mathbf{1})(x+\mathfrak{h})) = \lim_{t\to o} \frac{1}{t}(\varphi((g \exp tx)H) - \varphi(gH))$$

for $g \in G$ and $x \in \mathfrak{g}$. If now $x \in W$ then $\exp tx \in S_{\underline{W}}$ for $t \geqq 0$ and hence $H \leqq_{\underline{W}} (\exp tx)H$ so that $gH \leqq_{\underline{W}} (g \exp tx)H$. Thus, the monotonicity of φ shows that $d\varphi(gH)(d\lambda_g(\mathbf{1})(x+\mathfrak{h})) \geqq 0$, i.e. that $d\varphi(\underline{g})(d\lambda_{\underline{g}}(\underline{\mathbf{1}})\underline{x}) \geqq 0$ for all $\underline{x} \in \underline{W}$ so that $d\varphi(\underline{g})(\underline{y}) \geqq 0$ for all $\underline{y} \in \underline{W}(\underline{g})$ and hence $\varphi \in P(\underline{W})$.
Conversely, let $\gamma \in P(\underline{W})$ and $g_1H \leqq_{\underline{W}} g_2H$. Then $g_1^{-1}g_2 \in S_{\underline{W}}$ and we find a W-admissible curve $\gamma : [0,1] \to G$ such that $\gamma(a) = \mathbf{1}$, $\gamma(b) = g_1^{-1}g_2$. Therefore the curve $\gamma_1(t) = g_1\gamma(t)$ is a W-admissible curve from g_1 to g_2. By 6.3. the curve $\pi \circ \gamma_1$ is a $\underline{W}$-admissible in G/H and by 6.9. the function $\varphi \circ \pi \circ \gamma_1$ is non decreasing. Thus $\varphi(g_1H) \leqq \varphi(g_2H)$ and $\varphi \in M_{\leqq_{\underline{W}}}(G/H)$.

□

Sometimes it is much easier to find monotonic functions than positive functions. For example, if $\underline{W}$ is a causal cone and $\hat{S}_{\underline{W}}$ has nonempty interior

$$\chi_{\underline{W}} : G/H \to \mathbb{R} \ , \quad \chi_{\underline{W}}(gH) = \begin{cases} 1 & g \in \operatorname{int} \hat{S}_{\underline{W}} \\ 0 & g \notin \operatorname{int} \hat{S}_{\underline{W}} \end{cases}$$

is a monoton function. In fact $g_1H \leqq_{\underline{W}} g_2H$ implies $g_1^{-1}g_2 \in \hat{S}_{\underline{W}}$.
If $g_1 \in \operatorname{int} \hat{S}_{\underline{W}}$ this shows that $g_2 \in \operatorname{int} \hat{S}_{\underline{W}}$ by 2.5. so that $\chi_{\underline{W}}(g_2H) = \chi_{\underline{W}}(g_1H) = 1$. If $g_1 \notin \operatorname{int} \hat{S}_{\underline{W}}$ we have $\chi_{\underline{W}}(g_1H) = 0 \leqq \chi_{\underline{W}}(g_2H)$ anyway.

Therefore Proposition 6.22. suggests to try to obtain positive functions by regularisation of monotonic functions. To this end we consider a left Haar measure μ on G and assume that $\varphi \circ \pi : G \to \mathbb{R}$ is a μ-measurable essentially bounded monotonic function. If $f : G \to \mathbb{R}^+$ has compact support we calculate

$$f * (\varphi\circ\pi)(g) = \int_G f(g_1)\ \varphi\circ\pi\ (g_1^{-1}g)d\mu(g_1) =$$
$$= \int_{\text{supp } f} f(g_1)\ \varphi(g_1^{-1}gH)d\mu(g_1)$$

If now $g_2H \leq_W g_3H$ then $g_1^{-1}g_2H \leq_W g_1^{-1}g_3H$ and $\varphi(g_1^{-1}g_2H) \leq \varphi(g_1^{-1}g_2H)$. Since f was chosen to have positive values and $f * (\varphi\circ\pi)(g)$ depends only on gH we find that $f \ast \varphi : g/H \to \mathbb{R}$ defined by $f \ast \varphi(gH) = f * (\varphi\circ\pi)(g)$ is a monotonic function. If f is C^1 then $f * (\varphi\circ\pi)$ is C^1 and since $f \ast \varphi$ can be locally defined as $(f * (\varphi\circ\pi)) \circ \sigma$ with some local section $\sigma : G/H \cap \underline{U} \to G$, we see that $f \ast \varphi$ is C^1, hence by 6.22. $\underline{W}$-positive. Suppose that $\hat{S}_{\underline{W}}$ has nonempty interior then we have $f \ \ \chi_{\underline{W}} \in M_{\leq_W}(G/H)$ for any $f \in C_c^1(G,\mathbb{R}^+)$. Of course any constant function $\phi : G/H \to \mathbb{R}$ is positive so we are interested in deciding whether $f \ast \chi_{\underline{W}}$ is constant or not. Suppose that $g_o \in \operatorname{int} \hat{S}_{\underline{W}}$. Then we can choose a neighborhood U of $\mathbf{1}$ in G such that $U = U^{-1}$ and $\overline{U}g_o$ are contained in $\operatorname{int} \hat{S}_{\underline{W}}$. Then for $f \in C_c^1(G,\mathbb{R}^+)$ with $\operatorname{supp} f \subset U$ and $f(g_o) > 0$ we obtain

$$f \ast \chi_{\underline{W}}(g_o) = \int_U f(g_1)\chi_{\underline{W}}(g_1^{-1}g_oH)dg_1 = \int_U f(g_1)dg_1 > 0$$

Moreover $f \ast \chi_{\underline{W}}(g) = 0$ for all g satisfying $Ug \not\subset \hat{S}_{\underline{W}}$. Thus, if $\hat{S}_{\underline{W}}$ is not dense, such a $\overline{U}$ exists and we have proved

PROPOSITION IV.6.23. _Let W be a causal cone such that $\hat{S}_{\underline{W}}$ has nonempty interior but is not dense in G. Then there exists a non-constant $\underline{W}$-positive function on G/H._

□

Now let φ be a nonconstant $\underline{W}$-positive function on G/H. Then there exists a $g_o \in G$ such that $d\varphi(g_oH)$ is nonzero. Consider $\varphi_1 = \varphi \circ \underline{\lambda}_{g_o^{-1}}$ then φ_1 is still $\underline{W}$-positive since

$$d\varphi_1(gH)(\underline{W}(gH)) = d\varphi(g_o^{-1}gH) \circ d\underline{\lambda}_{g_o^{-1}}(g)(\underline{W}(gH)) = d\varphi(g_o^{-1}gH)(\underline{W}(g_o^{-1}gH)) \subseteq \mathbb{R}^+ .$$

Moreover we see that $d\varphi_1(H)$ is nonzero. Therefore 6.23. yields:

PROPOSITION IV.6.24. Let S be an infinitesimally generated subsemigroup of G with tangent wedge $\underline{L}(S) = W$ and maximal subgroup $H = S \cap S^{-1}$. If $G(S) = G$, then H is closed and there exists a $(W+\mathfrak{h})/\mathfrak{h}$- positive function $\varphi : G/H \to \mathbb{R}$ with $d\varphi(H) \neq 0$ where $\mathfrak{h} = \underline{L}(H)$ is the Lie algebra of H,

Proof. The first statement follows directly from 3.7, so it only remains to show that we can apply 6.23. to $W + \mathfrak{h}/\mathfrak{h}$. To this end note first that by 3.7 the group H is also connected and hence the admissibility of $\underline{W} = (W + \mathfrak{h})/\mathfrak{h}$ is equivalent to the fact that W is a Lie wedge with edge $\mathfrak{h}$. Moreover, by 6.6. the semigroup $S_{\underline{W}}$ is equal to the semigroup generated by all end-points of W-admissible curves in G. Therefore by [HL84] $S_W \subset S_{\underline{W}} \subset \bar{S}_W$ where S_W is the subsemigroup generated by $\exp W$. Since $S_W \subset \bar{S}$ and $\underline{L}(S) = W$ we conclude that $S_{\underline{W}}$ is infinitesimally generated with tangent wedge $\underline{L}(S_{\underline{W}}) = \underline{L}(S) = W$. Again from 3.7 ist follows that $S_{\underline{W}} \cap S_{\underline{W}}^{-1} = \bar{S} \cap S^{-1} = H$ so that $\underline{W}$ is causal by 6.21.
Note that $H \subset S_{\underline{W}}$ since it is connected so that $S_{\underline{W}} = \hat{S}_{\underline{W}}$ by 6.20. But then we know from 2.2 that $\hat{S}_{\underline{W}}$ has dense interior. Finally $\underline{L}(\hat{S}_{\underline{W}}) = \underline{L}(S_{\underline{W}}) = W$ implies that $\hat{S}_{\underline{W}}$ cannot be dense in G and we can apply 6.23. to $\underline{W}$. □

THEOREM IV.6.25. Let S be an infinitesimally generated subsemigroup of G with tangent wedge $\underline{L}(S) = W$ and W_1 a Lie wedge in $\mathfrak{g}$ such that $W_1 \subset\subset W$. If the analytic subgroup H_1 of G with Lie algebra $\mathfrak{h}_1 = W_1 \cap (-W_1)$ is closed then there exists a subsemigroup S_1 of G such that $\underline{L}(S_1) = W_1$.

Proof. By 6.24. we find a $\underline{W}$-positive function $\varphi : G/H \to \mathbb{R}$, where $H = S \cap S^{-1}$, $\mathfrak{h} = \underline{L}(H)$ and $\underline{W} = (W + \mathfrak{h})/\mathfrak{h}$. Moreover we can assume that $\ker(d\varphi(H))$ is a hyperplane $E/\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$ where E is a hyperplane in such that $E \cap \operatorname{int} W = \emptyset$. Note that 6.8 implies that $\varphi \circ \pi$ is W-positive where $\pi : G \to G/H$ is the canonical projection. Let $\pi_1 : G \to G/H_1$ be the canonical projection onto G/H_1 then $\varphi \circ \pi = \varphi_1 \circ \pi_1$ for some C^1-function $\varphi_1 : G/H_1 \to \mathbb{R}$. We remark that $\underline{W}_1 = W_1/\mathfrak{h}_1$ is H_1-admissible since H_1 is connected and W_1 is a Lie wedge with $W_1 \cap (-W_1) = \mathfrak{h}_1 = \underline{L}(H_1)$. Again by 6.8 we conclude that φ_1 is $\underline{W}_1$-positive. Moreover, since $\ker(d\varphi(H)) = E/\mathfrak{h}$

we get $\ker(d\varphi_1(H_1)) = E/\mathfrak{z}_1$. Therefore $d\varphi_1(H_1)(w + \mathfrak{z}_1) > 0$ for all $w \in W_1 \setminus \mathfrak{z}_1$ since $W_1 \setminus \mathfrak{z}_1 \subset \operatorname{int} W$. Thus $\varphi_1 \in P_s(\underline{W}_1)$ and Theorem 6.10 implies the existence of the subsemigroup S_1 of G such that $\underline{L}(S_1) = W_1$. □

The following example shows that the hypothesis $W_1 \subset\subset W$ can not be replaced by $W_1 \subset W$. Let G_1 be the Heisenberg group and K_1 be a cone in $\underline{L}(G_1) = \mathfrak{g}_1$ containing a central point in its interior. If $G = G_1 \times \mathbb{R}$ and $W_1 = K_1 \oplus \mathbb{R}^+$ then $W_1 \subset \mathfrak{g}_1 \oplus \mathbb{R}^+$ and $G + \mathbb{R}^+$ is a semigroup with tangent wedge $W = \mathfrak{g}_1 + \mathbb{R}^+$. Nevertheless there can not be a semigroup S_1 in G with $\underline{L}(S_1) = W_1$ since $\underline{L}(S_1) = W_1$ implies $\bar{S}_1 = G_1$ by Example 5.5.

Suppose that G is, as before, a connected Lie group and N is a closed normal subgroup. Let $\pi : G \to G/N$ be the canonical projection and S be a subsemigroup of G/N with $\underline{L}(S_\pi) = W_\pi$. We have seen in section 4 that for $d\pi(\mathbf{1})^{-1}W_\pi =: W$ we can find a subsemigroup S with $\underline{L}(S) = W$. After 6.25. there arises of course the question if it is possible to find subsemigroups S_1 of G with $\underline{L}(S_1) = W_1$ for any $W_1 \subset W$ satisfying $d\pi(\mathbf{1})W_1 = W_\pi$. We cannot apply 6.25. since we don't have $W_1 \subset\subset W$. Nevertheless we get the right conclusion if we make the right restrictions on W. First we need a few lemmas

LEMMA IV.6.26. Let $\mathfrak{g}$ be a Lie algebra and W a wedge in $\mathfrak{g}$. Let $V = W \cap -W$ be the edge of the wedge and E a vectorspace complement of V in $\mathfrak{g}$. Then there exists a proper cone K in $\mathfrak{g}$ with $K \cap V = \{0\}$ and a neighborhood B of zero in $\mathfrak{g}$ such that $x,y \in B \cap E$ implies $x*y \in W$ together with $x+y \in K$. Here $*$ denotes the Campbell-Hausdorff multiplication. Moreover, if F is a fixed subspace of E such that $F \cap W = \{0\}$, we may choose K such that $K \cap F = \{0\}$.

