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Dissertation

# Dynamics of Stochastic Partial Differential Equations with Dynamical Boundary Conditions

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**Deutsche Zusammenfassung**:
Zufällige dynamische Systeme spielen in vielen Anwendungen eine große Rolle. Wir untersuchen das Langzeitverhalten dieser Systeme. In dieser Arbeit geht es um die Dynamik von stochastischen partiellen Differentialgleichungen mit dynamischen Randbedingungen. Dabei werden zunächst parabolische Gleichungen, sowohl mit additivem, als auch mit multiplikativem Rauschen, auf zufällige Attraktoren untersucht. Das Hauptresultat dieses Abschnitts ist die Existenz eines Attraktors für das Boussinesqsystem mit dynamischen Randbedingungen. Danach betrachten wir inertiale Mannigfaltigkeiten dieser Gleichungen. Im letzten Abschnitt dieser Arbeit wird die Existenz eines zufälligen Attraktors einer hyperbolischen Gleichung mit multiplikativem Rauschen gezeigt. Hierzu wird in diesem Abschnitt auch ein neuer Ansatz mit milden Lösungen verwendet.

**Abstract**:
Random dynamical systems are very important in many applications. We are interested in the long–time–behaviour of these systems. In this work, the dynamics of stochastic partial differential equations with dynamical boundary conditions are investigated. At first, we consider random attractors of parabolic equations with additive and multiplicative noise. The main result of this section is the existence of an attractor of the Boussinesq system with dynamical boundary conditions. Then, we show the existence of an inertial manifold of these equations. Finally, the existence of a random attractor of a hyperbolic stochastic partial differential equation with multiplicative noise is proven. In this chapter, a new method, based on mild solutions, is additionally used.

# Contents

# Chapter 1

# Introduction

The intention of this work is to study the dynamics of randomly perturbed parabolic and hyperbolic partial differential equations with *dynamical boundary conditions*. The simplest parabolic problem of this type is given as follows

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= f(x,u) + \eta_0 \text{ on } D \times \mathbb{R}^+ \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \nu} + c(x)u &= g(x,u) + \eta_1 \text{ on } \partial D \times \mathbb{R}^+ \\ u(0,x) &= u_0(x), \quad x \in D, \qquad \xi \in \partial D \end{aligned} \tag{1.1}$$

with a smooth bounded domain $D \subset \mathbb{R}^n$ with boundary $\partial D$ and

$$u : \mathbb{R}^+ \times D \to \mathbb{R},$$

where $\nu$ is the outer normal and $\eta_i, i = 0, 1$ are white noise terms. Later, these terms are given by generalized temporal derivatives of a Wiener process with values in a function space.

This type of stochastic partial differential equation (spde) has interesting applications, for instance describing the dynamics of the ocean–atmosphere system. We can use this type of equations to model the interaction of atmosphere and ocean. So we assume, that the dynamics of the ocean takes place in $D$, while the dynamics of the atmosphere takes place on the surface of the ocean denoted by $\partial D$. The time scales for the atmosphere are much *faster* than the time scales of the *slow* ocean. Often all these short time influences of the atmosphere are modeled by a noise term. A deterministic meteorological problem can be found in [5]. The main difference to previous theory is that on the boundary not only the noise is acting, but we have also a differential operator there.

However, our mathematical model is more general. It also allows a noise acting in the domain $D$. In addition, we are going to study more general strongly elliptic differential operators and its boundary conditions

$$\frac{\partial u}{\partial t} - \sum_{k,j=1}^{n} \partial_{x_k}(a_{kj}(x)\partial_{x_j} u) + a_0(x)u = f(x,u) + \eta_0 \text{ on } D \times \mathbb{R}^+$$

$$\frac{\partial u}{\partial t} + \sum_{k,j=1}^{n} \nu_k a_{kj}(x)\partial_{x_j} u + c(x)u = g(x,u) + \eta_1 \text{ on } \partial D \times \mathbb{R}^+$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal to $\partial D$ with sufficiently regular coefficients. We will transform this stochastic partial differential equation into a non–autonomous differential equation with random coefficients, which has the advantage to avoid stochastic differentials for technical reasons. For basic results about differential operators with dynamical boundary conditions, we refer to Amann and Escher [2], Escher [26], [25].

Our intention is to describe the dynamics of such a system, which is influenced by some noise terms. We use the theory of random dynamical systems to describe the stochastic dynamics of this kind of equations. This theory is explained in detail in the monograph by Arnold, [3]. There are different approaches to investigate the long-time dynamics of a spde. One way is to consider *attractors* of this spde. Attractors of these problems have been investigated in [16] and [6].

