FOUNDATIONS OF K-THEORY FOR C*-ALGEBRAS

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TABLE OF CONTENTS

INTRODUCTION

Let X be a compact space and Y a closed subset of X. For M_k, the complex $k \times k$-matrices, consider the C*-algebra of continuous functions $f : X \to M_k$ with the property that $f(x)$ is a diagonal matrix for all $x \in Y$. We shall study the K-theory of this C*-algebra and some closely related C*-algebras for various spaces X and Y. The tools used in this study are a Mayer-Vietoris Sequence and a Puppe Sequence for K-theory of C*-algebras, both of which reduce to the respective sequence in K-theory of locally compact spaces if the involved C*-algebras are commutative.

First we set up K-theory of unital C*-algebras, following the approach of Karoubi. We define relative K-groups $K_\alpha(\phi)$ for unital C*-morphisms ϕ and prove two excision theorems, which will allow us to define K-theory of non-unital C*-algebras. Moreover, we show that the K-functors do not distinguish between homotopic C*-morphisms. This will enable us to define K_n of a C*-algebra for all $n \in \mathbb{N}$ and to establish a long exact sequence in K-theory associated to a short exact sequence of C*-algebras. We also define a cup product in K-theory of C*-algebras, which will be a $\mathbb{Z}_2$-graded bilinear map $K_*(A) \times K_*(B) \to K_*(A \,\overline{\otimes}\, B)$, give some of its basic properties, and use it to define module structures on the K-groups.

Finally we prove a non-commutative splitting principle which generalizes the well known splitting ~~principle~~ principle for vector bundles over compact spaces.

CHAPTER I: K-THEORY FOR UNITAL C*-ALGEBRAS

This chapter is devoted largely to the introduction of notations and terminology which will be used throughout. We also give the definitions and establish some basic properties of the K-groups for unital C*-algebras.

I.1. Definition (cf. [K]: II.2.1). Let C be an additive category. Let $C(E,F)$ denote the set of C-morphisms $E \to F$. A Banach structure on C is given by a completely normable topological vector space structure (over $\mathbb{C}$) on all $C(E,F)$ such that the composition of morphisms $C(E,F) \times C(F,G) \to C(E,G)$ is bilinear and continuous. A Banach category is an additive category provided with a Banach structure.

I.2. Definition (cf. [K]: II.2.1). Let C and C' be additive categories and $\phi : C \to C'$ an additive functor. Then ϕ is called quasi-surjective if every object of C' is a direct factor of an object isomorphic to an object of the form $\phi(E)$ with $E \in Ob(C)$.

ϕ is called full if $C(E,F) \to C'(\phi(E),\phi(F))$ is surjective. If C and C' are Banach categories, the functor ϕ is called a Banach functor if $C(E,F) \to C'(\phi(E),\phi(F))$ is linear and continuous.

I.3. Lemma (cf. [K]: II.2.9). Let $P(B)$ be the category of finitely generated projective (left) modules over B and module maps. Then $P(B)$ is a Banach category.

Proof: The proof is done in several steps. First we consider an object $E \in Ob(P(B))$ and endow it with a completely normable topological vector space structure. If $E \in Ob(P(B))$ is free, we can give it the product norm of B^n. If $E \in Ob(P(B))$ is not free, then there is a projection of B-modules $p : B^n \to E$ onto E. Equip E with the quotient topology. Then E is complete.

Next, we show that this topology does not depend on the particular choice of p. Let $q : B^m \to E$ be another B-module projection onto E, then we get the following commutative diagrams

$$\begin{tikzcd} & B^n \arrow[dl, "u"'] \arrow[d, "p"] \\ B^m \arrow[r, "q"'] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B^n \arrow[d, "q"] \\ B^m \arrow[ur, "v"] \arrow[r, "q"'] & E \end{tikzcd}$$

where the existence of the module maps u and v follows from the projectivity of E. Now u and v are automatically continuous, since they are implemented by $m \times n$, respectively, $n \times m$ matrices with entries in B in the usual way. Thus, the two quotient topologies agree.

We call the topology on E the canonical topology. The canonical topology on E is actually the same as the induced topology given by the injection $j : E \to B^n$ which inverts the projection p on the right, i.e., satisfies $p \circ j = 1_E$. Indeed, consider $f := j \circ p : B^n \to B^n$, then f is a module homomorphism, hence is continuous. Therefore, j is continuous, too, since E carries the quotient topology with respect to p. Clearly, p is continuous by definition, and $p \circ j = 1_E$. Define $g : j(E) \to E$ by $g(x) = p(x)$. If $U \subset E$ is open, then $p^{-1}(U)$ is open in B^n and $g^{-1}(U) = p^{-1}(U) \cap j(E)$. Hence $g^{-1}(U)$ is open in $j(E)$ with respect to the subspace topology, thus g is continuous. Moreover, if $j' : E \to j(E)$ denotes the corestriction of j, we have $g \circ j' = 1_E$ and $j' \circ g = 1_{j(E)}$. This implies that j' is a homeomorphism and thus j is an embedding, which proves our claim.

The next step is to equip $P(B)(E,F) = \mathrm{Hom}_B(E,F)$ with the structure of a completely normable vector space. The topology of uniform convergence on bounded sets turns the vector space $L(E,F)$ of all continuous linear operators $E \to F$ with respect to the canonical topologies of E and F into a complete topological vector space. This topology is compatible with the operator norm $\|f\| = \sup_{\|m\|_E \le 1} \|f(m)\|_F$ for any pair of norms on E and F compatible with the canonical topologies on E, F. We show that $P(B)(E,F)$ is a closed vector subspace of $L(E,F)$:

It is clear that $\mathrm{Hom}_B(E,F)$ is a vector subspace of $L(E,F)$. Pick norms on E and F, which are compatible with the canonical topologies. Endow $L(E,F)$ with the corresponding operator norm. To prove that $\mathrm{Hom}_B(E,F)$ is closed in $L(E,F)$, it is enough to show that if $f_i \in \mathrm{Hom}_B(E,F)$ converges to f in $L(E,F)$, f has to be in $\mathrm{Hom}_B(E,F)$, i.e., $f(bm) = bf(m)$ $b \in B$, $m \in E$. But

$\|f(bm) - bf(m)\|_F \le \|f(bm) - f_i(bm)\|_F + \|f_i(bm) - bf_i(m)\|_F$
$+ \|bf_i(m) - bf(m)\|_F \le \|f(bm) - f_i(bm)\|_F +$
$\|b\|_B \|f_i(m) - f(m)\|_F$. Now uniform continuity proves that $\|f(bm) - bf(m)\| = 0$, i.e., $f(bm) = bf(m)$. To complete the proof of the lemma we have to show that the composition of morphisms $P(B)(E,F) \times P(B)(F,G) \to P(B)(E,G)$ is bilinear and continuous. But this is clear since the composition of linear operators $L(E,F) \times L(F,G) \to L(E,G)$ is bilinear and continuous with respect to the topologies of uniform convergence on bounded sets. □

Note that there is no canonical norm on an $E \in \mathrm{Ob}(P(B))$, so we don't ask for a Banach space structure on E, as it might seem natural. This problem does not occur in [K] II.2.9, because Karoubi gives no proof.

<u>I.4 Lemma</u> (cf. [K]: II.2.9). Let B and A be unital C^*-algebras. Let $\phi : B \to A$ be a unital C^*-homomorphism. Consider A as a right B-module via $a \cdot b = a\phi(b)$ and let $A \otimes_B E$ be the algebraic tensor product of the right

B-module A and the left B-module E. Then $A \otimes_B E$ is a left A-module with $a \cdot (a' \otimes x) = aa' \otimes x$. Then the assignment $\phi_* : P(B) \to P(A)$ defined by $\phi_*(E) = A \otimes_B E$ on objects and by $\phi_*(f) = id_A \otimes_B f$ on morphisms (cf. [M], §2 for these definitions) is a quasisurjective Banach functor. Further, ϕ_* is full iff ϕ is surjective.

Proof: It is easy to see that ϕ_* is a functor. Moreover, is ϕ_* clearly quasisurjective, since $\phi_*(B^n) = A^n$. Now let $f : E \to F$ be a morphism in $P(B)(E,F)$. We first consider the case $E = B^n$, $F = B^m$. Then $f : B^n \to B^m$ may be identified with that $m \times n$-matrix $C = (c_{nk})_{\substack{j=1\ldots m\\ k=1\ldots n}}$ with entries in B for which $pr_j f(x) = \sum_{k=1}^{n} c_{jk} p_k^r(x)$, $j = 1,\ldots,m$. Then $\phi_*(E) = A \otimes_B B^n = (A \otimes_B B)^n$ may be canonically identified with A^n via the isomorphism $a \longmapsto a \otimes 1 : A \to A \otimes_B B$. Likewise, we identify $\phi_* F$ with A^m. It is readily observed that the matrix associated with $\phi_*(f) : \phi_* E \to \phi_* F$ after identifying $\phi_*(E)$ with A^n and $\phi_*(F)$ with A^m is $\phi^{\#}(C) := (\phi(c_{jk}))_{\substack{j=1\ldots m\\ k=1\ldots n}}$. This shows the continuity and linearity of $\phi_* : P(B)(E,F) \to P(A)(\phi_* E, \phi_* F)$. Now, let $E,F \in Ob(P(B))$ be arbitrary. Select projections $p_E : B^n \to E$, $p_F : B^m \to F$ and corresponding coprojections $j_E : E \to B^n$, $j_F : F \to B^m$. The commutative diagram

$$\begin{tikzcd}
B^n \arrow[rrr, "C_f"] \arrow[ddd, "\phi^n"'] & & & B^m \arrow[dl, "p_F"'] \arrow[ddd, "\phi^m"] \\
& E \arrow[ul, "j_E"'] \arrow[r, "f"] \arrow[d] & F \arrow[d] & \\
& \phi_*E \arrow[r, "\phi_*f"'] \arrow[dl, "\phi_*j_E"] & \phi_*F & \\
\phi_*B^n = A^n \arrow[rrr, "\phi^*(C_f)"'] & & & A^m = \phi_*B^m \arrow[ul, "\phi_*p_F"]
\end{tikzcd}$$

may be rephrased in the commutative diagram

$$\begin{tikzcd}
\mathcal{P}(B)(E,F) \arrow[r, "\phi_*"] \arrow[d, tail, "\mathcal{P}(B)(p_E,j_F)"] & \mathcal{P}(A)(\phi_*E,\phi_*F) \arrow[d, tail, "\mathcal{P}(A)(\phi_*p_E,\phi_*j_F)"] \\
\mathcal{P}(B)(B^n,B^m) \arrow[r, "C \mapsto \phi^*(C)"'] & \mathcal{P}(A)(A^n,A^m)
\end{tikzcd}$$

This proves linearity and continuity of ϕ_*, provided one can show that the injections $\mathcal{P}(B)(p_E,j_F)$ and $\mathcal{P}(A)(\phi_*p_E,\phi_*j_F)$ are embeddings. But $\mathcal{P}(B)(j_E,p_F) \circ \mathcal{P}(B)(p_E,j_F) = \mathcal{P}(B)(p_Ej_E,p_Fj_F) = \mathcal{P}(B)(id_E,id_F) = id_{\mathcal{P}(B)(E,F)}$. Thus $\mathcal{P}(B)(p_E,j_F)$ is a coretraction of completely normable spaces, hence is an embedding. The same argument works for $\mathcal{P}(A)(\phi_*p_E,\phi_*j_F)$. This concludes the proof that ϕ_* is a Banach functor. The assertion that ϕ_* is full iff ϕ is surjective is clear if E and

F are free, since for ϕ surjective one can lift any A-matrix to a B-matrix. The general case follows easily from the described embeddings. $\square$

We are now ready to define the K_0 and K_1 groups for unital C*-algebras and pairs of unital algebras. We follow Karoubi's approach.

<u>I.5 Definition</u> (cf. [K] II.1.7). Let B be a unital C*-algebra. Consider the set Γ_B of isomorphism classes of modules in $P(B)$. For $E \in Ob(P(B))$ denote the class of E by $[E]$. Define an equivalence relation on Γ_B by setting $[E] \sim [F]$ if there exists a $G \in Ob(P(B))$ such that $E \oplus G \cong F \oplus G$. Denote the class of $[E]$ in $\Gamma_{B/\sim}$ by $\overline{[E]}$. Then $\Gamma_{B/\sim}$ is a cancellative monoid with respect to the addition $\overline{[E]} + \overline{[F]} = \overline{[E \oplus F]}$. Let $K_0(B)$ be the Grothendieck group of $\Gamma_{B/\sim}$, i.e., the group of formal differences of elements of $\Gamma_{B/\sim}$.

Note that this definition is based only on the ring structure of B. The full C*-algebra structure of B does not enter.

<u>I.6 Definition</u> (cf. [K] 2.13). Let $\phi : B \to A$ be a unital C*-homomorphism. Consider the set of triples

$$\Gamma(\phi) := \{(E,F,\alpha) : E,F \in Ob(P(B)),\ \alpha : \phi_*E \to \phi_*F \text{ an isomorphism.}\}$$

Two triples (E,F,α) and (E',F',α') are called <u>isomorphic</u>, written $(E,F,\alpha) \cong (E',F',\alpha')$, if there exist isomorphisms $f : E \to E'$ and $g : F \to F'$ which make the following square commute

$$\begin{tikzcd} \phi_*E \arrow[r, "\alpha"] \arrow[d, "\phi_*f"] & \phi_*F \arrow[d, "\phi_*g"] \\ \phi_*E' \arrow[r, "\alpha'"'] & \phi_*F' \end{tikzcd}$$

A triple (E,F,α) is called elementary if $E = F$ and α is homotopic to 1_{ϕ_*E} within $\mathrm{Aut}(\phi_*E)$. Define an addition on $\Gamma(\phi)$ by setting $(E,F,\alpha) + (E',F',\alpha') := (E \oplus E', F \oplus F', \alpha \oplus \alpha')$. The definition makes $\Gamma(\phi)$ into a commutative monoid. Define a congruence relation on $\Gamma(\phi)$ by setting $\sigma \sim \sigma'$ for $\sigma, \sigma' \in \Gamma(\phi)$, if there exist elementary triples τ and τ' such that $\sigma + \tau \cong \sigma' + \tau'$. Denote the equivalence class of (E,F,α) by $d(E,F,\alpha)$. Then $K_0(\phi)$ is defined as the quotient monoid of $\Gamma(\phi)$ modulo $\sim$. It turns out that $K_0(\phi)$ is a group.

Note that $A = 0$ is viewed as a unital C*-algebra. Then, for $\phi : B \to 0$, we can identify $K_0(\phi)$ and $K_0(B)$ (cf. [K] II.2.13).

The following lemmas are stated and proved in [K] and are stated here only for the sake of completeness. They will enable us to give an alternative description of $K_0(\phi)$, which will be useful in actual calculations.

I.7 Lemma (cf. [K] II.2.14). $K_0(\phi)$ is an abelian group. The inverse of $d(E,F,\alpha)$ is $d(F,E,\alpha^{-1})$.

I.8 Lemma (cf. [K] II.2.15). For $d(E,F,\alpha)$ and $d(E,F,\alpha')$ in $K_0(\phi)$ with α homotopic to α' within the space of isomorphisms from ϕ_*E to ϕ_*F, we have $d(E,F,\alpha) = d(E,F,\alpha')$.

I.9 Lemma (cf. [K] II.2.16). For $d(E,F,\alpha)$ and $d(F,G,\beta)$ in $K_0(\phi)$, we have that $d(E,F,\alpha) + d(F,G,\beta) = d(E,G,\beta\alpha)$.

I.10 Lemma (cf. [K] II.2.20). The maps $j^* : K_0(\phi) \to K_0(B)$, given by $d(E,F,\alpha) \longrightarrow \overline{[E]} - \overline{[F]}$ and $\phi^* : K_0(B) \to K_0(A)$, given by $\overline{[E]} - \overline{[F]} \longrightarrow \overline{[\phi_*E]} - \overline{[\phi_*F]}$ are well-defined group homomorphisms yielding the exact sequence $K_0(\phi) \xrightarrow[j^*]{} K_0(B) \xrightarrow[\phi^*]{} K_0(A)$. Moreover, if there exists a C^*-homomorphism $\psi : A \to B$ such that $\phi \circ \psi = id_A$, we get a split exact sequence

$$0 \longrightarrow K_0(\phi) \xrightarrow[j^*]{} K_0(B) \xrightarrow[\phi^*]{} K_0(A) \longrightarrow 0.$$

I.11 Lemma (cf. [K] II.2.25). Let $\phi : B \to A$ be a surjective unital C^*-homomorphism. If $\tau = (E,E,\alpha') \in \Gamma(\phi)$ is an elementary triple, then $\tau \cong (E,E,id_{\phi_*E})$.

I.12 Lemma (cf. [K] II.2.26). Let $\phi : B \to A$ be as in I.11. If we replace elementary triples in the definition of $K_0(\phi)$ by triples of the form $(E,E,id_{\phi*E})$ and proceed in the same fashion otherwise, we get the same group $K_0(\phi)$.

<u>I.13 Theorem</u> (cf. [K] II.2.28). Let ϕ be as in I.11, then $(E,F,\alpha) = 0$ in $K_0(\phi)$ iff there is a $G \in Ob(P(B))$, which can be chosen to be free, and a module isomorphism $\beta : E \oplus G \to F \oplus G$ such that $\psi_*\beta = \alpha \oplus id_{\phi_* G}$.

Note that this description of $K_0(\phi)$ does no longer involve the topological structure of A and B.

We now turn to the definition of K_1. Here we can rely only partly on previous work. The notion of relative K_1-groups has, to my knowledge, not been used before in the context of C*-algebras.

<u>~~I.14 Defi~~nition</u>. Let $\phi : B \to A$ be a unital C*-algebra homomorphism. Consider the set of pairs

$$\Gamma_1(\phi) := \{(E,\alpha) : E \in Ob(P(B)),\ \alpha \in \mathrm{Aut}\ E,\ \phi_*\alpha = id_{\phi_* E}\}$$

Two pairs (E,α) and (E',α') are called <u>isomorphic</u>, written $(E,\alpha) \cong (E',\alpha')$, if there is an isomorphism $h : E \to E'$ which makes the following square commute:

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "\alpha"'] & E' \arrow[d, "\alpha'"] \\ E \arrow[r, "h"'] & E' \end{tikzcd} .$$

A pair (E,α) is called <u>elementary</u> if α is homotopic to id_E in Aut E relative to A, written $\alpha \simeq id_E$ rel A. This means that if σ_t is the homotopy between α and id_E, we have $\phi_*\sigma_t = id_{\phi_* E}$ for all $t \in I$. Define an addition on $\Gamma_1(\phi)$ by $(E,\alpha) + (E',\alpha') := (E \oplus E', \alpha \oplus \alpha')$. For $\sigma,\sigma' \in \Gamma_1(\phi)$, define a relation $\sim$ by $\sigma \sim \sigma'$ if there exist elementary pairs τ and τ' such that $\sigma + \tau \cong \sigma' + \tau'$. It is easy to check that $\sim$ is a contruence. Denote the equivalence class of (E,α) by $d(E,\alpha)$. Now set $K_1(\phi) := \Gamma_1(\phi)/\sim$. For $A = 0$, $\phi(B) = 0$, we set $K_1(B) := K_1(\phi)$.

It is easy to see that $K_1(\phi)$ is a monoid with zero as neutral element. In the following we shall show that $K_1(\phi)$ is an abelian group and give an alternative description of $K_1(\phi)$, which will prove useful in calculations.

<u>I.15 Lemma</u>. With the notation of I.14, we have that $d(E,\alpha) + d(E,\alpha^{-1}) = 0$. Thus $K_1(\phi)$ is a group.

<u>Proof</u>: It suffices to show that $\alpha \oplus \alpha^{-1} \simeq id_{E \oplus E}$ rel A.

$$\text{Let } \sigma_t := \begin{pmatrix} 1-t^2 & -(2-t^2)t \\ t & 1-t \end{pmatrix}^{-1} \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix}$$

then we have $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_E \oplus \mathrm{id}_E$ and

$$\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \alpha \oplus \alpha^{-1}.$$

Moreover, we see from I.4 and $\phi_*(\alpha) = \mathrm{id}_{\phi_* E}$ that

$$\phi_*(\sigma_t) = \begin{pmatrix} 1-t^2 & -(2-t^2)t \\ t & 1-t \end{pmatrix}^{-1} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

now with entries in $\mathbb{C} \cdot 1_A$ instead of $\mathbb{C} \cdot 1_B$. $\square$

<u>I.16 Lemma</u>. Let B, A, ϕ, E be as in I.14. Let $\alpha, \alpha' \in \mathrm{Aut}\, E$ be such that $\phi_*(\alpha) = \mathrm{id}_{\phi_* E} = \phi_*(\alpha')$ and $\alpha \simeq \alpha'$ rel A. Then $d(E,\alpha) = d(E,\alpha')$.

<u>Proof</u>: $d(E,\alpha') - d(E,\alpha) = d(E,\alpha') + d(E,\alpha^{-1}) = d(E \oplus E, \alpha' \oplus \alpha^{-1})$. The last term is zero, since $\alpha' \oplus \alpha^{-1} \simeq \alpha \oplus \alpha^{-1} \simeq \mathrm{id}_E \oplus \mathrm{id}_E$ rel A by I.15.

<u>I.17 Lemma</u>. Let $\phi : B \to A$ be a unital C*-morphism. Then

a) $K_1(\phi)$ is abelian,

b) $d(E,\alpha) + d(E,\beta) = d(E,\alpha\beta) = d(E,\beta\sigma)$ for all $\alpha, \beta \in \mathrm{Aut}\, E$.

<u>Proof</u>: a) We want to show $d(E \oplus F, \alpha \oplus \beta) = d(F \oplus E, \beta \oplus \alpha)$. Let $h : E \oplus F \to F \oplus E$ be the isomorphism which simply interchanges summands. Then the following square commutes:

$$\begin{array}{ccc} E \oplus F & \xrightarrow{h} & F \oplus E \\ \Big\downarrow{\alpha \oplus \beta} & & \Big\downarrow{\beta \oplus \alpha} \\ E \oplus F & \xrightarrow[h]{} & F \oplus E \end{array}$$

Thus $(E \oplus F, \alpha \oplus \beta)$ is isomorphic to $(F \oplus E, \beta \oplus \alpha)$ which proves the first claim.

b) By adding elementary pairs, we get $d(e,\alpha\beta) = d(E \oplus E, \alpha\beta \oplus id_E)$. Thus, it suffices to show $\alpha\beta \oplus id_E \simeq \alpha \oplus \beta \simeq \beta \oplus \alpha \simeq \beta\alpha \oplus id_E$ rel A. By I.15, we have $(\alpha \oplus \beta)^{-1}(\alpha\beta \oplus id_E) = \beta \oplus \beta^{-1} \simeq id_E \oplus id_E$ rel A. Multiplying the homotopy from the left with $\alpha \oplus \beta$, we obtain $\alpha\beta \oplus id_E \simeq \alpha \oplus \beta$ rel A. Similarly $(\alpha\beta \oplus id_E)(\beta \oplus \alpha)^{-1} \simeq id_E \oplus id_E$ rel A and $\alpha\beta \oplus id_E \simeq \beta \oplus \alpha$ rel A. Interchanging the roles of α and β now proves the claim.

<u>I.18 Lemma</u>. Let B, A, ϕ, E, α be as in I.14. Then $d(E,\alpha) = 0$ in $K_1(\phi)$ iff there is a $G \in Ob(P(b))$ such that $\alpha \oplus id_E \simeq id_{E\oplus G}$ rel A in $Aut(E \oplus G)$. We can choose G to be free.

<u>Proof</u>: If $d(E,\alpha) = 0$, then there exist elementary pairs (G,η) and (G',η') in $K_1(\phi)$ and an isomorphism $h : E \oplus G \to G'$ which satisfies $h \circ (\alpha \oplus \eta) = \eta' \circ h$. By adding another elementary pair to (G,η) and (G',η'), if necessary, we can choose G to be free. Hence, we have that $\alpha \oplus id_G \simeq \alpha \oplus \eta$ rel A and $\alpha \oplus \eta = h^{-1} \circ \eta' \circ h \simeq h^{-1} \circ id_{G'} \circ h$ rel A. Thus $\alpha \oplus id_G \simeq id_{E\oplus G}$ rel A.

The converse is clear. $\square$

Note that this description of $K_1(\phi)$ does still depend on the notion of homotopy.

Before we turn to yet another way to view K_1, let us note that K_0 and K_1 are covariant functors from the category of unital C*-algebras and unital C*-morphisms into the category of abelian groups. The proof is routine, so we only describe how K_0 and K_1 act on a C*-morphism $\phi : B \to A$. Since the notation $K_i(\phi)$, for $i = 0,1$, is already in use, we denote the image of ϕ under K_i by ϕ_i^*. Then ϕ_i^* is the group homomorphism from $K_i(B)$ to $K_i(A)$ defined by $\phi_0^*(\overline{[E]} - \overline{[F]}) = \overline{[\phi_* E]} - \overline{[\phi_* F]}$ and $\phi_1^*(d(E,\alpha)) = d(\phi_* E, \phi_* \alpha)$, respectively.

We now give another description of $K_1(\phi)$, which is extremely useful in relating homotopy and K-theory as well as in many calculations. First, we describe $Gl(A)$ for a unital C*-algebra A. Let $Gl_n(A) \subset M_n(A)$ be the set of $n \times n$ matrices with entries in A. It is well-known that $Gl_n(A)$ is a topological group, which is open in $M_n(A)$. Let $Gl_n^0(A)$ be the connected component of 1 in $Gl_n(A)$. Denote the quotient group $Gl_n(A)/Gl_n^0(A)$ by G_n. For each $n \in \mathbb{N}$, we obtain a map from $Gl_n(A)$ to $Gl_{n+1}(A)$ sending $a \in Gl_n(A)$ to $a \oplus 1_A = \begin{pmatrix} a & 0 \\ 0 & 1_A \end{pmatrix}$. Note that this map sends $Gl_n^0(A)$ into $Gl_{n+1}^0(A)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
G_1 & \to & G_2 & \to \cdots & G_n & \to \cdots \to & \varinjlim G_n & =: & G_\infty \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
Gl_1(A) & \to & Gl_2(A) & \to \cdots & Gl_n(A) & \to \cdots \to & \varinjlim Gl_n(A) & =: & Gl(A) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
Gl_1^0(A) & \to & Gl_2^0(A) & \to \cdots & Gl_n^0(A) & \to \cdots \to & \varinjlim Gl_n^0(A) & =: & Gl_\infty^0(A)
\end{array}$$

Here $\varinjlim$ denotes the direct limit. The diagram clearly commutes, and since the direct limit commutes with exact sequences, we get the following short exact sequence of groups

$$0 \to Gl_\infty^0(A) \to Gl(A) \to G_\infty \to 0.$$

Give $Gl(A)$, $Gl_\infty^0(A)$ and G_∞ the inductive limit topology, i.e., a set $C \subset Gl(A)$ is closed iff $C \cap Gl_n(A)$ is closed in $Gl_n(A)$ for all $n \in \mathbb{N}$.