Proof. Let F' be a vectorspace complement of F in E such that $W \cap F' \neq \{0\}$. Let $0 \neq x_o \in W \cap F'$ and E_1 be a hyperplane in E containing F such that $W \cap E_1 = \{0\}$. Now choose an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\mathbb{R}x_o \oplus E_1 \oplus V$ is an orthogonal decomposition. Let $K_E = W \cap E$, then K_E is a proper cone with $K_E \cap (E_1 \oplus V) = \{0\}$ and hence there exists an $\alpha_o \in\ ]0,\pi/2[$ such that $\alpha(x_o,x) < \alpha_o$ for all $x \in K_E$ where $\alpha(x_o,x)$ denotes the angle between x_o and x:

$$\cos(\alpha(x,y)) = \frac{\langle x,y\rangle}{|x|\,|y|}$$

Recall that $x * y = x + y + \frac{1}{2}[x,y] + \dots$ so that there exists a neighborhood B_o of zero in $\mathfrak{g}$ and a constant such that $|x + y - x * y| \leq c|x|\,|y|$ for all $x,y \in B_o$.

If we decompose $x * y = e + z$ where $e \in E$ and $z \in V$ then we get $|x + y - x * y|^2 = |x + y - e|^2 + |z|^2$ since E and V are orthogonal.

We want to calculate $\cos(\alpha(x_o,x+y)) = \dfrac{\langle x_o,\ x+y\rangle}{|x_o|\ |x+y|}$.

Note that $e \in W$ so that

$$\frac{\langle x_o,\ x+y\rangle}{|x_o|\,|x+y|} = \frac{\langle x_o,(x+y) - x*y\rangle + \langle x_o,e\rangle + \langle x_o,z\rangle}{|x_o|\quad|x+y|} =$$

$$= \frac{\langle x_o,(x+y) - x*y\rangle + \langle x_o,z\rangle}{|x_o|\quad|x+y|} + \frac{\langle x_o,e\rangle}{|x_o|\,|x+y|} =$$

$$= \frac{\langle x_o,(x+y) - x*y\rangle + \langle x_o,z\rangle}{|x_o|\quad|x+y|} + \frac{\langle x_o,e\rangle}{|x_o|}\left(\frac{1}{|e|} - \frac{1}{|x+y|}\right) + \frac{\langle x_o,e\rangle}{|x_o|\,|x+y|}$$

We get

$$\left|\frac{\langle x_o,(x+y) - x*y\rangle + \langle x_o,z\rangle}{|x_o|\quad|x+y|}\right| \leq \frac{|x_o|\ c\ |x|\ |y| + |x_o|\ |z|}{|x_o|\ |x+y|}$$

$$\leq 2\ c\ \frac{|x|\ |y|}{|x+y|}$$

and

$$\left|\frac{\langle x_o,e\rangle}{|x_o|}\left(\frac{1}{|e|} - \frac{1}{|x+y|}\right)\right| \leq \frac{|x_o|\ |e|}{|x_o|}\left(\frac{\big|\,|x+y| - |e|\,\big|}{|e|\ |x+y|}\right) \leq$$

$$\leq \frac{|x+y - e|}{|x+y|} \leq \frac{c\,|x|\,|y|}{|x+y|}$$

Therefore

$$|\cos(\alpha(x_o,x+y) - \cos(\alpha(x_o,e))| \leq 3\ c\ \frac{|x|\ |y|}{|x+y|}$$

and hence we can find an ε such that $|x|, |y| < \varepsilon$ implies $\alpha(x_o,x+y) < \alpha_1 \in]0, \pi/2[$. Thus we define $B = B_\varepsilon = \{x \in \mathfrak{g} : |x| < \varepsilon\}$ and $K = \{x \in \mathfrak{g} : \alpha(x_o,x) < \alpha_1\}$

□

LEMMA IV.6.27. *Let V be a finite dimensional vectorspace and W a proper cone in V. If $V = V_1 \oplus V_2$ and $W \cap V_2 = \{0\}$, then $p_1(W)$ is a proper cone in V_1 where $p_1 : V \to V_1$ is the projection along V_2. Moreover, for any neighborhood B of zero in V we find a neighborhood B_1 of zero in V_1 such that $x = x_1 \oplus x_2$, $x_1 \in B_1$, $x_2 \in V_2$ and $x \in W$ imply $x_1, x_2 \in B$.*

Proof. Enlarging W, if necessary we may assume that there is an inner product on V such that $V_1 \oplus V_2$ is an orthogonal decomposition and W is of the form $W = \{x \in V : 0 \leqq \alpha(x_o,x) < \alpha_1 < \frac{\pi}{2}\}$ for some $x_o \in V_1$, where again $\alpha(x_o,x)$ is the angle between x_o and x. But then, for $x = x_1 \oplus x_2$ we obtain

$$0 < (\arccos \alpha_o)^2 \leqq \frac{|\langle x_o,x\rangle|^2}{\|x_o\|^2\,\|x\|^2} = \frac{|\langle x_o,x_1\rangle|^2}{\|x_o\|^2(\|x_1\|^2+\|x_2\|^2)} \leqq$$

$$\leqq \frac{\|x_o\|^2\,\|x_1\|^2}{\|x_o\|^2(\|x_1\|^2+\|x_2\|^2)} = \left(1+\frac{\|x_2\|^2}{\|x_1\|^2}\right)^{-1}$$

If we set $c = (\arccos \alpha_o)^2$ then we get

$$\|x_2\|^2 \leqq \left(\frac{1-c}{c}\right)\|x_1\|^2$$

and hence

$$\|x\|^2 \leqq \frac{1}{c}\|x_1\|^2$$

□

LEMMA IV.6.28. *Let G be a Lie group and W be a Lie wedge in $\mathfrak{g} = \underline{L}(G)$ such that the analytic subgroup H of G, corresponding to $\mathfrak{h} = W \cap -W$, is closed. Let S_W be the subsemigroup G generated by $\exp(W)$. Suppose that for any local subsemigroup (S_U,U) of G with tangent wedge W we can find a neighborhood U_1 of $\mathbf{1}$ in G such that $S_W \cap U_1 \subset S_U H$. Then $\underline{L}(S_W) = W$.*

Proof. Let (S_{U_o},U_o) be a fixed local semigroup in G with tangent wedge W. Then it suffices to show that we can find a neighborhood U_1 such that $S_W \cap U_1 \subset S_{U_o}$. Let V_o be a symmetric neighborhood of $\mathbf{1}$ in G such that

$V_o^2 \subset U_o$. We set $S_{V_o} = S_{U_o} \cap V_o$, then (S_{V_o}, V_o) is a local subsemigroup of G with tangent wedge W. Thus we find a neighborhood U_1 of $\mathbf{1}$ in G such that $S_W \cap U_1 \subset S_{V_o} H$. We may also assume that $U_1 \;\; V_o$. But then $g = sh \in S_W \cap U_1$ with $s \in S_{V_o}$ and $h \in H$ implies that $h = s^{-1} g \in V_o U_1 \subset V_o^2 \subset U_o$ so that $h \in S_{U_o}$. Therefore s and h are contained in S_{U_o} and $sh \in U_o$ implies $g \in S_{U_o}$. Thus $S_W \cap U_1 \subset S_{U_o}$.

We can now prove the anounced theorem:

THEOREM IV.6.29. *Let G be a Lie group and N be a closed normal subgroup of G. Let $\pi : G \to G/N$ be the canonical projection and W be a generating Lie wedge in $\mathfrak{g}$ with $W \cap -W = \mathfrak{h}$. Suppose that*

(i) *The analytic subgroup H of G with $\underline{L}(H) = \mathfrak{h}$ is closed.*

(ii) $W \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}$ *where* $\mathfrak{n} = \underline{L}(N)$.

(iii) *There exists an infinitesimally generated subsemigroup S_π of G/N with $\underline{L}(S_\pi) = d\pi(\mathbf{1})W$.*

Then there exists a subsemigroup S of G such that $\underline{L}(S) = W$.

Proof: Let $H_\pi = S_\pi \cap S_\pi^{-1}$ then by 3.7 the group H_π is closed and connected with Lie algebra $\mathfrak{h}_\pi = W_\pi \cap -W_\pi$, where $W_\pi = d\pi(\mathbf{1})W$. Therefore (ii) implies $\mathfrak{h}_\pi = (\mathfrak{h} + \mathfrak{n})/\mathfrak{n}$.

We can decompose $\mathfrak{g}$ as a vectorspace into $\mathfrak{g} = F' \oplus F \oplus \mathfrak{h}$ where $E = F' \oplus F$ is any vectorspace complement of $\mathfrak{h}$ in $\mathfrak{g}$, $F = E \cap d\pi(\mathbf{1})^{-1}(\mathfrak{h}_\pi) = E \cap (\mathfrak{h} + \mathfrak{n})$ and finally F' is any vectorspace complement of F in E. Note that $K_E = W \cap E$ is a proper cone such that $\ker(d\pi(\mathbf{1})) \cap K_E \subset \ker(d\pi(\mathbf{1})) \cap W \cap E \subseteq \mathfrak{h} \cap \mathfrak{n} \cap E = \{0\}$.

Let K_o be cone in E such that $K_E \subset\subset K_o$ and $K_o \cap F = \{0\}$. This is possible since $K_E \cap F \subseteq E \cap (\mathfrak{h} + \mathfrak{n}) \cap W = \{0\}$. In fact $x + y = w$, where $x \in \;$, $y \in \;$, $w \in W \cap E$ implies $y \in W \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}$ so that $w \in \mathfrak{h} \cap E = \{0\}$. Finally we set $W_o = \mathfrak{h} + K_o$ and obtain that W_o is a wedge with $W_o \cap -W_o =$ and $W \subset\subset W_o$.

We apply Lemma 6.26. to W_o and find a proper cone K in $\mathfrak{g}$ such that $K \cap \mathfrak{h} = \{0\}$ and a neighborhood B of zero in $\mathfrak{g}$ such that $x*y \in W_o$ together with $x, y \in B \cap E$ implies $x+y \in K$. Moreover we may assume that $K \cap F = \{0\}$ since $W_o \cap F = W_o \cap E \cap F = K_o \cap F = \{0\}$.

Consider $L_\pi = d\pi(\mathbf{1}) : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$. Then $F'_\pi = L_\pi(F') = (F'+\mathfrak{n})/\mathfrak{n}$ and $\mathfrak{h}_\pi = (\mathfrak{h}+\mathfrak{n})/\mathfrak{n}$ yield a direct vectorspace decomposition $\mathfrak{g}/\mathfrak{h} = F'_\pi \oplus \mathfrak{h}_\pi$. In fact $\mathfrak{g}/\mathfrak{h} = L_\pi(\mathfrak{g}) = L_\pi(F'+F+\mathfrak{h}) \subset L_\pi(F') + \mathfrak{h}_\pi = F'_\pi + \mathfrak{h}_\pi$. Moreover $x+\mathfrak{n} = y+\mathfrak{n}$ for $x \in F'$ and $y \in \mathfrak{h}$ implies $x \in (\mathfrak{h}+\mathfrak{n}) \cap F' = F \cap F' = \{0\}$ hence $y \in \mathfrak{n}$, so that $F'_\pi \cap \mathfrak{h}_\pi = \{\mathfrak{n}\}$.

Next we study the local decompositions of the groups G and $G_\pi = G/N$ arising from the decompositions on the Lie algebra level. It follows from the closedness of H and H_π that we can find open neighborhoods $B_{F'}$ and B_F of zero in F' and F respectively as well as an open eighborhood $B_{F'_\pi}$ of zero in F'_π such that the following maps are diffeomorphisms onto their (open) images in G and G_π respectively:

$$\phi : B_{F'} \times B_F \times H \to (\exp_G B_{F'})(\exp_G B_F)H$$
$$(x,y,h) \to (\exp_G x)(\exp_G y)h$$

$$\phi_\pi : B_{F'_\pi} \times H_\pi \to (\exp_{G_\pi} B_{F_\pi})H$$
$$(x_\pi,h_\pi) \to (\exp_{G_\pi} x_\pi)h_\pi$$

Note, that we may assume that $L_\pi(B_{F'}) = B_{F'_\pi}$ since $L_\pi|_{F'} : F' \to F'_\pi$ is an isomorphism of vector spaces. Moreover we may assume that $B_{F'} \times B_F \subset B$.