The dynamics of stochastic parabolic partial differential equations with dynamical boundary conditions have been investigated in [15, 16, 55].

This will be also done for *hyperbolic equations* in Chapter 8. Attractors of stochastic hyperbolic pdes have been considered in [33] and [18].

Another way is to consider *inertial manifolds* to examine the finite dimensional long–time dynamics of the equation (1.1). Such a manifold represents a set, which is positively invariant and attracts all states of the phase space. In this sense, the inertial manifold describes the essential long–time behavior of our random dynamical system. In addition, this manifold is defined on a finite dimensional linear subspace of the phase space. This allows us to model the dynamics of the original system on the inertial manifold by a finite dimensional dynamical system which is called *inertial form* of the original system.

Invariant manifolds for finite dimensional random dynamical systems have been introduced by Wanner [52], see also Arnold [3] and for deterministic partial differential equations by Chow et al. [11], [12]. We also refer to Duan et al. [23], [22], Lu and Schmalfuß [36], for the existence of *unstable* manifolds for stochastic parabolic and hyperbolic differential equations with additive and multiplicative noise.

This work is organized as follows. In Section 2 we introduce $C_0$-semigroups and the basic theory of Sobolev spaces. In Chapter 3 we present main properties of dynamical boundary problems. In the following the basic ideas of stochastic analysis in infinite dimensions are given in Chapter 4 and are extended in Chapter 5 to random dynamical systems. Then we consider attractors of parabolic equations in Chapter 6 with additive as well as multiplicative noise. In this chapter, both simple reaction-diffusion equations and a coupled problem (Boussinesq) are considered. In Chapter 7, we deal with inertial manifolds of random dynamical systems. Finally, in Chapter 8, we investigate second order in time spdes.

# Chapter 2

# Functional analysis

At the beginning, we give some basic tools in functional analysis. The first part of this chapter is about semigroups of linear operators on Hilbert spaces. They are important in the solution theory of partial differential equations. Then, we introduce the Lax-Milgram theory, which connects bilinear forms and linear operators. Later on, we deal with elliptic boundary value problems and introduce the Courant-Fisher principle, which is essential to estimate the asymptotic behaviour of eigenvalues, which is needed in Chapter 7.
The second part is about Sobolev spaces and their embedding theorems. The derivatives are understood in this setting in the sense of distributions. We also introduce Sobolev spaces of fractional order, the Sobolev-Slobodetski spaces. Next, we introduce the trace operator on the boundary as well for the function itself as for the normal derivative. After that, we give the connection between semigroups and differential operators. In particular, we show that semigroups are generated by linear operators. At last, we discuss the generalized Poincaré inequality, which is essential to create the energy estimates in Chapter 6.

## 2.1 Basic concepts

### 2.1.1 Semigroups of operators

In this section, we give basic definitions of the theory of semigroups of operators. More details can be found in [41] and [37].

**Definition 2.1.1**
*Let $X$ be a Banach space. A one parameter family $T(t), 0 \leq t < \infty$, of bounded linear operators from $X$ into $X$ is a semigroup of linear operators on $X$ if*

$$T(0) = \mathrm{Id}$$

*and*

$$T(t+s) = T(t) \circ T(s) \quad \forall \quad t, s \geq 0.$$

The generator $A$ of a semigroup is defined as follows:

**Definition 2.1.2 (Generator)**
*The linear operator $A$ defined on*

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists}\right\}$$

*and given by*

$$Ax = \lim_{t \searrow 0} \frac{T(t)x - x}{t} \text{ for } x \in D(A)$$

*is called the infinitesimal generator of the semigroup $T$. $D(A)$ is called the domain of $A$.*

We can define special types of semigroups, which have useful properties.

**Definition 2.1.3 ($C_0$-Semigroup)**
*A semigroup $T$ of bounded linear operators on $X$ is called $C_0$-semigroup of bounded linear operators if*

$$\lim_{t \to 0} T(t)x = x \quad \forall x \in X.$$

These semigroups have the following property:

**Theorem 2.1.4**
*Let $T$ be a $C_0$-semigroup. There exist constants $\delta \geq 0$ and $M \geq 1$, such that*

$$\|T(t)\| \leq Me^{\delta t} \text{ for } 0 \leq t < \infty.$$

*Here, $\|\cdot\|$ denotes the operator norm from $X$ to $X$. The space of linear operators from $X$ to $X$ is denoted by $L(X)$.*

We extend this definition to uniformly continuous semigroups. These semigroups are generated by a bounded operator.