I.20 Lemma. Let X_n be a directed system of Hausdorff spaces such that $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$. Let $X = \varinjlim X_n$ be the inductive limit with the inductive limit topology. Then any compact set $K \subset X$ is contained in an X_n for some $n \in \mathbb{N}$.

Proof: Suppose K is not contained in any X_n. We may assume $X_{n+1} \neq X_n$. Then there exists a sequence k_n of points $k_n \in (X_{n+1} \setminus X_n) \cap K$. Since K is compact, $\{k_n\}$ has a cluster point $k \in K \subseteq X$. Now $k \in X_\ell$ for some

$\ell \in \mathbb{N}$. The set $\{k, k_1, k_2, \ldots\}$ is closed since it intersects each X_n in a finite, hence closed, set; thus it is compact. On the other hand, $\{k, k_1, k_2, \ldots\}$ is the inductive limit of the discrete subspaces $\{k, k_1, \ldots, k_n\}$, hence it is discrete as subspace of X. This is a contradiction. □

<u>I.21 Proposition</u>. For a unital C*-algebra A, the group $Gl_\infty^0(A)$ is the connected component of 1 in $Gl(A)$.

<u>Proof</u>: By the preceding lemma, we see that any path in $Gl(A)$ is actually a path in $Gl_n(A)$ for some $n \in \mathbb{N}$. Thus, any $a \in Gl^0(A)$ is in $Gl_n^0(A)$ for some n. The reverse inclusion is clear. □

For a topological ~~group G we denote~~ its connected component by G^0 and the quotient G/G^0 by $\pi_0(G)$.

<u>I.22 Theorem</u>. Let B and A be unital C*-algebras and $\phi : B \to A$ a unital C*-morphism. Then ϕ induces a natural group homomorphism $\phi^\# : Gl(B) \to Gl(A)$ and we have that $K_1(\phi) \cong \pi_0(\ker \phi^\#)$.

<u>Proof</u>: The map $\phi^\#$ is the map induced by the $\phi^\# : Gl_n(B) \to Gl_n(A)$ (cf. I.4 for the definition) on the direct limits. Note first that for $A = 0$, $Gl(A) = 0$ and $\phi^\#$ is the zero map. Thus, in particular, we prove that $K_1(B) \cong \pi_0(Gl(B))$. Now we define the map $\tau : K_1(\phi) \to \pi_0(\ker \phi^\#)$ which will be the desired isomorphism.

Let $p_i : B^n \to E$ be a projection onto E and $i : E \to B^n$ a corresponding co-projection. Let E' be the complement of E in B^n with respect to (p_i,i). For any $\alpha \in \mathrm{Aut}(E)$ with $\phi_*\alpha = \mathrm{id}_{\phi_*E}$, we define $\alpha_i := \alpha \oplus \mathrm{id}_{E'}$. Then $\alpha_i \in Gl_n(B) \subset Gl(B)$. In fact, $\alpha_i \in \ker \phi^{\#}$, since $\phi^{\#}(\alpha_i) = \phi_*\alpha_i = \phi_*\alpha \oplus \phi_*\mathrm{id}_{E'} = \mathrm{id}_{B^n}$ (cf. I.4). Denote the class of α_i in $\pi_0(\ker \phi^{\#})$ by $[\alpha_i]$. Now, for $d(E,\alpha) \in K_1(\phi)$, we set $\tau(d(E,\alpha)) = [\alpha_i]$. To show that τ is well-defined, we have to show that $[\alpha_i]$ does not depend on the embedding. Suppose $p_j : B^m \to E$, $j : E \to B^m$ is another pair of projection and co-projection. Let E'' be the complement of E in B^m with respect to (p_j,j). Then $d(B^n,\alpha_i) = d(E,\alpha) = d(B^m,\alpha_j)$. Thus, by I.19, there exists a $G \in \mathrm{Ob}(P(B))$ such that $\alpha_i \oplus \mathrm{id}_{B^m} \oplus \mathrm{id}_G \simeq \mathrm{id}_{B^m} \oplus \alpha_j \oplus \mathrm{id}_G$ rel A. We can assume that G is free, say $G = B^k$. This is the same as saying $\alpha_i \oplus 1_{B^m} \oplus 1_{B^k}$ is path connected to $1_{B^n} \oplus \alpha_j \oplus 1_{B^k}$ in $\ker \phi^{\#}$. Since $1_{B^n} \oplus \alpha_j$ is path connected to $\alpha_j \oplus 1_{B^n}$ in $\ker \phi^{\#}$, we have $[\alpha_j] = [\alpha_i]$. The same kind of argument shows in general that $\tau(d(E,\alpha)) = \tau(d(F,\beta))$ if $d(E,\alpha) = d(F,\alpha)$. Thus τ is well-defined. For any $\alpha \in \ker \phi^{\#}$, there is a number n_α such that $\alpha \in Gl_{n_\alpha}(B)$. Define $\tau' : \pi_0(\ker \phi^{\#}) \to K_1(\phi)$ by $\tau([\alpha]) = d(B^{n_\alpha},\alpha)$. Using the same methods as above, it is now routine to check τ' is well-defined, a group homomorphism, and the inverse of τ. This concludes the proof. □

We now state a lemma which actually has been a key ingredient in the proof of Theorem I.13 and which will be used again and again in the sequel.

<u>I.23 Lemma</u> (cf. [K]: II.2.21). Let A and B be unital Banach algebras and $\phi : B \to A$ a continuous surjective ring homomorphism. If $\gamma : I \to A$ is a path such that $\gamma(t) \in Gl_1(A)$ for all $t \in I$ and $\gamma(0) = \gamma(b)$ for some $b \in Gl_1(B)$. Then there exists a $b' \in Gl_1(B)$ such that $\phi(b') = \gamma(1)$ and b' is connected to b in $Gl_1(B)$.

<u>Proof</u>: Let $V := \{y \in A : \|y-1\| < 1\}$, then we can define a logarithm on V. Find a partition $0 = t_0 < \ldots < t_n = 1$ of I such that $\gamma(t_i)^{-1}\gamma(t_{i+1}) \in V$ for $i = 0 \ldots n-1$. Define $a_i := \log(\gamma(t_i)^{-1}\gamma(t_{i+1}))$. We get $\gamma(1) = \gamma(0) \cdot \exp(a_1) \cdot \ldots \cdot \exp(a_{n-1})$. Choose $b_i \in B$ such that $\phi(b_i) = a_i$ and define b' by $b' := b \cdot \exp(b_1) \cdot \ldots \cdot \exp(b_{n-1})$. But $\phi(b') = \phi(b) \cdot \phi(\exp b_1) \cdot \ldots \cdot \phi(\exp b_{n-1})$ and since ϕ is continuous, $\phi(\exp b_i) = \exp(\phi(b_i))$. Thus, $\phi(b') = \gamma(1)$. Moreover, b' is connected to b in $Gl_1(B)$ via the path $t \longmapsto b \cdot \exp(tb_1) \cdot \ldots \cdot \exp(tb_{n-1})$ with $t \in [0,1]$. $\square$

<u>I.24 Lemma</u>. Let B and A be unital C^*-algebras and ϕ a surjective C^*-morphism. For $E \in Ob(P(B))$ and $\alpha \in Aut(E)$ such that $\phi_*(\alpha) \simeq_{\sigma_t} id_{\phi_* E}$, there exists a $\beta \in Aut(E)$ with $\beta \simeq \alpha$ in $Aut(E)$ and $\phi_*(\beta) = id_{\phi_* E}$.

Proof: By I.3, the set End E can be given a Banach space structure and $\phi_* : \mathrm{End}\, E \to \mathrm{End}\, \phi_* E$ is continuous. By I.4, the map ϕ_* is surjective. Clearly, ϕ_* is a ring homomorphism. We apply I.23 to obtain a $\beta \in \mathrm{Aut}(E)$ with $\phi_*(\beta) = 1_{\mathrm{End}(\phi_* E)} = \mathrm{id}_{\phi_* E}$ and β connected to α in $\mathrm{Aut}(E)$, i.e., $\beta \simeq \alpha$ in $\mathrm{Aut}(E)$. □

Next, we prove the analogue of I.10 for K_1. Note the important role Lemma I.24 plays in the proof. First, to simplify language, we introduce the notion of a retract.

I.25 Definition. A C*-algebra A is called a retract of the C*-algebra B if there exists a C*-surjection $\phi : B \to A$ and a C*-morphism $\psi : A \to B$ such that $\phi \circ \psi = \mathrm{id}_A$. The map ψ is required to be unital if B, A and ϕ are.

I.26 Proposition. Let B and A be unital C*-algebra and $\phi : B \to A$ a unital C*-surjection. Then we get an exact sequence $K_1(\phi) \xrightarrow[\pi_1^*]{} K_1(B) \xrightarrow[\phi_1^*]{} K_1(A)$. Moreover, if A is a retract of B, we get a split exact sequence

$$0 \longrightarrow K_1(\phi) \xrightarrow[\pi^*]{} K_1(B) \underset{\phi_1^*}{\overset{\psi_1^*}{\rightleftarrows}} K_1(A) \longrightarrow 0.$$

Proof: The maps ϕ_1^* and ψ_1^* are the images of ϕ and ψ under the functor K_1. The map $\pi_1^* : K_1(\phi) \to K_1(B)$ is defined by $\pi_1^*(d(E,\alpha)) = d(E,\alpha)$. Note that the right hand $d(E,\alpha)$ denotes the class of (E,α) in $K_1(B)$. It is routine to verify that π^* is a well-defined group homomorphism. From the definition of $K_1(\phi)$, it follows that $\phi_1^* \circ \pi_1^* = 0$. Now, let $d(E,\alpha) \in \ker \phi_1^*$. Then $d(\phi_* E, \phi_* \alpha) = 0$. By I.19, there exists a free $G \in Ob(P(B))$, say, $G = B^n$, such that $\phi_*\alpha \oplus id_G \simeq id_{\phi_* E \oplus G}$ in $Aut(\phi_* E \oplus G) = Aut(\phi_*(E \oplus B^n))$. Lemma I.24 now shows the existence of $\beta \in Aut(E \oplus B^n)$ such that $\alpha \oplus id_{B^n} \simeq \beta$ in $Aut(E \oplus B^n)$ and $\phi_*\beta = id_{\phi_*(E \oplus B^n)}$. Thus, $d(E,\alpha) = d(E \oplus B^n, \alpha \oplus id_{B^n}) = d(E \oplus B^n, \beta)$. But $(E \oplus B^n, \beta)$ defines an element in $K_1(\phi)$. Thus $d(E,\alpha)$ is in the image of π_1^*.

If A is a retract of B, the retraction induces a splitting for ϕ_1^*. Thus ϕ_1^* is surjective and it only remains to show that π_1^* is an injection. To this end, view $K_1(\phi)$ as $\pi_0(\ker \phi^\#)$ and $K_1(B)$ as $\pi_0(Gl\ B)$. Then π_1^* maps the class of $a \in \ker \phi^\#$ to its class in $Gl(B)$. Suppose now that $\pi_1^*([a]) = 0$, i.e., that there is a path $\gamma : I \to Gl(B)$ connecting a and $1_{Gl(B)}$. Consider $\gamma'(t) := \gamma(t) \cdot \psi^\#(\phi^\#(\gamma(t)^{-1}))$, then $\phi^\#(\gamma'(t)) = \phi^\#(\gamma(t)) \cdot \phi^\#\psi^\#\phi^\#(\gamma(t)^{-1}) = \phi^\#(\gamma(t)) \cdot \phi^\#(\gamma(t)^{-1}) = 1_{G\ell(A)}$. Thus, $\gamma'(t)$ is a path in $\ker \phi^\#$. We have $\gamma'(1) = 1_{G(B)}$ and $\gamma'(0) = \gamma(0) \cdot \psi^\#\phi^\#\gamma(0)^{-1} = a\ \psi^\#\phi^\#(a^{-1}) = a\psi^\#(1) = a$. Thus a is actually connected to $1_{Gl(B)}$ inside $\ker \phi^\#$, hence $a \in (\ker \phi^\#)^\circ$ and $[a] = 0$. □

CHAPTER II: EXCISION THEOREMS

The purpose of this chapter is to prove two theorems, called the excision theorems, which will allow us to define a K-theory for non unital C*-algebras. Alain Connes proved a result which is analogous to our excision theorem for K_0, using the notion of classes of stably homotopic quasi-isomorphisms. This notion is essentially the same as our $K_0(\phi)$. In his proof Connes uses, however, analytic as well as algebraic techniques, whereas the present proof shows that the excision theorem for K_0 is a purely algebraic theorem once I.13 is achieved.

Let B and A be unital rings and $\phi : B \to A$ a unital ring homomorphism. Then A becomes a right B-module with respect to $a \cdot b := a\phi(b)$. For $E \in \mathrm{Ob}(FGP(B))$, we can form the tensor product $A \otimes_B E$. Then $A \otimes_B E$ is a finitely generated projective left A-module, i.e., $A \otimes_B E \in \mathrm{Ob}(P(A))$. For any $f \in P(B)(E,F)$, we have $\mathrm{id}_A \otimes f \in P(B)(A \otimes_B E, A \otimes_B F)$. The assignment $\phi_* : P(B) \to P(A)$ defined by $\phi_*(E) = A \otimes_B E$ and $\phi_*(f) = \mathrm{id}_A \otimes f$ is precisely the functor we used already in I.4. There is a canonical map $\phi_E : E \to \phi_* E$ given by $\phi_E(m) = 1_A \otimes m$. This ϕ_E is a generalized module map, i.e., $\phi_E(bm) = \phi(b)\phi_E(m)$ (cf. [M] §2 for these definitions and properties).

II.1 Proposition. Let

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[dr, "\delta"'] & A \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

be a commutative triangle of unital rings on unital ring homomorphisms. Then for any $E \in Ob(P(B))$, the modules $\pi_*(\phi_*E)$ and δ_*E are canonically isomorphic and this isomorphism makes the following diagram commute:

$$\begin{tikzcd} E \arrow[r, leftrightarrow, "\phi_E"] \arrow[d, "\delta_E"'] & \phi_*E = A \otimes E \arrow[d, "\pi_{\phi_*E}"] \\ C \otimes E = \delta_*E \arrow[r, leftrightarrow, "\cong"'] & \pi_*(\phi_*E) = C \otimes (A \otimes E) \end{tikzcd}$$

Proof: First consider the case where E is free, say B^n. Then $\pi_*(\phi_*E) = C \otimes_A (A \otimes_B E) \cong C \otimes_B E = \delta_*E$ via the map that sends $c \otimes_A (a \otimes_B m)$ to $c\pi(a) \otimes_B m$. For E, E' such that $E \oplus E' = B$, the distributive law for tensor products and direct sums shows that the map $C \otimes_A (A \otimes_B E) \to C \otimes_B E$ given by restricting and corestricting the canonical isomorphism between $\pi_*(\phi_*B^n)$ and δ_*B^n is a well-defined module isomorphism. It is easy to check that the square commutes.

□

Next, we introduce the map which the excision theorem will show to be an isomorphism.

<u>II.2 Proposition</u>. Let

$$\begin{tikzcd} D \arrow[r, "\rho"] \arrow[d, "j_1"'] & C \arrow[d, "j_2"] \\ B \arrow[r, "\phi"] & A \end{tikzcd}$$

be a commutative square of unital C*-algebras. Moreover, assume ρ and ϕ to be surjective. Then there exists a natural group homomorphism $j^* : K_0(\rho) \to K_0(\phi)$ given by $j^*(d(E,F,\alpha)) = d(j_{1*}E, j_{1*}F, j_{2*}\alpha) = d(B \otimes E,\ B \otimes F,\ id_A \otimes \alpha)$.

<u>Proof</u>: By II.1 we know that $A \otimes_B (B \otimes_D E)$ is canonically isomorphic to $A \otimes_C (C \otimes_D E)$ for $E \in Ob(P(D))$. Under this identification, $id_A \otimes_C \alpha$ is an isomorphism from $A \otimes_B (B \otimes_D E)$ to $A \otimes_B (B \otimes_D F)$, so $d(B \otimes_D E,\ B \otimes_D F,\ id_A \otimes \alpha)$ defines an element in $K_0(\phi)$. It is clear that j^* is additive, so in order to show that j^* is well-defined, it suffices to show that $d(E,F,\alpha) = 0$ implies that $d(B \otimes_D E,\ B \otimes_D F,\ id_A \otimes \alpha) = 0$. Suppose that $d(E,F,\alpha) = 0$ in $K_0(\rho)$, then by I.13, there exists $G \in Ob(P(B))$ and an isomorphism $h : E \oplus G \to F \oplus G$ such that the following square commutes:

$$\begin{tikzcd}
C \otimes_D (E \oplus G) \arrow[r, "id_C \otimes h"] \arrow[d, "\alpha \oplus id"] & C \oplus_D (F \oplus G) \arrow[d, "id"] \\
C \otimes_D (E \oplus G) \arrow[r, "id_C \otimes h"'] & C \otimes_D (F \oplus G)
\end{tikzcd}$$

We apply the functor j_{2*} to this square to obtain, with the obvious identifications, a commutative square:

$$\begin{tikzcd}
A \otimes_B (B \otimes_D (E \oplus G)) \arrow[r, "id_A \otimes (id_B \otimes h)"] \arrow[d, "(id_A \otimes \alpha) \oplus id"] & A \otimes_B (B \otimes_D (F \oplus G)) \arrow[d, "id"] \\
A \otimes_B (B \otimes_D (E \oplus G)) \arrow[r, "id_A \otimes (id_B \otimes h)"'] & A \otimes_B (B \otimes_D (F \oplus G)).
\end{tikzcd}$$

Since $id_B \otimes_D h : B \otimes_D (E \oplus G) \to B \otimes_D (F \oplus G)$ is an isomorphism, this proves that $d(B \otimes_D E,\ B \otimes_D F,\ id_A \otimes_C \alpha) = 0$. □

Now we describe a method of constructing projective modules over a pullback, which will be essential in what follows. Let

$$\begin{tikzcd}
D \arrow[r, "\rho"] \arrow[d, "j_1"'] & C \arrow[d, "j_2"] \\
B \arrow[r, "\phi"'] & A
\end{tikzcd}$$

be a pullback square of unital rings. Further, let $\hat{E} \in Ob(P(B))$ and $M \in Ob(P(C))$ be such that there exists an isomorphism $\beta : A \otimes_C M \to A \otimes_B \hat{E}$. We define a module E over D as the pullback of the following diagram

$$\begin{tikzcd} E \arrow[r, dashed, "p_C"] \arrow[d, dashed, "p_B"'] & M \arrow[d, "\beta \circ j_{2M}"] \\ \hat{E} \arrow[r, "\phi_{\hat{E}}"'] & A \otimes_B \hat{E} \end{tikzcd}.$$

Then $E = \{(\hat{e},m) \in \hat{E} \oplus M : \beta \circ j_{2M}(m) = \phi_{\hat{E}}(\hat{e})\}$. The module structure is given by $d \cdot (\hat{e},m) = (j_1(d)\hat{e}, \rho(d)m)$.

<u>II.3 Theorem</u> (cf. [M] §2). Assume that, in addition to these circumstances, ϕ is surjective. Then $E \in Ob(P(D))$. Moreover, $B \otimes_D E$ is naturally ismorphic to $\hat{E}$ and $C \otimes_D E$ is naturally isomorphic to M.

<u>Proof</u>: We only give the natural maps which the theorem proves to be isomorphisms. After identifying $B \otimes_B \hat{E}$ with $\hat{E}$ and $C \otimes_C M$ with M, we note that they are given by $id_B \otimes p_B : j_{1*}E \to \hat{E}$ and $id_C \otimes p_C ; \rho_*E \to M$. □

<u>II.4 Theorem</u> (Excision for K_0). Given is a pullback square

$$\begin{tikzcd} D \arrow[r, "\rho"] \arrow[d, "j_1"] & C \arrow[d, "j_2"] \\ B \arrow[r, "\phi"'] & A \end{tikzcd}$$

of unital C*-algebras with ϕ surjective. Then the map $j^* : K_0(\rho) \to K_0(\phi)$, defined in II.1 is an isomorphism.

Proof: We split the proof into two lemmas.

II.5 Lemma. The map $j^* : K_0(\rho) \to K_0(\phi)$ is surjective.

Proof: Let $d(\hat{E},\hat{F},\hat{\alpha}) \in K_0(\phi)$, i.e., $\hat{E}, \hat{F} \in \mathrm{Ob}(FGP(B))$ and $\hat{\alpha} : A \otimes_B \hat{E} \to A \otimes_B \hat{F}$ is an isomorphism. By adding an elementary triple if necessary, we can assume without loss of generality that $\hat{F}$ is free, say, $\hat{F} = B^n$. Then $A \otimes_B \hat{F} = A^n$. Define a D-module E via the pullback

$$\begin{tikzcd} E \arrow[r, dashed, "p_C"] \arrow[d, "p_B"'] & C^n \arrow[d, "j_{2C^n}"] \\ \hat{E} \arrow[r, "\hat{\alpha} \circ \phi_{\hat{E}}"'] & A^n \end{tikzcd}$$

Theorem II.3 applies; thus $E \in \mathrm{Ob}(P(D))$. Moreover, $\alpha := \mathrm{id}_C \otimes p_C : C \otimes_D E \to C^n$ is an isomorphism. Thus, (E, D^n, α) defines an element of $K_0(\rho)$. We want to show that $(B \otimes_D E, B \otimes_D D^n, \mathrm{id}_A \otimes \alpha) \cong (\hat{E}, B^n, \hat{\alpha})$. For $h := \mathrm{id}_B \otimes p_C : j_{1*}E \to \hat{E}$, the natural isomorphism from II.3, we consider the diagram:

$$\begin{array}{ccc}
A \otimes_C (C \otimes_D E) = A \otimes_B (B \otimes_D E) & \xrightarrow[\cong]{id_A \otimes h} & A \otimes_B E \\
\Big\downarrow{\scriptstyle id_A \otimes \alpha} & & \Big\downarrow{\scriptstyle \hat{\alpha}} \\
A \otimes_C (C \otimes_D D^n) = A \otimes_B (B \otimes_D D^n) & \xrightarrow[id_{A^n}]{} & A \otimes_B B^n
\end{array} .$$

We check the commutativity of the square on elementary tensors in $A \otimes_B (B \otimes_D E)$, which we can, without loss of generality, assume to be of the form $a \otimes_C (1_B \otimes_D e)$. Then $\hat{\alpha}(\phi_* h(a \otimes_B (1_B \otimes_D e))) = \hat{\alpha}(a \otimes_B p_B(e)) = a \cdot \hat{\alpha}(\phi_{\hat{E}} \circ p_B(e)) = a \cdot (j_{2^{C^n}} \circ p_C(e)) = a \otimes_C p_C(e) = j_{2*}\alpha(a \otimes_C (1_C \otimes_D e))$. The commutativity of the square implies that $(B \otimes_D E, B \otimes_D D^n, id_A \otimes \alpha) \cong (\hat{E}, B^n, \hat{\alpha})$. This concludes the proof. □

<u>II.6 Lemma</u>. The map $j^* : K_0(\rho) \to K_0(\phi)$ is injective.

<u>Proof</u>: Suppose $j^*(d(E,F,\alpha)) = d(B \otimes_D E, B \otimes_D F, id_A \otimes \alpha) = 0$. As before, we can assume F to be free, say $F = D^n$. By I.13, we can find a $T \in Ob(P(B))$, which also can be assumed free, say $T = B^m$, and an isomorphism $\beta : B \otimes_D E \oplus B^m \to B^{n+m}$ such that the following square commutes:

$$
\begin{tikzcd}
B \otimes_D E \oplus B^m \arrow[rr, "\beta"] \arrow[d, "\phi_{B \otimes E} \oplus \phi_{B^m}"] & & B^n \oplus B^m \arrow[dd, "\phi_{B^{n+m}}"] \\
A \otimes_B (B \otimes_D E) \oplus A^m \arrow[d, equal] & & \\
A \otimes_C (C \otimes_D E) \oplus A^m \arrow[rr, "(id_B \otimes \alpha) \oplus id"'] & & A^n \oplus A^m
\end{tikzcd}
$$

Consider the following commutative diagram:

$$
\begin{tikzcd}
 & j_{1*}E \oplus B^m \arrow[rr, "\beta"] \arrow[dd, "\phi_{j_{1*}E \oplus B^m}"] & & B^{n+m} \arrow[dd, "\phi_{B^{n+m}}"] \\
E \oplus D^m \arrow[ur, "j_{1E \oplus D^m}"] \arrow[rr, dashed, "\beta'"] \arrow[dd, "\rho_{E \otimes D^m}"'] & & D^{n+m} \arrow[ur, "j_{1D^{n+m}}"] \arrow[dd, "\rho_{D^{n+m}}"'] & \\
 & j_{2*}(\rho_* E) \oplus A^m \arrow[rr, "j_{2*}\alpha \oplus id_{2m}"] & & A^{n+m} \\
\rho_* E \oplus C^m \arrow[ur, "j_{2}\rho_* E \oplus C^m"'] \arrow[rr, "\alpha \oplus id_{C^m}"'] & & C^{n+m} \arrow[ur, "j_{2C^{n+m}}"'] &
\end{tikzcd}
$$

Note that both the right and the left square are pullback squares. This implies the existence of the map β', induced by $\beta \circ j_{1E \oplus D^n}$ and $(\alpha \oplus id_{C^m}) \circ \rho_{E \oplus D^m}$ and of β'^{-1}, induced by $\beta \circ j_{1D^{n+m}}$ and $(\alpha^{-1} \oplus id_{C^m}) \circ \rho_{D^{n+m}}$. Now with β' being an isomorphism, the commutativity of the front square proves, again by I.13, that $d(E,D^n,\alpha) = 0$. Thus j^* is injective. This concludes the proof of the lemma and thereby the proof of Theorem II.4. □

Now we turn to K_1. We are going to prove a completely analogous result as for K_0. We shall, however, have to use an argument which is not purely algebraic. This was to be expected since we lack an analogue of I.13 for K_1.

<u>II.7 Theorem</u> (Excision for K_1). Consider a pullback square of unital C*-algebras

$$\begin{array}{ccc} D & \xrightarrow{\rho} & C \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ B & \xrightarrow[\phi]{} & A \end{array}$$

with surjective ϕ. Then there is a natural group isomorphism $j^* : K_1(\rho) \to K_1(\phi)$, given by $j^*(d(E,\alpha)) = (j_{1*}E, j_{1*}\alpha) = d(B \otimes_D E, \mathrm{id}_B \otimes_D \alpha)$.