Let (S_{U_o}, U_o) be a local subsemigroup of G with tangent wedge W. Making U_o smaller if necessary we may assume that $U_o \subset \phi(B_{F'} \times B_F \times H) \cap \exp_G B$ and that $\exp_G|_{\exp^{-1}(U)} : \exp_G^{-1}(U_o) \to U_o$ is a diffeomorphism. Moreover we may assume that $S_{U_o} \subset \exp_G(W_o \cap (\exp^{-1} U_o))$ since $W \subset\subset W_o$. Now let U be neighborhood of $\mathbf{1}$ in G such that $U^2 \subset U_o$ and we set $S_U = S_{U_o} \cap U$. By Lemma 6.28. it suffices to show that we can find a neighborhood U_W of $\mathbf{1}$ in G such that $S_W \cap U_W \subset S_U H$, where S_W is the subsemigroup of G generated by $\exp_G W$.
Let $g = \phi(x,y,h)$ with $x \in B_{F'}$, $y \in B_F$, $h \in H$. Then

$$\pi(g) = \pi(\exp_G x)\pi(\exp_G y)h = \exp_{G_\pi}(L_\pi(x))\exp_{G_\pi}(L_\pi(y))\pi(h).$$

Since H is connected and hence generated by $\exp_G \mathfrak{h}$ we know that $\pi(h) \in H_\pi$. Similarly $\exp_{G_\pi}(L_\pi(y)) \in H_\pi$ since $L_\pi(y) \in (\mathfrak{h}+\mathfrak{n})/\mathfrak{n}$. Thus $\pi(g) = \phi\pi(L_\pi(x),h_\pi)$ with $h_\pi = \exp_{G_\pi}(L_\pi(y))\pi(h)$ and $L_\pi(x) \in B_{F'_\pi}$. Suppose that $g \in S_U$

and $(\exp_G x)(\exp_G y) \in U$ as well a $h \in U$, then $h^{-1} \in S_U$ (we may assume

U and U_o to be symmetric), hence $(\exp_G x)(\exp_G y) \in S_U S_U \cap U \subseteq S_U$.

Therefore we find $x*y = \exp_G^{-1}((\exp_G x)(\exp_G y)) \in W_o$ (we may assume $B_{F'} \times B_F$

to be contained in a Campbell-Hausdorff neighborhood). Since we have

$\exp_G^{-1} U \supset B \supset B_{F'} \times B_F$ we get $x+y \in K \cap E$.

Let U_1 be a symmetric neighborhood of $\mathbf{1}$ in G such that $U_1^5 \subset U$. Now

we apply Lemma 6.27. to $E = F' \oplus F$, $K \cap E$ and $(\exp_G^{-1} U_1) \cap E$. What we find

is a neighborhood $B'_{F'}$ such that $x = x_1 + x_2$, $x_1 \in B'_{F'}$ and $x_2 \in F$

with $x \in K$ imply $x_2, x_2 \in \exp^{-1}(U_1)$.

Of course we may assume that $L_\pi(B'_{F'}) = B'_{F'_\pi}$ $B_{F'_\pi}$. But by 3.6 we may also

assume that $S_\pi \setminus (S_\pi \cap ((\exp_{G_\pi}(B'_{F'_\pi}))H_\pi))$ is a left ideal in S_π. (here we

have to use left ideals since we use $\exp_{G_\pi}(B'_{F'_\pi})H_\pi$ instead of $H_\pi \exp_{G_\pi}(B'_{F'_\pi})$).

Now we choose a neigborhood U_W of $\mathbf{1}$ in G such that

$\phi(B_{F'} \times B_F \times H) \cap U \supset U_W$ and $\pi(U_W) \subset (\exp_{G_\pi} B'_{F'_\pi})H_\pi$. If now

$g = (\prod_{k=1}^{n} \phi(x_k, y_k, h_k)) \in U_W$, where $x_k * y_k \in W$, $x_k \in B_{F'} \cap \exp_G^{-1}(U_1)$,

$y_k \in B_F \cap \exp_G^{-1}(U_1)$ and $h_k \in U_1$, then

$$\pi(g) = \prod_{k=1}^{n} \phi\pi(L_\pi(x_k),(h_k)_\pi) \in S_\pi \cap (\exp_{G_\pi} B'_{F'_\pi})H_\pi,$$

since $\pi(S_W) \subset S_\pi$. The left-ideal-property now shows that for $g_m = \prod_{k=m}^{n} \phi(x_k, x_k, h_k)$

the elements $\pi(g_m)$ are contained in $S_\pi \cap (\exp_{G_\pi} B'_{F'_\pi})H_\pi$. We claim that

$g_m \in S_U H$ and prove this claim by induction:

Clearly $g_n = \phi(x_n, y_n, h_k) \in S_U H$ since $x_n * y_n \in W$ so that

$(\exp_G x_n)(\exp_G y_n) = \exp(x_n * y_n) \in \exp_G(W \cap \exp_G^{-1} U)$ S_U (Taking the relative

closure of S_U in U we may always assume that $\exp(W \cap (\exp^{-1} U)) \subset S_U$).

If now $g_m \in S_U H$, then g_m is of the form $\phi(x^{(m)}, y^{(m)}, h^{(m)})$ where

$(x^{(m)}, y^{(m)}) \in B_{F'} \times B_F$. Moreover there exists an $\hat{h}^{(m)} \in H$ such that $x^{(m)} + y^{(m)} \in K$

$\hat{g}_m = \phi(x^{(m)}, y^{(m)}) \in S_U$. By the argument given above we find that

and, since $\pi(g_m) = \phi_\pi(L_\pi(x^{(m)}), h_\pi^{(m)}) \in \exp_{G_\pi} B'_{F'_\pi}$, we get $L_\pi(x^{(m)}) \in B'_{F'_\pi}$.

Thus $x^{(m)} \in B'_{F'}$ which shows that $x^{(m)}, y^{(m)} \in \exp_G^{-1}(U_1)$.

But then $\exp_G(y^{(m)})^{-1}\exp_G(x^{(m)})^{-1}\hat{g}_m \in U_1^2 U \subset U^2 \subset U_o$ so that $\hat{h}^{(m)} \in U_o$, hence $\hat{h}^{(m)} \in S_{U_o}$. Therefore we get $(\exp_G x^{(m)})(\exp_G y^{(m)}) = \hat{g}_m(\hat{h}^{(m)})^{-1} \in S_{U_o} S_{U_o} \cap U_1^2 \subset S_{U_o} \subset U_1^2$.

From this we conclude

$$\begin{aligned} g_{m-1} &= (\exp_G x_{m-1})(\exp_G y_{m-1})h_{m-1} \exp_G x^{(m)} \exp_G y^{(m)} h^{(m)} = \\ &= \exp_G(x_{m-1} * y_{m-1})h_{m-1}((\exp_G x^{(m)}(\exp_G y^{(m)}))h^{(m)} \in (S_{U_o} \cap U_1^5) H \\ &\subseteq (S_{U_o} \cap U)H = S_U H . \end{aligned}$$

Finally we note that the elements of the form $\phi(x,y,h)$ with $x*y \in W$, $x \in B_{F'}$, $y \in B_F$. $h \in U_1$ generate S_W so that we have shown $S_W \cap U_W \subset S_U H$.

□

The Theorems 6.25. and 6.29 describe methods how one can obtain semigroups with prescribed tangent wedges starting from certain semigroups whose tangent wedge we know. If we don't have any semigroups to start with, we go back to the examples, and find that we may have different reasons for the nonexistence of subsemigroups with a prescribed tangent wedge. For instance the invariant cone in $sl(2,\mathbb{R})$ is the tangent cone of a subsemigroup of $\tilde{SL}(2,\mathbb{R})$, but not the tangent cone of a subsemigroup of $SL(2,\mathbb{R})$. Here the obstruction is of topological nature. On the other hand we have seen that no cone in the Heisenberg algebra containing a central element in its interior is the tangent cone of a subsemigroup of the Heisenberg group. In this case the obstruction was of algebraic nature. It is possible to study the different kinds of obstructions separately. In fact, suppose that G is a connected Lie group and $\tilde{G}$ is its simply connected covering group. We have seen in 4.1 that any preanalytic semigroup S in G which generates G can be pulled back to a semigroup $\tilde{S}$ in $\tilde{G}$ with the same tangent wedge. Thus any algebraic obstruction to the existence can be detected by considering just $\tilde{G}$. This was to be expected since $\tilde{G}$ is completely determined by $\mathfrak{g} = \underline{L}(G) = \underline{L}(\tilde{G})$. We are lead to the following definitions:

DEFINITION IV.6.30. Let W be a Lie wedge in a finite dimensional Lie algebra $\mathfrak{g}$. We say that W is *global in* $\mathfrak{g}$ if in the simply connected Lie group $\tilde{G}$ with $\mathfrak{g} = \underline{L}(\tilde{G})$ there is an infinitesimally generated subsemigroup S

with $W = \underline{L}(S)$. We say that W is *global* if it is global in the Lie algebra $\langle\langle W \rangle\rangle$ generated by W in $\mathfrak{g}$.

Note that for any preanalytic semigroup S in G the tangent wedge $\underline{L}(S)$ is global. Hence any wedge in an abelian Lie algebra is global.

Note also that 3.7 implies that the analytic subgroup H of $\tilde{G}$ corresponding to the edge of a global wedge W in $\mathfrak{g}$ is necessarily closed. We call a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ *admissible* if the corresponding analytic subgroup H of $\tilde{G}$ with $\underline{L}(H) = \mathfrak{h}$ is closed. Thus the above statement can be reformulated as:
The edge of a global wedge W in $\mathfrak{g}$ is an admissible subalgebra.
We remark that the admissibility of a subalgebra is, contrary to the first impression, an algebraic property.

Proposition 4.1 yields the following pullback Lemma:

LEMMA IV.6.31. *Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a morphism of Lie algebras. If W is a wedge in $\mathfrak{g}_2$ which is global in $\mathfrak{g}_2$, then $f^{-1}(W)$ is global in $\mathfrak{g}_1$.*

Proof. Let G_1 and G_2 be the simply connected Lie groups with $\underline{L}(G_1) = {}_1$ and $\underline{L}(G_2) = \mathfrak{g}_2$ respectively. The simple connectivity yields the existence of a Lie group morphism $F : G_1 \to G_2$ with $dF(\mathbf{1}) = f$. If now S_2 is an infinitesimally generated subsemigroup of G_2 satisfying $\underline{L}(S) = W$ then S_2 has inner points by Theorem 2.2. and hence we can apply Proposition 4.1 to obtain the desired result. □

This lemma shows that any wedge W in $\mathfrak{g}$ containing the commutator algebra is global (cf.chapter II). More generally we obtain the following lemma, which deserves independent mentioning since it is frequently useful.

LEMMA IV.6.32. *If W is a Lie wedge in a Lie algebra $\mathfrak{g}$ and if W contains an ideal $\mathfrak{i}$ of $\mathfrak{g}$ such that $W/\mathfrak{i}$ is global in $\mathfrak{g}/\mathfrak{i}$, then W is global in $\mathfrak{g}$.* □

Another consequence of the Pullback-Lemma is the fact that globality does not depend on the embedding Lie algebra .

PROPOSITION IV.6.33. *Let W be a Lie wedge in a Lie algebra $\mathfrak{g}$. Then W is global in $\mathfrak{g}$ if and only if it is global.*

Proof: If W is global in $\langle\langle W\rangle\rangle$ it is obviously global in $\mathfrak{g}$. Conversely if j is the embedding of $\langle\langle W\rangle\rangle$ in $\mathfrak{g}$ then Lemma 6.31 shows that W global in $\mathfrak{g}$ implies that $W = j^{-1}(W)$ is global in $\langle\langle W\rangle\rangle$, hence global. □

Note that the truth of Proposition 6.33. heavily depends on the fact that we required the semigroup S with $\underline{L}(S) = W$ only to be infinitesimally generated. Had we required that it be closed in G, then the situation would be different. It is perhaps good in this context to contemplate the follwing example:

EXAMPLE IV.6.34. Let $\mathfrak{g}_1$ be a compact semisimple Lie algebra of rank at least 2. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}_1$ and set $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{R}$. Let $\mathfrak{f}$ be a vector subspace of $\mathfrak{a}$ which generates in the simply connected Lie group G_1 with $\underline{L}(G_1) = \mathfrak{g}_1$ a non-closed dense analytic subgroup of the maximal torus T generated by $\mathfrak{a}$.

Now $W = V \oplus \mathbb{R}^+$ is a global Lie wedge for any wedge V in $\mathfrak{f}$ by Proposition 6.33. But if V is generating in $\mathfrak{f}$ (i.e. $\mathfrak{f} = V - V$), then no infinitesimally generated subsemigroup S in $G_1 \oplus \mathbb{R}$ with $\underline{L}(S) = W$ is closed; in fact its closure contains $T \oplus \mathbb{R}$ and $\underline{L}(T \oplus \mathbb{R})$ is $\mathfrak{a} \oplus \mathbb{R}$. □

One can also obtain global Lie wedges as the intersection of global Lie wedges.

LEMMA IV.6.35. Let W_j, $j \in J$ be a family of global Lie wedges in a Lie algebra $\mathfrak{g}$ and W be their intersection. Then W is global.