**Definition 2.1.5**
*A semigroup of bounded linear operators $T(t)$ is uniformly continuous if*

$$\lim_{t \searrow 0} \|T(t) - \mathrm{Id}\,\| = 0.$$

**Example 2.1.6**
*Let $X$ be a Banach space, $A \in L(X)$ a bounded linear operator. Define*

$$T(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

*This semigroup is uniformly continuous. Conversely, every uniformly continuous semigroup $T$ on a Banach space $X$ is of the form*

$$T(t) = e^{tA}.$$

We have the following perturbation theorem from [41].

**Theorem 2.1.7 (Perturbation theorem)**
*Let $B$ be the generator of a $C_0$-semigroup of operators. If $L$ is a bounded linear operator on $X$, then $A = B + L$ is the infinitesimal generator of a $C_0$-semigroup on $X$.*

We want to know, whether a linear operator $A$ is a generator of a $C_0$-semigroup. At first, we define the resolvent operator.

**Definition 2.1.8 (Resolvent operator/Resolvent set)**
*We define the resolvent operator of $A$ for $\lambda \in \rho(A)$ as*

$$R_\lambda = (\lambda \operatorname{Id} - A)^{-1}.$$

*The resolvent set $\rho(A)$ of $A$ is the set of all $\lambda \in \mathbb{C}$ for which the range of $(\lambda \operatorname{Id} - A)$ is dense in $X$ and the resolvent operator defined in Definition (2.1.8) is continuous. This set is denoted as $\rho(A)$.*

We need some additional definitions from [27] to state the Hille-Yoshida theorem.

**Definition 2.1.9**
*A linear operator is called closed, if we have for $u_k \in D(A)$ and*

$$u_k \to u, \quad Au_k \to v \text{ for } k \to \infty$$

*that*

$$u \in D(A) \text{ with } v = Au.$$

**Remark 2.1.10**
*A linear operator between two Banach spaces $X$ and $Y$ is closed, if its graph is closed.*

**Definition 2.1.11**
*A linear operator $A$ between two Banach spaces $X$ and $Y$ is called densely defined, if $D(A)$ is dense in $X$.*

With these definitions, we can formulate the Hille-Yoshida theorem, which can be found in [51, p.51].

**Theorem 2.1.12 (Hille-Yoshida)**
*A linear operator $A : D(A) \subset X \mapsto X$ is the infinitesimal generator of a $C_0$-semigroup of contractions if and only if*

- *$A$ is densely defined and closed and*
- *$(0, \infty) \subset \rho(A)$ and for each $\lambda > 0$*

$$\|R_\lambda\|_{L(X)} \leq \frac{1}{\lambda}.$$

The concept of dissipativity is also very important because dissipative operators generate under additional assumptions $C_0$-semigroups, see Theorem 2.1.17. We take the definitions from [41, p.13]. At first, we need to define the duality set.

**Definition 2.1.13**
*Let $X$ be a Banach space and let $X^*$ be its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$. So we can define for every $x \in X$ the duality set $F(x) \subset X^*$ by*

$$F(x) = \{x^* : x^* \in X^* \text{ and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

*Note that from the Hahn–Banach theorem it follows that $F(x) \neq \emptyset$ for every $x \in X$.*

In this case, $\|\cdot\|$ denotes the norm on $X$ or respectively $X^*$.

**Definition 2.1.14**
*A linear operator $A$ is dissipative, if for every $x \in D(A)$ there is a $x^* \in F(x)$ such that $Re\langle Ax, x^*\rangle \leq 0$.*

In a Hilbert space setting, we have the following theorem [51, p.59], which characterizes dissipativity.

**Theorem 2.1.15**
*A linear operator $A : D(A) \subset X \mapsto X$ is dissipative if and only if, for each $x \in D(A)$ and $\lambda > 0$ we have*

$$\lambda\|X\| \leq \|(\lambda I - A)x\|.$$

We have also the following remark from [51, p.59].

**Remark 2.1.16**
*Note that, if $X = H$ is a real Hilbert space with inner product $(\cdot, \cdot)$, a linear operator is dissipative, if for each $x \in D(A)$, we have*

$$(x, Ax) \leq 0.$$

It is now possible to state the Lumer-Phillips theorem [51, p.60]:

**Theorem 2.1.17 (Lumer-Phillips)**
*Let $A : D(A) \subset X \mapsto X$ be a densely defined operator. Then $A$ generates a $C_0$-semigroup of contractions, if and only if*

$$A \text{ is dissipative,} \tag{2.1}$$

*and*

$$\text{there exists } \lambda > 0 \text{ such that } \lambda I - A \text{ is surjective.} \tag{2.2}$$

We now recall some basic classifications of operators, we cite [51, Def. 1.6.3].