<u>Proof</u>: Note first that if $\sigma_t : E \to E$ is a homotopy in End E, then $\mathrm{id}_B \otimes_D \sigma_t : B \otimes_D E \to B \otimes_D E$ is a homotopy in End $B \otimes_D E$. First, we show that j^* is surjective. Let $d(\hat{E},\hat{\alpha}) \in K_1(\phi)$. By adding an elementary pair if necessary, we can assume without loss of generality that $\hat{E}$ is free, say, $\hat{E} = B^n$. The fact that D is a pullback allows us to establish a map $\alpha : D^n \to D^n$ via the commutative diagram:

$$\begin{tikzcd}
 & B^n \arrow[rr, "\hat{\alpha}"] \arrow[dd] & & B^n \arrow[dd] \\
D^n \arrow[ur] \arrow[rr, dashed, "\alpha"] \arrow[dd] & & D^n \arrow[ur] \arrow[dd] & \\
 & A^n \arrow[rr, "id"] & & A^n \\
C^n \arrow[ur] \arrow[rr, "id"] & & C^n \arrow[ur] &
\end{tikzcd}$$

The map α is induced by $\hat{\alpha} \circ j_{1D^n}$ and ρ_{D^n}. As in II.6, we see that α is invertible. The commutativity of the front square now implies that (D^n, α) defines an element of $K_1(\rho)$. Moreover, the commutativity of the top square implies that $j_{1*}\alpha = \hat{\alpha}$ which shows that $d(j_{1*}D^n, j_{1*}\alpha) = d(\hat{E}, \hat{\alpha})$. Thus j^* is surjective. Now suppose that $j^*(d(E,\alpha)) = 0$ for some $d(E,\alpha) \in K_1(\rho)$. As before, we can assume that E is free, say $E = B^n$. By I.19, we can find a $G \in Ob(P(B))$, without loss of generality G free, say, $G = B^m$, and a homotopy σ_t in $Aut(B^{n+m})$ such that $j_{1*}\alpha \oplus id_{B^m} \simeq_{\sigma_t} id_{B^{n+m}}$ rel A. From the following commutative diagram we derive as before the existence of a family of automorphisms $\sigma'_t : D^{n+m} \to D^{n+m}$:

$$\begin{tikzcd}
 & B^{n+m} \arrow[rr, "\sigma_t"] \arrow[dd] & & B^{n+m} \arrow[dd] \\
D^{n+m} \arrow[ur] \arrow[rr, dashed, "\sigma'_t"] \arrow[dd] & & D^{n+m} \arrow[ur] \arrow[dd] & \\
 & A^{n+m} \arrow[rr, "id"] & & A^{n+m} \\
C^{n+m} \arrow[ur] \arrow[rr, "id"] & & C^{n+m} \arrow[ur] &
\end{tikzcd}$$

The family of maps σ_t' is a homotopy as follows directly from the fact that the family σ_t is a homotopy. Thus $\alpha \oplus \mathrm{id}_{D^m} \simeq_{\sigma_t'} \mathrm{id}_{D^{n+m}}$ rel A and therefore $d(D^n,\alpha) = 0$ by I.19. This concludes the proof. □

CHAPTER III: K-THEORY FOR NON-UNITAL C*-ALGEBRAS

In this chapter we define $K_0(L)$ and $K_1(L)$ for non-unital C*-algebras. We shall also examine the functorial properties of K_0 and K_1. Moreover, for a short exact sequence of C*-algebras $0 \to L \to B \to A \to 0$, we define a connecting homomorphism $K_1(A) \to K_0(L)$ which will allow us to put the sequences from I.10 and I.26 together.

<u>III.1 Lemma</u> (cf. [K], II.3.22). Let B and A be unital C*-algebras and $\phi : B \to A$ a unital C*-morphism. Then there is a natural group homomorphism $\partial_\phi : K_1(A) \to K_0(\phi)$ which makes the following sequence exact:

$$K_1(B) \xrightarrow{\phi_1^*} K_1(A) \xrightarrow{\partial_\phi} K_0(\phi) \xrightarrow{\pi_0^*} K_0(B) \xrightarrow{\phi_0^*} K_0(A)$$

<u>Proof</u>: The maps ϕ_1^*, π_0^* and ϕ_0^* have been defined in I.10 and I.26. We give the definition of ∂_ϕ : Let $d(E',\alpha')$ be in $K_1(A)$. Then there exists an $F' \in Ob(FGP(A))$ such that $E' \oplus F'$ is free over A, say $E' \oplus F' = A^n$. Then $\partial_\phi(d(E',\alpha')) := d(B^n, B^n, \alpha' \oplus id_{F'})$. This makes sense because $\phi_* B^n = A^n = E' \oplus F'$. The proof that ∂_ϕ is well-defined and satisfies the desired properties can be found in [K], II.3.22. □

III.2 Lemma. Consider a commutative square of unital C*-algebras

$$\begin{tikzcd} D \arrow[r, two heads, "\rho"] \arrow[d, "j_1"] & C \arrow[d, "j_2"] \\ B \arrow[r, two heads, "\phi"'] & A \end{tikzcd} .$$

Then for $j^{*(1)}: K_1(\rho) \to K_1(\phi)$ and $j^{*(0)}: K_0(\rho) \to K_0(\phi)$, the maps defined in II.7 and II.2, the following diagram is commutative:

$$\begin{tikzcd} K_1(\rho) \arrow[r, "{}_\rho\pi_1^*"] \arrow[d, "j^{*(1)}"] & K_1(D) \arrow[r, "\rho_1^*"] \arrow[d, "j_1^*"] & K_1(C) \arrow[r, "\partial_\rho"] \arrow[d, "j_2^*"] & K_0(\rho) \arrow[r, "{}_\rho\pi_0^*"] \arrow[d, "j^{*(0)}"] & K_0(D) \arrow[r, "\rho_0^*"] \arrow[d, "j_1^*"] & K_0(C) \arrow[d, "j_2^*"] \\ K_1(\phi) \arrow[r, "{}_\phi\pi_1^*"'] & K_1(B) \arrow[r, "\phi_1^*"] & K_1(A) \arrow[r, "\partial_\phi"'] & K_0(\phi) \arrow[r, "{}_\phi\pi_0^*"'] & K_0(B) \arrow[r, "\phi_0^*"'] & K_0(A) \end{tikzcd} .$$

Proof: All the maps have been defined before. Subscripts ρ and ϕ only indicate for which morphism we construct the natural maps. The commutativity of the second and the fifth squares follows from the functoriality of K_1 and K_0. Let $d(E,\alpha) \in K_1(\rho)$. Then $j_1^* \circ {}_\rho\pi_1^*(d(E,\alpha)) = j_1^*(d(E,\alpha)) = d(j_{1*}E, j_{1*}\alpha) = {}_\phi\pi_1^* \circ j^{*(1)}(d(E,\alpha))$, where the middle terms mean equivalence classes in $K_1(D)$ and $K_1(B)$, respectively. Let $d(E',\alpha') \in K_1(C)$ and $E' \oplus F' = C^n$.

Then $j_{2*}E' \oplus j_{2*}F' = A^n$. Thus $j^{*(0)} \circ \partial_\rho(d(E',\alpha')) = j^{*(0)}(d(D^n,D^n,\alpha' \oplus id_{F'})) = d(B^n,B^n,j_{2*}\alpha' \oplus id_{j_{2*}F'}) = \partial_\phi d(j_{2*}E',j_{2*}\alpha') = \partial_\phi \circ j_2^*(d(E',\alpha'))$. Finally, let $d(E,F,\alpha) \in K_0(\rho)$. Then $j_1^* \circ {}_\rho\pi_0^*(d(E,F,\alpha)) = j_k^*(\overline{[E]} - \overline{[F]}) = \overline{[j_{1*}E]} - \overline{[j_{1*}F]} = {}_\phi\pi_0^*(d(j_{1*}E,j_{1*}F,j_{2*}\alpha)) = {}_\phi\pi_0^* \circ j^{*(0)}(d(E,F,\alpha))$. $\square$

III.3 Lemma. Let $0 \longrightarrow L \xrightarrow{\hat{j}} B \xrightarrow{\phi} A \longrightarrow 0$ be a short exact sequence of C*-algebras such that B, A and ϕ are unital. Let $\tilde{L}$ be the C*-algebra we obtain from L by adjoining an identity. Let $j_1 : \tilde{L} \to B$ be the unital C*-morphism induced by $\hat{j}$. Then we get a commutative diagram of C*-algebras

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \tilde{L} \arrow[r, "\rho"] \arrow[d, "j_1"] & \mathbb{C} \arrow[r] \arrow[d, "j_2"] & 0. \\
0 \arrow[r] & L \arrow[r, "\hat{j}"'] & B \arrow[r, "\phi"'] & A \arrow[r] & 0
\end{tikzcd}$$

Moreover, the right square is a pullback square.

Proof: The map ρ is the canonical surjection $\tilde{L} \to \tilde{L}/L = \mathbb{C}$. The map j_2 is the canonical injection which sends $\lambda \in \mathbb{C}$ to $\lambda \cdot 1_A$. With these maps the right square is clearly commutative. Let $\hat{L} := \{(b,\lambda) \in B \oplus \mathbb{C} : \phi(b) = \lambda \cdot 1_A\}$ be the pullback of ϕ and j_2, then it is easy to check the map $h : \tilde{L} \to \hat{L}$, defined by $h(\ell + \lambda 1) := (\hat{j}(\ell) + \lambda 1, \lambda)$ is an isomorphism of C*-algebras. □

Now we are ready to define K_0 and K_1 for arbitrary C*-algebras. This definition makes sense also for unital C*-algebras and we shall show that the two definitions coincide for those unital C*-algebras. Note first that for any C*-algebra L there is a short exact sequence $0 \longrightarrow L \longrightarrow \tilde{L} \xrightarrow{\rho} \mathbb{C} \longrightarrow 0$. By the preceding lemma and the excision theorems, we see that, for any short exact sequence $0 \longrightarrow L \longrightarrow B \xrightarrow{\phi} A \longrightarrow 0$ with A, B and ϕ unital, $K_i(\phi) \cong K_i(\rho)$. If L is also unital we get a pullback square of unital C*-algebras

$$\begin{tikzcd} \tilde{L} \arrow[r, "\rho"] \arrow[d, "\cong"] & \mathbb{C} \arrow[d, "id"] \\ L \oplus \mathbb{C} \arrow[r, "pr_2"] & \mathbb{C} \end{tikzcd} .$$

The isomorphism is given by $(a,c) \longmapsto a + c1_L \oplus c$. Thus $K_i(\rho) \cong K_i(pr_2)$. But we can view pr_2 as $0 \oplus id_{\mathbb{C}}$: $L \oplus \mathbb{C} \to 0 \oplus \mathbb{C}$ and relative K-theory preserves direct sums as the reader can show easily. Thus $K_i(pr_2) = K_i(0) \oplus K_i(id_{\mathbb{C}}) = K_i(L) \oplus 0$. □

<u>III.4 Definition</u>. Let L be any C*-algebra such that we have a short exact sequence of C*-algebras $0 \longrightarrow L \longrightarrow B \xrightarrow{\phi} A \longrightarrow 0$ with B, A and ϕ unital.

Then define $K_i(L) := K_i(\phi)$ for $i = 0,1$.

The above remarks make sure that the definition III.4 makes sense and does not create ambiguity for unital C^*-algebras.

For any C^*-morphism $\phi : B \to A$ we can define a unital C^*-morphism $\phi : B \to A$ that sends an $(b,\lambda) \in \tilde{B}$ to $(\phi(b),\lambda) \in \tilde{A}$. If B, A and ϕ are already unital, then the composition of $\tilde{\phi} : \tilde{B} \to \tilde{A}$ with the isomorphisms $B \oplus \mathbb{C} \to \tilde{B}$ and $\tilde{A} \to A \oplus \mathbb{C}$ described above, is the isomorphism $\phi \oplus id_{\mathbb{C}} : B \oplus \mathbb{C} \to A \oplus \mathbb{C}$. Thus, we see as before that in this case $K_i(\tilde{\phi}) = K_i(\phi) \oplus K_i(id_{\mathbb{C}}) = K_i(\phi)$.

<u>III. 5 Definition</u>. Let B and A be C^*-algebras and $\phi : B \to A$ be a C^*-morphism. If $\tilde{\phi} : \tilde{B} \to \tilde{A}$ is the unital C^*-morphism induced by ϕ, we define $K_i(\phi) := K_i(\tilde{\phi})$ for $i = 0,1$.

Now we can assign to each C^*-morphism $\phi : B \to A$ group homomorphisms $\phi_i^* : K_i(B) \to K_i(A)$ where $\phi_i^* = (\tilde{\phi})_i^*|_{K_i(B)}$ and $(\tilde{\phi})_i^* : K_i(\tilde{B}) \to K_i(\tilde{A})$ is the map defined in I.10 and I.26, respectively. The fact that ϕ_i^* maps $K_i(B)$ actually into $K_i(A) \subset K_i(\tilde{A})$ follows from the following commutative diagram, III.2, I.10 and I.26.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\phi"] & \tilde{B} \arrow[r] \arrow[d, "\tilde{\phi}"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{A} \arrow[r] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

□

III.6 Theorem. The assignment K_i which sends a C*-algebra A to $K_i(A)$ and a C*-morphism $\phi : B \to A$ to $\phi_i^* : K_i(B) \to K_i(A)$ is a covariant functor from the category of C*-algebras into the category of abelian groups.

Proof: The proof is routine and left to the reader as an easy exercise. □

For any short exact sequence of C*-algebras $0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\phi} A \longrightarrow 0$ we get a short exact sequence $0 \longrightarrow L \xrightarrow{\pi} \tilde{B} \xrightarrow{\tilde{\phi}} \tilde{A} \longrightarrow 0$. By III.1, we obtain an exact sequence of abelian groups

$$K_1(\tilde{\phi}) \longrightarrow K_1(\tilde{B}) \longrightarrow K_1(\tilde{A}) \xrightarrow{\partial_\phi} K_0(\tilde{\phi}) \longrightarrow K_0(\tilde{B}) \longrightarrow K_0(\tilde{A}).$$

Note that $Gl(\mathbb{C})$ is path connected and thus $K_1(\mathbb{C}) = 0$. Hence we have that $K_1(A) = K_1(\tilde{A})$. Morever, by definition $K_i(\tilde{\phi}) = K_i(\phi)$. So we get a commutative diagram

$$\begin{array}{ccccccccccc}
K_1(\tilde{\phi}) & \longrightarrow & K_1(\tilde{B}) & \longrightarrow & K_1(\tilde{A}) & \xrightarrow{\partial_\phi} & K_0(\tilde{\phi}) & \xrightarrow{\pi_0^*} & K_0(\tilde{B}) & \xrightarrow{(\tilde{\phi})_0^*} & K_0(\tilde{A}) \\
\| & & \| & & \| & & \| & & \uparrow & & \uparrow \\
K_1(\phi) & \longrightarrow & K_1(B) & \longrightarrow & K_1(A) & \xrightarrow{\partial_\phi} & K_0(\phi) & \xrightarrow{\pi_0^*} & K_0(B) & \xrightarrow{\phi_0^*} & K_0(A)
\end{array}$$

Since ϕ_0^* is just the restriction of $(\tilde{\phi})_0^*$, we obtain:

III.7 Proposition. Let $0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\phi} A \longrightarrow 0$ be a short exact sequence of C*-algebras. Then the following sequence is exact:

$$K_1(L) \xrightarrow{\pi_1^*} K_1(B) \xrightarrow{\phi_1^*} K_1(A) \xrightarrow{\partial_\phi} K_0(L) \xrightarrow{\pi_0^*} K_0(B) \xrightarrow{\phi_0^*} K_0(A).$$

III.8 Proposition. Consider a commutative diagram of C^*-algebras:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\pi"] & B \arrow[r, "\phi"] & A \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, "\tau"] \arrow[u] & 0 \arrow[r, "\rho"] \arrow[u] & C \arrow[r] \arrow[u] & 0
\end{tikzcd} .$$

Let the rows be exact. Then the diagram

$$\begin{tikzcd}
K_1(L) \arrow[r, "\pi_1^*"] & K_1(B) \arrow[r, "\phi_1^*"] & K_1(A) \arrow[r, "\partial_\phi"] & K_0(L) \arrow[r, "\pi_0^*"] & K_0(B) \arrow[r, "\phi_0^*"] & K_0(A) \\
K_1(K) \arrow[r, "\tau_1^*"] \arrow[u] & K_1(D) \arrow[r, "\rho_1^*"] \arrow[u] & K_1(C) \arrow[r, "\partial_\rho"] \arrow[u] & K_0(K) \arrow[r, "\tau_0^*"] \arrow[u] & K_0(D) \arrow[r, "\rho_0^*"] \arrow[u] & K_0(C) \arrow[u]
\end{tikzcd}$$

obtained from III.7, commutes.

Proof: First consider the case where all algebras and maps in the right square are unital. Then we are in the situation of Lemma III.2 which gives the commutativity of the diagram. In the general case we replace the right square of the algebra diagram by

$$\begin{tikzcd}
\tilde{B} \arrow[r, "\tilde{\phi}"] & \tilde{A} \\
\tilde{D} \arrow[r, "\tilde{\rho}"'] \arrow[u] & \tilde{C} \arrow[u]
\end{tikzcd} .$$

Since the diagram for the general case is gotten from the diagram for the unital case by just restricting some maps, it is clear that it commutes. □

CHAPTER IV: K-THEORY AS HOMOTOPY FUNCTOR

In this chapter we introduce the notion of homotopic C*-morphisms and various other concepts arising naturally from homotopy theory of topological spaces. These ideas have been used more or less implicitly by many authors. It seems, however, that nobody ever bothered to write up the powerful consequences in K-theory in a concise form. The main result is that the K-functors do not distinguish between homotopic C*-morphisms.

<u>IV.1. Definition</u>. Let A and B be C*-algebras and $\phi_i : A \to B$ for $i = 0,1$ be two C*-morphisms. The maps ϕ_0 and ϕ_1 are called homotopic, written $\phi_0 \simeq \phi_1$, if there exists a family $\Phi_t : A \to B$ of C*-morphisms for $t \in I$ such that $\Phi : I \times A \to B$ defined by $\Phi(t,a) = \Phi_t(a)$ is jointly continuous and $\Phi_0 = \phi_0$ as well as $\Phi_1 = \phi_1$.

<u>IV.2. Definition</u>. Let A and B be C*-algebras. A C*-morphism $\phi : A \to B$ is called a homotopy equivalence if there exists a C*-morphism $\psi : B \to A$ such that $\phi \circ \psi \simeq id_B$ and $\psi \circ \phi \simeq id_A$.

<u>IV.3. Definition</u>. A C*-algebra C is called contractible if $id_C \simeq 0$. Here 0 denotes the map $C \to C$ that sends everything to zero.

IV.4. Lemma. Let B be a unital C*-algebra. Let $0 \neq E, F \in \mathrm{Ob}(P(B))$ and $p_E : B^n \to E$, $j_E : E \to B^n$, $p_F : B^n \to F$, $j_F : F \to B^n$ be pairs of projections and coprojections for the modules E and F. Endow B^n with the product norm and E and F with the subspace norm with respect to j_E and j_F. Then if

$$\| j_E \circ p_E - j_F\, p_F \| < \frac{1}{\max\{\|p_E\|, \|p_F\|\}} ,$$

$p_E \circ j_F : F \to E$ is an isomorphism of topological vector spaces.

Proof: First, we show that $p_F \circ j_E \circ p_E \circ j_F$ is an automorphism of F. The space $\mathrm{End}(F)$ of endomorphisms of F is a unital Banach algebra with respect to the operator norm. Then $\| id_F - p_F \circ j_E \circ p_E \circ j_F \| = \| p_F \circ j_F \circ p_F \circ j_F - p_F \circ j_E \circ p_E \circ j_F \| = \| p_F \circ (j_F \circ p_F - j_E \circ p_E) \circ j_F \| \leq \|p_F\| \, \| j_F \circ p_F - j_E \circ p_E \| \, \|j_F\| \leq \|p_F\| \, \| j_F \circ p_F - j_E \circ p_E \| < 1$ since $\|j_F\| = 1$. Thus $p_F \circ j_E \circ p_E \circ j_F$ is invertible in $\mathrm{End}(F)$. But $(p_F \circ j_E \circ p_E \circ j_F)^{-1} \circ p_F \circ j_E$ is a left inverse for $p_E \circ j_F$, thus $p_E \circ j_F$ is injective. Similarly we now show that $p_E \circ j_F \circ p_F \circ j_E \in \mathrm{Aut}(E)$ and thus $p_E \circ j_F$ has a right inverse $p_F \circ j_E \circ (p_E \circ j_F \circ p_F \circ j_E)^{-1}$. Hence $p_E \circ j_F$ is surjective. □

IV.5. Theorem. Let A and B be C*-algebras and $\Phi_t : B \to A$ be a homotopy between the C*-morphisms

$\phi : B \to A$ and $\psi : B \to A$. Then the induced maps $\phi_0^* : K_0(B) \to K_0(A)$ and $\psi_0^* : K_0(B) \to K_0(A)$ are equal.

For a given free module B^n over a unital C*-algebra B we can identify projective retracts E of B^n, given by a pair of projection and co-projection (p_E, j_E), with a projection P_E in $M_n(B)$, namely the matrix associated with $j_E \circ p_E$. If A is another unital C*-algebra and $\phi : B \to A$ is a unital C*-morphism, then the module $\phi_* E = A \otimes_B E$ is given by the projection $\phi^{\#}(P_E)$. Thus, if $\phi : B \to A$ is homotopic to $\psi : B \to A$ via a unital homotopy Φ_t, for any $t \in I$, there exists an open neighborhood U_t of t such that $\|\Phi_t^{\#}(P_E) - \Phi_s^{\#}(P_E)\| < \dfrac{1}{1 + \|\Phi_t^{\#}(P_E)\|} \leq 1$ for all $s \in U_t$. If $J = \{s \in I \mid \phi_*(E) \cong \Phi_{s*}(E)\}$, this shows by IV.4 that $J \neq \emptyset$ and open. But if $t \in \overline{J}$, we find an $s \in U_t \cap J$, hence, again by IV.4, $t \in J$. Since I is connected, this implies $J = I$.

<u>Proof</u>: First we assume that A, B, ψ, ϕ and Φ_t are unital. The map $(\Phi_t)_0^* : K_0(B) \to K_0(A)$ is given by $(\Phi_t)_0^*(\overline{[E]} - \overline{[F]}) = \overline{[\Phi_{t*}(E)]} - \overline{[\Phi_{t*}(F)]}$, which does not depend on t as we saw above. In the general case, we replace A and B by $\tilde{A}$ and $\tilde{B}$, and ϕ, ψ and Φ_t by $\tilde{\phi}, \tilde{\psi}$ and $\tilde{\Phi}_t$. Thus we get that the maps $\tilde{\phi}_0^* : K_0(\tilde{B}) \to K_0(\tilde{A})$ and $\tilde{\psi}_0^* : K_0(\tilde{B}) \to K_0(\tilde{A})$ are equal. Therefore, also their restriction-corestrictions $\phi_0^* : K_0(B) \to K_0(A)$ and $\psi_0^* : K_0(B) \to K_0(A)$ are equal. □

<u>IV.6. Theorem</u>. If $\phi, \psi : B \to A$ are homotopic C*-morphisms, the maps $\phi_1^* : K_1(B) \to K_1(A)$ and $\psi_1^* : K_1(B) \to K_1(A)$ are equal.

<u>Proof</u>: First we assume that all algebras and maps are unital. View $K_1(B)$ as $\pi_0(Gl(B))$ and $K_1(A)$ as $\pi_0(Gl(A))$. For a $b \in Gl(B)$ and $[b]$ its class in $\pi_0(Gl(B))$, the map $\Phi^*_{t_1} : K_1(B) \to K_1(A)$ is given by $\Phi^*_{t_1}([b]) = [\Phi^\#_t(b)]$. But $\Phi^\#_t(b)$ and $\Phi^\#_s(b)$ are path-connected in $Gl(A)$ via $\gamma : I \to Gl(A)$ defined by $\gamma(r) = \Phi^\#_{rt+(1-r)s}(b)$. Thus, $\Phi^*_{t_1}[b] = \Phi^*_{s_1}[b]$ for all $s,t \in I$. In particular, $\phi_1^* = \psi_1^*$. The general case follows easily from replacing A and B by $\tilde{A}$ and $\tilde{B}$ and all the maps by the corresponding unital maps, just as in IV.5. □

<u>IV.7. Corollary</u>. Let A and B be C*-algebras and $\phi : A \to B$ a homotopy equivalence. Then the induced map $\phi_i^* : K_i(A) \to K_i(B)$ is an isomorphism.

<u>Proof</u>: Note that the identity on A, B induce the identity on $K_i(A)$ and $K_i(B)$, respectively. Now the claim follows directly from the preceding theorems and the definition of a homotopy equivalence via the usual argument. □

<u>IV.8. Corollary</u>. Let B be a contractible C*-algebra. Then $K_0(B)$ and $K_1(B)$ are zero.

<u>Proof</u>: Note that the zero map on B induces the zero map on $K_0(B)$. Thus the identity map on $K_0(B)$ is equal to the zero map. □

CHAPTER V: SUSPENSIONS AND HIGHER K-GROUPS

In this chapter we give the definition of the suspension of a C*-algebra. We use it to define $K_n(A)$ for any $n \in \mathbb{N}$. The main result will be a long exact sequence in K-theory associated with a short exact sequence of C*-algebras.

V.1. Definition. Let B be a C*-algebra. The cone CB over B is defined by $CB = \{f : I \to B : f$ continuous and $f(1) = 0\}$. The suspension SB of B is defined by $SB = \{f \in CB : f(0) = 0\}$.

It is easy to check that CB and SB are C*-algebras. In fact, cone and suspension can be viewed as functors from the category of C*-algebras into itself. The image of a morphism $\phi : B \to A$ under these functors is given by $C\phi : CB \to CA$ with $C\phi(f) = \phi \circ f$ and $S\phi : SB \to SA$ with $S\phi(f) = \phi \circ f$, respectively. Note that if $ev : CB \to B$ denotes the evaluation at 0, we get a short exact sequence $0 \longrightarrow SB \hookrightarrow CB \longrightarrow B \longrightarrow 0$.

V.2. Lemma. The cone CB is contractible for any C*-algebra B.

Proof: Consider the family of C*-morphisms $\Phi_t : CB \to CB$ for $t \in I$, defined by $\Phi_t(f)(s) = f(1 - (1-t)(1-s))$. Then $\Phi_0 = id_{CB}$ and $\Phi_1 = 0$. It is clear that Φ_t is a homotopy. □

V.3. Proposition. Let B be a C*-algebra. Then we have a natural isomorphism $K_1(B) \cong K_0(SB)$.

Proof: The above remarks and Theorem III.7 show that we have an exact sequence in K-theory

$$K_1(CB) \longrightarrow K_1(B) \xrightarrow{\partial} K_0(SB) \longrightarrow K_0(CB).$$

But CB is contractible by V.2, hence $K_i(CB) = 0$ by IV.8. Thus ∂ is an isomorphism.