Proof. Note first that for any Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ the wedges $\mathfrak{m} \cap W_j$ areglobal by the pullback Lemma. If we set $\mathfrak{m} = \langle\langle W\rangle\rangle$ we thus may assume without loss of generality that $\mathfrak{g} = \langle\langle W\rangle\rangle$. Let now G be the corresponding simply connected Lie group with exponential function $\exp : \mathfrak{g} \to G$ and S_j be the subsemigroup of G generated by $\exp W_j$. Then, by hypothesis $\underline{L}(S_j) = W_j$ and therefore $\underline{L}(G(S_j)) = \langle\langle W_j\rangle\rangle \supset \langle\langle W\rangle\rangle = \mathfrak{g}$ so that $G(S_j) = G$. Now Theorem 2.8. shows that $\underline{L}(\bar{S}_j) = \underline{L}(S_j) = W_j$ where $\bar{S}_j$ is the closure of S_j. Let S be the semigroup generated by $\exp W$. Then S is a ray semigroup with $G(S) = G$ by Theorem 2.2. Thus $x \in \underline{L}(S)$ if and only if $\exp \mathbb{R}^+ x \subseteq \bar{S}$. But $S \subset S_j$ for any $j \in J$ hence $x \in \underline{L}(S)$ implies $\exp \mathbb{R}^+ x \subseteq \bar{S}_j$ for any $j \in J$. Thus $\underline{L}(S) \subset \bigcap_{j \in J} \underline{L}(S_j) = \bigcap_{j \in J} W_j = W$. Conversely $W \subset \underline{L}(S)$ by definition. □

Note that Theorems 6.25 and 6.29 also yield information on global Lie wedges:

PROPOSITION IV.6.36. Let W be a global wedge in the Lie algebra $\mathfrak{g}$ and W_1 be a Lie wedge such that $W_1 \ll W$ and $W_1 \cap -W_1$ is an admissible subalgebra, then W_1 is global. □

PROPOSITION IV.6.37. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ an ideal in $\mathfrak{g}$. Suppose that $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ is the quotient map and $W \subset \mathfrak{g}$ is a Lie wedge such that

(i) $\pi(W)$ is a global Lie wedge.

(ii) $\mathfrak{i} \cap W \subset W \cap -W$.

(iii) $W \cap -W$ is an admissible subalgebra.

Then W is global. □

There are very few general methods known at the moment to decide whether a given wedge is global. One was to compare the wedge with global ones. Given this approach it is reasonable to now study special cases, if possible big ones. The biggest kind of wedges are the halfspaces. They are Lie wedges if and only if the bounding hyperplane is a subalgebra, a situation which is well understood (cf.[Ho65]):

LEMMA IV.6.38. Let $\mathfrak{g}$ be a (real) Lie algebra and $\mathfrak{a}$ a subalgebra in $\mathfrak{g}$ of codimension one which contains no nontrivial ideal of $\mathfrak{g}$. Then one of the following three cases occurs.

(i) $\mathfrak{g} = \mathbb{R}$ and $\mathfrak{a} = \{0\}$.

(ii) $\mathfrak{g} = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, a,b \in \mathbb{R}\}$ is the abelian algebra of dimension two and $\mathfrak{a}$ is any line in $\mathfrak{g}$, different from the one dimensional ideal.

(iii) $\mathfrak{g} = sl(2,\mathbb{R})$ and $\mathfrak{a}$ is any Borelsubalgebra of $\mathfrak{g}$. □

From this lemma we can conclude that *any* Lie wedge bounded by a hyperplane is global:

PROPOSITION IV.6.39. Let $\mathfrak{g}$ be a Lie algebra and W a halfspace in $\mathfrak{g}$ which is a Lie wedge. Then W is global.

Proof. Let W be a halfspace Lie wedge in $\mathfrak{g}$ and $\mathfrak{a} = H(W)$ be the edge of W. Moreover let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$ contained in $\mathfrak{a}$ and maximal with respect to that property. Then $\mathfrak{g}/\mathfrak{i}$ and $\mathfrak{a}/\mathfrak{i}$ satisfy the hypothesis of Lemma 6.38. But the examples of section 5 show that in either of the three cases $W/\mathfrak{i}$ is global. Thus Lemma 6.32. shows that W is global. □

Now from 6.35. we obtain

COROLLARY IV.6.40. Any wedge in a Lie algebra which is the intersection of a family of halfspace Lie wedges is a global Lie wedge. □

In the sequel we give a number of examples which can be treated with the methods developed in this section. We start with an immediate corollary to 6.40.

COROLARRY IV.6.41. Any Lie semialgebra of dimension at most three is global.

Proof. Up to dimension three semialgebras are intersections of halbspace semialgebras. □

At this point we should note that not all semialgebras are intersections of halfspace semialgebras. If one wants to prove globality for more general semialgebras one needs to make more hypotheses on the Lie algebra in question. For example it is sufficient to require that the Lie algebra be exponential (cf.chapter II). In this context the following proposition is just a reformulation of Corollary II.1.30.

PROPOSITION IV.6.42. Let $\mathfrak{g}$ be an exponential Lie algebra and W be a semialgebra in $\mathfrak{g}$ then W is global. □

A Lie group G is usually called exponential if the exponential function $\exp : \underline{L}(G) \to G$ is surjective. Note that this does not imply that $\underline{L}(G)$ is exponential in the sense of Definition II.1.29. In fact any nonabelian compact Lie algebra shows that this is not so. Thus Proposition 6.42. does not apply to compact Lie algebras. Nevertheless we can prove the analogous result for a class of Lie algebras which contains the compact Lie algebras:

Recall that a *motion algebra* is a Lie algebra which is the sum of a compactly embedded subalgebra and an abelian ideal.

PROPOSITION IV.6.43. Let $\mathfrak{g}$ be a motion algebra then any generating semialgebra W in $\mathfrak{g}$ is global.

Proof. Recall (cf.chapter II) that W is of the from $W = \mathfrak{j} + \mathbb{R}^+(a + C)$ where $\mathfrak{j}$ is an ideal, C is a compact convex neighborhood of zero in a subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ and a is in the centralizer of $\mathfrak{n}$. Lemma 6.3 shows that we may assume that $\mathfrak{j} = 0$. But then W is a cone and $\mathfrak{n}$ is an ideal in $\mathfrak{g} = \mathbb{R}a + \mathfrak{n}$. Moreover $(W + \mathfrak{n})/\mathfrak{n} = \mathbb{R}^+a$ which is global. Thus Proposition 6.37. shows that W is global. □

We have to be a little more careful if we want to detect also the topological obstructions to the existence of global semigroups in motion groups. Here we call a connected Lie group a *motion group* if its Lie algebra is a motion algebra.

LEMMA IV.6.44. Let G be a connected Lie group whose Lie algebra $L = \underline{L}(G)$ is compact and let W be a generating invariant cone in L. Then for the maximal compact subgroup K of G the following statement are equivalent:

(1) There exists a subsemigroup S of G such that $\underline{L}(S) = W$.

(2) $W \cap L(K) = \{0\}$.

Proof: (1) => (2). Let $x \in W \cap L(K)$ then we may assume that $\exp x \in S$ since $\underline{L}(S) = \underline{L}(\bar{S})$. Since $(\exp \mathbb{R}x)^-$ is compact this implies $(\exp \mathbb{R}x)^- \subset \bar{S}$ so that $\mathbb{R}x \subset W$ whence $x = 0$.

(2) => (1). Note first that $G \simeq K \bullet V$ where V is a vectorgroup. Let L_M be a hyperplane in $\underline{L}(G)$ containing $\underline{L}(K)$ and satisfying $L_M \cap W = \{0\}$. This is possible by (2). Then L_M is an ideal in $\underline{L}(G)$ whose corresponding analytic subgroup M is closed and contains K. Now consider $G/M \simeq \mathbb{R}$ and the cone $(W + L_M)/L_M$ in $\underline{L}(G/M)$. Identifying G/M with $\underline{L}(G/M)$ we see that $(W + L_M)/L_M$ is a subsemigroup of G/M, so that Theorem 6.29 shows that there is a subsemigroup S of G with $\underline{L}(S) = W$. □

Using this result we obtain:

THEOREM IV.6.45. Let G be a motion group and W be a generating semialgebra in $L(G)$. If A is the analytic subgroup corresponding to $H(W)$ and K is a maximal compact subcroup of G then the following statement are equivalent.

(1) There exists an infinitesimally generated subsemigroup S of G such that $\underline{L}(S) = W$.

(2) The group A is closed and $W \cap L(K) \subset H(W)$.

Proof. (1) => (2). The group A is closed by 3.7. and $W \cap L(K) \subset H(W)$ follows as in the preceding lemma.
(2) => (1). Conversely, if A is closed we can consider G/A since A is a normal subgroup and find from the description of all semialgebras in motion algebras given in chapter II that $L(G/A)$ is compact and $W/H(W)$ is a generating invariant cone in $\underline{L}(G/A)$. Let K_1 denote the maximal compact subgroup of G/A and $\pi : G \to G/A$ the quotient map. Then $K \subset \pi^{-1}(K_1)$ and by [Hoch 65] even $\pi^{-1}(K_1) = KA$ since K is also a maximal compact subgroup of $\pi^{-1}(K_1)$. Hence $L_\pi^{-1}L(K_1) = L(KA) = L(K) + H(W)$ and $L_\pi^{-1}(W/H(W) \cap L(K_1)) = W \cap (L(K) + H(W)) = W \cap L(K) \subset H(W)$ by (2). Thus Lemma 6.44. applies to $W/H(W)$ and yields a subsemigroup S_1 of G/A such that $\underline{L}(S_1) = W/H(W)$. But then Proposition 4.1. shows that $S = \pi^{-1}(S_1)$ has tangent wedge W so that the Theorem is proven in view of 2.8. □

Next we show that any Lorentzian semialgebra is global. In fact we can even determine all the groups which admit subsemigroups with Lorentzian semialgebras as Lie wedges. First we recall the classification theorem for invariant Lorentzian cones.

THEOREM IV.6.46. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and W be an invariant Lorentzian cone in $\mathfrak{g}$, then W is defined by a Lorentzian form $q : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that $(\mathfrak{g}, q)$ is isomorphic to the orthogonal direct sum of a compact Lie algebra (K,p) with a positive definite form p and a Lie algebra $(\mathfrak{g}_1, q_1)$ which is isomorphic to one of the following types:

(i) $\mathfrak{g}_1 = \mathbb{R}$, $q_1(x,y) = -xy$

(ii) $\mathfrak{g}_1 = sl(2,\mathbb{R})$, q_1 is the Killing form

(iii) $\mathfrak{g}_1 = o_m$, $q_1 = q_m$ for some $m = 1,2,3,\ldots$ (cf. Example 5.20)

Moreover all the forms are invariant in the sense that $q([x,y],z) = q(x,[y,z])$ for all $x,y,z \in \mathfrak{g}$. □

PROPOSITION IV.6.47. Let G be a Lie group and W be an invariant Lorentzian cone in $L(G)$. If $L(G) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the decomposition provided by Theorem 6.46. and $G = G_1 \bullet G_2$ where G_1 and G_2 are the analytic subgroups of G corresponding to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then the following statements are equivalent:

(i) G_1 is simply connected.

(ii) There exists a subsemigroup S of G such that $\underline{L}(S) = W$.

Proof. (i) => (ii). Note that we may apply Theorem 6.29 as soon as we have established the existence of subsemigroups of the groups $\mathbb{R}$, $SL(2,\mathbb{R})$ and O_m with the right tangent object. But this follows from the examples of section 5.
(ii) => (i). Consider $S \cap G_1$ then $\underline{L}(S \cap G_1)$ contains the invariant cone in $\mathfrak{g}_1$ given by q_1. Thus again the examples show that G_1 needs to be simply connected if $\underline{L}(S \cap G_1)$ is to be a proper cone. □

Recall that a Lorentzian semialgebra is an invariant cone unless $\mathfrak{g}$ is almost abelian in which case the Lorentzian semialgebra is global by Example 5.8. But since the almost abelian groups do have trivial center there cannot be any topological obstructions. Thus we have indeed

PROPOSITION IV.6.48. Let W be a Lorentzian semialgebra in a Lie algebra then W is global. □

We conclude this section with a special case which has not been studied extensively yet, but certainly deserves attention. It is motivated by examples like 5.4.

DEFINITION IV.6.49. A subset W in an associative algebra A is called on *associative semialgebra* if and only if it is a wedge with $WW \subset W$.

LEMMA IV.6.50. Let A be an associative algebra with unit $\mathbf{1}$. Let $\mathfrak{g} = A$ be the Lie algebra obtained from A by taking the multiplication $[\ ,\ ] : A \times A \to A$ defined by $[x,y] = xy - yx$ for all $x,y \in A$. Then for any associative semialgebra W in A we have

(i) $\mathbf{1} + W$ is a semigroup.

(ii) $\underline{L}((\mathbf{1} + W) \cap Gl_1(A)) = W$ where $Gl_1(A)$ is the group of units in A.