**Definition 2.1.18**
*Let $H$ be a real Hilbert space indentified with its own topological dual. An operator $A : D(A) \subset H \mapsto H$ is called selfadjoint, if*

$$A = A^*.$$

*An operator $A : D(A) \subset H \mapsto H$ is called symmetric, if*

$$(Ax, y) = (x, Ay) \text{ for } x, y \in D(A).$$

*Note that a selfadjoint operator is always symmetric.*

Another method to prove the existence of an infinitesimal generator of a $C_0$-semigroup is the Stone theorem, which can be applied in the context of hyperbolic problems. The theorem and the definitions are taken from [51, p. 72].

**Definition 2.1.19**
*An operator $U \in L(H)$ is called unitary, if*

$$UU^* = U^*U = \mathrm{Id}\,.$$

**Theorem 2.1.20 (Stone)**
*The necessary and sufficient condition such that $A : D(A) \subset H \mapsto H$ is the infinitesimal generator of a $C_0$-group of unitary operators on $H$ is that $iA$ is self-adjoint.*

The concept of $C_0$– semigroups can be extended to analytic semigroups.

**Definition 2.1.21 (Analytic semigroups)**
*Let $\Delta = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \Delta$ is an analytic semigroup in $\Delta$, if*

- *$z \mapsto T(z)$ is analytic in $\Delta$.*
- *$T(0) = \mathrm{Id}$ and $\lim_{z\to 0, z\in\Delta} T(z)x = x$ for every $x \in X$.*
- *$T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.*

We state some estimates from [41, p.72] on (operator–) norms of powers of operators, whose infinitesimal generator is an analytic semigroup.

**Lemma 2.1.22**
*There exist constants $M > 0$, $M_m > 0$, $\delta > 0$ such that for $t > 0$ and $m \in \mathbb{N}_0$*

- *$\|T(t)\| \le Me^{-\delta t}$,*
- *$\|AT(t)\| \le M_1 t^{-1} e^{-\delta t}$,*
- *$\|A^m T(t)\| \le M_m t^{-m} e^{-\delta t}$*

*for $M, M_i > 0$.*

From [27, p. 435], we have the following Lemma about differential properties of semigroups:

**Lemma 2.1.23**
*Assume $u \in D(A)$. Then*

- *$T(t)u \in D(A)$ for each $t \ge 0$.*
- *$AT(t)u = T(t)Au$ for each $t \ge 0$.*
- *The mapping $t \mapsto T(t)u$ is differentiable for each $t > 0$.*
- *$\frac{d}{dt}T(t)u = AT(t)u \quad (t > 0)$.*

### 2.1.2 Courant-Fisher principle

The Courant-Fisher principle is important to get estimates of the eigenvalue expansion of linear operators, which is important in the gap conditon later in Chapter 7. This principle is also known as the Minimax principle, see also [53].

**Theorem 2.1.24**
*Let $H$ be a Hilbert space and $T$ a positive selfadjoint (see Definition 2.1.18) operator. The eigenvalues of $T$ are ordered as follows*

$$\lambda_1 \geq \lambda_2 \geq \ldots > 0.$$

*Then, we have*

$$\lambda_n = \min_V \max_{x \in V^\perp \setminus \{0\}} \frac{(Tx, x)}{(x, x)}.$$

*The minimum is considered over all $(n-1)$-dimensional subspaces $V$. $V^\perp$ denotes the orthogonal complement of $V$.*

## 2.2 Theory of linear differential operators

### 2.2.1 Sobolev spaces

**Definition 2.2.1**
*In this work $D \subset \mathbb{R}^n$ is an open set and $\Gamma := \partial D$ is its boundary. For technical reasons, we assume that $D$ is bounded, i.e. $D$ is a bounded domain, otherwise one can assume that the Poincaré inequality or the generalized Poincaré inequality, see Theorem 2.3.9, holds on the domain.*

On these sets, we can define some function spaces. These spaces are used for example in problem (1.1) mentioned in the introduction. The functional analytic background is important to understand the theory of existence and uniqueness. The theory of Galerkin-approximation is also based on theory of embeddings and inclusions of $L^p$-spaces. Another important topic in this context is the measureability of a solution of a partial differential equation. This measureability leads to a random dynamical system. To prove this measureability, we also need the theory of $L^p$- and Sobolev spaces.
At first, we give the definition of the usual $L^p$- and Sobolev spaces. Sobolev spaces are introduced to investigate the derivatives of a function $u$.