If A is another C*-algebra and $\phi : B \to A$ is a C*-morphism, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SA & \longrightarrow & CA & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle S\phi} & & \uparrow{\scriptstyle C\phi} & & \uparrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & SB & \longrightarrow & CB & \longrightarrow & B & \longrightarrow & 0 . \end{array}$$

By III.8 we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 = K_1(CA) & \longrightarrow & K_1(A) & \xrightarrow{\partial_A} & K_0(SA) & \longrightarrow & K_0(CA) = 0 \\ & & \uparrow{\scriptstyle \phi_1^*} & & \uparrow{\scriptstyle (S\phi)_0^*} & & \\ 0 = K_1(CB) & \longrightarrow & K_1(B) & \xrightarrow{\partial_B} & K_0(SB) & \longrightarrow & K_0(CB) = 0 . \end{array}$$

□

V.4. Definition. Let B be a C*-algebra. Define the n-th K-group of B by $K_n(B) = K_{n-1}(SB) = \ldots = K_0(S^nB)$. Here S^n means the n-fold application of the functor S to B.

Proposition V.4 shows that there is no ambiguity in this definition. Note that the functor S is exact, i.e., it sends exact sequences to exact sequences. In particular, for a short exact sequence of C*-algebras $0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\phi} A \longrightarrow 0$, we get a short exact sequence $0 \longrightarrow SL \xrightarrow{S\pi} SB \xrightarrow{S\phi} SA \longrightarrow 0$. By III.7, this induces an exact sequence

$$K_1(SL) \xrightarrow{(S\pi)_1^*} K_1(SB) \xrightarrow{(S\phi)_1^*} K_1(SA) \xrightarrow{\partial_{S\phi}}$$
$$K_0(SL) \xrightarrow{(S\pi)_0^*} K_0(SB) \xrightarrow{(S\phi)_0^*} K_0(SA),$$

which we rephase in the following manner

$$K_2(L) \xrightarrow{\pi_2^*} K_2(B) \xrightarrow{\phi_2^*} K_2(A) \xrightarrow{\partial_2} K_0(SL) \xrightarrow{(S\pi)_0^*} K_0(SB) \xrightarrow{(S\phi)_0^*} K_0(SA).$$

The naturality of the isomorphism from V.3 shows that the following diagram commutes.

$$\begin{tikzcd} K_0(SL) \arrow[r, "(S\pi)_0^*"] & K_0(SB) \arrow[r, "(S\phi)_0^*"] & K_0(SA) \\ K_1(L) \arrow[u, "\cong"] \arrow[r, "\pi_1^*"] & K_1(B) \arrow[u, "\cong"] \arrow[r, "\phi_1^*"] & K_1(A) \arrow[u, "\cong"] \end{tikzcd}$$

Thus, we can put together the above sequence and

$$K_1(L) \longrightarrow K_1(B) \longrightarrow K_1(A) \xrightarrow{\partial} K_0(L) \longrightarrow K_0(B) \longrightarrow K_0(A).$$

We obtain the following theorem.

<u>V.5. Theorem</u>. Let $0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\phi} A \longrightarrow 0$ be a short exact sequence of C*-algebras. Then we have a long exact sequence in K-theory as follows: for $n \geq 1$

$$\longrightarrow K_n(L) \xrightarrow{(S^n\pi)_0^*} K_n(B) \xrightarrow{(S^n\phi)_0^*} K_n(A) \xrightarrow{\partial_{S^n\phi}}$$

$$K_{n-1}(L) \xrightarrow{(S^{n-1}\pi)_0^*} K_{n-1}(N) \xrightarrow{(S^{n-1}\phi)_0^*} K_{n-1}(A).$$

We denote $(S^n\pi)_0^*$ by π_n^* and $(S^n\phi)_0^*$ by ϕ_n^*. Moreover, we denote ∂_{S^n} by ∂_n if the map ϕ is clear from the context.

CHAPTER VI: BOTT PERIODICITY AND THE SIX-TERM-SEQUENCE

In this chapter we shall describe the Bott periodicity theorem which is of great importance. It will, among other things, enable us to install the so-called six term sequence, which is a different form of expressing the long exact sequence.

<u>VI.1. Definition</u>. Let G be a topological group. Define $\pi_1(G)$ to be the first homotopy group of G with respect to homotopies and loops based at the identity.

It is well known that in this case the multiplication in $\pi_1(G)$ can be described by pointwise multiplication of loops just as well as by composition of loops. For a unital C*-algebra A, let $L_n(A)$ be the group of loops in $Gl_n(A)$, based at 1, under pointwise multiplication. Let $N_n(A)$ be the subgroup of loops which are homotopic to a constant loop. $N_n(A)$ is normal in $L_n(A)$. There is a canonical injection $L_{n-1}(A) \to L_n(A)$ which maps $f \in L_{n-1}(A)$ to $f \oplus 1 = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \in L_n(A)$. This map sends $N_{n-1}(A)$ into $N_n(A)$. Thus we obtain the following commutative diagram.

$$\begin{array}{ccccccccc}
\pi_1(Gl_1(A)) & \longrightarrow & \cdots & \longrightarrow & \pi_1(Gl_n(A)) & \longrightarrow & \cdots & \longrightarrow & \varinjlim \pi_1(Gl_n(A)) \\
\uparrow & & & & \uparrow & & & & \uparrow \\
L_1(A) & \longrightarrow & \cdots & \longrightarrow & L_n(A) & \longrightarrow & \cdots & \longrightarrow & \varinjlim L_n(A) \\
\uparrow & & & & \uparrow & & & & \uparrow \\
N_1(A) & \longrightarrow & \cdots & \longrightarrow & N_n(A) & \longrightarrow & \cdots & \longrightarrow & \varinjlim N_n(A)
\end{array}$$

If we let $L(Gl(A))$ be the group of loops in $Gl(A)$ based at 1 and $N(Gl(A))$ the subgroup of $L(Gl(A))$ consisting of the contractible loops, Lemma I.20 shows that $L(Gl(A)) = \varinjlim L_n(A)$ and $N(Gl(A)) = \varinjlim N_n(A)$. Thus we get the following proposition.

<u>VI.2. Proposition</u>. Let A be a unital C*-algebra. Then $\pi_1(Gl(A)) = \varinjlim \pi_1(Gl_n(A))$.

<u>VI.3. Proposition</u>. Let A be a unital C*-algebra. Then $\pi_0(Gl(\tilde{S}A)) \cong \pi_1(Gl(A))$.

<u>Proof</u>: Let G be any topological group and H be a closed subgroup of H. Denote the group of continuous functions from the one-sphere S^1 into G which sends the base point of S^1 into H by $C(S^1,G,H)$. Then we know that $\pi_0(C(S^1,G,H)) = \pi_1(G,H)$, the relative homotopy group. Moreover, if H is path-connected, we have $\pi_1(G,H) = \pi_1(G,1)$. Now note that $\tilde{S}A = \{f : I \xrightarrow[\text{cont.}]{} A : f(1) = f(0) \in \mathbb{C} \cdot 1_A\}$. We identify $Gl(\tilde{S}A)$ with $C(S^1, Gl(A), Gl(\mathbb{C} \cdot 1_A))$ in the obvious way. Then, since $Gl(\mathbb{C} \cdot 1_A)$ is path-connected, we get $\pi_0(Gl(\tilde{S}A)) = \pi_1(Gl(A), Gl(\mathbb{C} \cdot 1_A)) = \pi_1(Gl(A), 1_{GL(A)})$ and since we defined $\pi_1(Gl(A))$ as $\pi_1(Gl(A), 1_{GL(A)})$, this proves the claim. $\square$

<u>VI.4. Theorem</u> (Bott Periodicity, cf. [K] III.1.11). Let A be a unital Banach algebra. Then the map

$\gamma_A : K_0(A) \to \pi_1(\mathrm{Gl}(A))$ induced by the assignment that sends the isomorphy class $[E]$ of a finitely generated projective A-module to the homotopy class of the loop $t \longmapsto z(t)P_E + 1 - P_E$, where the projection $P_E \in M_\infty(A)$ is as in IV.4 and $z(t) = e^{2\pi it}$, is an isomorphism, called the Bott isomorphism.

Note that $z(t)P_E + 1 - P_E = \exp(t \cdot P_E)$. Note also that the Bott isomorphism is natural in the sense that for A and B unital C*-algebras, and $\phi : B \to A$ a unital C*-morphism, the map $\phi_1^{\#} : \pi_1(\mathrm{Gl}(B)) \to \pi_1(\mathrm{Gl}(A))$ induced by the map $\phi^{\#} : \mathrm{Gl}(B) \to \mathrm{Gl}(A)$ makes the following diagram commute

$$\begin{tikzcd} K_0(B) \arrow[r, "\phi_0^*"] \arrow[d, "\gamma_B"] & K_0(A) \arrow[d, "\gamma_A"] \\ \pi_1(\mathrm{Gl}(B)) \arrow[r, "\phi_1^{\#}"] & \pi_1(\mathrm{Gl}(A)) \end{tikzcd} .$$

<u>VI.5. Corollary</u>. Let A be a unital C*-algebra. Then $K_0(A) \cong K_1(SA)$ via the Bott map.

<u>Proof</u>: We have seen in the proof of III.7 that $K_1(B) = K_1(\tilde{B})$ for any C*-algebra B. Thus $K_1(SA) = K_1(\widetilde{SA})$ and the claim follows from VI.4. □

<u>VI.6. Corollary</u>. Let A be a C*-algebra. Then we have that $K_0(S^2A) \cong K_0(A)$.

Proof: If A is unital, we have $K_0(S^2A) = K_1(SA)$ by V.3 and $K_1(SA) = K_0(A)$ by VI.5. In the general case we have a split exact sequence $0 \to S^2A \to S^2\tilde{A} \to S^2\mathbb{C} \to 0$. If $\rho : K_0(\tilde{A}) \to K_0(S^2\tilde{A})$ denotes the composition of the Bott map $\gamma_{\tilde{A}}$ and the map from V.3 and $\rho' : K_0(\mathbb{C}) \to K_0(S^2\mathbb{C})$ the corresponding map for $\mathbb{C}$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(S^2A) \arrow[r] & K_0(S^2\tilde{A}) \arrow[r] & K_0(S^2\mathbb{C}) \arrow[r] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] & K_0(\tilde{A}) \arrow[r] \arrow[u, "\rho"', "\cong"] & K_0(\mathbb{C}) \arrow[r] \arrow[u, "\rho'"', "\cong"] & 0 .
\end{tikzcd}$$

Since the rows are exact, we see that $\rho|_{K_0(A)}$ is an isomorphism between $K_0(A)$ and $K_0(S^2A)$. $\square$

For a short exact sequence of C*-algebras $0 \longrightarrow L \xrightarrow{\pi} B \xrightarrow{\phi} A \longrightarrow 0$, we can now write down the so-called six term sequence.

$$\begin{tikzcd}
K_2(A) \arrow[r, "\partial_2"] & K_1(L) \arrow[r, "\pi_1^*"] & K_1(B) \arrow[r, "\phi_1^*"] & K_1(A) \arrow[d, "\partial"] \\
 & K_0(A) \arrow[ul, "\rho", "\cong"'] \arrow[u, dashed, "\partial_2 \circ \rho"'] & K_0(B) \arrow[l, "\phi_0^*"] & K_0(A) \arrow[l, "\pi_0^*"]
\end{tikzcd}$$

The map $\partial_2 \circ \rho$ is often referred to as the exponential map because of the structure of the Bott map, which is an essential part in ρ.

<u>VI.7. Theorem</u>. The six term sequence is exact.

<u>Proof</u>: We only have to show exactness at $K_0(A)$. First, consider the case where B, A and ϕ are unital. Then, by writing down all the maps whose composition $\partial_2 \circ \rho$ is, we get the following commutative diagram.

$$\begin{array}{ccccccccccc}
K_0(A) & \xrightarrow{\gamma_A} & \pi_1(Gl(A)) & \xrightarrow{\cong} & \pi_0(Gl(\tilde{SA})) & \xrightarrow{\cong} & K_1(SA) & \xrightarrow{\cong} & K_0(S^2A) & \xrightarrow{\partial_{2\phi}} & K_1(L) \\
\uparrow \phi_0^* & & \uparrow \phi_1^{\#} & & \uparrow (S\phi)^{\#} & & \uparrow \phi_1^* & & \uparrow \phi_2^* & & \\
K_0(B) & \xrightarrow[\gamma_B]{} & \pi_1(Gl(B)) & \xrightarrow{\cong} & \pi_0(Gl(SB)) & \xrightarrow{\cong} & K_1(SB) & \xrightarrow{\cong} & K_0(S^2B) & & .
\end{array}$$

We condense this to the commutative diagram

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow[\cong]{\rho_A} & K_0(S^2A) & \xrightarrow{\partial_{2\phi}} & K_1(L) \\
\uparrow \phi_0^* & & \uparrow \phi_2^* & & \\
K_0(B) & \xrightarrow[\rho_B]{\cong} & K_0(S^2B) . & &
\end{array}$$

Since $K_0(S^2B) \xrightarrow{\phi_2^*} K_0(S^2A) \xrightarrow{\partial_2} K_1(L)$ is part of the long exact sequence, this proves that the six term sequence is exact at $K_0(A)$. In the general case, we replace A and B by $\tilde{A}$ and $\tilde{B}$, respectively, to get a diagram

$$\begin{tikzcd}
K_0(\tilde{A}) \arrow[r, "\rho_{\tilde{A}}"] & K_0(S^2\tilde{A}) \arrow[r, "\partial_{2\phi}"] & K_1(L) \\
K_0(\tilde{B}) \arrow[u, "(\tilde{\phi})_0^*"] \arrow[r, "\rho_{\tilde{B}}"'] & K_0(S^2\tilde{B}) \arrow[u, "(\tilde{\phi})_2^*"] &
\end{tikzcd} .$$

We saw in VI.6 that the restrictions $\rho_{\tilde{A}}\big|_{K_0(A)}$ and $\rho_{\tilde{B}}\big|_{K_0(B)}$ are isomorphisms from $K_0(A) \to K_0(S^2A)$ and $K_0(B) \to K_0(S^2B)$, respectively. Moreover, from III.7, we know that $\phi_0^* = (\tilde{\phi})_0^*\big|_{K_0(B)}$. If we can show that $\phi_2^* = (\tilde{\phi})_2^*\big|_{K_0(S^2B)}$ and $\partial_{2\phi} = \partial_{2\tilde{\phi}}\big|_{K_0(S^2\tilde{A})}$, we get a diagram as in the unital case, which proves the exactness of the six term sequence at $K_0(A)$. But we see as in III.7, since $0 \to S^2A \to S^2\tilde{A} \to S^2\mathbb{C} \to 0$ and $0 \to S^2B \to S^2\tilde{B} \to S^2\mathbb{C} \to 0$ are split exact, that $\phi_2^* = (S^2\phi)_0^* = (S^2\tilde{\phi})_0^*\big|_{K_0(S^2B)}$. Moreover, out of a similar reasoning, $\partial_{2\phi} = \partial_{2\tilde{\phi}}\big|_{K_0(S^2A)}$.

CHAPTER VII: A MAYER-VIETORIS SEQUENCE

In this chapter we present a Mayer-Vietoris Sequence which seems to be a useful tool in calculating the K-theory of C*-algebras which are gotten as pullbacks. Jonathan Rosenberg uses a Mayer-Vietoris Sequence for certain continuous trace algebras, but to the authors knowledge, nobody treated the general case.

Let B_1, B_2 and C be C*-algebras and $f_i : B_i \to C$ be C*-morphisms for $i = 1,2$. Consider the pullback

$$\begin{tikzcd} & B_1 \arrow[dr, "f_1"] & \\ D \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & C \\ & B_2 \arrow[ur, "f_2"'] & \end{tikzcd} .$$

The C*-algebra D can be written as $\{(b_1,b_2) \in B_1 \oplus B_2 : f_1(b_1) = f_2(b_2)\}$. Then there is a natural inclusion $j : D \to B_1 \oplus B_2$. The map j induces group homomorphisms $j_n^* : K_n(D) \to K_n(B_1) \oplus K_n(B_2)$. We define group homomorphisms $\nu_n : K_n(B_1) \oplus K_n(B_2) \to K_n(C)$ as $\nu_n := (f_1)_n^* - (f_2)_n^*$. This means, for $A_i \in K_n(B_i)$, that $\nu_n(A_1 \oplus A_2) = (f_1)_n^*(A_1) - (f_2)_n^*(A_2) \in K_n(C)$. There are two more maps which will play an important role in the

Mayer-Vietoris sequence. We construct $\tau_0 : K_0(C) \to K_1(D)$, the map $\tau_1 : K_1(C) \to K_0(D)$ will be constructed analogously. Note first that there is a natural isomorphism between $\ker f_2$ and $\ker g_1$. Let $\ell : \ker f_2 \xrightarrow{\ell} D$ be the inclusion induced by that isomorphism. We get the following commutative diagram with exact rows if f_2 is surjective:

$$\begin{tikzcd}
0 \arrow[r] & \ker f_2 \arrow[r, "\ell"] \arrow[d, equal] & D \arrow[r, "g_1"] \arrow[d, "g_2"] & B_1 \arrow[r] \arrow[d, "f_1"] & 0 \\
0 \arrow[r] & \ker f_2 \arrow[r] & B_2 \arrow[r, "f_2"] & C \arrow[r] & 0 .
\end{tikzcd}$$

This diagram induces by III.8 the following commutative diagram with exact rows:

$$\begin{tikzcd}
K_0(D) \arrow[r, "(g_1)_0^*"] \arrow[d, "(g_2)_0^*"] & K_0(B_1) \arrow[r, "\partial_g"] \arrow[d, "(f_1)_0^*"] & K_1(\ker f_2) \arrow[r, "\ell_1^*"] \arrow[d, equal] & K_1(D) \arrow[r, "(g_1)_1^*"] \arrow[d, "(g_2)_1^*"] & K_1(B_1) \arrow[d, "(f_1)_1^*"] \\
K_0(B_2) \arrow[r, "(f_2)_0^*"'] & K_0(C) \arrow[r, "\partial_f"'] & K_1(\ker f_2) \arrow[r, "i_1^*"'] & K_1(B_2) \arrow[r, "(f_2)_1^*"'] & K_1(C)
\end{tikzcd}$$

Now $\tau_0 : K_0(C) \to K_1(D)$ is defined as $\tau_0 := \ell_1^* \circ \partial_f$.

<u>VII.1. Theorem</u> (Mayer-Vietoris Sequence). Let B_1, B_2 and C be as above. Then the following sequence is exact if the map f_2 is surjective.

$$\begin{array}{ccccc} K_0(D) & \xrightarrow{j_0^*} & K_0(B_1) \oplus K_0(B_2) & \xrightarrow{\nu_0} & K_0(C) \\ \uparrow \tau_1 & & & & \downarrow \tau_0 \\ K_1(C) & \xleftarrow[\nu_1]{} & K_1(B_1) \oplus K_1(B_2) & \xleftarrow[j_1^*]{} & K_1(D) \end{array}$$

Proof: First we show that $\mathrm{im}\ j_n^* \subset \ker \nu_n$. For $A \in K_n(D)$, we have $\nu_n(j_n^*(A)) = \nu_n((g_1)_n^*(A) \oplus (g_2)_n^*(A)) = (f_1)_n^*((g_1)_n^*(A)) - (f_2)_n^*((g_2)_n^*(A)) = (f_1 \circ g_1)_n^*(A) - (f_2 \circ g_2)_n^*(A) = 0$. The reverse inclusion we have to do separately for $n = 0,1$. We show that $\ker \nu_1 \subset \mathrm{im}\ j_1^*$. To do this, we first consider the case where all algebras and morphisms are unital. The general case can then be easily derived from this using the methods of Chapter III. For any of the involved algebras, call it A, we view $K_1(A)$ as $\pi_0(\mathrm{Gl}(A))$. Now let $b_i \in \mathrm{Gl}(B_i)$ and $[b_i]$ be its component in $\mathrm{Gl}(B_i)$. Suppose that $\nu_1([b_1] \oplus [b_2]) = (f_1)_1^*([b_1]) - (f_2)_1^*([b_2]) = 0$, then for $f_i^{\#} : \mathrm{Gl}(B_i) \to \mathrm{Gl}(C)$ the f_i - induced map, we have that $[f_1^{\#}(b_1)] = [f_2^{\#}(b_2)]$ in $\pi_0(\mathrm{Gl}(C))$. Thus, there exists a path $\gamma : I \to \mathrm{Gl}(C)$ connecting $f_1^{\#}(b_1)$ and $f_2^{\#}(b_2)$. As was shown in VI.2, the path γ is actually a path in $\mathrm{Gl}_k(C)$ for some $k \in \mathbb{N}$. Thus, we can apply Lemma I.23 to the algebras $M_k(B_2)$ and $M_k(C)$ to obtain an element $b_2' \in \mathrm{Gl}(B_2)$ with $f_1^{\#}(b_1) = f_2^{\#}(b_2')$ and such that b_2' and

b_2 are pathconnected in $Gl(b_2)$. This shows that (b_1, b_2') is an element in $Gl(D)$ and moreover, $j_1^*([b_1, b_2')]) = [b_1] \oplus [b_2'] = [b_1] \oplus [b_2]$. Thus $\ker \nu_1 \subset \operatorname{im} j_1^*$.

To see that $\ker \nu_0 \subset \operatorname{im} j_0^*$, note first that the suspension, being an exact, additive functor, respects pullbacks. Hence, we have the following pullback diagram:

$$\begin{tikzcd} & SB_1 \arrow[dr, "Sf_1"] & \\ SD \arrow[ur, "Sg_1"] \arrow[dr, "Sg_2"'] & & SC \\ & SB_2 \arrow[ur, "Sf_2"'] & \end{tikzcd}$$

The preceding shows that the following sequence is exact:

$$K_1(SD) \xrightarrow{(Sj)_1^*} K_1(SB_1) \oplus K_1(SB_2) \xrightarrow{(Sf_1)_1^* - (Sf_2)_1^*} K_1(SC).$$

But from Chapter V, we see that the following diagram commutes.

$$\begin{tikzcd} K_0(D) \arrow[r, "j_0^*"] \arrow[d, "\cong"] & K_0(B_1) \oplus K_0(B_2) \arrow[r, "\nu_0"] \arrow[d, "\cong"] & K_0(C) \arrow[d, "\cong"] \\ K_1(SD) \arrow[r, "(Sj)_1^*"] & K_1(SB_1) \oplus K_1(SB_2) \arrow[r, "(Sf_1)_1^* - (Sf_2)_1^*"] & K_1(SC) \end{tikzcd}$$

The exactness of the lower row shows that the upper row is exact since the vertical maps are isomorphisms.

It remains to be shown that the Mayer-Vietoris sequence is exact at the corners. We show that for the right side, the left side is proved analogously. To see that $\operatorname{im}\nu_0 \subset \ker \tau_0$ calculate for $A_i \in K_0(B_i)$ that $\tau_0((f_1)_0^*(A_1) - (f_2)_0^*(A_2)) = \tau_0((f_1)_1^*(A_1)) - \tau_0((f_2)_0^*(A_2)) = \ell_1^* \circ \partial_g(A_1) - \ell_1^* \circ \partial_f \circ (f_2)_0^*(A_2) = 0$. The reverse inclusion is done by a little diagram chasing. Suppose, for $c \in K_0(C)$, that $\tau_0(c) = 0$. Then $\ell_1^* \circ \partial_f(c) = 0$ and there exists an $A_1 \in K_0(B_1)$ with $\partial_g(A_1) = \partial_f(c)$. Therefore, $\partial_f((f_1)_0^*(A_1) - c) = \partial_g(A_1) - \partial_f(c) = 0$. This in turn implies that there exists an $A_2 \in K_0(B_2)$ with $(f_2)_0^*(A_2) = (f_1)_0^*(A_1) - c$, thus $c = \nu_0(A_1 \oplus A_2)$. The inclusion $\operatorname{im}\tau_0 \subset \ker j_1^*$ we see from the following calculation for $c \in K_0(C)$: we have $j_1^*(\tau_0(c)) = j_1^*(\ell_1^* \circ \partial_f(c)) = (g_1^*)_1 \circ \ell_1^* \circ \partial_f(c) \oplus (g_2)_1^* \circ \ell_1^* \circ \partial_f(c) = 0 \oplus i_1^* \circ \partial_f(c) = 0$. Finally, we get the reverse inclusion again by diagram chasing. Note that $\ker j_1^* = \ker(g_1)_1^* \cap \ker(g_2)_1^*$. Thus, for $d \in \ker j_1^*$, there exists an $A \in K_1(\ker f_2)$ with $\ell_1^*(A) = d$. We get $i_1^*(A) = (g_2)_1^* \circ \ell_1^*(A) = 0$ and hence, there exists a $c \in K_0(C)$ with $\partial_f(c) = A$. This implies that $\tau_0(c) = \ell_1^* \circ \partial_f(c) = \ell_1^*(A) = d$. This concludes the proof. □

CHAPTER VIII: MULTIPLICATIVE STRUCTURES

In this chapter we give a few canonical multiplicative structures relating the K-theory of two nuclear C*-algebras B_1 and B_2 to the K-theory of their tensor product. Karoubi described those for unital algebras in a fairly abstract manner in [K_2]. We give a more concrete description, also for nonunital algebras. Moreover, we describe a way of providing $K_n(B)$ with a module structure.

For two C*-algebras B_1 and B_2, we can form the algebraic tensor product $B_1 \otimes_{\mathbb{C}} B_2$. We can provide $B_1 \otimes_{\mathbb{C}} B_2$ with C*-crossnorms. If the algebras are nuclear, all possible C*-crossnorms agree. Denote the completion of $B_1 \otimes_{\mathbb{C}} B_2$ with respect to this norm by $B_1 \overline{\otimes} B_2$. The tensor product $\overline{\otimes}$ is natural with respect to morphisms $B_1 \to A_1$ and $B_2 \to A_2$. Moreover, tensoring with a fixed nuclear algebra is an exact functor as we see from [G]. From now on all C*-algebras are assumed to be nuclear.