Proof: Note first that we may assume that A carries a norm that makes it a Banachalgebra. If we let $\exp : \mathfrak{g} \to Gl_1(A)$ be the standard exponential function then the open unit ball U around zero allows us to define the logarithm $\log : \mathbf{1} + U \to B$ with a suitable open neighborhood B of 0 in $\mathfrak{g}$. We note that $\mathfrak{g}$ is the Lie algebra of $Gl_1(A)$ and $\exp$ is the accompanying exponential function. Now we set $S = \mathbf{1} + W$. Then $\mathbf{1} \in S$ and $SS = (\mathbf{1}+W)(\mathbf{1}+W) = \mathbf{1} + W + W + WW \subset \mathbf{1} + W = S$. Thus S is a closed subsemigroup of the multiplicative subsemigroup of A and $T = S \cap Gl_1(A)$ is a closed subsemigroup of $Gl_1(A)$ which is an open neighborhood of $\mathbf{1}$ in S. Since $W - W$ is an associative algebra we may assume that W is generating. Thus T is preanalytic and $\underline{L}(T)$ is defined. The obvious relation $\exp(W) \subseteq (\mathbf{1}+W) \cap Gl_1(A)$ shows that $W \subset \underline{L}(T)$. In order to prove the converse, we let $x \in \underline{L}(T)$. Then $\exp tx \subset T \subset \mathbf{1} + W$ for all $t \geq 0$. This means that $\frac{1}{t}(\exp tx - \mathbf{1}) \in W$ for all $t \geq 0$. Passing to the limit $t \to 0$ through positive t we obtain $x \in W$ which we had to show. □

COROLLARY IV.6.51. Let $\mathfrak{g} = \mathfrak{g}_A$ be the associated Lie algebra of an associative algebra A. Then any associative semialgebra W in A is a global Lie wedge in $\mathfrak{g}$.

□

COROLLARY IV.6.52. If $f : \mathfrak{g} \to gl(n,\mathbb{R})$ is a respresentation of a Lie algebra $\mathfrak{g}$ and if there is an associative semialgebra V in the associative algebra $M_n(\mathbb{R})$ of all $n \times n$ matrices then $W = f^{-1}(V)$ is global.

Proof. 6.51 and 6.31.

□

Section 7: Maximal semigroups in Lie groups

Maximal semigroups are important for various reasons. On the one hand they provide potential upper bounds for subsemigroups of a Lie group. On the other hand they may be used to decide globality questions as we saw in Theorem 6.25. Moreover it turns out that maximal semigroups are suited ideally to solve controllability problems on Lie groups.

DEFINITION IV.7.1. Let S be a preanalytic semigroup in a connected Lie group G. Assume that $S \neq G(S)$.

(i) S is called *maximal proper in* $G(S)$ if the only closed proper subsemigroup of $G(S)$ containing S is S. It is called *maximal* if $G(S) = G$ and S is maximal in $G(S)$.

(ii) S is called *maximal open proper in* $G(S)$ if S is open in $G(S)$ and S is the only open proper subsemigroup of $G(S)$ containing S. It is called *maximal open proper* if $G(S) = G$ and S is maximal open proper in $G(S)$.

Again there are some technical complications attached to this idea which need to be understood. Consider the Lie group $\mathbb{R}$. The axion of choice allows us to write $\mathbb{R}$, considered as abelian group, as the direct sum of a subgroup V

and the subgroup $\mathbb{Q}$ of rational numbers, since $\mathbb{Q}$ is divisible. The subsemigroup $V \bullet \mathbb{Q}^+$ (with the semigroup $\mathbb{Q}^+$ of non-negative rationals) is a maximal subsemigroup of $\mathbb{R}$ in the algebraic sense that there is no semigroup properly between it and the whole group. But semigroups of this type are fairly useless in the context of Lie theory and for the applications of the Lie theory of semigroups.

The dense winding semigroup in the torus is maximal proper in $G(S)$, but is not maximal proper according ot our definition since $G(S) \neq G$. But notice that there is no closed subsemigroup of G properly between S and G. For the most part, by considering $G(S)$ instead of G we will be able to consider maximal (open) proper semigroups.

In order to prove the existence of maximal (open) proper semigroups we need the following Lemma.

LEMMA IV:7.2. *Let S be a proper subsemigroup of a connected topological group G and U be a non-empty open subset of S. Then* $U^{-1} \cap S = \emptyset$.

Proof: Suppose $s \in S \cap U^{-1}$, then $s^{-1} \in U \subseteq S$. Thus $\mathbf{1} = ss^{-1} \subseteq sU \subseteq sS \subseteq S$. But then S contains the neighborhood sU of $\mathbf{1}$ which generates all of G since G is connected. Hence $S = G$, a contradiction. Thus we have the claim. □

PROPOSITION IV.7.3. *Let S be an open proper subsemigroup of a connected topological group G. Then S is contained in a maximal open proper subsemigroup of G.*

Proof: Let $\mathfrak{M}$ be a maximal tower of open proper subsemigroups of G containing S, and let M be their union. Lemma 7.2 implies that $S^{-1} \cap T = \emptyset$ for all $T \in \mathfrak{M}$, hence $M \cap S^{-1} = \emptyset$, so that M is open proper. By the maximality of the tower M is then maximal open proper. □

Lemma 7.2 also shows that some of the problems described before vanish if we talk about semigroups with nonempty interior:

REMARK IV.7.4. *Let M be a subsemigroup of a connected Lie group G, such that*

(i) $\operatorname{int} M \neq \emptyset$.

(ii) *M is the only proper subsemigroup of G containing M.*

Then M is maximal proper and in particular closed.

Proof: Let $U = \operatorname{int} M$, then by Lemma 7.2 we have $M \cap U^{-1} = \emptyset$ and hence also $\tilde{M} \cap U^{-1} = \emptyset$. Thus $\tilde{M}$ is a proper subsemigroup containing M, so by maximality $M = \tilde{M}$. This also shows that M is maximal proper, since $G(M) = G(\mathrm{int}M) = G$.

□

PROPOSITION IV.7.5. _Every infinitesimally generated semigroup S which is not a group is contained in a maximal proper semigroup in $G(S)$._

Proof: Note first that we may assume without loss of generality that $G(S) = G$. If we now let T be the ray semigroup generated by $\underline{L}(S)$ then $G(T) = G$ and $\mathrm{int}\, T \neq \emptyset$ hence also $\mathrm{int}\, S \neq \emptyset$ by Theorem 2.2. By Zorn's Lemma there exists a maximal tower $\mathfrak{M}$ of proper semigroups containing S. Let $M = \bigcup\{R \in \mathfrak{M}\}$, then M is proper, since for U open in S Lemma 7.2 shows that $R \cap U^{-1} = \emptyset$ for all $R \in \mathfrak{M}$, hence $M \cap U^{-1} = \emptyset$. Moreover, by construction there is no proper subsemigroup of G containing M. But then Remark 7.4 shows that M is maximal proper.

□

Before we turn to more special situations we list a few properties of maximal open subsemigroups in topological groups which will be useful in the sequel.

PROPOSITION IV.7.6 _Let G be a connected topological group and S be a maximal open proper subsemigroup of G and N a closed normal subgroup of G. Then_

(i) $N \cap S = \emptyset$ _if and only if_ $NS = SU = S$.

(ii) $N \cap S = \emptyset$ _implies that_ SN/N _is maximal open proper in_ G.

(iii) _There exists a unique largest normal subgroup_ N_S _of_ G _such that_ $N_S \cap S = \emptyset$. _Moreover_ N_S _is closed._

Proof: (i) Since N is normal, $NS = SN$ is an open subsemigroup of G. Thus NS is either S or G by the maximality of S. If $NS = G$, then we have $\mathbf{1} \in NS$, so that we find $n \in N$ and $s \in S$ with $\mathbf{1} = ns$. But then $s = n^{-1} \in N \cap S$. Conversely, let $g \in N \cap S$. Then $\mathbf{1} = g^{-1}g \in NS$ so that $NS = G$.

(ii) Since $N \cap S = \emptyset$ implies $NS = SN = S$ and G/N carries the quotient topology, SN/N is an open proper semigroup in G/N. If $\pi : G \to G/N$ is the quotient map and T_π is an open proper semigroup in G/N containing SN/N then $\pi^{-1}(T_\pi)$ is an open proper semigroup in G containing S, hence $\pi^{-1}(T_\pi) = S$ so that $T_\pi = \pi(\pi^{-1}(T_\pi) = \pi S = SN/N$. Thus SN/N is maximal open proper in G/N.

(iii) Consider the monoid $S^* = \{g \in G : gS \subset S\}$ in G.

If N is *any* normal subgroup of G with $N \cap S = \emptyset$ then by (i) we have

$NS = S$ so that $N \subset S^*$. Conversely, any normal subgroup N which is contained in S^* satisfies $NS \subset S$, hence $N \cap S = \emptyset$. Thus N_S is the unique largest normal subgroup in G that is contained in $S^* \cap (S^*)^{-1}$. Since S is open then $N_S \cap S = \emptyset$ implies $\bar{N}_S \cap S = \emptyset$. Thus $\bar{N}_S \subset S^*$ whence $\bar{N}_S = N_S$. □

The biggest possible tangent wedge of a subsemigroup in a Lie group is a half-space. Thus semigroups having halfspaces as tangent wedge appear naturally in the study of maximal subsemigroups:

DEFINITION IV.7.7. A preanalytic semigroup S in a connected Lie group G is called a *halfspace semigroup* if $\underline{L}(S)$ is a halfspace.

Halfspace semigroups turn out to have many nice properties. They are strictly infinitesimally generated, their interiors are maximal open proper semigroups and their closures are manifolds with boundary. We start with a characterization theorem.

THEOREM IV.7.8. Let S be a closed subsemigroup in a connected Lie group G and let $\partial S = S \setminus \mathrm{int} S$ be the boundary of S. Then the following statements are equivalent.

(1) S is a halfspace semigroup.

(2) ∂S is a group and distinct from S.

Proof: (2) => (1). Note first that $\mathrm{int}\, S \neq \emptyset$ so that $\emptyset \neq (\mathrm{int}\, S)^- \cap \partial S \subset S$. Let $s \in (\mathrm{int}\, S)^- \cap \partial S$ then for any open neighborhood U of $\mathbf{1}$ we get $\emptyset \neq U \cap s^{-1}(\mathrm{int}\, S) \subset U \cap (\mathrm{int}\, S)$. Suppose that $\underline{L}(S)$ is not a halfspace. Then $H(\underline{L}(S))$ can not be a hyperplane. In fact if $H(\underline{L}(S))$ is a hyperplane then $\underline{L}(\partial S) = H(\underline{L}(S))$ since $\exp \underline{L}(S) \subset S$ and $\underline{L}(\partial S) \neq \underline{L}(G)$. Therefore, if Pick a neighborhood U of $\mathbf{1}$ in G which is of the form $(\exp B)(\exp B_F)$ where F is a complement of $\mathfrak{g} = \underline{L}(\partial S)$ in $\underline{L}(G)$ and $\phi : B \times B_F \to U$ given by $\phi(x,y) = (\exp x)(\exp y)$ is a diffeomorphism of an open neighborhood $B \times B_F$ in $\mathfrak{g} \times F = \underline{L}(G)$ onto U, then we know that $U \cap \mathrm{int}\, S \neq \emptyset$, i.e. there is a $\phi(x,y) \in \mathrm{int}\, S$ with $y \neq 0$. But $\exp x \in \partial S \subset S \cap S^{-1}$ so that also $\exp y \in S$. Since this works for arbitrarily small B_F we find by 1.6 that a halfline of F has to be contained in $\underline{L}(S)$. Thus $\underline{L}(S)$ is a halfspace contradicting our assumptions.

Now we know that there exists a wedge W with $\underline{L}(S) \setminus H(\underline{L}(S)) \subset \mathrm{int}\, W$ i.e. $\underline{L}(S) \subset\subset W$. By 1.6 we know that $S_B = B \cap \exp^{-1} S$ is a local semigroup with respect to B for any Campbell-Hausdorff-neighborhood B in $\underline{L}(G)$.

Now [HL 83] implies that we can find an open neighborhood of zero $B_1 \subset B$ such that $S_{B_1} = B_1 \cap \exp^{-1}(S) \subset W$. If now $x \in \text{int } S_{B_1}$ and $w \in \partial W \cap B_1 \setminus H(W)$ then there exists a $t \in [0,1]$ such that $x_o = x + t(w-x) \in (S_{B_1} \cap B_1)\setminus \text{int } S_{B_1}$ and hence $\exp x_o \in \partial S$ since $\exp_{|B_1}$ is a diffeomorphism onto its image. But then $\exp -x_o \in S$ so that $-x_o \in S_{B_1} \subset W$ whence $x_o \in H(W)$ contradicting the fact that $w \in W \setminus H(W)$. This proves the implication (2) => (1).
To prove the converse note first that $S \cap S^{-1}$ is closed, hence a Lie subgroup. Moreover $\exp(H(\underline{L}(S))) \subset S \cap S^{-1}$ so that $\underline{L}(S) \neq \underline{L}(G)$ implies that $S \cap S^{-1}$ is a Lie subgroup of codimension one. Lemma 7.2 shows that $S \cap S^{-1} \subset \partial S$ so that it suffices to show that $\partial S \subset S^{-1}$. By [HM 68] Theorem 1 and 2 we can find a one-parameter group $\exp \mathbb{R}x$ in G such that the map $\mathbb{R}x(S \cap S^{-1}) \to G$ given by $(t,s) \to (\exp tx)s$ is surjective. But then $x \notin H(\underline{L}(S))$, hence $x \in \text{int}(\underline{L}(S)$(replacing x by $-x$ if necessary). Therefore $\exp ts \in \text{int } S$ for all $t > 0$. If now $g \in \partial S$ and $g = (\exp tx)s$ then Lemma 2.5 shows that $t \leqq 0$. If $t < 0$ then $s = (\exp -tx)(\exp tx)s = \exp(-tx)g \in \text{int } S$ again by Lemma 2.5, contradicting the fact that $S \cap S^{-1} \subset \partial S$. Thus $t = 0$ and $g = s \in S \cap S^{-1}$.