**Definition 2.2.2**
*We denote by the Banach space $L^p(D)$ of the classes of functions $u : D \mapsto \mathbb{R}^n$ for $1 \leq p < \infty$ the space of functions with norm*

$$\|u\|_{L^p(D)} := \Big(\int_D \|u(x)\|^p \, dx\Big)^{1/p} < \infty, \quad 1 \leq p < \infty.$$

*The space $L^\infty(D)$ is a set of functions defined on $D$ with*

$$\operatorname*{ess\,sup}_{x \in D} \|u(x)\| < \infty.$$

$\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$, and $\operatorname{ess\,sup}_{x\in D}$ the essential supremum, that is the supremum on almost all $x \in D$. We now introduce the concept of Sobolev spaces. We start with integer $k$, later, we generalize this concept to Sobolev-Slobodetski spaces for general non-integer $k$. These spaces are equivalent to the spaces generated by an interpolation method, see [38]. The derivatives are considered in the sense of distributions.
In general, we assume that $D$ fullfills the cone condition (see [1, p.82]).

**Definition 2.2.3**
*$D$ satisfies the cone condition, if there exists a finite cone $C$ such that each $x \in D$ is the vertex of a finite cone $C_x$ contained in $D$ and congruent to $C$.*

One can say, that for each $x \in D$ there exists a cone completely in $D$. This cone is contained in a cone $C_x$ depending on the whole domain $D$.
At first, we introduce the multi–index notation.

**Definition 2.2.4**
*Let $k$ be a nonnegative integer number, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j$ nonnegative integer numbers with*

$$\|\alpha\| = \alpha_1 + \ldots + \alpha_n, \quad D_k = \frac{\partial}{\partial x^k}, D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_n^{\alpha_n}.$$

We introduce the concept of the theory of weak derivatives. At first, we have to introduce the concept of distributions, see [51, p.16]

**Definition 2.2.5**
*Let $\mathcal{D}(D)$ the set of $C^\infty$ functions from $D$ to $\mathbb{R}$ with compact supports included in $D$. Let $\alpha \in \mathbb{Z}^n$ be a multi-index, $\alpha = (\alpha_1, \cdots, \alpha_n)$. We again set*

$$D^\alpha \varphi = \frac{\partial^{\alpha_1+\cdots+\alpha_n}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

**Definition 2.2.6**
*By a distribution on $\mathcal{D}(D)$, we mean a real-valued, linear continuous functional defined on $\mathcal{D}(D)$ .*

Now, we can define the exact meaning of weak derivatives.

**Definition 2.2.7**
*Let $\alpha \in \mathbb{N}^n$ be a multi-index and $u : D \mapsto \mathbb{R}$ a locally integrable funtion, i.e. $u \in L^1(O)$ for every open subset $O$ of $D$. By definition the derivative of order $\alpha$ of the function $u$ in the sense of distributions over $\mathcal{D}(D)$ is the distribution $\mathcal{D}^\alpha u$ defined by*

$$(\mathcal{D}^\alpha u, \varphi) = (-1)^{\|\alpha\|} \int_D u D^\alpha \varphi \, dx$$

*for each $\varphi \in \mathcal{D}(D)$, where $\|\alpha\| = \alpha_1 + \cdots + \alpha_n$ is the length of the multi-index $\alpha$.*

**Remark 2.2.8**
*If $u$ is a.e. differentiable of order $\alpha$ on $D$ in the classical sense and $\mathcal{D}^\alpha u$ is locally integrable, then $\mathcal{D}^\alpha u$ coincides with $D^\alpha u$ with the following equality*

$$(\mathcal{D}^\alpha u, \varphi) = \int_D D^\alpha u \varphi \, dx$$

*for each $\varphi \in \mathcal{D}(D)$.*

**Definition 2.2.9** *Then, u is in $W_p^k(D)$ if*

$$\|u\|_{W_p^k} := \Big(\int_D \sum_{\|\alpha\|\leq k} \|D^\alpha u(x)\|^p\Big)^{\frac{1}{p}} dx < \infty.$$

We concentrate on the case $p = 2$. Therefore, we set

**Definition 2.2.10**

$$W_2^k(D) = H^k(D).$$

*$H^k(D)$ is a Hilbert space with scalar product*

$$((u, v)) = \sum_{\|\alpha\|\leq k} (D^\alpha u, D^\alpha v).$$

**Definition 2.2.11**
*The space $C^\infty(\overline{D})$ is the space of infinitely differentiable functions on the closure of D.*

In this context, there are several embedding theorems about the relation between $L^p$-spaces und $W_p^k$-spaces. Later, we consider the embeddings on the boundary. We have the following Sobolev embedding theorem (see [43, Theorem 5.26]).