Let $E_i \in Ob(P(B_i))$ for $i = 1,2$. If the E_i are free, say $E_i = B^{n_i}$, then we have a canonical isomorphism between $E_1 \otimes_{\mathbb{C}} E_2$ and $(B_1 \overline{\otimes}_{\mathbb{C}} B_2)^{n_1 \cdot n_2}$. Define $E_1 \overline{\otimes}_{\mathbb{C}} E_2$ to be the closure of $E_1 \otimes_{\mathbb{C}} E_2$ in $(B_1 \otimes B_2)^{n_1 \cdot n_2}$ with the product norm. Now suppose that E_i is embedded in $B_i^{n_i}$ as a retract. Let $j_i : E_i \to B_i^{n_i}$ be the embedding, and $p_i : B_i^{n_i} \to E_i$ the retractions. We topologize $E_1 \otimes_{\mathbb{C}} E_2$ with quotient topology of the map $p_1 \otimes p_2$ which is clearly surjective. As in I.3, we see that $j_1 \otimes j_2$ is an embedding with respect to this topology. Thus, we can define $E_1 \bar{\otimes} E_2$ as the closure of $E_1 \otimes_{\mathbb{C}} E_2$ in $B_1^{n_1} \bar{\otimes} B_2^{n_2} = (B_1 \bar{\otimes} B_2)^{n_1 \cdot n_2}$. We have to show that this definition does not depend on the particular embeddings. In fact, since $E_1 \otimes_{\mathbb{C}} E_2$ is finitely generated projective over $B_1 \otimes_{\mathbb{C}} B_2$, we see as in I.3 that the topology on $E_1 \otimes_{\mathbb{C}} E_2$ does not depend on the choice of embeddings and projections. But the closure in $B_1^{n_1} \bar{\otimes} B_2^{n_2}$ is just the completion of $E_1 \otimes_{\mathbb{C}} E_2$ with respect to that topology, since $j_1 \otimes j_2$ is an embedding. We show that $E_1 \bar{\otimes} E_2 \in \mathrm{Ob}(P(B_1 \bar{\otimes} B_2))$. The map $p_1 \otimes p_2 : B_1^{n_1} \otimes B_2^{n_2} \to E_1 \otimes E_2$ is continuous, so there exists a unique extension to the completions $p_1 \bar{\otimes} p_2 : B_1^{n_1} \bar{\otimes} B_2^{n_2} \to E_1 \bar{\otimes} E_2$.

Similarly, we get a unique map $j_1 \bar{\otimes} j_2 : E_1 \bar{\otimes} E_2$ $B_1^{n_1} \bar{\otimes} B_2^{n_2}$. But we have $(p_1 \bar{\otimes} p_2) \circ (j_1 \bar{\otimes} j_2) = \mathrm{id}_{E_1 \bar{\otimes}_{\mathbb{C}} E_2}$ Thus, $p_1 \bar{\otimes} p_2$ is onto, and $j_1 \bar{\otimes} j_2$ is one-to-one. Uniqueness of the extension also ensures that $(j_1 \bar{\otimes} j_2) \circ (p_1 \bar{\otimes} p_2)$ is an idempotent.

<u>VIII.1. Proposition</u>. Let B_1 and B_2 be unital nuclear C^*-algebras and $E_2 \in \mathrm{Ob}(P(B_2))$. Then, tensoring with E_2

is an additive, exact functor from $P(B_1)$ to $P(B_1 \mathbin{\overline{\otimes}} B_2)$ which is natural with respect to C*-morphisms $B_i \to A_i$ for $i = 1,2$.

Proof: In view of the above remarks, it is easy to check that it is a functor. It is enough to show that the functor transforms short exact sequences into short exact sequences. Let $0 \longrightarrow E \xrightarrow{\alpha} F \xrightarrow{\beta} H \longrightarrow 0$ be short exact in $P(B_1)$. The algebraic tensor product with E_2 is an exact functor, so we have an exact sequence

$$0 \longrightarrow E \otimes_{\mathbb{C}} E_2 \xrightarrow{\alpha \otimes id} F \otimes_{\mathbb{C}} E_2 \xrightarrow{\beta \otimes id} H \otimes_{\mathbb{C}} E_2 \longrightarrow 0.$$

Moreover, since H is projective, we have a splitting $\gamma : H \to F$ which induces a splitting $\gamma \otimes id : H \otimes_{\mathbb{C}} E_2 \to F \otimes_{\mathbb{C}} E_2$. We get a sequence

$$0 \longrightarrow E \mathbin{\overline{\otimes}} E_2 \xrightarrow{\alpha \overline{\otimes} id} F \mathbin{\overline{\otimes}} E_2 \xrightarrow{\beta \overline{\otimes} id} H \mathbin{\overline{\otimes}} E_2 \longrightarrow 0$$

and a map $\gamma \mathbin{\overline{\otimes}} id : H \mathbin{\overline{\otimes}} E_2 \to F \mathbin{\overline{\otimes}} E_2$. Uniqueness of the completion shows that $(\beta \mathbin{\overline{\otimes}} id) \circ (\gamma \mathbin{\overline{\otimes}} id) = id_{H \overline{\otimes} E_2}$, thus $\beta \mathbin{\overline{\otimes}} id$ is surjective and the sequence is exact at $H \mathbin{\overline{\otimes}} E_2$. But the splitting $\gamma \mathbin{\overline{\otimes}} id$ induces a splitting $\delta \mathbin{\overline{\otimes}} id : F \mathbin{\overline{\otimes}} E_2 \to E \mathbin{\overline{\otimes}} E_2$. So we see similarly as above that the sequence is exact at $E \mathbin{\overline{\otimes}} E_2$. Again, by the uniqueness of the completion of a map, we see that

$(\beta \mathbin{\overline{\otimes}} \mathrm{id}) \circ (\alpha \mathbin{\overline{\otimes}} \mathrm{id}) = 0$. Moreover, if $a \in F \mathbin{\overline{\otimes}} E_2$ is in the kernel of $\beta \mathbin{\overline{\otimes}} \mathrm{id}$, and $a_k \in F \otimes E_2$ tend to a, then $\beta \otimes \mathrm{id}(a_k) = \beta \mathbin{\overline{\otimes}} \mathrm{id}(a_k)$ tends to zero in $H \otimes_{\mathbb{C}} E_2$. Replacing a_k by $a_k - (\gamma \otimes \mathrm{id}) \circ (\beta \otimes \mathrm{id})(a_k)$, we can assume that $a_k \in \ker(\beta \otimes \mathrm{id}) = \mathrm{im}(\alpha \otimes \mathrm{id})$. Then the sequence $b_k := \delta \otimes \mathrm{id}(a_k)$ converges and $(\alpha \otimes \mathrm{id})(b_k) = a_k$ since $\delta \otimes \mathrm{id}$ is a topological isomorphism from $\mathrm{im}(\alpha \otimes \mathrm{id})$ to $E \otimes E_2$ with inverse $\alpha \otimes \mathrm{id}$. Thus, $a = \alpha \mathbin{\overline{\otimes}} \mathrm{id}(\lim_k b_k)$ and the sequence is exact also at $F \mathbin{\overline{\otimes}} E_2$. It is an easy consequence of this that the tensor-product $\overline{\otimes}$ distributes over direct sums.

Now let $\phi_i : B_i \to A_i$ be unital C*-morphisms between nuclear C*-algebras. We want to show that for $E_i \in \mathrm{Ob}(P(B_i))$, the modules $(A_1 \mathbin{\overline{\otimes}} A_2) \otimes_{B_1 \overline{\otimes} B_2} (E_1 \mathbin{\overline{\otimes}} E_2)$ and $(A_1 \otimes_{B_1} E_1) \mathbin{\overline{\otimes}} (A_2 \otimes_{B_2} E_2)$ are equal. If E_1 and E_2 are free, this is clear. But since the modules are projective and all the involved tensor products distribute over direct sums, the general case follows easily. $\square$

The isomorphism classes of objects in $P(B_i)$ form commutative monoids S_i with respect to taking direct sums as addition. The assignment $\hat{\mu} : S_1 \times S_2 \to T$, induced by the tensor product, where T is the monoid of isomorphism classes of objects in $P(B_1 \mathbin{\overline{\otimes}} B_2)$, is bilinear. if $\mathbb{A}\mathrm{bsem}$ is the category of abelian monoids, $\mathbb{A}\mathrm{b}$ the category of abelian groups, and $G : \mathbb{A}\mathrm{bsem} \to \mathbb{A}\mathrm{b}$ the Grothendieck functor, then

$G(S_i) = K_0(B_i)$ and $G(T) = K_0(B_1 \bar{\otimes} B_2)$. Moreover, we have the following isomorphisms:

$$\begin{tikzcd}
\mathrm{Bil}(S_1 \times S_2, T) \arrow[r] & \mathcal{A}\mathrm{bsem}(S_1, \mathcal{A}\mathrm{bsem}(S_2,T)) \arrow[r] & \mathcal{A}\mathrm{bsem}(S_1, \mathcal{A}\mathrm{b}(G(S_2), G(T))) \arrow[d] \\
\mathrm{Bil}(G(S_1) \times G(S_2), G(T)) & & \mathcal{A}\mathrm{b}(G(S_1), \mathcal{A}\mathrm{b}(G(S_2), G(T))) . \arrow[ll]
\end{tikzcd}$$

Thus, μ induces a bilinear map $\mu : K_0(B_1) \times K_0(B_2) \to K_0(B_1 \bar{\otimes} B_2)$. The formula is $\mu(\overline{[E_1]} - \overline{[F_1]}, \overline{[E_2]} - \overline{[F_2]}) = \overline{[E_1 \bar{\otimes} E_2]} - \overline{[E_1 \bar{\otimes} F_2]} - \overline{[F_1 \bar{\otimes} E_2]} + \overline{[F_1 \bar{\otimes} F_2]}$. Thus we get the following lemma.

<u>VIII.2. Lemma</u>. Let B_i be unital C*-algebras. Then the tensor product $\bar{\otimes} : \mathrm{Ob}(P(B_1)) \times \mathrm{Ob}(P(B_2)) \to \mathrm{Ob}(P(B_1 \bar{\otimes} B_2))$ induces a bilinear map $\mu : K_0(B_1) \times K_0(B_2) \to K_0(B_1 \bar{\otimes} B_2)$ which is natural with respect to unital C*-morphisms $\phi_1 : B_i \to A_i$. $\square$

Let $\phi_i : B_i \to A_i$ be unital C*-morphisms. We define a unital C*-algebra $P(\phi_1, \phi_2)$ as the following pullback

$$\begin{tikzcd}
& B_1 \bar{\otimes} A_2 \arrow[dr, "\phi_1 \bar{\otimes} \mathrm{id}"] & \\
P(\phi_1,\phi_2) \arrow[ur] \arrow[dr] & & A_1 \bar{\otimes} A_2 \\
& A_1 \bar{\otimes} B_2 \arrow[ur, "\mathrm{id} \bar{\otimes} \phi_2"'] &
\end{tikzcd}$$

.

The maps $\mathrm{id}_{B_1} \bar{\otimes} \phi_2 : B_1 \bar{\otimes} B_2 \to B_1 \bar{\otimes} A_2$ and $\phi_1 \bar{\otimes} \mathrm{id}_{B_2}$: $B_1 \bar{\otimes} B_2 \to A_1 \bar{\otimes} B_2$ induce a map $\chi : B_1 \bar{\otimes} B_2 \to P(\phi_1,\phi_2)$.

<u>VIII.3. Lemma</u>. For $\phi_i : B_i \to A_i$ surjective unital C*-morphisms, the following sequence is exact:

$$0 \longrightarrow \ker \phi_1 \bar{\otimes} \ker \phi_2 \longrightarrow B_1 \bar{\otimes} B_2 \xrightarrow{\chi} P(\phi_1,\phi_2) \longrightarrow 0.$$

<u>Proof</u>: Let $L_i := \ker \phi_i$. Tensor the exact sequence $0 \to L_2 \to B_2 \to A_2 \to 0$ with L_1 to get the exact sequence $0 \to L_1 \bar{\otimes} L_2 \to L_1 \bar{\otimes} B_2 \to L_1 \bar{\otimes} A_2 \to 0$. Similarly, we get $0 \to L_1 \bar{\otimes} B_2 \to B_1 \bar{\otimes} B_2 \to A_1 \bar{\otimes} B_2 \to 0$. So we get an exact sequence

$$0 \to \frac{L_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2} \to \frac{B_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2} \to \left(\frac{B_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2}\right) \Big/ \left(\frac{L_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2}\right) \to 0$$

Using the second isomorphism theorem, we can rephrase this to

$$0 \to L_1 \bar{\otimes} A_2 \to \frac{B_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2} \to A_1 \bar{\otimes} B_2 \to 0.$$

The map χ induces a map

$$\bar{\chi} : \frac{B_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2} \to P(\phi_1,\phi_2).$$

Moreover, there is a map $\tau : L_1 \bar{\otimes} A_2 \to P(\phi_1,\phi_2)$ induced by $L_1 \bar{\otimes} A_2 \hookrightarrow B_1 \bar{\otimes} A_2$ and $L_1 \bar{\otimes} A_2 \xrightarrow{0} A_1 \bar{\otimes} B_2$. It is easy to see that $A_1 \bar{\otimes} B_2$ is the cokernel of τ, using the fact that $P(\phi_1,\phi_2)$ is the pullback of $A_1 \bar{\otimes} B_2$ and $B_1 \bar{\otimes} A_2$. Thus we obtain the following diagram, which is easily checked to be commutative:

$$\begin{tikzcd}
0 \arrow[r] & L_1 \bar{\otimes} A_2 \arrow[r] \arrow[d, equal] & \dfrac{B_1 \bar{\otimes} B_2}{L_1 \bar{\otimes} L_2} \arrow[r] \arrow[d, "\bar{\chi}"] & A_1 \bar{\otimes} B_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_1 \bar{\otimes} A_2 \arrow[r, "\tau"] & P(\phi_1,\phi_2) \arrow[r] & A_1 \bar{\otimes} B_2 \arrow[r] & 0.
\end{tikzcd}$$

This proves that $\bar{\chi}$ is an isomorphism, whence the claim. □

We now turn to the case where $\phi_i : \tilde{B}_i \to \dfrac{\tilde{B}_i}{B_i} = \mathbb{C}$ is the canonical surjection.

<u>VIII.4. Lemma</u>. Let B_i be C*-algebras and $\phi_i : \tilde{B}_i \to \mathbb{C}$ be the canonical surjections. For $P := P(\phi_1,\phi_2)$ and $\chi : \tilde{B}_1 \bar{\otimes} \tilde{B}_2 \to P$ the natural map, the induced map $\chi_1^* : K_1(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) \to K_1(P)$ is surjective.

<u>Proof</u>: The algebra P is given by the following pullback

$$\begin{tikzcd}
 & \tilde{B}_1 \arrow[dr, "\phi_1"] & \\
P \arrow[ur] \arrow[dr] & & \mathbb{C} \\
 & \tilde{B}_2 \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

An element in $K_1(P)$ is therefore given by invertible $\Delta_i = (b_i^{(\alpha\beta)})_{\alpha,\beta=1,\dots n} \in Gl_n(\tilde{B}_i)$ for some $n \in \mathbb{N}$, such that

$$\Lambda := (\phi_1(b_1^{(\alpha\beta)}))_{\alpha,\beta=1\dots n} = (\phi_2(b_2^{(\alpha\beta)}))_{\alpha,\beta=1\dots n} \in Gl_n(\mathbb{C}).$$

Define matrices in $Gl_n(\tilde{B}_1 \bar{\otimes} \tilde{B}_2)$ by $\Lambda_1 := (b_1^{(\alpha\beta)} \bar{\otimes} 1_{B_2})_{\alpha,\beta=1\dots n}$ and $\Lambda_2 := (1_{B_1} \bar{\otimes} b_2^{(\alpha\beta)})_{\alpha,\beta=1\dots n}$. It is now easy to check that $\chi^{\#} : Gl_n(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) \to Gl_n(P)$ maps $\Lambda_1 \cdot \Lambda^{-1} \cdot \Lambda_2$ to the pair (Δ_1, Δ_2), which proves the claim. $\square$

We extend now our definition of the cup product to nonunital algebras. The lemma and the long exact sequence yield an exact sequence

$$0 \longrightarrow K_0(B_1 \bar{\otimes} B_2) \longrightarrow K_0(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) \xrightarrow{\chi_0^*} K_0(P).$$

Moreover, the Mayer-Vietoris sequence for P shows that the map $\gamma_0^* : K_0(P) \to K_0(\tilde{B}_1 \bar{\otimes} \mathbb{C}) \oplus K_0(\mathbb{C} \bar{\otimes} \tilde{B}_2)$, induced by the natural map $\gamma : P \to (\tilde{B}_1 \bar{\otimes} \mathbb{C}) \oplus (\mathbb{C} \bar{\otimes} \tilde{B}_2)$, is injective. We get the following diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & K_0(B_1 \bar{\otimes} B_2) & \longrightarrow & K_0(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) & \xrightarrow{\chi_0^*} & K_0(P) & \xrightarrow{\gamma_0^*} & K_0(\tilde{B}_1 \bar{\otimes} \mathbb{C}) \oplus K_0(\mathbb{C} \bar{\otimes} \tilde{B}_2) \\
& & & & & \uparrow{\scriptstyle \mu} & & \\
0 \longrightarrow & K_0(B_1) \oplus K_0(B_2) & \xrightarrow{j} & K_0(\tilde{B}_1) \oplus K_0(\tilde{B}_2) & & \longrightarrow & & K_0(\mathbb{C}) \oplus K_0(\mathbb{C}).
\end{array}$$

In order to define the cup product $\mu : K_0(B_1) \oplus K_0(B_2) \to K_0(B_1 \bar{\otimes} B_2)$ as the restriction of the product on $K_0(\tilde{B}_1) \oplus K_0(\tilde{B}_2)$, we have to show that $\chi_0^* \circ \mu \circ j = 0$. But the maps $pr_1 \circ \gamma_0^* \circ \chi_0^*$ and $pr_2 \circ \gamma_0^* \circ \chi_0^*$ are given by $(id_{\tilde{B}_1} \bar{\otimes} \phi_2)_0^*$ and $(\phi_1 \bar{\otimes} id_{\tilde{B}_2})_0^*$, respectively, if $pr_i : K_0(\tilde{B}_1) \oplus K_0(\tilde{B}_2) \to K_0(\tilde{B}_i)$ is the projection on the i-th summand. Moreover, the cup product is natural, i.e., the following diagram commutes

$$\begin{tikzcd}
K_0(\tilde{B}_1 \bar{\otimes} \tilde{B}_2) \arrow[r] & K_0(\tilde{B}_1 \bar{\otimes} \mathbb{C}) \\
K_0(\tilde{B}_1) \oplus K_0(\tilde{B}_2) \arrow[r] \arrow[u, "\mu"] & K_0(\tilde{B}_1) \oplus K_0(\mathbb{C}) \arrow[u, "\mu"']
\end{tikzcd}$$

Thus $pr_1 \circ \gamma_0^* \circ \chi_0 \circ \mu \circ j = 0$. Similarly, $pr_2 \circ \gamma_0^* \circ \chi_0 \circ \mu \circ j = 0$, and hence $\gamma_0^* \circ \chi_0 \circ \mu \circ j = 0$ and since γ_0^* is injective, we have $\chi_0 \circ \mu \circ j = 0$.

Recall that $K_0(B_i)$ is defined as $K_0(\phi_i)$ and $K_0(B_1 \bar{\otimes} B_2)$ is equal to $K_0(\chi)$. Thus we have a cup product on the relative K_0-groups $\mu : K_0(\phi_1) \times K_0(\phi_2) \to K_0(\chi)$. We want to define such a cup product for arbitrary unital C*-surjections $\phi_i : B_i \to A_i$ and the induced map $\chi_B : B_1 \bar{\otimes} B_2 \to P(\phi_1, \phi_2)$. If $L_i := \ker \phi_i$ and $\rho_i : \tilde{L}_i \to \tilde{L}_i/L_i = \mathbb{C}$ are the natural surjections, our construction applies and we get a cup product

$\mu : K_0(\rho_1) \times K_0(\rho_2) \to K_0(\chi_L)$ where $\chi_L : \tilde{L}_1 \bar{\otimes} \tilde{L}_2 \to P(\rho_1,\rho_2)$ is the induced map. The following lemma, together with the excision theorem will establish a natural isomorphism $j : K_0(\chi_L) \to K_0(\chi_B)$. Thus we can define a cup product using the following diagram

$$\begin{tikzcd}
K_0(\rho_1) \times K_0(\rho_2) \arrow[r, "\mu"] \arrow[d, "\cong"] & K_0(\chi_L) \arrow[d, "j"] \\
K_0(\phi_1) \times K_0(\phi_2) \arrow[r, dashed] & K_0(\chi_B)
\end{tikzcd}$$

<u>VIII.5. Lemma</u>. For $i = 1,2$, let the following diagram be a pullback square of unital C*-algebras:

$$\begin{tikzcd}
D_i \arrow[r, two heads, "\rho_i"] \arrow[d, tail, "\delta_i"] & C_i \arrow[d, tail, "\gamma_i"] \\
B_i \arrow[r, two heads, "\phi_i"'] & A_i
\end{tikzcd} .$$

Moreover, let ρ_i and ϕ_i be surjective and δ_i and γ_i be injective. Then the following square of unital C*-algebras is a pullback square:

$$\begin{tikzcd}
D_1 \bar{\otimes} D_2 \arrow[r, two heads, "\chi_0"] \arrow[d, "\delta_1 \bar{\otimes} \delta_2"] & P(\rho_1,\rho_2) \arrow[d, "(\delta_1 \bar{\otimes} \gamma_2{,}\gamma_1 \bar{\otimes} \delta_2)"] \\
B_1 \bar{\otimes} B_2 \arrow[r, two heads, "\chi_B"'] & P(\phi_1,\phi_2)
\end{tikzcd} .$$

<u>Proof</u>: The proof is achieved in two steps. First we show that for any C*-algebra S, the following square is a pullback:

$$\begin{tikzcd} S \overline{\otimes} D_i \arrow[r, "\mathrm{id} \overline{\otimes} \rho_i"] \arrow[d, "\mathrm{id} \overline{\otimes} \delta_i"'] & S \overline{\otimes} C_i \arrow[d, "\mathrm{id} \overline{\otimes} \gamma_i"] \\ S \overline{\otimes} B_i \arrow[r, "\mathrm{id} \overline{\otimes} \phi_i"'] & S \overline{\otimes} A_i \end{tikzcd} .$$

To see this, note first that $I_i := \ker \rho_i$ is naturally isomorphic to $\ker \phi_i$. Moreover, tensoring over $\mathbb{C}$ with a fixed C*-algebra is an exact functor. If Q is the pullback of $\mathrm{id}_S \overline{\otimes} \gamma_i$ and $\mathrm{id}_S \overline{\otimes} \phi_i$, we get the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & S \overline{\otimes} I \arrow[r, dashed, "f"] \arrow[d, equal] & Q \arrow[r] & S \overline{\otimes} C_i \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & S \overline{\otimes} I \arrow[r] & S \overline{\otimes} D_i \arrow[r] \arrow[u, dashed, "g"'] & S \overline{\otimes} C_i \arrow[r] & 0 \end{tikzcd} .$$

The map $f : S \overline{\otimes} I_i \to Q$ is induced by the inclusion $S \overline{\otimes} I_i \hookrightarrow S \overline{\otimes} B_i$ and the zero map $S \overline{\otimes} I_i \to S \overline{\otimes} C_i$. The map $g : S \overline{\otimes} D_i \to Q$ is induced by $\mathrm{id} \overline{\otimes} \delta_i$ and $\mathrm{id} \overline{\otimes} \rho_i$. Thus the diagram commutes and g is an isomorphism, which proves the claim.

Now let L be a unital C*-algebra such that the following is a commutative square of unital C*-algebras:

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "g"'] & P(\rho_1,\rho_2) \arrow[d, "(\delta_1 \bar{\otimes} \gamma_2, \gamma_2 \bar{\otimes} \delta_2)"] \\
B_1 \bar{\otimes} B_2 \arrow[r, "\chi_B"'] & P(\phi_1,\phi_2)
\end{tikzcd} .$$

We have to show that this square induces a unique map σ from L to $D_1 \bar{\otimes} D_2$ such that $f = (id_S \bar{\otimes} \rho_i) \circ \sigma$ and $g = (id_S \bar{\otimes} \delta_i) \circ \sigma$. Consider the followimg commutative diagram

$$\begin{tikzcd}
C_1 \bar{\otimes} D_2 \arrow[d, tail] & D_1 \otimes D_2 \arrow[l, two heads] \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "PB"] & D_1 \bar{\otimes} C_2 \arrow[d, tail, "id \bar{\otimes} \gamma_2"] & L \arrow[lll, bend right, "p_1 \circ f"'] \arrow[l, "p_2 \circ f"'] \arrow[ddll, bend left] \\
C_1 \otimes B_2 \arrow[d, tail] \arrow[dr, phantom, "PB"] & D_1 \bar{\otimes} B_2 \arrow[l, two heads] \arrow[r, two heads, "id \bar{\otimes} \phi_2"] \arrow[d, tail, "\delta_1 \bar{\otimes} id"] & D_1 \bar{\otimes} A_2 \arrow[d, tail, "\delta_1 \bar{\otimes} id"] & \\
A_1 \bar{\otimes} B_2 & B_1 \bar{\otimes} B_2 \arrow[l, two heads] \arrow[r, two heads] & B_1 \bar{\otimes} A_2 &
\end{tikzcd}$$

The preceding shows that the lower left and the upper right squares are pullbacks. From the lower one we get a unique map $h : L \to D_1 \bar{\otimes} B_2$ fitting in the commutative diagram. The fact that $\delta_1 \bar{\otimes} id_{B_2}$ and $\delta_1 \bar{\otimes} id_{A_2}$ are monic now implies that $(id_{D_1} \overline{\otimes} \phi_2) \circ h = (id_{D_1} \overline{\otimes} \gamma_2) \circ (p_2 \circ f)$. Using the upper pullback square, we now get the unique map $\sigma : L \to D_1 \overline{\otimes} D_2$ with the desired properties. □

The cup product induces naturally several other multiplications. To define those, we need the following well known fact.

<u>VIII.6. Lemma</u>. Let A and B be C*-algebras. Then we have a natural isomorphism $\ell_{np} : S^n(A) \overline{\otimes} S^p(P) \to S^{n+p}(A \overline{\otimes} B)$.

<u>Proof</u>: Note that $S^n(A) = C_0(S^n) \overline{\otimes} A$ and $S^p(B) = C_0(S^p) \overline{\otimes} B$. Here S^n denotes the n-dimensional sphere with base point and $C_0(S^n)$ the complex valued continuous functions on S^n vanishing at the base point. Moreover, there is a natural isomorphism $\lambda_{n,p} : C_0(S^n) \overline{\otimes} C_0(S^p) \to C_0(S^n \wedge S^p) = C_0(S^{n+p})$ where $\wedge$ denotes the wedge product. Now the map ℓ_{np} is given by the following composition of maps:

$$C_0(S^n) \overline{\otimes} A \overline{\otimes} C_0(S^p) \overline{\otimes} B \longrightarrow$$

$$\longrightarrow C_0(S^n) \otimes C_0(S^p) \overline{\otimes} (A \overline{\otimes} B) \xrightarrow{\lambda_{np} \overline{\otimes} id_{A\overline{\otimes}B}} C_0(S^{n+p}) \otimes (A \overline{\otimes} B).$$

□

Now we can define a cup product $\mu_{np} : K_n(A) \times K_p(B) \to K_{n+p}(A \overline{\otimes} B)$ by $\mu_{np} = (\ell_{np})_0^* \circ \mu$ where we identify

$K_n(A)$ with $K_0(S^nA)$, $K_p(B)$ with $K_0(S^pB)$ and $K_{n+p}(A \bar{\otimes} B)$ with $K_0(S^{n+p}(A \bar{\otimes} B))$.

$$\begin{tikzcd} K_0(S^nA) \times K_0(S^pB) \arrow[r, "\mu"] \arrow[dr, "\mu_{np}"'] & K_0(S^nA \bar{\otimes} S^pB) \arrow[d, "(\ell_{np})_0^*"] \\ & K_0(S^{n+p}(A \bar{\otimes} B)). \end{tikzcd}$$

We know that $A \bar{\otimes} B$ is isomorphic to $B \bar{\otimes} A$, thus the question arises how it will affect the product μ_{np} if we switch the factors $K_n(A)$ and $K_p(B)$. To answer this question, we have to study a group action of the symmetric group of order k on $K_k(B)$ for any $k \in \mathbb{N}$ and an arbitrary C*-algebra B.