□

Note that, in the notation of 7.8., [HM 68] also implies that $(\exp \mathbb{R}x)(S \cap S^{-1})_o = G$ where $(S \cap S^{-1})_o = \langle \exp H(\underline{L}(S))\rangle$ is the identity component of $S \cap S^{-1}$. Since $\exp tx \in \text{int } S$ for all $t > 0$ this shows that $(\exp tx)S \in \partial S$ if and only if $t = 0$ so that $S \cap S^{-1} = (S \cap S^{-1})_o$. Thus $S \cap S^{-1}$ is a connected Lie group hence algebraically generated by $\exp(H(\underline{L}(S)))$. But then S is algebraically generated by $\exp(H(\underline{L}(S)) + \mathbb{R}^+x) = \exp(\underline{L}(S))$. We summarize.

REMARK IV.7.9. Let S be a closed halfspace semigroup in a connected Lie Subgroup of codimension one and S is strictly infinitesimally generated.

From this remark we may draw a few more conclusions:

COROLLARY IV.7.10. Let S be a closed half space semigroup in a connected Lie group G, then

(i) S is maximal proper.

(ii) int S is maximal open proper.

(iii) $G = (\text{int } S) \cup \partial S \cup (\text{int } S)^{-1}$ where the union is disjoint.

Proof: Since int $S \neq \emptyset$ any proper semigroup T containing int S is
preanalytic and $\underline{L}(T)$ contains $\underline{L}(S)$. But since $\underline{L}(S)$ is a halfspace
this shows that T is a halfspace semigroup. Thus $\bar{T}$ is strictly infini-
tesimally generated by $\underline{L}(S)$ by Remark 7.9. and hence $\bar{T} = S$. Moreover
int T int $\bar{T}$ = int S. To show the last assertion we note that S^{-1} is a
halfspace semigroup with $\underline{L}(S^{-1}) = - \underline{L}(S)$. Thus the equality $(\exp \mathbb{R}x)\partial S = G$
for some $x \in \underline{L}(S)$ shows that $G =$ (int S) ∂S (int $S)^{-1}$. Clearly the
union is disjoint. □

We want to prove a classification theorem for halfspace semigroups. To do this
we need two more lemmas. The first is well known (cf.[HM 68]).

LEMMA IV.7.11. *Let* S *be a open proper semigroup in a topological group* G.
If K *is a compact subgroup of* G , *then* $S \cap K = \emptyset$. □

LEMMA IV.7.12. *Let* S *be a closed halfspace semigroup in a connected Lie
group* G *such that* $H(\underline{L}(S))$ *does not contain any nontrivial ideal. Then* G
is simply connected.

Proof: Let K be a maximal compact subgroup of G, then Lemma 7.11. shows
that $K \cap$ int $S = K \cap$ (int $S)^{-1} = \emptyset$. Hence $K \subset \partial S$ and thus the group C
generated by all the maximal compact subgroups of G is also contained in ∂S.
But C is a normal analytic group since all the K are connected (cf.[Hoch 65]).
Therefore $\underline{L}(C)$ is an ideal contained in $(H(\underline{L}(S)))$, hence $C = \mathbf{1}$. But G
is topologically isomorphic to $K \times \mathbb{R}^n$, hence to $\mathbb{R}^n$ and thus is simply
connected. □

THEOREM IV.7.13. *Let* S *be a closed halfspace semigroup in a connected
Lie group* G. *Then* ∂S *contains a maximal closed normal subgroup* N_S *and
for* S/N_S *one of the following cases occurs:*

(i) $(G/N_S, S/N_S) \simeq (\mathbb{R}, \mathbb{R}^+)$

(ii) $(G/N_S, S/N_S) \simeq (\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, a > 0, b \in \mathbb{R}\}, \{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, a > 0, b \geq 0\})$

(iii) $(G/N_S, S/N_S) \simeq (SL(2, \mathbb{R}), \Omega^+)$ where Ω^+ is defined as in section 5.

Proof: Since int S is maximal open proper by Corollary 7.10. the existence of N_S is assured by Proposition 7.6. Then Lemma 6.37. determines the Lie algebras of G/N_S and $\partial S/N_S$ so that S/N_S is determinde by Lemma 7.12. But then Remark 7.9. shows that S/N_S is the semigroup generated by a halfspace bounded by $\underline{L}(\partial S/N_S)$. Thus Lemma 7.12. shows that the proof is finished by inspecting the examples in section 5. □

We have seen in section 5 that maximal semigroups need not be halfspace semigroups in general. But for certain classes of Lie groups this is true:

THEOREM IV.7.14. Let S be a maximal open proper subsemigroup of a connected nilpotent Lie group G. Then $\bar{S}$ is a halfspace semigroup containing the commutator subgroup.

Proof: Let (G,S) be a counterexample to the claim, in which G is of minimal dimension with respect to this property. We may assume that the largest normal subgroup N_S that does not intersect S is the identity since $(G/N_S,\ S/N_S)$ is also a counterexample by Proposition 6.6. and the fact that the preimage under a quotient of a halfspace semigroup is clearly a halfspace semigroup. If now G is abelian and T is a maximal torus in G, then $T \quad S = \emptyset$ by Lemma 6.11. But T is normal in G, hence $T = \{\mathbf{1}\}$ and so G is a vectorgroup. But then $S_{\mathbb{R}} = \{ts \in G : t > 0 \text{ and } s \in S\}$ is an open semigroup in G containing S. Therefore $S_{\mathbb{R}}$ is G or S. If $S_{\mathbb{R}} = G$ then for any line M in G the open semigroup $M \cap S$ in M contains elements in both connected components of $M \setminus \{0\}$ and hence is equal to M. Thus S contains the identity contradicting the fact that S is proper. We conclude $S_{\mathbb{R}} = S$ so that S is a wedge, and hence by maximality a halfspace. This contradiction shows that G cannot be abelian.

Let now $L = \underline{L}(G)$ and L^n be the last nonzero element of the descending central series. Then one finds some $x \in L$ and $y \in L^{n-1}$ in L such that $z = [x,y] \neq 0$. Note that z is central in L so that $\exp \mathbb{R}z \cap S \neq \emptyset$ since $N_S = \{\mathbf{1}\}$. Let K be the analytic subgroup of G whose Lie algebra $\underline{L}(K)$ is $\mathbb{R}x + \mathbb{R}y + \mathbb{R}z$. Then $\underline{L}(K)$ is a Heisenberg algebra and, equipped with the Campbell-Hausdorff-multiplication, the universal covering group of K. Since $\exp_{|\underline{L}(K)} : \underline{L}(K) \to K$ is the covering morphism, then $\exp^{-1}(S \cap K)$ is an open subsemigroup of $\underline{L}(K)$ which meets the center of $\underline{L}(K)$. The calculation given in Lemma 5.5. then shows that it must be all of $\underline{L}(K)$.

Hence $\exp^{-1}(S \cap K)$ and thus also $S \cap K$ contains the identity, which is impossible since S is proper. This final contradiction proves the claim that $\underline{L}(S)$ is a halfspace. Since any hyperplane subalgebra in a nilpotent Lie algebra contains the commutator algebra and $\underline{L}(S)$ is a Lie wedge the last assertion follows. □

We want extend Theorem 7.14 to the case where G is the product of a compact subgroup with a nilpotent normal subgroup. To do this we will need the following

LEMMA IV.7.15. _Let_ V _be a finite dimensional vectorspace and_ C _be a compact connected subgroup of_ $\mathrm{Aut}\, V$ _that operates on_ U _without nonzero fixed points. If_ S _is an open subsemigroup of_ $G = V \rtimes C$ _which intersects_ $V \rtimes \{\mathbf{1}\}$ _then_ $S = G$.

Proof: Let $(v,\mathbf{1}) \in V \rtimes \{\mathbf{1}\}$ S then $\bar{v} = \int_C c \cdot v \, d\mu$, with μ Haarmeasure on C, is a ficed point of C in V, hence $\bar{v} = 0$. Now consider the orbit $M = \{c \cdot v : c \in C\}$ of v under the action of C. Since C is compact, M is also compact and therefore the closed convex hull $\overline{\mathrm{conv}}\, M$ of M is just the convex hull $\mathrm{conv}\, M$ of M. Moreover $\bar{v}$ is contained in $\overline{\mathrm{conv}}\, M$, i.e. $0 \in \mathrm{conv}\, M$. Thus there exist $c_1 \ldots c_k \in C$ and $\lambda_1 \ldots \lambda_k \in]0,1]$ with $0 = \sum_{i=1}^{k} \lambda_i c_i \cdot v$. We claim that there is an $r \in \mathbb{R}^+$ such that $r\lambda_i c_i \cdot v \in S$ for all i. If this claim is true we can conclude

$$0 = \sum_{i=1}^{k} r\lambda_i c_i \cdot v \quad \text{and hence} \quad (0,\mathbf{1}) = \prod_{i=1}^{k} (r\lambda_i c_i \cdot v, \mathbf{1}) \in S$$

so that $S = G$.

To prove the claim it suffices to prove the following two statements:

(i) For all $(x,\mathbf{1}) \in S$ there $r_x \in \mathbb{R}^+$ with $(rx,\mathbf{1}) \in S$ for all $r > r_x$.

(ii) For all $c \in C$ there is an $m_c \in \mathbb{N}$ with $(m_c\, c \cdot v, \mathbf{1}) \in S$.

In fact if (i) and (ii) hold the claim is true for

$$r = (\max\{r_{c_i \cdot v}\})(\min\{\lambda_i\})^{-1} \prod_{i=1}^{k} m_{c_i}\ .$$

To prove (i) we only need to remark that the complement of an open subsemigroup of $(\mathbb{R}^+,+)$ is always bounded. In order to prove (ii) we note that $(V \rtimes \{\mathbf{1}\}) \cap S \neq \emptyset$ implies $(V \rtimes \{\mathbf{1}\})S = G$ so we can find elements v_c and $v_{c^{-1}} \in V$

with $(v_c,c) \in S$ and $(v_{c^{-1}},c^{-1}) \in S$ hence also $(v_c,c)(v_{c^{-1}},c^{-1}) = (v_{c^{-1}} + c^{-1}\cdot v_c, \mathbf{1}) \in S$. Since S is open there exists an $m_c \in \mathbb{N}$ such that $s_o := (v - \frac{1}{m_c}(v_{c^{-1}} + c^{-1}\cdot v_c), \mathbf{1}) \in S$.
Thus $s_o^{m_c} = (mv - v_{c^{-1}} + c^{-1}\cdot v_c, \mathbf{1}) \in S$ and finally $(v_c,c)\, s_o^{m_c}(v_{c^{-1}},c^{-1}) =$
$= (m_c c\cdot v, \mathbf{1}) \in S$.

□

We are now ready to describe the tangent wedge of maximal open subsemigroups in semidirect products of compact groups and vector groups.

PROPOSITION IV.7.16. *Let V be a finite dimensional vector space and C a compact connected group of automorphisms of V. If S is a maximal open subsemigroup of $G = V \rtimes C$ then the tangent wedge $\underline{L}(S)$ of S is a halfspace in $\underline{L}(G)$ bounded by an ideal.*

Proof: Let G be a counterexample of minimal dimension. Then for any C-invariant subspace I of V we have $(I \times \{\mathbf{1}\}) \cap S \neq \emptyset$. In fact $N = I \times \{\mathbf{1}\}$ is a closed normal subgroup of G hence $N \cap S = \emptyset$ implies $SN = NS = S$ by Proposition 7.6. Moreover the subsemigroup S/N is maximal open in G/N. Since G was a counter‿example of minimal dimension $\underline{L}(S/N)$ is a halfspace bounded by an ideal in $\underline{L}(G/N) = \underline{L}(G)/\underline{L}(N)$. But Proposition 4.1 implies that $\underline{L}(S) = \pi^{-1}(\underline{L}(S/N))$ where $\pi : \underline{L}(G) \to \underline{L}(G/N)$ is the canonical projection. Thus $\underline{L}(S)$ is a halfspace bounded by an ideal in $\underline{L}(G)$ contradicting our hypotheses. Now consider $G_1 = [\underline{L}(G),V] \times C$ and note that the "Fitting Decomposition of motion algebras given in Chapter II shows that C operates without nonzero fixed points on $[\underline{L}(G),V]$. Moreover $S_1 = G_1 \cap S$ is an open proper semigroup that intersects $[\underline{L}(G),V] \times \{\mathbf{1}\}$ by the above, so that Lemma 7.15 applies and we see that $(0,\mathbf{1}) \in S_1 \subset S$ whence $S = G$. This final contradiction to our assumptions proves the propositions.