**Theorem 2.2.12**
*Let $D \subset \mathbb{R}^n$ be a bounded $C^k$ domain. If $u \in H^k(D)$ and $k < n/2$, then $u \in L^{\frac{2n}{n-2k}}(D)$ with*

$$\|u\|_{L^{\frac{2n}{n-2k}}(D)} \leq C\|u\|_{H^k(D)}.$$

This theorem is generalized in theorem 2.2.19. From now on, we consider bounded domains $D$ and give the main results for space dimension $n = 2, 3$ by applying Theorem 2.2.12.

**Corollary 2.2.13**
*Assume that D is an open bounded set and $n = 2$. Then, we have the following continuous embedding with a constant k depending on D*

$$H^1(D) \subset L^q(D) \text{ for } 1 \leq q < \infty,$$

*and*

$$\|u\|_{L^q(D)} \leq k\|u\|_{H^1(D)}.$$

*In the case $n = 3$, we have the continuous embedding with a constant k depending on D*

$$H^1(D) \subset L^6(D) \text{ for } 1 \leq q < \infty,$$

*and*

$$\|u\|_{L^6(D)} \leq k\|u\|_{H^1(D)}.$$

We have the following Rellich-Kondrachov Compactness Theorem from [43, Theorem 5.32]:

**Theorem 2.2.14**
*Let D be a bounded $C^1$ domain. Then $H^1(D)$ is compactly embedded in $L^2(D)$.*

This theorem provides us, that a bounded subset of $H^1$ is a compact subset of $L^2$.

### 2.2.2 Sobolev spaces with non-integer order

At first, we introduce from [49] the concept of interpolation spaces. The main goal of this section is to define Sobolev spaces of non-integer order. They have a close connection to fractional spaces defined later in Chapter 3.1.1.

**Theorem 2.2.15 (interpolation space)**
*Assume that $X$ and $Y$ are two Hilbert spaces, $X \subset Y$, $X$ dense in $Y$ and the injection being continuous. Then we have a family of Hilbert spaces $[X,Y]_\theta$, $0 \le \theta \le 1$ such that*

$$[X,Y]_0 = X \text{ and } [X,Y]_1 = Y.$$

*We have the following relation*

$$X \subset [X,Y]_\theta \subset Y$$

*with continuous injections and a norm on $[X,Y]_\theta$ with the following properties*

$$\|u\|_{[X,Y]_\theta} \le c(\theta)\|u\|_X^{1-\theta}\|u\|_Y^\theta, \quad \forall u \in X, \quad \forall \theta \in [0,1].$$

*Proof.* The proof can be found in [35], the interpolation is understood in the complex sense, see [1, p. 247]. □

Applying this theorem to our Sobolev space setting, we get the following interpolation spaces from [49, p. 49], their definition is included in the corollary.

**Corollary 2.2.16**
*Let $\alpha \in (0,1)$ and $m \in \mathbb{N}$. We define*

$$H^{m+\alpha}(D) = [H^{m+1}(D), H^m(D)]_{1-\alpha}.$$

*For $m = 0$, we set $H^0(D) = L^2(D)$ and the definition is complete and consistent.*

Alternatively, we define the Sobolev spaces with non–integer order by

**Definition 2.2.17 (Sobolev-Slobodetski spaces)**
*Assume that $s \in (0,1), k \in \mathbb{N}$. Then*

$$H^{k+s}(D) := \{u \in H^k(D) : \|u\|_{H^{k+s}(D)} < \infty\},$$

*where*

$$\|u\|^2_{H^{k+s}(D)} = \|u\|^2_{H^k(D)} + \sum_{\|\alpha\|=k} \int_D \int_D \frac{\|D^\alpha u(x) - D^\alpha u(y)\|^2}{\|x-y\|^{n+2s}}\, dx\, dy.$$

**Remark 2.2.18**
*The interpolation spaces in Corollary 2.2.16 and the Slobodetski spaces are equivalent spaces.*

*Proof.* The proof can be found in [38, p. 81, p. 328]. □

We need also a Sobolev embedding theorem for fractional Sobolev spaces. Note that $m$ is not necessary integer–valued in the next theorem. The theorem is taken from [24, Theorem 2.4.5] and generalizes theorem 2.2.12.