<u>VIII.7. Lemma</u>. Let B be a C*-algebra. Define a map $T : SB \to SB$ by $T(f)(t) = f(1-t)$. Then the induced map in K-theory $T_n^* : K_n(SB) \to K_n(SB)$ is given by $T_n^*(u) = -u$.

<u>Proof</u>: Define $\check{B} := \{f : I \to B,$ continuous$\}$ as the C*-algebra of paths in B. Then we get a short exact sequence $0 \longrightarrow SB \xrightarrow{j} \check{B} \xrightarrow{ev} B \oplus B \longrightarrow 0$ where ev denotes evaluation at the endpoints. Thus we have $K_n(SB) \cong K_n(ev)$. It is easy to see that $\check{B}$ is homotopy equivalent to B and that after identifying $K_n(B)$ with $K_n(B)$ the map $ev_n^* : K_n(\check{B}) \to K_n(B \oplus B)$ is given by the diagonal map. From the long exact sequence we get a commutative diagram

$$\begin{array}{ccc} K_{n+1}(B \oplus B) & \xrightarrow{\partial_n} & K_n(SB) \\ \downarrow s & & \downarrow T_n^* \\ K_{n+1}(B \oplus B) & \xrightarrow{\partial_n} & K_n(SB) \end{array} .$$

Here s denotes the map that switches summands. Moreover, we have

$$K_{n+1}(B) \xrightarrow{ev_{n+1}^*} K_{n+1}(B \oplus B) \xrightarrow{\partial_n} K_n(SB) \xrightarrow{j_n^*} K_n(B) \xrightarrow{ev_n^*} K_n(B \oplus B)$$

exact. From the above we see that ev_n^* is injective, thus j_n^* is the zero map and hence ∂_n is surjective. For $u \in K_n(SB)$, we find a $v \in K_{n+1}(B \oplus B)$ such that $u = \partial_n(v)$. Let v be $v_1 \oplus v_2$ with $v_1, v_2 \in K_{n+1}(B)$, then $u + T_n^*(u) = \partial_n(u) = \partial_n v + \partial_n(s(v)) = \partial_n(v + s(v)) = \partial_n(v_1 \oplus v_2 + v_2 \oplus v_1) = \partial_n((v_1 + v_2) \oplus (v_1 + v_2)) = \partial_n \circ ev_{n+1}^*(v_1 + v_2) = 0$. Thus $T_n^*(u) = -u$. $\square$

Let Σ_n be the symmetric group of order n. It acts on $S^n = S^1 \wedge \ldots \wedge S^1$ by sending $[x_1 \ldots x_n] \in S^1 \wedge \ldots \wedge S^1$ to $[x_{\sigma^{-1}(1)} \ldots x_{\sigma^{-1}(n)}] \in S^1 \wedge \ldots \wedge S^1$. This action induces a group homomorphism $\tau_n : \Sigma_n \to \mathrm{Aut}(C_0(S^n))$, given by $\tau_n(\sigma)(g[t_1 \ldots t_n]) = g[t_{\sigma(1)} \ldots t_{\sigma(n)}]$ for $\sigma \in \Sigma_n$ and $g \in C_0(S^n)$. For any C*-algebra B we get a group homomorphism $\hat{\ } : \Sigma_n \to \mathrm{Aut}(S^n B) = \mathrm{Aut}(C_0(S^n) \bar{\otimes} B)$

by sending $\sigma \in \Sigma_n$ to $\hat{\sigma} = \tau_n(\sigma) \bar{\otimes} \mathrm{id}_B$. This in turn induces a group homomorphism $\hat{}^* : \Sigma_n \to \mathrm{Aut}(K_0(S^n B))$.

<u>VIII.8. Proposition</u>. Let B be a C*-algebra. Then the group homomorphism $\hat{}^* : \Sigma_n \to \mathrm{Aut}(K_0(S^n B))$ is given by $\hat{\sigma}^* = \mathrm{sign}(\sigma) \cdot \mathrm{id}_{K_0(S^n B)}$.

<u>Proof</u>: It clearly suffices to show that $\hat{\sigma}^* = -\mathrm{id}$ for any transposition σ. So let σ_{ij} be the transposition that interchanges the i-th and j-th coordinates. We can assume that $i = 1$ and $j = 2$ since for $\alpha = (i1)(j2) \in \Sigma_n$, $\sigma_{ij} = \alpha\sigma_{12}\alpha^{-1}$ and if $\hat{\sigma}^*_{12} = -\mathrm{id}$, then $\hat{\sigma}^*_{ij} = \hat{\alpha}^*(-\mathrm{id})\hat{\alpha}^{*-1} = -\mathrm{id}$. The automorphism $\tau_n(\sigma_{ij}) : C_0(S^n) = C_0(S^2) \bar{\otimes} C_0(S^{n-2}) \to C_0(S^2) \bar{\otimes} C_0(S^{n-2})$ is given by $\tau_2(\sigma_{ij}) \bar{\otimes} \mathrm{id}_{C_0(S^{n-2})}$. Thus, replacing B by $S^{n-2}(B)$, we see that it suffices to consider the case $n = 2$. Now we view $S^2(B)$ as the continuous functions from $I^2 \to B$ vanishing on the boundary. Then the action of the transposition σ on $S^2(B)$ is given by interchanging the arguments, i.e., $\hat{\sigma}(g)(x,y) = g(y,x)$ for any $g \in S^2(B)$. Consider the homeomorphism $f : \mathbb{R} \to \mathrm{int}\, I$ given by $f(x) = \frac{1}{2} + \frac{x}{2(1+|x|)}$. It satisfies $f(-x) = 1 - f(x)$. Define two endomorphisms α_1 and α_2 of $S^2(B)$ by setting $\alpha_1(g)(x,y) = g(f(f^{-1}(x) + f^{-1}(y)), f(f^{-1}(y) - f^{-1}(x)))$ and $\alpha_2(g)(x,y) = g(f(f^{-1}(y) + f^{-1}(x)), f(f^{-1}(x) - f^{-1}(y)))$ for $(x,y) \in \mathrm{int}\, I^2$ and $\alpha_2(g)(x,y) = 0$

if $(x,y) \in \partial I^2$. It is routine to check that this definition makes sense. Note that α_1 is homotopic to the identity via $\Phi_t(g)(x,y) = g(f(f^{-1}(x) + tf^{-1}(y),\ f(f^{-1}(x) + tf^{-1}(y)))$. Similarly, we see that α_2 is homotopic to $\hat{\sigma}$. If we define $T : S^2(B) = S(SB) \to S(SB) = S^2(B)$ by $T(g)(x,y) = g(x, 1-y)$, then $\alpha_2 = T \circ \alpha_1$ and Lemma VIII.7 combined with the homotopy invariance of the functor K_0 show that $\hat{\sigma}^* = -\mathrm{id}_{K_0(S^2B)}$. $\square$

VIII.9. Proposition. Let A and B be C*-algebras and $s : A \bar{\otimes} B \to B \bar{\otimes} A$ the canonical isomorphism. Then the following diagram commutes,

$$\begin{array}{ccc}
K_0(S^nA) \times K_0(S^pB) & \xrightarrow{\mu_{np}} & K_0(S^{n+p}(A \bar{\otimes} B)) \\
 & & \downarrow (-1)^{np} \\
\downarrow \text{switch} & & K_0(S^{n+p}(A \bar{\otimes} B)) \\
 & & \downarrow (S^{n+p}_{(s)})^*_0 \\
K_0(S^pB) \times K_0(S^nA) & \xrightarrow{\mu_{pn}} & K_0(S^{n+p}(B \bar{\otimes} A))
\end{array} .$$

Proof: The diagram is induced by the following diagram

$$\begin{array}{ccccc}
(C_0(S^n)\bar{\otimes}A)\times(C_0(S^p)\bar{\otimes}B) & \to & (C_0(S^n)\bar{\otimes}A\bar{\otimes}C_0(S^p)\bar{\otimes}B & \to & C_0(S^n\wedge S^p)\bar{\otimes}(A\bar{\otimes}B) \\
 & & & & \downarrow \hat{\sigma}\bar{\otimes}\mathrm{id}_{A\bar{\otimes}B} \\
\downarrow \text{switch} & & & & C_0(S^p\wedge S^n)\bar{\otimes}(A\bar{\otimes}B) \\
 & & & & \downarrow \mathrm{id}\bar{\otimes}s \\
(C_0(S^p)\bar{\otimes}B)\times(C_0(S^n)\bar{\otimes}A) & \to & C_0(S^p)\bar{\otimes}B\bar{\otimes}C_0(S^n)\bar{\otimes}A & \to & C_0(S^p\wedge S^n)\bar{\otimes}(B\bar{\otimes}A)
\end{array}$$

which is commutative if σ is the permutation that sends $[x_1 \dots x_n, y_1 \dots y_p]$ to $[y_1 \dots y_p, x_1 \dots x_n]$. The claim follows because $\operatorname{sign} \sigma = (-1)^{np}$. □

The cup product can be used to provide the K-group of an algebra with multiplicative structures. In fact, if, for two- C*-algebras A and B, there exists a C*-morphism $m : A \bar{\otimes} B \to B$, then $m \circ \mu : K_0(A) \times K_0(B) \to K_0(B)$ is a bilinear map. If $A = B$, it is in fact a ring multiplication if m is associative.

Now identify $K_{2n}(B)$ with $K_0(B)$ and $K_{2n+1}(B)$ with $K_1(B)$. We obtain a $\mathbb{Z}_2$-graded multiplication μ on $K_*(A) = \bigoplus_{i=0,1} K_i(A)$ cross $K_*(B) = \bigoplus_{i=0,1} K_i(B)$ as follows: for elements $a = (a_0 \oplus a_1) \in K_*(A)$ and $b = (b_0 \oplus b_1) \in K_*(B)$, define

$$\begin{aligned}\mu(a,b) = {} & (\mu_{00}(a_0,b_0) + \mu_{11}(a_1,b_1)) \oplus \\ & (\mu_{10}(a_1,b_0) + \mu_{01}(a_0,b_1)) \in K_*(A \bar{\otimes} B).\end{aligned}$$

This multiplication is a bilinear map as follows from the bilinearity of the μ_{ij}.

<u>VIII.9. Proposition</u>. Let B be a C*-algebra and A a subalgebra of the center $Z(B)$ of B. Then $K_*(A)$ is a $\mathbb{Z}_2$-graded ring and $K_*(B)$ is a $\mathbb{Z}_2$-graded $K_*(A)$ module.

<u>Proof</u>: After the preceding remarks, it suffices to show

that there is a C*-morphism $m : A \mathbin{\overline{\otimes}} B \to B$ such that the image of the restriction of m to $A \mathbin{\overline{\otimes}} A$ is contained in A. Since $A \subset Z(B)$, we have a ring homomorphism $A \otimes B \xrightarrow{m} B$ given on elementary tensors by $m(a \otimes b) = ab$. By the universal property of the maximal cross norm on $A \otimes B$, this map extends to the maximal tensor product $A \otimes_\gamma B$ which is equal to $A \mathbin{\overline{\otimes}} B$ since the algebras are nuclear. The rest is clear. $\square$

Finally, note that for a C*-morphism $\phi : B \to D$ and A a subalgebra of $Z(B)$, and $\phi(A) \subset Z(D)$, the induced map $\phi^* : K_*(B) \to K_*(D)$ is a module map with respect to the rings $K_*(A)$ and $K_*(\phi A)$. This follows from the commutativity of the following diagram:

$$\begin{tikzcd}
K_*(A) \times K_*(B) \arrow[r, "\mu"] \arrow[d, "\phi^* \times \phi^*"] & K_*(A \mathbin{\overline{\otimes}} B) \arrow[r] \arrow[d, "(\phi \mathbin{\overline{\otimes}} \phi)^*"] & K_*(B) \arrow[d, "\phi^*"] \\
K_*(\phi A) \times K_*(D) \arrow[r, "\mu"] & K_*(\phi A \mathbin{\overline{\otimes}} D) \arrow[r] & K_*(D)
\end{tikzcd}$$

CHAPTER IX: THE PUPPE SEQUENCE

In this chapter we define a mapping cylinder and a mapping cone for C*-algebras and use it to establish a Puppe Sequence which generalizes the well known Puppe Sequence from K-theory of locally compact spaces.

<u>IX.1. Definition</u>. Let A and B be C*-algebras and $\phi : B \to A$ be a C*-morphism. Define $\check{A} := \{f : I \to A$ continuous$\}$ to be the algebra of paths in A and let $\rho_t : \check{A} \to A$ be the evaluation at t. Then we define the mapping cylinder of ϕ, denoted by M_ϕ as the following pullback:

$$\begin{tikzcd} & B \arrow[dr, "\phi"] & \\ M_\phi \arrow[ur, "p_B"] \arrow[dr, "p_{\check{A}}"'] & & A \\ & \check{A} \arrow[ur, "\rho_0"'] & \end{tikzcd}$$

Note that $M = \{(b,f) \in B \oplus \check{A} : \phi(b) = f(0)\}$.

<u>IX.2. Lemma</u>. In the above situation, the map $\psi = \rho_1 \circ p_{\check{A}} : M_\phi \to A$ is surjective. Moreover, the map $p_B : M_\phi \to B$ is a homotopy equivalence and $\psi \simeq \phi \circ p_B$.

<u>Proof</u>: For any $a \in A$, we have $(0, t\cdot a) \in M_\phi$ and $\psi((0,ta)) = a$, thus ψ is surjective. Define for $t \in I$ a family of maps $\psi_t : M_\phi \to A$ by $\psi_t := \rho_t \circ p_{\check{A}}$, then

$\psi_0 = \rho_0 \circ p_A^\vee = \phi \circ p_B$ and $\psi_1 = \psi$. It is clear that ψ_t is a homotopy, thus $\psi \simeq \phi \circ p_B$. To show that p_B is a homotopy equivalence, we define a map $q_B : B \to M_\phi$ which will turn out to be a homotopy inverse to p_B. Set $q_B(b) = (b, f_b)$, where $f_b : I \to A$ is defined by $f_b(t) = \phi(b)$. With this we have $p_B \circ q_B = id_B$ and $q_B \circ p_B((b,f)) = (b, f_b)$. For $s \in I$ define a family of morphisms $\Phi_s : M_\phi \to M_\phi$ by $\Phi_s((b,f)) = (b, f_s)$, where $f_s(t) = f(st)$. Then $f_s(0) = f(0) = \phi(b)$ which shows that Φ_s is well defined. It is clear that Φ_s is a homotopy. Moreover, we have that $\Phi_0(b,f) = (b, f_0)$ with $f_0(t) = f(0) = \Phi(b)$ and $\Phi_1(b,f) = (b,f)$. Thus Φ_s is a homotopy between $q_B \circ p_B$ and id_{M_ϕ}. This concludes the proof. □

<u>IX.3. Corollary</u>. The map $(p_B)_n^* : K_n(M_\phi) \to K_n(B)$ is an isomorphism and the following diagram is commutative.

$$\begin{tikzcd} K_n(B) \arrow[dr, "\phi_n^*"] & \\ & K_n(A) \\ K_n(M\ ) \arrow[uu, "(p_B)_n^*"] \arrow[ur, "\psi_n^*"'] & \end{tikzcd}$$

<u>Proof</u>: This follows directly from IV.5, IV.6 and IV.7. □

<u>IX.4. Definition</u>. In the above situation we define the mapping cone, denoted by C_ϕ, to be the kernel of $\rho_1 \circ p_A^\vee$. This means $C_\phi = \{(b,f) \in B \oplus A : \phi(b) = f(0), f(1) = 0\}$.

Given the map $i : SA \to C_\phi$, defined by $i(f) = (0,f)$, we get a sequence of C*-algebras, which we call the Puppe Sequence:

$$SB \xrightarrow{S_\phi} SA \xrightarrow{i} C_\phi \xrightarrow{p_B|C_\phi} B \xrightarrow{\phi} A.$$

IX.5. Theorem. The Puppe sequence induces an exact sequence in K-theory:

$$K_n(SB) \xrightarrow{(S_\phi)_n^*} K_n(SA) \xrightarrow{i_n^*} K_n(C_\phi) \xrightarrow{(p_B|C_\phi)_n^*} K_n(B) \xrightarrow{\phi_n^*} K_n(A).$$

Proof: We start with the exactness at $K_n(B)$. Consider the following diagram:

$$\begin{tikzcd} K_n(C_\phi) \arrow[r, "(p_B|C_\phi)_n^*"] \arrow[d, equal] & K_n(B) \arrow[r, "\phi_n^*"] & K_n(A) \arrow[d, equal] \\ K_n(C_\phi) \arrow[r, "j_n^*"'] & K_n(M_\phi) \arrow[u, "(p_B)_n^*"'] \arrow[r, "\psi_u^*"'] & K_n(A) \end{tikzcd} .$$

Here $j : C_\phi \to M_\phi$ denotes the inclusion. Thus the left square is induced by the commutative triangle

$$\begin{tikzcd} & B \\ C_\phi \arrow[ur, "p_B|C_\phi"] \arrow[dr, "j"'] & \\ & M_\phi \arrow[uu, "p_B"'] \end{tikzcd}$$

and hence is commutative. The right square is also commutative as we see from IX.3. The lower row of the diagram is part of the long exact sequence induced by $0 \longrightarrow C_\phi \xrightarrow{j} M_\phi \xrightarrow{\psi} A \longrightarrow 0$ and hence exact. But all the vertical maps are isomorphisms, so the upper row is also exact. To see the exactness at $K_n(C_\phi)$, note that for any $b \in B$, we have that $(b,(1-t)\phi(b)) \in C_\phi$ and $p_B|_{C_\phi}((b,(1-t)\phi(b))) = b$. Hence $p_B|_{C_\phi}$ is surjective. The kernel of $p_B|_{C_\phi}$ is given by

$$\{(b,f) \in B \oplus \check{A} : b = 0,\ f(0) = \phi(b) = 0,\ f(1) = 0\} = i(SA).$$

Thus we have an exact sequence of C*-algebras

$0 \longrightarrow SA \xrightarrow{i} C_\phi \xrightarrow{p_B|_{C_\phi}} B \longrightarrow 0$ which shows that $K_n(SA) \xrightarrow{i_n^*} K_n(C_\phi) \xrightarrow{(p_B|_{C_\phi})_n^*} K_n(B)$ is exact. Finally, we show that the Puppe Sequence is exact at $K_n(SA)$. We define $\check{B} := \{g : [-1,0] \to B \text{ continuous}\}$ and $\gamma_t : \check{B} \to B$ to be the evaluation at $t \in [-1,0]$. Consider the mapping cylinder $M_{(\phi\circ\gamma_0)}$ which is given by the following pullback:

$$\begin{tikzcd} & \check{B} \arrow[dr, "\phi\circ\gamma_0"] & \\ M_{(\phi\circ\gamma_0)} \arrow[ur] \arrow[dr] & & A \\ & \check{A} \arrow[ur, "\rho_0"'] & \end{tikzcd}$$

.

We define a C*-subalgebra D of $M_{(\phi\circ\gamma_0)}$ by setting $D = \{(g,f) \in \check{B} \oplus \check{A} : \phi(g(0)) = f(0), f(1) = 0, g(-1) = 0\}$. For $CB = \{g : [-1,0] \to B : g(-1) = 0\}$, the (slightly modified) cone over B, we get a C*-morphism $\mu : D \to CB$ defined by $\mu((g,f)) = g$. Since $(g,(1-t)\phi(g(0))) \in D$ for any $g \in CB$, we see that μ is surjective. Define a map $k : SA \to D$ by $k(f) = (0,f)$, then k is clearly injective. Moreover, it is easily checked that $k(SA) = \ker \mu$. Hence we have an exact sequence $0 \longrightarrow SA \xrightarrow{k} D \xrightarrow{\mu} CB \longrightarrow 0$. By IV.8, we see that $k_n^* : K_n(SA) \to K_n(D)$ is an isomorphism since CB is contractible. Now define a map $\nu : D \to C_\phi$ by $\nu((g,f)) = (g(0),f)$. Note that for any $(b,f) \in C_\phi$ the pair $((1+t)b,f)$ is in D and thus ν is surjective. For $\ker \nu = \{(g,0) : g(0) = 0 = g(-1)\}$, we get that $K_n(\ker \nu) \to K_n(D) \to K_n(C_\phi)$ is exact. The last C*-algebra we need to consider is $S'B$, defined by $S'B := \{(g_1,g_2) \mid g_1 : [-1,0] \to B,\ g_2 : [0,1] \to B,\ g_1(0) = 0 = g_2(0)\ g_1(-1) = 0 = g_2(1)\}$. We get a map $\chi : S'B \to D$ by $\chi((g_1,g_2)) = (g_1, \phi\circ g_2)$. Note that we also have maps $\ell_1 : \ker \nu \to S'B$ defined by $\ell_1(g,0) = (g,0)$ and $\ell_2 : SB \to S'B$ defined by $\ell_2(g) = (0,g)$. Putting all these maps together, we obtain a diagram which is easily checked to be commutative:

$$\begin{tikzcd}
SB \arrow[r, "\phi"] \arrow[d, "\ell_2"] & SA \arrow[r, "i"] \arrow[d, "k"] & C_\phi \arrow[d, equal] \\
S'B \arrow[r, "\chi"] & D \arrow[r, "\nu"] \arrow[d, equal] & C_\phi \arrow[d, equal] \\
\ker \nu \arrow[u, "\ell_1"] \arrow[r, hook] & D \arrow[r] & C_\phi .
\end{tikzcd}$$

Define a map $\sigma_1 : S'B \to CB = \{g : [0,1] \to B,\ g(1) = 0\}$ by $\sigma_1((g_1,g_2)) = g_2$. Note that for any $g \in CB$, the pair $((1+t)g(0),g)$ is in $S'B$, hence σ_1 is surjective. Moreover, $\ker \sigma_1 = \{(g_1,0) : g_1(-1) = 0 = g_1(0)\} = \ell_1(\ker \nu)$. Thus we get a short exact sequence

$0 \longrightarrow \ker \nu \xrightarrow{\ell_1} S'B \xrightarrow{\sigma_1} CB \longrightarrow 0$. This proves, by IV.8, that $(\ell_1)_n^* : K_n(\ker \nu) \to K_n(S'B)$ is an isomorphism. Define a map $\sigma_2 : S'B \to CB$ by $\sigma_2((g_1,g_2)) = g_1'$ where $g_1' : I \to B$ is defined by $g_1'(t) = g_1(-t)$. Note that for any $g \in CB$, the pair $(g',g(0)(1-t))$ is in $S'B$ and thus σ_2 is surjective. Moreover, $\ker \sigma_2 = \{(0,g_2) : g_2(0) = 0 = g_2(1)\} = \ell_2(SB)$ and we get a short exact sequence $0 \longrightarrow SB \xrightarrow{\ell_2} S'B \xrightarrow{\sigma_2} CB \longrightarrow 0$. As before we see that $(\ell_2)_n^* : K_n(SB) \to K_n(S'B)$ is an isomorphism. We collect all our information in the following diagram which is commutative and whose vertical maps are all isomorphisms:

$$\begin{tikzcd}
K_n(SB) \arrow[r, "(S_\phi)_n^*"] \arrow[d, "(\ell_2)_n^*"] & K_n(SA) \arrow[r, "i_n^*"] \arrow[d, "K_n^*"] & K_n(C_\phi) \arrow[d, equal] \\
K_n(S'B) \arrow[r, "\chi_n^*"] & K_n(D) \arrow[r, "\nu_n^*"] \arrow[d, equal] & K_n(C_\phi) \arrow[d, equal] \\
K_n(\ker \nu) \arrow[r] \arrow[u, "(\ell_1)_n^*"] & K_n(D) \arrow[r, "\nu_n^*"] & K_n(C_\phi) .
\end{tikzcd}$$

The bottom row is exact as we saw before, therefore, so are the middle and top rows. This concludes the proof. □

Note that we get also a Puppe Sequence

$$K_n(S^2B) \to K_n(S^2A) \to K_n(C_{S\phi}) \to K_n(SB) \to K_n(SB)$$

for $S\phi : SB \to SA$. Recall that suspensions respect pullbacks, so

$$\begin{tikzcd}
 & SB \arrow[dr, "S_\phi"] & \\
SM_\phi \arrow[ur] \arrow[dr] & & SA \\
 & SA \arrow[ur, "S\rho_0"'] &
\end{tikzcd}$$

is a pullback. But it is easy to check that $S\check{A}$ is canonically isomorphic to $\check{S}A : \{f : I \to SA \text{ continuous}\}$, and thus that SM_ϕ is canonically isomorphic to $M_{S\phi}$. From this we get a canonical isomorphism from $C_{S\phi}$ to SC_ϕ and a commutative diagram with exact rows and isomorphisms as vertical maps:

$$\begin{array}{ccccccccc}
K_n(S^2B) & \to & K_n(S^2A) & \to & K_n(C_{S_\phi}) & \to & K_n(SB) & \to & K_n(SA) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & K_n(SC_\phi) & & & & \\
 & & & & \downarrow & & & & \\
K_{n+1}(SA) & \to & K_{n+1}(SA) & \to & K_{n+1}(C_\phi) & \to & K_{n+1}(B) & \to & K_{n+1}(A)
\end{array} .$$

Thus we get the following theorem:

<u>IX.6. Theorem</u>. The Puppe Sequence induces the following long exact sequence in K-theory:

$$\to K_{n+1}(A) \to K_n(C_\phi) \to K_n(B) \to K_n(A) \to \ldots \to K_n(A) \to K_0(C_\phi) \to K_0(B) \to K_0(A).$$

$\square$

<u>IX.7. Corollary</u>. The Puppe Sequence induces the following six-term-sequence in K-theory

$$\begin{array}{ccccc}
K_0(C_\phi) & \to & K_0(B) & \to & K_0(A) \\
\uparrow & & & & \downarrow \\
K_1(A) & \leftarrow & K_1(B) & \leftarrow & K_1(C_\phi)
\end{array} .$$

Proof: The only problem is the map $K_0(A) \to K_1(C_\phi)$. It is gotten by the following triangle:

$$\begin{tikzcd} K_0(A) \arrow[r, "\cong"] \arrow[dr, dashed] & K_2(A) \arrow[d] \\ & K_1(C_\phi) \end{tikzcd} .$$

It can be checked as in VI.8 that the sequence is actually exact. □

Let B, A and ϕ be as above. If $Z_B \subset Z(B)$ and $Z_A \subset Z(A)$ are two C*-subalgebras such that $\phi(Z_B) \subset Z_A$, then $\phi : B \to A$ is a module map with respect to the module structure described in VII.5. Hence $\phi_n^* : K_n(B) \to K_n(A)$ is a module map as we saw in Chapter VII. Moreover, it is quite clear that SB is an SZ_B module and SA is an SZ_A module and $S\phi$ is a module homomorphism. We can endow C_ϕ with a module structure, such that the Puppe Sequence is a module sequence, as follows. For $\phi|_{Z_B} : Z_B \to Z_A$ consider $C_{(\phi|_{Z_B})} \subset M_{\phi|_{Z_B}} \subset Z_B \oplus \check{Z}_A$. Then $C_{(\phi|_{Z_B})}$ is a C*-subalgebra of $Z(C_\phi)$. This gives the module structure. Thus we get the following proposition:

<u>IX.8. Proposition</u>. In the above situation the Puppe exact sequence is an exact sequence of modules with respect to the rings

$$\cdots K_n(SZ_B) \to K_n(SZ_A) \to K_n(C_{\phi|Z_B}) \to K_n(Z_B) \to K_n(Z_A). \qquad \square$$

CHAPTER X: EXAMPLES

In this chapter we apply our exact sequences to calculate the K-theory of certain C*-algebras. A typical example of an application of the Mayer-Vietoris Sequence is the algebra of functions from a disk into the matrix algebra $M_n := M_n(\mathbb{C})$ with boundary conditions exemplified by the following: All matrices on the boundary have to be diagonal. A typical example for applying the Puppe Sequence is the algebra of functions on projective space into M_n taking restricted values on a submanifold.