□

More generally we obtain

PROPOSITION IV.7.17. *Let G be a finite dimensional Lie group, C a connected compact subgroup of G and A an abelian analytic normal subgroup of G such that $G = C\bar{A}$. If S is a maximal open subsemigroup of G then $\underline{L}(S)$ is a halfspace bounded by an ideal.*

Proof: Let $\bar{A} = TV$ where T is the maximal torus in $\bar{A}$ and V is a vector group. Note that T is characteristic in $\bar{A}$ so it is normal in G. Moreover $S \cap T = \emptyset$ since otherwise $S \supset T$ which would imply $S = G$. Thus $ST = S$ and S/T is a maximal open subsemigroup of G/T by Proposition 7.6. But $G/T = (C\bar{A})/T \simeq ((CT)/T)(\bar{A}/T)$ with $CT/T \simeq C/_{C \cap \bar{A}}$ and $\bar{A}/T \simeq V$. Therefore $(CT/T) \cap (\bar{A}/T) = \{T\}$ and we may apply Proposition 7.16 to G/T and S/T. Thus $\underline{L}(S/T)$ is a halfspace bounded by an ideal and as before Proposition 4.1 shows that $\underline{L}(S)$ is a halfspace bounded by an ideal. □

Finally we obtain

THEOREM IV.7.18. Let G be a connected finite dimensional Lie group, C a compact subgroup of G and N a nilpotent analytic normal subgroup of G such that $G = C\bar{N}$. If S is a maximal open subsemigroup of G then $\underline{L}(S)$ is a halfspace bounded by an ideal in $\underline{L}(G)$.

Proof: Note first that we may assume that C is connected. In fact if C_m is a maximal compact subgroup of G containing C then C_m is connected and $G = C_m\bar{N}$. Thus we can replace C by C_m. Moreover we may assume that N is closed.

Now consider the commutatorgroup N' of N. If $S \cap N' \neq \emptyset$ then $S \cap N$ is contained in some maximal open subsemigroup S_N of N. But then by 7.14 we have that $\underline{L}(S_N)$ is a halfspace bounded by an ideal which must then contain $L(N') = [L(N),L(N)]$. Hence $\exp L(N') \subset \bar{S}$ so that $N' \subset \bar{S}$. Therefore $SN' = N'S = S$ since $\bar{S}S,\ S\bar{S} \subset S$ and the identity cannot be in S. Thus Proposition 7.6 shows that $S \cap N' = \emptyset$. Since S is open we have also $H \cap S = \emptyset$ where H is the closure of N' in G. Again we apply Proposition 7.6 to see that $SH/H = S/H$ is a maximal open subsemigroup of G/H. Note that $G/H \simeq (CH/H)(N/H)$ where CH/H is compact and N/H is abelian. Hence Proposition 7.17 shows that $\underline{L}(S/H)$ is a halfspace bounded by an ideal and consequently $\underline{L}(S)$ is a halfspace bounded by an ideal in $\underline{L}(G)$. □

It now only remains to translate the information we have on the tangent wedges of maximal open semigroup into information on the semigroups themselves:

REMARK IV.7.19. <u>Let G be a connected Lie group then for a non empty open subsemigroup S of G the following statements are equivalent:</u>

(1) <u>$\underline{L}(S)$ is a halfspace bounded by an ideal.</u>

(2) $S = \varphi^{-1}(\mathbb{R}^+ \setminus \{0\})$ <u>for some continuous homomorphism</u> $\varphi : G \to \mathbb{R}$.

<u>These properties imply that S is maximal open.</u>

Proof: (1) => (2). Let I be the ideal that bounds $\underline{L}(S)$ and N be the subgroup generated by $\exp I$ where $\exp : \underline{L}(G) \to G$ is the exponential function. Then $N \cap S = \emptyset$ hence $\bar{N} \cap S = \emptyset$ so that $\dim N \leq \dim \bar{N} < \dim S$ and thus N is closed. Moreover Proposition 7.6 implies $SN = NS = S$ or, in other words, $S = \pi^{-1}(SN/N)$ where $\pi : G \to G/N$ is the quotient map. Therefore $G/N \simeq \mathbb{R}$ since there are no proper open subsemigroups in the torus. Moreover Proposition 4.1 shows that $\underline{L}(S) = (\underline{L}(\pi))^{-1}\ \underline{L}(S/N)$. Thus $\underline{L}(S/N)$ is a halfline and the claim is proved.

(2) => (1). Conversely since $S \neq \emptyset$ the map φ is a quotient map and hence Proposition 4.1 implies that $\underline{L}(S)$ is a halfspace bounded by $\underline{L}(S)$ is a half-space bounded by $\underline{L}(\ker\varphi)$. Note finally that any open subsemigroup T, which contains $S = \varphi^{-1}(\mathbb{R}^+ \setminus \{0\})$ strictly, must satisfy $\varphi(T) = \mathbb{R}$ since any open subsemigroup of $\mathbb{R}$ containing positive and negative elements must be all of $\mathbb{R}$. By the first part of the proof we may assume that $\ker\varphi$ is connected and contained in $\bar{S}$ hence in $\bar{T}$. Thus $\bar{T} = \varphi^{-1}(\varphi(\bar{T})) = G$ and T is dense open in G. But then T^{-1} is open dense and also $T \cap T^{-1}$ is open dense, hence $T \cap T^{-1} = G$ since $T \cap T^{-1}$ is a group. Thus $T = G$. $\square$

In the local theory of semigroups nilpotent and complex Lie algebras had in common the striking fact that they admit only trivial semialgebras. It is too much to expect that Theorem 7.14 will hold also for complex groups in general. It does however hold for complex solvable groups. We start with a lemma which is similar to Lemma 7.15 and also plays a similar role in the context of complex groups:

LEMMA IV.7.20. <u>Let G be a Lie group acting linearly on a finite dimensional vector space V. If G contains an element $g \in G$ which leaves no ray in V fixed, then no open subsemigroup S of the semidirect product $G \rtimes V$ intersects</u> $\{(\mathbf{1},v) : v \in V\}$ <u>where $\mathbf{1}$ is the identity of G.</u>

Proof: We may assume without loss of generality that S is maximal open proper. Now suppose that the set $S_V = \{v \in V : (\mathbf{1},v) \in S\}$ is non-empty. Then $\mathbb{R}^+S_V$ is a convex set. In fact, since S is an open semigroup in $G \ltimes V$ we know that S_V is an open semigroup in V. But this implies that for any $v \in S_V$ there is an $r(v) > 0$ such that $rv \in S_V$ for all $r > r(v)$. If now $v_1, v_2 \in S_V$ and $r_1, r_2 > 0$ then

$$r_1v_1 + r_2v_2 = \frac{m}{n}\left(\frac{n}{m} r_1v_1 + \frac{n}{m} r_2v_2\right) \in \mathbb{R}^+S_V$$

if we set $m = \min(r_1,r_2)$ and $n = \max(r(v_1),r(v_2))$, so that $\frac{n}{m} r_i > r(v_i)$ for $i = 1,2$.

Note that $\mathbb{R}^+S_V$ is contained in some halfspace since S cannot contain the identity $(\mathbf{1},0)$ of $G \ltimes V$. But then the quotient space $S_V/\sim$, where $v_1 \sim v_2$ if $\mathbb{R}^+v_1 = \mathbb{R}^+v_2$, is homeomorphic to an open disk. If now g is an element of G which satisfies $g \cdot (\mathbb{R}^+S_V) \subset \mathbb{R}^+S_V$ then Brouwer's Fix-Point-Theorem shows that there is a $v \in \overline{S_V}$ such that $g \cdot v \subset \mathbb{R}^+v$.
Thus, in order to finish the proof, it suffices to show that for any $g \in G$ and any $v \in S_V$ we have $g \cdot v \subset \mathbb{R}^+S_V$. To do this, note first that $\{\mathbf{1}\} \times V$ is a closed normal subgroup of $G \ltimes V$ so that $S \cap (\{\mathbf{1}\} \ltimes V) \neq \emptyset$ implies that $(\{\mathbf{1}\} \ltimes V)S = G$ by Proposition 7.6 and hence there are elements v_g and $v_{g^{-1}}$ in V such that (g,v_g) and $(g^{-1},v_{g^{-1}})$ are in S. Therefore also $(g,v_g)(g^{-1},v_{g^{-1}}) = (\mathbf{1},v_{g^{-1}} + g^{-1} \cdot v_g) \in S$ which shows that there exists a $k \in \mathbb{N}$ with $s_o = (\mathbf{1},v - \frac{1}{k}(v_{g^{-1}} + g^{-1} \cdot v_g)) \in S$ since S is open. But then $s_o^n = (\mathbf{1},kv + (v_{g^{-1}} + g^{-1} \cdot v_g)) \in S$ so that $(\mathbf{1},kg \cdot v) = (g,v_g)(g^{-1},kv - g^{-1} \cdot v_g) = (g,v_g)s_o^n(g^{-1},v_{g^{-1}}) \in S$. Thus $kg \cdot v \in S_V$ or, in other words, g leaves the set S_V fixed since it operates linearly. □

Note that an analytic group is called <u>weakly exponential</u> (cf. [HMu 78]) if the image of its exponential function is dense.

LEMMA IV.7.21. <u>Let G be a connected Lie group with a weakly exponential subgroup H and S be an open subsemigroup of G with $S \cap H \neq \emptyset$. If $\Pi : \tilde{G} \to G$ is the universal covering morphism of G and $\tilde{H}$ is the analytic subgroup of $\tilde{G}$ corresponding to H, then $\Pi^{-1}(S) \cap \tilde{H} \neq \emptyset$.</u>

Proof: Since $S \cap H \neq \emptyset$ and $\exp L(H)$ is dense in H, where $L(H)$ is the Lie algebra of H' and $\exp : L(G) \to G$ so the exponential function, we also $S \cap \exp(L(H)) \neq \emptyset$. If now $\mathrm{Exp} : L(G) \to \tilde{G}$ is the exponential function of $\tilde{G}$ we have $S \quad \exp L(H) = S \cap (\Pi \circ \mathrm{Exp})(L(H))$ and therefore $\emptyset \neq \Pi^{-1}(S) \cap \mathrm{Exp}\, L(H) \subset \Pi^{-1}(S) \cap \tilde{H}$. □

We are now ready to prove the analogue of Theorem 7.14 for complex solvable groups.

THEOREM IV.7.22. Let G be a connected complex solvable Lie group and S be a maximal open proper semigroup in G then $\bar{S}$ is a halfspace semigroup containing the commutator subgroup G' of G.

Proof: It suffices to show that $G' \cap S = \emptyset$. In fact $G/(G')^-$ is an abelian Lie group and Proposition 7.6 then implies that $S(G')^- = S$ and $S/_{(G')^-}$ is a maximal open proper semigroup in $G/(G')^-$ since $(G')^- \cap S = \emptyset$ follows trivially from $G' \cap S = \emptyset$. But then $S/(G')^-$ is a halfspace semigroup by 7.14 and hence S is a halfspace semigroup with $\underline{L}(S) \supset \underline{L}(G')$ so that $G' \subset S$.

Thus let G be a counterexample of minimal dimension to the claim $G' \cap S = \emptyset$. Since G is solvable, $\underline{L}(G)$ is solvable. Therefore the derived series $\underline{L}(G)^{(n)}$ decreases to zero. Let A be the abelian analytic subgroup generated by the last non-vanishing $\underline{L}(G)^{(n)}$. Then $S \cap A \neq \emptyset$. In fact, if $S \cap A = \emptyset$ then $S \cap \bar{A} = \emptyset$ since S is open, and Proposition 7.6 hence implies that $S/\bar{A}$ is a maximal open proper subsemigroup of $G/\bar{A}$. Now we conclude that $S/\bar{A} \cap G'/A^- = \emptyset$ by the minimality of the counterexample, since $(G/\bar{A})' = G'/A$. But then also $S\bar{A} \cap G'\bar{A} = \emptyset$ and consequently $S \cap G' = \emptyset$ contradicting our assumptions. Thus we have indeed $S \cap A \neq \emptyset$.

Next we claim that A must be central. For, if it is not we find an $x \in L$ such that Lie group B generated by $\exp \mathbb{C}x$ and $\bar{A}$ is non abelian. Since $S \cap \bar{A} \neq \emptyset$ we have a fortiori $S \cap B \neq \emptyset$. But $\bar{A}$ is a connected abelian Lie group hence it is exponential so that Lemma 7.21 applies. Since the universal covering group of B is just the semidirect product $\mathbb{C} \ltimes L(\bar{A})$ where action of $\mathbb{C} = \mathbb{C}x$ on $L(\bar{A})$ is given by $cx \cdot v = e^{\mathrm{ad}\, cx} v$. This group satisfies the hypotheses of Lemma 7.20 and we can conclude that $S \cap \bar{A}$ must be empty. This contradiction now shows that A is central in G.