**Theorem 2.2.19**
*Assume, that $D \subset \mathbb{R}^n$ is bounded and sufficiently smooth. Let additionally $m \ge n(1/2 - 1/q)$. Then we have a continuous embbedding,*

$$H^m(D) \subset L^q(D).$$

### 2.2.3 $L^p(0,T;E)$ spaces

To consider evolution equations, we need the following function spaces, the definitions are taken from Zeidler [56, p. 407]. These spaces are spaces involving time, which are very useful in the context of evolution equations.

**Definition 2.2.20**
*Let $E$ be a Banach space with norm $\|\cdot\|_E$, $1 < p < \infty$ and $0 < T \leq \infty$. Then $L^p(0,T;E)$ is the space of all measurable functions $u : (0,T) \mapsto E$, with*

$$\|u\|_{L^p(0,T;E)} := \Big(\int_0^T \|u(t)\|^p \, dt\Big)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we have the following definition:

**Definition 2.2.21**
*The space $L^\infty(0,T;E)$ consists of all measurable functions $u : (0,T) \mapsto E$, which are bounded for almost all $t \in (0,T)$. We denote these functions as essentially bounded. This means, there exists a number $B$, such that*

$$\|u(t)\| \leq B \text{ for almost all } t \in (0,T).$$

These concepts can be extended to continuous functions.

**Definition 2.2.22**
*$C^m([0,T];E)$, $0 < T < \infty$ is the space of continuous functions $t \mapsto u(t) \in E$, which have continuous derivatives up to order $m$, with norm*

$$\|u\|_{C^m([0,T];E)} = \sum_{i=0}^{m} \max_{t\in[0,T]} \|u^{(i)}(t)\|.$$

*$u^{(i)}$ are the $i$-th distribution derivatives.*

For $m = 0$ we set $C^m([0,T];E) = C([0,T];E)$.
From [56, Prop 23.2] we take the following proposition:

**Proposition 2.2.23**
*$C([0,T];E)$ is dense in $L^p(0,T;E)$ and the embedding*

$$C([0,T];E) \subset L^2(0,T;E)$$

*is continuous. Furthermore, if the embedding $E \subset F$, both Banach spaces, is continuous, then the embedding*

$$L^r(0,T;E) \subset L^q(0,T,F), \quad 1 \leq q \leq r \leq \infty$$

*is also continuous.*

We have also embedding theorems, which give connections between $L^p(0,T;E)$ and $C^m([0,T];E)$ spaces. One important theorem is the following theorem of Dubinskij, see [50, p. 123].

**Theorem 2.2.24**
*We have*

$$E_0 \subset E \subset E_1$$

*a continuous embedding, which are all reflexive Banach spaces, and the first embedding from $E_0$ into $E$ is compact.*
*Let $1 < q < \infty$ and $M$ bounded in $L^q(0,T;E_0)$, which is equicontinuous in $C([0,T];E_1)$. Then $M$ is relatively compact in $L^q(0,T;E)$ and $C([0,T];E_1)$.*

We cite another theorem from [47, Theorem III.2.3] which is used in Chapter 8 to obtain the existence of weak solutions of an evolution equation.

**Theorem 2.2.25**
*Assuming that*

$$E_0 \subset E \subset E_1$$

*are Hilbert spaces with continuous injections, and the injection of $E_0$ into $E$ is compact. Then the injection of*

$$Y = \{u \in L^2(0,T;E_0), u' \in L^1(0,T;E_1)\}$$

*into $L^2(0,T;E)$ is compact.*

## 2.3 Spaces on the boundary

We extend the theory of Sobolev spaces on product spaces. At first, we introduce the trace operator and give some embedding theorems.

### 2.3.1 Trace operator

Now, we explain the meaning of the trace operator $\gamma$. $\gamma$ can be seen as a projection of a function from the inner domain $D$ on the boundary $\partial D$. We have the following definition from [43, p. 129]:

**Definition 2.3.1**
*We say that $D \subset \mathbb{R}^m$ is a bounded domain of class $C^k$ or a bounded $C^k$ domain provided that at each point $x_0 \in \partial D$ there is an $\epsilon > 0$ and a $C^k$-diffeomorphism $\varphi$ of $B(x_0, \epsilon)$ onto a subset $\tilde{B}$ of $\mathbb{R}^m$, such that*

- $\varphi(x_0) = 0$,
- $\varphi(B \cap D) \subset \mathbb{R}^m_+$ *and*
- $\varphi(B \cap \partial D) \subset \partial\mathbb{R}^m_+$,

*where*

$$\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_m > 0\}.$$

Moreover, we have the following theorem, which implies the definition.