We shall need the fact that for any C*-algebra A, the inclusion $\phi : A \to M_n(A)$, which maps $a \in A$ to the matrix $(a_{ij})_{i,j=1\ldots n} \in M_n(A)$ with $a_{11} = a$ and $a_{ij} = 0$ otherwise, induces an isomorphism $\phi_\alpha^* : K_\alpha(A) \to K_\alpha(M_n(A))$ for $\alpha = 0,1$. To show this, we need a device which will allow us to describe maps in K-theory which are induced by non-unital C*-morphisms.

We have seen earlier that for two unital C*-algebras A and B the map $\phi_0^* : K_0(A) \to K_0(B)$, induced by a unital C*-morphism $\phi : A \to B$, can be described as follows: Any $E \in \mathrm{Ob}(\mathcal{P}(A))$ can be embedded as a retract in A^k for some $k \in \mathbb{N}$. If $p : A^k \to E$ is a corresponding retraction and $j : E \to A^k$ the embedding, $j \circ p$ can be viewed as a $k \times k$-matrix with entries in A, say $(p_{ij})_{i,j=1\ldots k}$.

Then the image $(\phi(p_{ij}))_{i,j=1\ldots k}(B^k)$ of the B-module endomorphism of B^k which is defined by the matrix $(\phi(p_{ij}))_{i,j=1\ldots k}$ with entries in B is an object in $\mathcal{P}(B)$. Its class in $K_0(B)$ is the image of $\overline{[E]} \in K_0(A)$ under ϕ_0^*. We want to show that this description of ϕ_0^* is also valid if ϕ is non-unital. For this purpose, suppose that A and B are unital C*-algebras and $\phi : A \to B$ is a C*-morphism. We adjoin a unit to A and B and define a unital C*-morphism $\tilde{\phi} : \tilde{A} \to \tilde{B}$ by $\tilde{\phi}((a,\lambda)) \to (\phi(a),\lambda)$. The fact that A and B are unital implies that $\tilde{A} \cong A \oplus \mathbb{C}$ via the isomorphism $(a,\lambda) \longmapsto (a + \lambda 1_A) \oplus \lambda$. Similarly $\tilde{B} \cong B \oplus \mathbb{C}$. Now it is easy to check that the composition $A \oplus \mathbb{C} \longrightarrow \tilde{A} \longrightarrow \tilde{B} \longrightarrow B \oplus \mathbb{C}$ is the map $\mathring{\phi}$ given by $\mathring{\phi}(a \oplus \lambda) = (\phi(a) + \lambda(1_B - \phi(1_a))) \oplus \lambda$. Thus we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & A \oplus \mathbb{C} & \xrightarrow{pr_2} & \mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathring{\phi}} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & B & \longrightarrow & B \oplus \mathbb{C} & \xrightarrow{pr_2} & \mathbb{C} & \longrightarrow & 0 .
\end{array}$$

Note that any $E \in Ob(\mathcal{P}(A \oplus \mathbb{C}))$ is the direct sum of $(1_A \oplus 0)\cdot E$ and $(0 \oplus 1)\cdot E$. If E is isomorphic to $(p_{ij} \oplus \lambda_{ij})_{i,j=1\ldots k}((A \oplus \mathbb{C})^k)$, then $(1_A \oplus 0)\cdot E$ is isomorphic to $(p_{ij} \oplus 0)_{i,j=1\ldots k}((A \oplus \mathbb{C})^k)$. Moreover, it is not hard to see that $\overline{[E]} - \overline{[F]} \in K_0(A \oplus \mathbb{C})$ is in the

kernel of $(pr_2)_0^* : K_0(A \oplus \mathbb{C}) \to K_0(\mathbb{C})$ iff $(0 \oplus 1)E$ is stably isomorphic to $(0 \oplus 1)F$. In other words, $[E] - [F] \in \ker(pr_2)_0^*$ iff $\overline{[E]} - \overline{[F]} = \overline{[(1_A \oplus 0)E]} - \overline{[(1_A \oplus 0)F]}$. It is clear how we identify $K_0(A)$ with $\ker(pr_2)_0^*$. We see that if $E' \in Ob(P(A))$ is isomorphic to $(p_{ij})_{i,j=1\ldots k}(A^k)$, where $p_{ij} \in A$, then its class in $K_0(A \oplus \mathbb{C})$ under this identification is the class of $(p_{ij} \oplus 0)_{i,j=1\ldots k}((A \oplus \mathbb{C})^k)$. This gets mapped under $(\mathring{\phi})_0^*$ to the class of
$(\mathring{\phi}(p_{ij} \oplus 0)_{i,j=1\ldots k})((B \oplus \mathbb{C})^k) =$
$(\phi(p_{ij}) \oplus 0)_{i,j=1\ldots k}((B \oplus \mathbb{C})^k)$ in $K_0(B \oplus \mathbb{C})$ which is in the kernel of $(pr_2)_0^*$ and gets identified with $(\phi(p_{ij}))_{i,j=1\ldots k}(B^k)$ in $K_0(B)$. This proves our claim.

Now we are ready to prove the following lemma.

X.1. Lemma. Let A be a C*-algebra and $\phi : A \to M_k(A)$ the map sending a to $(a_{ij})i,j=1,\ldots,n$ with $a_{11} = a$ and $a_{ij} = 0$ otherwise, then $\phi_\alpha^* : K_\alpha(A) \to K_\alpha(M_k(A))$ is an isomorphism.

Proof: First, consider the case where A is unital. We show that ϕ_0^* is surjective. Let $F \in Ob(P(M_k(A)))$, then F is isomorphic to the image of some projection $p_F : (M_k(A))^n \to (M_k(A))^n$. With respect to the canonical basis for $(M_k(A))^n$, the projection p_F is given as an $n \times n$-matrix with entries in $M_k(A)$, say, $(p_{ij})_{1,j=1,\ldots,n}$. Each entry p_{ij} is a $k \times k$-matrix with entries in A, say $p_{ij} = (p_{ij}^{\mu\nu})_{\mu,\nu=1,\ldots,k}$. Consider the $kn \times kn$-matrix

$$(\phi(p_{ij}^{\mu,\nu}))_{i,j=1\ldots n}^{\mu,\nu=1\ldots k} = \begin{pmatrix} \phi(p_{11}^{11}) & \cdots & \phi(p_{11}^{1k}) & \phi(p_{12}^{11}) & \cdots \\ \vdots & & \vdots & & \\ \phi(p_{11}^{k1}) & \cdots & \phi(p_{11}^{kk}) & & \\ & & & \ddots & \\ \phi(p_{21}^{11}) & & & & \\ \vdots & & & & \end{pmatrix}$$

with entries in $M_k(A)$. It defines a map $q_F : M_k(A)^{nk} \to M_k(A)^{nk}$ by matrix multiplication. We see that q_F is a projection and it is not hard to see that $p_F((M_k(A))^n) = q_F((M_k(A))^{nk})$. In fact, $(M_k(A))^{nk} = (M_k(A))^n \oplus (M_k(A))^{k(n-1)}$ and the image of p_F is isomorphic to the image of $p_F \oplus 0 : (M_k(A))^n \oplus (M_k(A))^{k(n-1)} \to (M_k(A))^n \oplus (M_k(A))^{k(n-1)}$. But $p_F \oplus 0$ is the composition of q_F and inner automorphisms of $(M_k(A))^{nk}$ given by permutations of rows and columns. Now note that the $kn \times kn$-matrix with entries in A

$$(p_{ij}^{\mu\nu})_{i,j=1,\ldots,k}^{\mu,\nu=1,\ldots,k} = \begin{pmatrix} p_{11}^{11} & \cdots & p_{11}^{1k} & p_{12}^{11} & \cdots \\ p_{11}^{k1} & \cdots & p_{11}^{kk} & & \\ & & & \ddots & \\ p_{21}^{11} & & & & \\ \vdots & & & & \end{pmatrix}$$

defines a projection $p_E : A^{nk} \to A^{nk}$ by matrix multiplication. Moreover, by the preceding description of ϕ_0^*, we see that

$$\phi_0^*(\overline{[p_E(A^{nk})]}) = \overline{[q_F((M_k(A)^{nk}))]} = \overline{[p_F((M_k(A))^n)]} = \overline{[F]}.$$

Thus ϕ_0^* is surjective.

To show that ϕ_0^* is injective, suppose that $E,F \in Ob(P(A))$ and $\phi_0^*([E] - [F]) = 0$ in $K_0(M_k(A))$. We can find projections p_E and p_F from A^n into A^n such that $E = p_E(A^n)$ and $F = p_F(A^n)$. With respect to the canonical basis for A^n, the projections are given by $n \times n$-matrices with entries in A. Say $p_E = (p_{ij}^E)_{i,j=1\ldots n}$ and $p_F = (p_{ij}^F)_{i,j=1\ldots n}$. By hypothesis, the two $M_k(A)$ modules, given as the images of the projections $(\phi(p_{ij}^E))_{i,j=1\ldots n} : (M_k(A))^n \to (M_k(A))^n$ and $(\phi(p_{ij}^F))_{i,j=1\ldots n} : (M_k(A))^n \to (M_k(A))^n$ are stably isomorphic. This means that there is a number m such that the projections

$$q_E := (\phi(p_{ij}^E))_{i,j=1\ldots n} \oplus id_{(M_k(A))^m} : (M_k(A))^{n+m} \to (M_k(A))^{n+m}$$

and

$$q_F := (\phi(p_{ij}^F))_{i,j=1\ldots n} \oplus id_{(M_k(A))^m} : (M_k(A))^{n+m} \to (M_k(A))^{n+m}$$

have isomorphic images. If we view $id_{(M_k(A))^m}$ as the identity matrix in $M_{k(n+m)}(A)$ and $\phi(p_{ij}^F)$ and $\phi(p_{ij}^E)$ as $k \times k$-matrices with entries in A, we see that q_E and

q_F define projections $A^{k(n+m)} \to A^{k(n+m)}$ with isomorphic images. Similarly as above, we see that $q_E(A^{k(n+m)}) \cong p_E(A^n) \oplus A^{mk}$ and $q_F(A^{k(n+m)}) \cong p_F(A^n) \oplus A^{mk}$. Thus E and F are stably isomorphic and hence $\overline{[E]} - \overline{[F]} = 0$. This proves that ϕ_0^* is injective.

Now it is easy to generalize this result to non-unital A. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{A} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi_A} & & \Big\downarrow{\scriptstyle \phi_{\tilde{A}}} & & \Big\downarrow{\scriptstyle \phi_{\mathbb{C}}} & & \\ 0 & \longrightarrow & M_k(A) & \longrightarrow & M_k(\tilde{A}) & \longrightarrow & M_k(\mathbb{C}) & \longrightarrow & 0 \end{array}.$$

The maps ϕ_A, $\phi_{\tilde{A}}$ and $\phi_{\mathbb{C}}$ are the previously described ones for the resepctive algebras. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(\tilde{A}) & \longrightarrow & K_0(\mathbb{C}) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle (\phi_A)_0^*} & & \Big\downarrow{\scriptstyle (\phi_{\tilde{A}})_0^*} & & \Big\downarrow{\scriptstyle (\phi_{\mathbb{C}})_0^*} & & \\ 0 & \longrightarrow & K_0(M_k(A)) & \longrightarrow & K_0(M_k(\tilde{A})) & \longrightarrow & K_0(M_k(\mathbb{C})) & \longrightarrow & 0. \end{array}$$

The first part of the proof shows that $(\phi_{\tilde{A}})_0^*$ and $(\phi_{\mathbb{C}})_0^*$ are isomorphisms, thus $(\phi_A)_0^*$ is an isomorphism. This concludes the proof of the lemma for K_0. To extend the result to the other K-groups, we only need to note that

$S(M_n(A))$ is canonically isomorphic to $M_n(SA)$ and that $S_{\phi_A} : SA \to S(M_k(A))$ becomes $\phi_{SA} : SA \to M_k(SA)$ under this isomorphism. Thus we have a commutative diagram

$$\begin{array}{ccc} K_1(A) & \xrightarrow{(\phi_A)_1^*} & K_1(M_k(A)) \\ \cong \uparrow & & \uparrow \cong \\ K_0(SA) & \underset{\cong}{\xrightarrow{(\phi_{SA})_0^*}} & K_0(M_k(SA)) \end{array}$$

which proves that $(\phi_A)_1^*$ is an isomorphism. □

Now let $Y \subset X$ be compact spaces. Define a C*-algebra D as the following pullback.

$$\begin{tikzcd} & M_{nk}(C(X)) \arrow[dr, "r"] & \\ D \arrow[ur] \arrow[dr] & & M_{nk}(C(Y)). \\ & M_k(C(Y)) \arrow[ur, "\Delta_n"'] & \end{tikzcd}$$

Here r simply denotes the restriction to Y and Δ_n is the map that assigns the block diagonal matrix

$\begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$ to an $f \in M_k(C(Y))$. For the sake of brevity we define $B := M_{nk}(C(X))$, $A := M_k(C(Y))$ and $C := M_{nk}(C(Y))$.

We obtain an exact sequence

$$\begin{tikzcd} K_1(D) \arrow[r] & K_1(B) \oplus K_1(A) \arrow[r, "\nu_1"] & K_1(C) \\ K_0(C) \arrow[u] & K_0(B) \oplus K_0(A) \arrow[l, "\nu_0"'] & K_0(D) \arrow[l] \arrow[u] \end{tikzcd}$$

The map $(\Delta_n)^* : K_\alpha(A) \to K_\alpha(C)$ is, if we identify $K_\alpha(A)$ with $K_\alpha(C)$ under $(\phi_{M_k(C(Y))})^*_\alpha$, just multiplication by n. If X is a contractible space and $y_0 \in Y$, the map $ev : B \to M_{nk}$ given as the evaluation at y_0 is a homotopy equivalence. Thus with the canonical embedding $j : M_{nk} \hookrightarrow M_{nk}(C(Y)) = C$, we get a commutative triangle up to homotopy:

$$\begin{tikzcd} M_{nk} \arrow[rr, hook, "j"] & & C \\ & B \arrow[ul, "ev"] \arrow[ur, "r"'] & \end{tikzcd}$$

Thus the triangle in K-theory induced by this triangle commutes; also $(ev)^*_\alpha$ is an isomorphicm; so we can replace $K_\alpha(B)$ by $K_\alpha(M_{nk})$ and r^*_α by j^*_α. If we set $\mathring{A} = \{f \in A : f(y_0) = 0\}$, we get a split exact sequence $0 \longrightarrow \mathring{A} \longrightarrow A \xrightarrow{ev} M_k \longrightarrow 0$, i.e., M_k is a retract of A, hence we get a split exact sequence in K-theory:

$$0 \longrightarrow K_\alpha(\mathring{A}) \longrightarrow K_\alpha(A) \longrightarrow K_\alpha(M_k) \longrightarrow 0.$$

Note that $K_1(M_{nk}) = K_1(M_k) = K_1(\mathbb{C}) = 0$ and $K_0(M_{nk}) = K_0(M_k) = K_0(\mathbb{C}) = \mathbb{Z}$; hence we get the exact sequence

$$\begin{tikzcd}
K_1(D) \arrow[r] & K_1(\mathring{A}) \arrow[r, "\nu_1"] & K_1(\mathring{A}) \arrow[d] \\
K_0(\mathring{A}) \oplus \mathbb{Z} \arrow[u] & \mathbb{Z} \oplus (K_0(\mathring{A}) \oplus \mathbb{Z}) \arrow[l, "\nu_0"'] & K_0(D), \arrow[l]
\end{tikzcd}$$

where the maps ν_1 and ν_0 are given as follows: $\nu_1(a) = -na$ for $a \in K_1(A)$; and for $m \oplus (a \oplus m') \in \mathbb{Z} \oplus (K_0(A) \oplus \mathbb{Z})$, we have $\nu_0(m \oplus (a \oplus m')) = (0 \oplus m) - n\cdot(a \oplus m') = (-na \oplus (m-nm'))$. If we assume that $K_1(A)$ is torsion-free, then ν_1 is injective and therefore $K_1(D) \cong K_0(A) \oplus \mathbb{Z}/\text{im } \nu$. But for $c,d \in K_0(A)$ and $m_c, m_d \in \mathbb{Z}$, we have that $c \oplus m_c - d \oplus m_d \in \text{im } \nu_0$ if and only if there is an $a \in K_0(A)$ and $m, m' \in \mathbb{Z}$ such that $c - d = -na$ and $m_c - m_d = m - nm'$. The condition on the integers is always satisfied thus $K_1(D) \cong K_0(A)/nK_0(A) = K_0(A) \otimes \mathbb{Z}/n\mathbb{Z}$. Further, we have the exact sequence

$$0 \longrightarrow K_1(A)/\text{im } \nu_1 \longrightarrow K_0(D) \longrightarrow \ker \nu_0 \longrightarrow 0.$$

If we continue to assume that $K_0(\mathring{A})$ is torsion-free, then $\ker \nu_0 = \{m \oplus (a \oplus m') \in \mathbb{Z} \oplus (K_0(\mathring{A}) \oplus \mathbb{Z}) :$

$-na = 0,\ m = nm'\} = \mathbb{Z}$. Thus the sequence splits, and since $\operatorname{im} \nu_1 = nK_1(\mathring{A})$, we have that $K_0(D) \cong (K_1(\mathring{A}) \otimes \mathbb{Z}/n\mathbb{Z}) \oplus \mathbb{Z}$. If we now observe that $K_\alpha(A) \cong K_\alpha(C(Y)) = K^\alpha(Y)$ and $K_\alpha(\mathring{A}) \cong K_\alpha(C_0(Y)) = \tilde{K}^\alpha(Y)$, we get the following theorem:

X.1. Theorem. Let $Y \subset X$ be compact spaces such that X is contractible and $\tilde{K}^\alpha(Y)$ is torsion-free for $\alpha = 0,1$ and let D be the C*-algebra of continuous functions from X into M_{nk} such that the values on Y are block diagonal matrices with identical blocks of size $k \times k$. Then $K_0(D) = (\tilde{K}^0(Y) \otimes \mathbb{Z}/n\mathbb{Z}) \oplus \mathbb{Z}$ and $K_1(D) = (\tilde{K}^1(Y) \otimes \mathbb{Z}/n\mathbb{Z})$.

The obvious question is of course: what happens if we don't require the block matrices to be identical. In that case we have the following result which, in contrast with previous results, requires no hypotheses for $\tilde{K}^\alpha(Y)$:

X.2. Theorem. Let $Y \subset X$ be compact spaces with X contractible and D the C*-algebra of continuous functions $X \to M_{nk}$ that map Y to block diagonal matrices with blocks of size $k \times k$. Then $K_0(D) = (\tilde{K}^0(Y))^{n-1} \oplus \mathbb{Z}^n$ and $K_1(D) = (K^1(Y))^{n-1}$.

Proof: If $\sigma : (M_k(C(Y)))^n \to (M_{nk}(C(Y)))$ is the map that sends $(a_i)_{i=1\dots n}$ to $\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$, then D can be expressed as the following pullback with notation as before

$$\begin{tikzcd}[column sep=small] & B \arrow[dr, "r"] & \\ D \arrow[ur] \arrow[dr] & & C \\ & A^n \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Again we replace B by $\mathbb{C}$ and obtain a Mayer-Vietoris Sequence:

$$\begin{tikzcd} K_1(D) \arrow[r] & \bigoplus_{i=1}^{n} K_1(\mathring{A}) \arrow[r, "\nu_1"] & K_0(\mathring{A}) \arrow[d] \\ K_0(\mathring{A}) \oplus \mathbb{Z} \arrow[u] & \mathbb{Z} \oplus \bigoplus_{i=1}^{n} (K_0(\mathring{A}) \oplus \mathbb{Z}) \arrow[l, "\nu_0"'] & K_0(D). \arrow[l] \end{tikzcd}$$

But this time, for $a_i \in K_1(\mathring{A})$, the map ν_1 is given by $\nu_1(\bigoplus_{i=1}^{n} a_i) = -\sum_{i=1}^{n} a_i$ and for $b_i \in K_0(\mathring{A})$ and $m, m_i \in \mathbb{Z}$ the map ν_0 is given by $\nu_0(m \oplus \bigoplus_{i=1}^{n} (b_i \oplus m_i)) = -\sum_{i=1}^{n} b_i \oplus (m - \sum_{i=1}^{n} m_i)$. It is easy to see that ν_0 as well as ν_1, is surjective. Thus $K_0(D) \cong \ker \nu_0$ and $K_1(D) \cong \ker \nu_1$. But it is clear that $\ker \nu_0 \cong \bigoplus_{i=1}^{n-1} K_0(\mathring{A}) \oplus \bigoplus_{i=1}^{n} \mathbb{Z}$ and $\ker \nu_1 \cong \bigoplus_{i=1}^{n-1} K_1(\mathring{A})$. This concludes the proof. □

The assumption that X be contractible has of course been made to avoid technical problems which arise from the fact that we did not know the map $r_\alpha^* : K_\alpha(B) \to K_\alpha(C)$ in general. In fact, this is the same as the map

$K^{\alpha}(X) \to K^{\alpha}(Y)$ induced by the inclusion $Y \to X$. If Y is a deformation retract of X, then $r : M_{nk}(X) \to M_{nk}(Y)$ is a homotopy equivalence and we can identify $K_{\alpha}(B)$ with $K_{\alpha}(C)$ and view $r_{\alpha}^{*} : K_{\alpha}(B) \to K_{\alpha}(C)$ as the identity map. For the pullback

$$\begin{tikzcd} & B \arrow[dr, "r"] & \\ D \arrow[ur] \arrow[dr] & & C \\ & A \arrow[ur, "\Delta_n"'] & \end{tikzcd},$$

we get the following exact sequence

$$\begin{tikzcd} K_1(D) \arrow[r] & K_1(A) \oplus K_1(A) \arrow[r, "\nu_1"] & K_1(A) \arrow[d] \\ K_1(A) \arrow[u] & K_0(A) \oplus K_0(A) \arrow[l, "\nu_0"'] & K_0(D) \arrow[l] \end{tikzcd}$$

with the maps ν_1 and ν_0 as follows: For $a_1, a_2 \in K_1(A)$, we have $\nu_1(a_1 \oplus a_2) = a_1 - na_2$ and for $b_1, b_2 \in K_0(A)$, we have $\nu_0(b_1 \oplus b_2) = b_1 - nb_2$. It is clear that ν_0 as well as ν_1 are surjective. Thus we have $K_0(D) \cong \ker \nu_0$ and $K_1(D) \cong \ker \nu_1$. But $\ker \nu_{\alpha}$ is isomorphic to $K_{\alpha}(A)$, hence we get the following theorem:

<u>X.3. Theorem</u>. Let $Y \subset X$ be compact spaces and Y a deformation retract of X. Let D be the C*-algebra of functions from X into M_{nk} such that the values on Y are block diagonal matrices with identical blocks of size $k \times k$. Then $K_0(D) \cong K^0(Y)$ and $K_1(D) \cong K^1(Y)$. □

Again we can consider the same algebra with the condition that the block matrices be the same dropped. We get:

<u>X.4. Theorem</u>. Let Y, X and D be as in X.3, except that we do not require the block matrices to be the same. Then $K_0(D) \cong (K^0(Y))^n$ and $K_1(D) \cong (K^1(Y))^n$.

<u>Proof</u>: The algebra D can be written as the pullback

$$\begin{tikzcd} & B \arrow[dr, "r"] & \\ D \arrow[ur] \arrow[dr] & & C \\ & A^n \arrow[ur, "\sigma"'] & \end{tikzcd}.$$

We get the following Mayer-Vietoris Sequence:

$$\begin{tikzcd} K_1(D) \arrow[r] & K_1(A) \oplus \bigoplus_{i=1}^{n} K_1(A) \arrow[r, "\nu_1"] & K_1(A) \arrow[d] \\ K_1(A) \arrow[u] & K_0(A) \oplus \bigoplus_{i=1}^{n} K_0(A) \arrow[l, "\nu_0"'] & K_0(D) \arrow[l] \end{tikzcd}.$$

For $a, a_i \in K_i(A)$ we have $\nu_1(a \oplus \bigoplus_{i=1}^{n} a_i) = a - \sum_{i=1}^{n} a_i$ and for $b, b_i \in K_0(A)$, we have $\nu_0(b \oplus \bigoplus_{i=1}^{n} b_i) = b - \sum_{i=1}^{n} b_i$. Again it is clear that ν_0 and ν_1 are surjective. Thus $K_0(D) \cong \ker \nu_0$ and $K_1(D) \cong \ker \nu_1$. Now it is easy to check that $\ker \nu_0 = \bigoplus_{i=1}^{n} K_0(A)$ and $\ker \nu_1 = \bigoplus_{i=1}^{n} K_1(A)$. □

We can generalize Theorem X.4 as follows:

<u>X.5. Theorem</u>. Let $Y \subset X$ be compact spaces such that $j^* : K^\alpha(X) \to K^\alpha(Y)$, the map induced by the inclusion $j : Y \hookrightarrow X$, is surjective. Let D be the C*-algebra of functions from X into M_{nk} such that the values on Y are block diagonal matrices with blocks of size $k \times k$. Then $K_0(D) \cong K^0(X) \oplus (K^0(Y))^{n-1}$ and $K_1(D) = K^1(X) \oplus (K^1(Y))^{n-1}$.