Note also that G must be metabelian. Hence $G'' = (\overline{G'})'$ does not intersect $S \cap (G')^-$ which is , by Proposition 7.3, contained in a maximal open proper subsemigroup S_1 of $(G')^-$. Since S is open we also have $(G'')^- \cap S = \emptyset$. Thus, again by Proposition 7.6, and the minimality of the counterexample we find $G'/(G'')^- \cap (S_1/_{(G'')^-}) = \emptyset$ and hence $G' \cap S = \emptyset$ contradicting our assumptions.

Finally we observe that $A = G'$ by the above so that G is nilpotent since we have shown that A is central. But then G cannot be a counterexample by Theorem 7.14 and the proof is finished. □

To conclude this section we describe how to apply results like 7.18 and 7.22 to control systems on Lie groups.

Here a control system will be simply a family F of left invariant vectorfields on G and the control system will be called controllable if for any point x in G we can find an integral curve for F connecting the identity with x.

It is well known (cf [JS 72]) that the system described by F is controllable if and only if the system given by the wedge $W = \overline{\mathrm{conv}}(\mathbb{R}^+F)$ is controllable. Therefore the question of controllability reduces to the question whether the semigroup S_W generated by $\exp W$ in G is all of G or not (cf also [BJKS 82]).

REMARK IV.7.23. *Let G be a connected Lie group such that for any maximal open subsemigroup S we have $\underline{L}(S)$ is a halfspace bounded by an ideal in $\underline{L}(G)$. If W is a wedge in $\underline{L}(G)$ which generates $\underline{L}(G)$ as a Lie algebra, then the following statements are equivalent:*

(1) *The semigroup S_w generated by $\exp W$ in G is not equal to G (hence not even dense).*

(2) *There exists a continuous non trivial homomorphism $\varphi : G \to \mathbb{R}$ such that $\underline{L}(\varphi)(W) \in \mathbb{R}^+$.*

Proof: (1) => (2) If $S_w \neq G$ then by the above $\mathring{S}_w$ is contained in a maximal open subsemigroup S of G. By hypothesis $\underline{L}(S)$ is a halfspace bounded by an ideal so that Rematk 7.19 shows the existence of $\varphi : G \to \mathbb{R}$ such that $S_w \subset S = \varphi^{-1}(\mathbb{R}^+ \setminus \{0\})$ and hence $W \subset \underline{L}(S_w) \subset \underline{L}(S) = (\underline{L}(\varphi))^{-1}(\mathbb{R}^+)$ by Proposition 4.1 .

(2) => (1) Conversely let $S := \varphi^{-1}(\mathbb{R}^+ \setminus \{0\})$ then again $\underline{L}(S) = (\underline{L}(\varphi))^{-1}(\mathbb{R}^+)$ and therefore $W \subset \underline{L}(S)$. Thus $\exp W \subset \exp \underline{L}(S) \subset \bar{S}$ which is a subsemgroup of G, strictly contained in G. □

Recall that $\underline{L}(\varphi) : \underline{L}(G) \to \mathbb{R}$ is a Lie algebra morphism and since $\mathbb{R}$ is abelian we know that $\underline{L}(G)' = [\underline{L}(G), \underline{L}(G)]$ is contained in the kernel of $\underline{L}(\varphi)$. Hence the relative interior $\mathrm{int}_{W-W} W$ in the vector space $W-W$ cannot inter_sect $\underline{L}(G)'$ unless $\underline{L}(\varphi)(W)$ contains positive and negative values or is completely contained in $\underline{L}(G)'$. Moreover if C is a maximal compact subgroup of G then $C \subset \ker \varphi$ since $\varphi(C)$ is a compact subgroup of $\mathbb{R}$. Thus in the situation of Remark 7.23 we have $\underline{L}(C) + \underline{L}(G)' \subset \ker \underline{L}(\varphi)$. Even more is true:

LEMMA IV.7.24 Let G be a connected Lie group, C a maximal compact subgroup of G and W a wedge in $\underline{L}(G)$ which generates $\underline{L}(G)$ as a Lie algebra. Then the following statements are equivalent.

(1) $\mathrm{int}_{W-W} W \cap (L(C) + L(G)') = \emptyset$

(2) There exists a continuous non trivial homomorphism $\varphi : G \to \mathbb{R}$ such that $\underline{L}(\varphi)(W) \subset \mathbb{R}^+$.

Proof: (2) => (1) We know that $L(C) + L(G)' \subset \ker \underline{L}(\varphi)$. Moreover W is not contained in $\ker \underline{L}(\varphi)$ since it generates $L(G)$.
If now $\mathrm{int}_{W-W} W \cap \ker \underline{L}(\varphi) = \emptyset$, then $\underline{L}(\varphi)(W)$ contains positive and negative values contradicting our hypothesis. Thus $\mathrm{int}_W W \cap (L(C) + L(G)') = \emptyset$.

(1) => (2) To show the converse note first that the analytic subgroup A of G with $L(A) = L(C) + L(G)'$ is normal and contains C since G, hence C, is connected. Therefore A contains all compact subgroups of G and hence A is closed (cf[Hoch.65]Ch. XVI). Moreover the quotient group G/A is a vectorgroup. In fact, since G/A is abelian connected it is isomorphic to $T \times V$ where T is a torus and V is a vectorgroup. If $\pi : G \to T \times V$ is the quotient map with kernel A and B is the identity component of $\pi^{-1}(V)$ then B is a closed connected normal subgroup of G and $\pi(B) = V$ since π was a quotient map. Thus G/B is compact and [Hoch.65] implies that $CB = G$ so that $G = CB \subset AB \subset B \subset G$. But this just means that $T = \{0\}$. Thus we may identify G/A with $L(G/A) = L(G)/L(A)$.

But now condition (1) implies that W is contained in a halfspace H with $L(A) \subset H$ so that we can find a linear functional $\bar{\varphi} : L(G)/L(A) \to \mathbb{R}$ with $H = \ker \bar{\varphi}$ and $\bar{\varphi}(W + L(A)/L(A)) \subset \mathbb{R}^+$. But then $\varphi = \bar{\varphi} \circ \pi : G \to \mathbb{R}$ is the desired homomorphism if we identify G/A and $L(G)/L(A)$. □

We can now summarize our results to

THEOREM IV.7.25. Let G be a connected Lie group such that for any maximal open semigroup S we have that $L(S)$ is a halfspace bounded by an ideal in $L(G)$. Moreover let C be any maximal compact subgroup of G. If W is a wedge in $L(G)$ which generates $L(G)$ as a Lie algebra, then the following statements are equivalent:

(1) $\mathrm{Int}_{w-w}(W) \cap (L(C) + [L(G), L(G)]) \neq \emptyset$

(2) $\exp W$ generates G as a semigroup.

(3) The system described by W is controllable. □

Section 8: Divisibility and local divisibility

A semigroup S is called *divisible* provided it satisfies the following condition:

For each $s \in S$ and each natural number n there is an $g \in S$ such that $g^n = s$.

We know from the local theory that for a local semigroup (S,U) in a Lie group the analogous statement holds if and only if $L(S)$ is a semialgebra (cf. chapter III). The example of $\tilde{S}l(2,\mathbb{R})$ shows that every semigroup generated by a semialgebra needs to be divisible. On the other hand we don't know any example of a divisible subsemigroup of a Lie group whose tangent wedge is not a semialgebra. It is the aim of this section to show that such an example cannot exist at least for semigroups with trivial group of units.

LEMMA IV.8.1. Let G be an abelian Lie group and $D \subseteq G$ be a divisible subgroup. Then $\bar{D}$ is divisible.

Proof: We may assume that D is dense in G, i.e. $\bar{D} = G$. Let now $\alpha_n : G \to G$ be the morphism defined by $\alpha_n(g) = ng$ and set $G_n = \ker \alpha_n$. Note that G is of the form $K \times \mathbb{R}^n \times \Delta$ where K is compact connected and Δ is discrete. Since any subgroup of G is invariant under α_n and $\alpha_n(D) = D$ is dense we have that $\alpha_n(K)$ is dense in K. But since K is compact this shows that $\alpha_n(K) = K$. On the other hand clearly $\alpha_n(\mathbb{R}^n) = \mathbb{R}^n$ and $\alpha_n(\Delta) = \Delta$ since $K \times \mathbb{R}^n \times \{\delta\}$ is open for any $\delta \in \Delta$. Thus $\alpha_n(G) = G$ and G is divisible. □

Note that using successive roots one observes that the divisibility of a semigroup S is equivalent to the existence of an algebraic homomorphism $f_s : \mathbb{Q}^+ \to S$ with $f_s(1) = s$ for any $s \in S$, where $\mathbb{Q}^+$ are the strictly positive rationals. We then find

PROPOSITION IV.8.2. Any closed divisible subgroup H in a connected Lie group G is connected.

Proof: Let $g \in H$. Then by the above remark we find an algebraic homomorphism $f_g : \mathbb{Q} \to H$ with $f_g(1) = g$. We set $A = \overline{f_g(\mathbb{Q})}$ then A is a closed abelian Lie subgroup of G with a dense divisible subgroup. Hence Lemma 8.1 shows that A is divisible. If A_o is the identity component in A, then [Mo 57] implies that A/A_o has finite rank. But A/A_o is also divisible so that it follows from the fundamental theorem of finitely generated abelian groups that $A = A_o$. Since H was closed and hence $A \subset H$ this implies that g is in the identity component of H. But g was chosen freely in H so that H is connected.

□

THEOREM IV.8.3. A closed subsemigroup S of a connected Lie group G is divisible if and only if $S = \exp L(S)$.

Proof: If $S = \exp \underline{L}(S)$, then S is clearly divisible, since $X \in \underline{L}(S)$ iff $\exp tX \in S$ for all $t \geq 0$

Now assume that S is divisible, and let $s \in S$. By using the divisibility of S we find an algebraic homomorphism $f^+ : \mathbb{Q}^+ \to S$ from the additive semigroup of positive rationals into S such that $f^+(1) = s$. This homomorphism extends to a unique group homomorphism $f : \mathbb{Q} \to G$. The group $A = f(\mathbb{Q})^-$ is divisible, hence a connected closed Lie subgroup of G by Proposition 8.2. Since it is abelian, it is of the form $K \times \mathbb{R}^n$ with a torus K. If $p : A \to \mathbb{R}^n$ is the projection, then $p \circ f : \mathbb{Q} \to \mathbb{R}^n$ is a group homomorphism with dense image; since it is clearly a $\mathbb{Q}$-vector space homomorphism we draw the conclusion $n = 0$ or $n = 1$. If $n = 0$, then A is a torus. Then $f(\mathbb{Q}^+)^-$ is a compact subsemigroup of the compact group A and is, therefore itself a group ([HM 66]). The definition of A implies $A = f(\mathbb{Q}^+)^- \subseteq S$. Since A is a torus, $s = f(1)$ lies on a one-parameter group of A which is contained in S. Hence $s = \exp X$ for some $X \in L(S)$. This settles the case $n = 0$.

Now we assume that $A = K \times \mathbb{R}$ and that $p \circ f(\mathbb{Q}^+)$ is dense in $\mathbb{R}^+$; we may indeed assume that $p \circ f(1) = 1$, whence in fact $p \circ f(q) = q$ for all $q \in \mathbb{Q}^+$. We now apply the techniques of the one-parameter semigroup theorem (cf.[Ho 60], [HM 66], [He 77]). Define $C = \bigcap\{f(\{q \in \mathbb{Q}^+ : q < t\})^- : 0 < t\}$. Then $C \subseteq S$, and the sets $f(\{q \in \mathbb{Q}^+ : q < t\})^-$ are contained in the compact set $p^{-1}([0,t])$ and are therefore compact. Hence C is a compact divisible abelian group (see e.g. [He 77]). (In fact $C = K$, but we do not need this here.) Thus C is a torus. There is a continuous one parameter semigroup $F : [0,\infty) \to f(\mathbb{Q}^+)^-$ with $p \circ F(r) = r$ for all $r > 0$, and $F(1) = cf(1) = cs$ for some $c \in C$. Let $g : \mathbb{R} \to C$ be a one parameter group in C with $g(1) = c$. Then $t \to g(t)^{-1}F(t) : [0,\infty[\to f(\mathbb{Q}^+)^- \subseteq S$ is a continuous one parameter semigroup mapping 1 to $g(1)^{-1}F(1) = c^{-1}cs = s$. Once again we have found an $X \in \underline{L}(S)$ with $\exp X = s$. □

This completes the proof.

THEOREM IV.8.4. *Let G be a Lie group and S be a closed divisible subsemigroup of G with trivial unit group, then $\underline{L}(S)$ is a semialgebra.*

Proof: Let U be a neighborhood of $\mathbf{1}$ in G satisfying the properties in the conclusion of Lemma 3.6, then $S \setminus (U \cap S)$ is a right semigroup ideal in S since $S \cap -S = \{\mathbf{1}\}$. But $S = \exp(\underline{L}(S))$ by Theorem 8.3. If now $s = \exp x \in U$ then $\exp[0,1]x \in U$ since $\exp tx \in S$ for all $t \geq 0$. Thus the local semigroup $(S \cap U, U)$ is divisible (cf. chapter III) and hence $\underline{L}(S)$ is a semialgebra. □

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