<u>Proof</u>: With the same notation as above, we have that $r_\alpha^* : K_\alpha(B) \to K_\alpha(C) = K_\alpha(A)$ is just the map j_α^*. Thus we get the following Mayer-Vietoris Sequence

$$\begin{tikzcd}
K_1(D) \arrow[r] & K_1(B) \oplus \bigoplus_{i=1}^{n} K_1(A) \arrow[r, "\nu_1"] & K_1(A) \arrow[d] \\
K_0(A) \arrow[u] & K_0(B) \oplus \bigoplus_{i=1}^{n} K_0(A) \arrow[l, "\nu_0"'] & K_0(D) \arrow[l]
\end{tikzcd}$$

For $b \in K_\alpha(B)$ and $a_i \in K_\alpha(A)$, we have $\nu_\alpha(b \oplus \bigoplus_{i=1}^{n} a_i) = r_\alpha^*(b) - \sum_{i=1} a_i$. The maps ν_α are surjective since the

r_α^* are. Thus $K_\alpha(D) \cong \ker \nu_\alpha$. But the map $\chi_\alpha : K_\alpha(B) \oplus (K_\alpha(A))^{n-1} \to \ker \nu_\alpha$ given by $\chi_\alpha(b \oplus (a_1 \oplus \ldots \oplus a_{n-1})) = b \oplus (a_1 \oplus \ldots \oplus a_{n-1} \oplus r_\alpha^*(b) - \sum_{i=1}^{n-1} a_i)$ is clearly an isomorphism. This concludes the proof. □

We can also generalize Theorem X.3 in this fashion, but the result will not be quite as nice:

<u>X.6. Theorem</u>. Let Y, X and D be as in X.5, except that the block matrices are required to be identical. Then $K_0(D)$ and $K_1(D)$ are given as the following pullbacks:

$$\begin{tikzcd} & K^0(X) \arrow[dr, "j_0^*"] & \\ K_0(D) \arrow[ur] \arrow[dr] & & K^0(Y) \\ & K^0(Y) \arrow[ur, "\text{mult.by } n"'] & \end{tikzcd} \qquad \begin{tikzcd} & K^1(X) \arrow[dr, "j_1^*"] & \\ K_1(D) \arrow[ur] \arrow[dr] & & K^1(Y) \\ & K^1(Y) \arrow[ur, "\text{mult.by } n"'] & \end{tikzcd}$$

<u>Proof</u>: The relevant Mayer-Vietoris Sequence is

$$\begin{tikzcd} K_1(D) \arrow[r] & K_1(B) \oplus K_1(A) \arrow[r, "\nu_1"] & K_1(A) \arrow[d] \\ K_0(A) \arrow[u] & K_0(B) \oplus K_0(A) \arrow[l, "\nu_0"'] & K_0(D) \arrow[l] \end{tikzcd}$$

and for $b \in K_\alpha(B)$ and $a \in K_\alpha(A)$, we have that

$\nu_\alpha(b \oplus a) = r_\alpha^*(b) - n\cdot a$. The maps ν_α are clearly surjective since r_α^* is. Thus $K_\alpha(D) \cong \ker \nu_\alpha$. It is clear that these kernels are described pullbacks. □

One can further modify the examples. For instance, we can demand certain block matrices to be zero. In this case we just replace as many summands of A^n by zero as we have zero block matrices. The calculations are clear. One can also combine the various boundary conditions. The resulting calculations follow the same scheme as the preceding ones.

Another way of generalizing these examples is to modify the maps from $M_k(C(Y))$ to $M_{nk}(C(Y))$. One can do this by letting automorphisms of M_k act on the block matrices. But all automorphisms of M_k are inner automorphisms, which do not affect the K-theory. Thus we have already described this case in the previous theorems.

We have seen that, if ν_1 and ν_0 in the Mayer-Vietoris Sequence are not surjective, torsion in the K-groups can cause trouble. In some cases, we can get around this using the Puppe Sequence. Let X and Y be compact spaces and $f : Y \to X$ be a continuous function. Consider the mapping cone C_f. We obtain a function $f' : Y \to C_f$ which is the composition of f and the canonical map $g : X \to C_f$. Now consider the C*-algebras $M_k(C(X))$ and $M_{nk}(C(Y))$. We get a map

$\phi : M_k(C(X)) \to M_{nk}(C(Y))$ if, for $a \in M_k(C(X))$, we set $\phi(a) = \Delta_n(a \circ f)$. Consider the mapping cone of ϕ. It is given by a subset of mapping cone M_ϕ

$$
\begin{tikzcd}
 & M_k(C(X)) \arrow[dr, "\phi"] & \\
M_\phi \arrow[ur, "p_x"] \arrow[dr, "p_y"'] & & M_{nk}(C(Y)) \\
 & M^{\vee}_{nk}(C(Y)) \arrow[ur, "\rho_0"'] &
\end{tikzcd}
$$

where $M^{\vee}_{nk}(C(Y))$ denotes the algebra paths in $M_{nk}(C(Y))$. An element $m \in M_\phi$ is in C_ϕ iff $\phi \circ p_x(m) = 0$. Note that $M^{\vee}_{nk}(C(Y))$ is canonically isomorphic to $M_{nk}(C(Y \times I))$ and $M_k(C(X))$ is canonically isomorphic to those functions from X into M_{nk} whose values are block diagonal matrices with identical blocks of size $k \times k$. Thus we see that C_ϕ is the C*-algebra of functions from C_f into M_{nk} whose values on $g(X)$ are block diagonal matrices with identical blocks of size $k \times k$ and which vanish on $y_0 \in C_f$, the vertex of the cone. Moreover, after identifying $K_\alpha(M_k(C(X)))$ with $K_\alpha(C(X)) = K^\alpha(X)$ and $K_\alpha(M_{nk}(C(Y)))$ with $K_\alpha(C(Y)) = K^\alpha(Y)$, we see that the map $\phi^*_\alpha : K_\alpha(M_k(C(X))) \to K_\alpha(M_{nk}(C(Y)))$ is given by $n \cdot f^*_\alpha : K^\alpha(X) \to K^\alpha(Y)$. With these identifications the Puppe Sequence of ϕ is

$$\begin{tikzcd} K_1(C_\phi) \arrow[r] & K^1(X) \arrow[r, "n\cdot f_1^*"] & K^1(Y) \arrow[d] \\ K^1(Y) \arrow[u] & K^0(X) \arrow[l, "n\cdot f_0^*"'] & K_0(C_\phi) \arrow[l] \end{tikzcd}.$$

We collect this information into the following theorem.

X.7. Theorem. Let X and Y be compact spaces, and $f : Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C*-algebra of functions from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with identical blocks of size $k \times k$. Let $\mathring{D}$ be the C*-subalgebra of D of functions which vanish on $y_0 \in C_f$, the vertex of the cone. Then $K_1(D) \cong K_1(\mathring{D})$, and $K_0(D) \cong \mathbb{Z} \oplus K_0(\mathring{D})$. Moreover, we have the following exact sequence:

$$\begin{tikzcd} K_1(\mathring{D}) \arrow[r] & K^1(X) \arrow[r, "n\cdot f_1^*"] & K^1(Y) \arrow[d] \\ K^0(Y) \arrow[u] & K^0(X) \arrow[l, "n\cdot f_0^*"'] & K_0(\mathring{D}) \arrow[l] \end{tikzcd}.$$

Proof: It only remains to be proved that $K_1(D) \cong K_1(\mathring{D})$ and $K_0(D) \cong \mathbb{Z} \oplus K_0(\mathring{D})$. To see this, consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathring{D} \arrow[r] & D \arrow[r, "ev(y_0)"] & M_{nk} \arrow[r] & 0 \\
0 \arrow[r] & \mathring{D} \arrow[r] \arrow[u, equal] & \tilde{\mathring{D}} \arrow[r] \arrow[u, hook] & \mathbb{C} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Here $\tilde{\mathring{D}}$ is viewed as functions on C_f whose values on y_0 are of the form $\begin{pmatrix} \lambda & 0 & \\ 0 & 0 & \\ & & \ddots \end{pmatrix}$, thus $\tilde{\mathring{D}} \subset D$ and i maps λ to $\begin{pmatrix} \lambda & 0 & \\ & \ddots & \\ & & 0 \end{pmatrix}$. The lower row is a retraction, hence gives rise to a split exact sequence in K-theory. Thus we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K_0(\mathring{D}) \arrow[r] & K_0(\mathring{D}) \arrow[r] & \mathbb{Z} \arrow[r] & K_1(\mathring{D}) \arrow[r] & K_1(\mathring{D}) \arrow[r] & 0 \\
0 \arrow[r] & K_0(\mathring{D}) \arrow[r] \arrow[u, equal] & K_0(\mathring{D}) \oplus \mathbb{Z} \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u, "id"'] \arrow[dr] & K_1(\mathring{D}) \arrow[r] \arrow[u, equal] & K_1(\mathring{D}) \arrow[r] \arrow[u] & 0. \\
& & & & 0 \arrow[u] & &
\end{tikzcd}$$

Thus we see that $K_0(D) \to \mathbb{Z}$ is surjective, hence $K_0(D) \cong K_0(\mathring{D}) \oplus \mathbb{Z}$ and $K_1(\mathring{D}) \cong K_1(D)$. $\square$

Again, we are interested in the case where we drop the condition that the block matrices be identical. This case is described by the mapping cone of $\phi : (M_k(C(X)))^n \to M_{nk}(C(Y))$ where ϕ sends an $(a_i)_{i=1\ldots n} \in M_k(C(X))^n$ to a matrix

$$\begin{bmatrix} a_1 \circ f & & & \\ & a_2 \circ f & & \\ & & \ddots & \\ & & & a_n \circ f \end{bmatrix}.$$

After making the appropriate identifications, we see that the maps $\phi_\alpha^* : (K^\alpha(X))^n \to K^\alpha(Y)$ is given by $\phi_\alpha^*(\bigoplus_{i=1}^n a_i) \to \sum_{i=1}^n f_\alpha^*(a_i)$ for $a_i \in K(X)$ and $f_\alpha^* : K^\alpha(X) \to K^\alpha(Y)$ is the map which is induced by $f : Y \to X$. We get a Puppe Sequence

$$\begin{tikzcd} K_1(C_\phi) \arrow[r] & (K^1(X))^n \arrow[r, "\phi_1^*"] & K^1(Y) \arrow[d] \\ K^1(Y) \arrow[u] & (K^0(X))^n \arrow[l] & K_0(C_\phi) \arrow[l] \end{tikzcd}.$$

Thus we have the following theorem:

<u>X.8. Theorem</u>. Let X and Y be compact spaces and $f : Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C*-algebra of functions from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with blocks of size $k \times k$. Let $\mathring{D}$ be the subalgebra of D whic consists of functions vanishing at y_0, the vertex of the cone. Then

$K_1(D) = K_1(\mathring{D})$ and $K_0(D) = \mathbb{Z} \oplus K_0(\mathring{D})$. Moreover, we have the following exact sequence:

$$\begin{array}{ccccc}
K_1(\mathring{D}) & \longrightarrow & \bigoplus_{i=1}^{n} K^1(X) & \xrightarrow{\sum_{i=1}^{n} f_1^*} & K^1(Y) \\
\uparrow & & & & \downarrow \\
K^0(Y) & \xleftarrow{\sum_{i=1}^{n} f_0^*} & \bigoplus_{i=1}^{n} K^0(X) & \longleftarrow & K_0(\mathring{D}).
\end{array}$$

<u>Proof</u>: The exact sequence has been shown already and the assertions concerning the connection of D and $\mathring{D}$ follow as in X.7. $\square$

We present an example which illustrates how this result may be of use in dealing with spaces with torsion. Let $X = Y = S^1$ and $f : Y \to X$ be the multiplication by 2 with respect to the $\mathbb{Z}$-module structure of the unit circle. Then C_f is the projective plane $\mathbb{R}P^2$, and the image of X is the projective plane $\mathbb{R}P^1$ sitting in $\mathbb{R}P^2$. Moreover, $f_0^* : K^0(S^1) = \mathbb{Z} \to K^0(S^1) = \mathbb{Z}$ is multiplication by 2. Thus in the situation of X.7, we have the sequence

$$\begin{array}{ccccc}
K_1(\mathring{D}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & & & \downarrow \\
\mathbb{Z} & \xleftarrow{2n} & \mathbb{Z} & \longleftarrow & K_0(\mathring{D}).
\end{array}$$

We get $K_1(D) \cong K_1(\mathring{D}) \cong \mathbb{Z}/2n\,\mathbb{Z}$ and $K_0(D) \cong K_0(\mathring{D}) \oplus \mathbb{Z}$ $\cong \mathbb{Z} \oplus \mathbb{Z}$. In the situation of X.8, the sequence is

$$\begin{tikzcd}
K_1(\mathring{D}) \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\
\mathbb{Z} \arrow[u] & \mathbb{Z}^n \arrow[l, "\Sigma f"'] & K_0(\mathring{D}) \arrow[l]
\end{tikzcd}$$

It is easy to check that the image of Σf is just $2\mathbb{Z}$, thus $K_1(D) \cong K_1(\mathring{D}) \cong \mathbb{Z}/2\mathbb{Z}$. The kernel of Σf is isomorphic to the direct sum of $n-1$ copies of $2\,\mathbb{Z}$. Thus we have $K_0(D) \cong K_0(\mathring{D}) \oplus \mathbb{Z} \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z} \cong \mathbb{Z}^n$. Note here that $K^0(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ and $K^0(\mathbb{R}P^1) = \mathbb{Z}$. Thus we can not obtain these results from any of the previous theorems.

CHAPTER XI: A NON-COMMUTATIVE SPLITTING PRINCIPLE

This chapter is motivated by the equivalence of the categories of vector bundles over a compact space X on the one hand and the category $P(C(X))$ on the other hand. For vector bundles the splitting principle says: for any vector bundle E over X, we can define the projectivisation $P(E)$ of E being based on the underlying set $\bigcup_{x \in X} P(E_x)$, where $P(E_x)$ is the projective space obtained from the fibre E_x over $x \in X$ as $P(E_x) = (E_x \backslash \{0\})/\mathbb{C}^x$. We provide $P(E)$ with the quotient topology with respect to the quotient map $E \backslash \bigcup_{x \in X} \{0 \in E_x\} \to P(E)$. There is a natural map $P(E) \to X$ induced by the projection $E \to X$. If we pull back E via this map the resulting bundle will be the direct sum of the canonical line bundle over $P(E)$ and another bundle over $P(E)$. If $\Gamma(X,E)$ is the right-$C(X)$-module of sections of E, the module of sections of the pullback bundle over $P(E)$ is $\Gamma(X,E) \otimes_{C(X)} C(P(E))$, where $C(P(E))$ is viewed as a left $C(X)$-module via the natural induced map $C(X) \to C(P(E))$. The direct sum decomposition of the pull-back bundle into the canonical line bundle ℓ and the bundle $\ell^{\perp}$ induces now a direct sum decomposition of $\Gamma(X,E) \otimes_{C(X)} C(P(E))$ into the section modules $\Gamma(P(E),\ell)$ and $\Gamma(P(E),\ell^{\perp})$.

What we propose to do here is to construct, for a given unital C*-algebra B and a right module $M \in P(B)$, a C*-algebra $P(M)$ and a natural map $B \to P(M)$ such that $M \otimes_B P(M)$ is the direct sum of a canonical right module $L \in P(P(M))$ and another module $L^{\perp} \in P(P(M))$. Moreover, if $B = C(X)$ and $M = \Gamma(X,E)$ for a vector bundle E over X, then $L = \Gamma(P(E),\ell)$.

To carry out the construction, we need the notion of a Hilbert module (cf. [D] for the definition). Note that for any closed ideal I of B, the module $M \otimes_B B/I$, induced by the quotient map $\phi_I : B \to B/I$, can be given a Hilbert module structure. If M is embedded in B^k as a retract, the inner product on M is induced by $\langle\ ,\ \rangle : B^k \times B^k \to B$, given by

$$\left\langle \sum_{i=1}^{k} b_1^{(i)}, \sum_{i=1}^{k} b_2^{(i)} \right\rangle = \sum_{i=1}^{k} b_1^{(i)*} b_2^{(i)} .$$

The induced inner product $M \otimes B/I \times M \otimes B/I \to B/I$ is denoted by $\langle\ ,\ \rangle_I$. If J is another closed ideal in B such that $I \subset J$ and $\phi_{IJ} : B/I \to B/J$ is the canonical map, we have that $\phi_{IJ}(\langle e,f\rangle_I) = \langle e \otimes_{B/I} 1_{B/J}, f \otimes_{B/I} 1_{B/J}\rangle_J$. In the following we shall identify $M \otimes_B B/I \otimes_{B/I} B/J$ and $M \otimes_B B/J$ (cf. [D]).

Note that for $B = C(X)$ and $M = \Gamma(X,E)$, a closed ideal I corresponds to a closed set $Y \subset X$ such that

$I = C_Y(X)$, the functions on X vanishing on Y. Then $B/I = C(Y)$ and the Hilbert module structure reduces to a metric on the bundle. Moreover, the induced mappings are all restrictions.

<u>XI.1. Definition</u>. Let B be a unital C*-algebra and let $M \in P(B)$ be provided with Hilbert module structure. For any closed ideal I in B define a subset M_I of $M \otimes B/I$ by $M_I := \{m_I \in M \otimes B/I : \langle m_I, m_I\rangle_I \in Gl_1(B/I)\}$. Denote $\langle m_I, m_I\rangle_I$ by $|m_I|^2$. Let $\overline{M}_I$ be the quotient space of M_I with respect to the group action $Gl_1(B/I) \times M_I \to M_I$ which is given by multiplication. Denote the class of m_i in $\overline{M}_I$ by $\overline{m}_I$. Note that M_I is open in $M \otimes B/I$. Moreover, for $I \subset J$ we have that $id \otimes \phi_{IJ} : M \otimes B/I \to M \otimes B/J$ maps M_I into M_J as we see from the above remarks. In the commutative case, M_I is the set of sections which are nonzero on the set Y corresponding to I and the orbits are the sets of collinear sections over Y.

<u>XI.2. Definition</u>. Let ζ be the set of closed ideals I in B for which $M_I \neq \emptyset$. Then define for every $I \in \zeta$, $P_I(M)$ as the set of continuous, bounded functions $f : \overline{M}_I \to B/I$ which satisfy the following compatibility condition: If $I \subset J$, let the relations $m_I, n_I \in M_I$ and $\overline{id \otimes \phi_{IJ}}(\overline{m}_I) = \overline{id \otimes \phi_{IJ}}(\overline{n}_I)$ imply $\phi_{IJ}(f(\overline{m}_I)) = \phi_{IJ}(f(\overline{n}_I))$. Here $\overline{id \otimes \phi_{IJ}} : \overline{M}_I \to \overline{M}_J$ denotes the map induced by $id \otimes \phi_{IJ}$ on equivalence classes. If $B = C(X)$,

$I = C_Y(X)$ and $J = C_Z(X)$ with $Z \subset Y \subset X$, then the compatibility condition just says that two classes of sections $\sigma, \sigma' \in \overline{\Gamma(Y,E|_Y)}$ with equal restrictions to Z get mapped to functions $f(\sigma), f(\sigma') \in C(Y)$ with equal restrictions to Z. Considering the special case $Z = \{y\}$ where $y \in Y$ is a point, we see that we can identify $P_I(M)$ with the set of continuous bounded functions on $P(E|_Y)$ because the set of images of the sections of $\Gamma(Y,E|_Y)$ is dense in $E|_Y$.

For closed ideals I and J in B, let $I \vee J$ be the closed ideal generated by I and J. Note that $I, J \in \zeta$ implies $I \vee J \in \zeta$.

<u>XI.3. Definition</u>. Provide $P_I(M)$ with the sup norm and define a normed space $P(M)$ in $\prod_{I \in \zeta} P_I(M)$ by $P(M) = \{(f_I : \overline{M}_I \to B/I)_I$: for $J \in \zeta$ and $m \in M_I \cap M_J$, $\phi_{I,I\vee J}(f_I(m)) = \phi_{J,I\vee J}(f_J(m))$ and (f_I) is bounded in the product norm$\}$.

Note that in the commutative case the compatibility condition reduces to saying that a family of functions on $(P(E|_Y))_{C_Y(X) \in M_I}$ is in $P(M)$ if the functions agree on overlaps. Thus we can identify $P(M)$ with the continuous functions on $P(E)$ via $(f_I : \overline{M}_I \to B/I)_I \longleftrightarrow f$ with $f(\overline{e}) = f_{C_{\{x\}}(X)}(\sigma)$ where $\sigma \in \Gamma(X,E)$, where $\sigma(x) = e$, and where $\overline{e}$ is the class of e in $P(E)$.

<u>XI.4. Lemma</u>. The normed space $P(M)$ is a C*-algebra under

pointwise multiplication and involution.

Proof: The only point that is not clear right away is the completeness of $P(M)$, but since we close the topology of uniform convergence on $P_I(M)$, this can also be seen easily. $\square$

Note that there is a canonical C*-morphism $\Pi : B \to P(M)$, given by $b \longmapsto (f_I : \bar{M}_I \to B/I)_I$ with $f_I(\bar{m}_I) = b + I = \phi_I(b)$. It is easy to see that in the commutative case this map is the one induced by the projection $P(E) \to E$.

We introduce some notation in order to make the subsequent calculations more transparent. If $\mathcal{B} = \{(b_I)_{I \in \zeta} \in \prod_{I \in \zeta} B/I : \sup \|b_I\| < \infty\}$, a C*-algebra, $\mathcal{N} = \prod_{I \in \zeta} M \otimes B/I$, and $\bar{\mathcal{N}} = \prod_{I \in \zeta} \bar{M}_I$, then an element of $P(M)$ is a continuous function from $\bar{\mathcal{N}}$ to $\mathcal{B}$. Moreover, we can view the elements of $M \otimes_B P(M)$ as continuous functions from $\bar{\mathcal{N}}$ to $\mathcal{B}$ by identifying $(n \otimes f_I)(x) = n \otimes f_I(x)$ for $n \in M$, $(f_I)_I \in P(M)$ and $x \in M_I$. Note that $\mathcal{N}$ is a $\mathcal{B}$-module.

XI.5. Definition. Let B, M and $P(M)$ be as above. Define a submodule L of $M \otimes_B P(M)$ by $L = \{\omega \in M \otimes_B P(M) : \forall\, x \in \mathcal{N},\ \exists\, b_x \in \mathcal{B}$ such that $\omega(\bar{x}) = xb\}$.

In the commutative case, $M \otimes_B P(M)$ is $\Gamma(X,E) \otimes C(P(E)) = \Gamma(X,P(E))$ and L is the set of sections of $P(E)$ of the form $\sum_{i=1}^{\ell} \sigma_i \otimes f_i \in \Gamma(X,E) \otimes C(P(E))$ such that for any nonzero point e in the fibre E_x over any x, the evaluation at $\bar{e} \in P(E)_x$ gives a scalar multiple of e. In other words, L is the set of sections of the canonical line bundle.

We define a formal inner product on $\mathcal{N}$ with values in $\mathcal{B}$ by $\langle (m_I)_I, (n_I)_I \rangle = (\langle m_I, n_I \rangle_I)_I$ for $m_I, n_I \in M \otimes B/I$.

<u>XI.6. Definition</u>. Let $L^\perp$ be the submodule of $M \otimes_B P(M)$ defined by $L^\perp = \{\omega \in M \otimes P(M) : \forall\, x \in \mathcal{N},\ \langle \omega(x), x \rangle = 0\}$.

We see that in the commutative case $L^\perp$ is just the orthogonal complement of the canonical line bundle with respect to a given metric on E.

<u>XI.7. Theorem</u>. Let B be a unital C*-algebra, $M \in \mathrm{Ob}(\mathcal{P}(B))$. If L and $L^\perp$ are the modules defined in XI.5 and XI.6, then $M \otimes_B P(M) = L \oplus L^\perp$. In particular, $L, L^\perp \in \mathrm{Ob}(\mathcal{P}(P(M)))$.

<u>Proof</u>: Let M be embedded as a retract in B^k and Δ_α, for $\alpha = 1 \ldots k$ be canonical basis for B^k. Then each $m_I \in M \otimes B/I$ can be written uniquely as

$\sum_{\alpha=1}^{k} \Delta_\alpha \bar{b}_\alpha^{(m_I)}$ where $b_\alpha^{(m_I)} \in B/I$. Thus, there is a function $c_\alpha : \mathcal{N} \to \mathcal{B}$ which sends $x = (m_I)_I$ to $b^{(x)} = (b_\alpha^{(m_I)})_I$.

For any $w \in M \otimes P(M)$ we define a function $P_\alpha(w) : N \to B$ by $P_\alpha(w)(x) = b^{(x)}\langle x\langle x,x\rangle^{-1}, w(\bar{x})\rangle$. Suppose now that $x, y \in N$ define the same element $\bar{x} = \bar{y}$ in $\bar{N}$. Then there exists an invertible $b \in B$ such that $x = yb$. Then the following calculation shows that $P_\alpha(w)(x) = P_\alpha(w)(y)$:

$$b^{(x)}\langle x\langle x,x\rangle^{-1}, w(\bar{x})\rangle = b^{(yb)}\langle yb\langle yb,yb\rangle^{-1}, w(\overline{yb})\rangle =$$

$$= b^{(y)}b\langle yb(b^{-1}\langle y,y\rangle^{-1}b^{*-1}, w(\bar{y})\rangle = b^{(y)}\langle y\langle y,y\rangle^{-1}, w(\bar{y})\rangle.$$

Thus we can view $P_\alpha(w)$ as an element of $M \otimes_B P(M)$. We now define a map $P_L : M \otimes_B P(M) \to M \otimes_B P(M)$ by $P_L(w) = \sum_{\alpha=1}^{k} p\Delta_\alpha \otimes_B P_\alpha(w)$ where $p : B^k \to M$ is the retraction. It is easy to check that P_L is well defined, and is a module homomorphism which is self adjoint with respect to the formal inner product $\langle\ ,\ \rangle$ on $M \otimes_B P(M)$. Moreover, P_L maps $M \otimes P(M)$ into L as we see from

$$\cancel{\sum_{\alpha=1}^{k} p\Delta_\alpha \otimes (b_\alpha^{(x)}\langle x\langle x,x\rangle^{-1}, w(\bar{x})\rangle)_I} =$$

$$= (\sum_{\alpha=1}^{k} p\Delta_\alpha \otimes (b_\alpha^{(x)})_I)(\langle x\langle x,x\rangle^{-1}, w(\bar{x})\rangle)_I =$$

$$x_I \cdot (\langle x\langle x,x\rangle^{-1}, w(\bar{x})\rangle)_I$$

in $M \otimes B/I$. Here a subscript I means the I-th component in the respective products. Again, it is easy to check that P_L is a projection and equal to the identity on L. Moreover $1 - P_L : M \otimes P(M) \to M \otimes P(M)$ is a projection onto $L^\perp$. This concludes the proof. $\square$

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BIOGRAPHY

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