**Theorem 2.3.2**
*Suppose that $D$ is a bounded $C^1$-domain. Then, there exists a bounded linear operator*

$$\gamma : H^1(D) \to L^2(\partial D),$$

*called the trace operator, if for all $u \in H^1(D) \cap C^0(\overline{D})$*

$$\gamma u = u|_{\partial D}.$$

The proof of this theorem can be found in [43]. We have to clarify, in which sense Sobolev spaces on the boundary are understood, see [43, p. 145]. We can define $H^k(\partial D)$ by identifying $\partial \mathbb{R}^m_+$ by $\mathbb{R}^{m-1}$.
We can extend this theorem to a trace theorem on Sobolev spaces.

**Theorem 2.3.3**
*Suppose that $D$ is a bounded $C^1$ domain. Then there exists a bounded linear operator*

$$\gamma : H^1(D) \to H^{1/2}(\partial D),$$

*called the trace operator, such that for all $u \in H^1(D) \cap C^1(\overline{D})$*

$$\gamma u = u|_{\partial D}.$$

The proof of this theorem can be found in [54, p. 130]. This theorem can also be generalized to normal derivatives. We get

**Theorem 2.3.4**
*Suppose that $D$ is a bounded $C^k$ domain and $j - m > \frac{1}{2}$ and $j \leq k-1$. Then there exists a bounded linear operator*

$$\gamma_m : H^j(D) \to \prod_{i=0}^{m} H^{j-i-\frac{1}{2}}(\partial D),$$

*called the trace operator, such that for all $u \in H^j(D) \cap C^0(\overline{D})$*

$$\gamma_m u = (u|_{\partial D}, \frac{\partial u}{\partial n}|_{\partial D}, \cdots, \frac{\partial^m u}{\partial n^m}|_{\partial D}).$$

*$\frac{\partial}{\partial n}$ is the derivative into the direction of the inner normal.*

The proof of this theorem can also be found in [54, p. 130] and we find there the following remark.

**Remark 2.3.5**
*The functions of $C^\infty(\overline{D})$ are dense in $H^j(D)$. Hence, the trace operator is uniquely defined.*

We use the above definition of the trace operator and extend these concepts to product spaces, see also Chapter 6.3.

**Remark 2.3.6**
*Assume that $U = (u, u_\Gamma) \in \mathbb{H} = L^2(D) \times L^2(\Gamma)$. Then we write*

$$\|U\|^2_{\mathbb{H}} = \|u\|^2_{L^2(D)} + \|u\|^2_{L^2(\Gamma)} := \|u\|^2_D + \|u_\Gamma\|^2_\Gamma.$$

*We often set $\|u\|_D = \|u\|$ and $\|u_\Gamma\|_\Gamma = \|u\|_\Gamma$.*

### 2.3.2 Generalized Poincaré inequality

We have the following lemma as a special case of the Generalized Poincaré inequality from [51, p.19]. This theorem is very important to obtain the energy estimates with dynamical boundary conditions.

**Theorem 2.3.7 (Friedrichs)**
*Let $D$ be a domain in $\mathbb{R}^n$ whose boundary is of class $C^1$. Then there exists $k_1 > 0$, such that for each $u \in H^1(D)$, we have*

$$\|u\|^2_{H^1(D)} \leq k_1\big(\|\nabla u\|^2_{L^2(D)} + \|u\|^2_{L^2(\Gamma)}\big).$$

This theorem has the following consequence:

**Theorem 2.3.8**
*Let $D$ be a bounded domain in $\mathbb{R}^n$ whose boundary is of class $C^1$. Then $\|\cdot\|_\gamma : H^1(D) \to \mathbb{R}^+$, defined by*

$$\|u\|_\gamma := \big(\|\nabla u\|^2_{L^2(D)} + \|u\|^2_{L^2(\Gamma)}\big)^{\frac{1}{2}}$$

*for each $u \in H^1(D)$ is a norm on $H^1(D)$ equivalent with the usual one.*

The Friedrichs inequality can be generalized in the following way:

**Theorem 2.3.9 (Generalized Poincaré inequality)**
*Let $D$ be a bounded and Lipschitz set in $\mathbb{R}^n$, i.e. $\Gamma = \partial D$ is locally the graph of a Lipschitz function and let $p$ be a continuous seminorm on $H^m(D)$, which is a norm on $P_{m-1}$(the polynomials of order lower or equal $(m-1)$). Then there exists a constant $c(D)$, such that*

$$\|u\|_{H^{m-1}(D)} \leq c(D)\big(\sum_{\|\alpha\|=m} \|D^\alpha u\|_{L^2(D)} + p(u)\big) \quad \forall u \in H^m(D). \tag{2.3}$$

*Proof.* We adapt the proof from [54, p. 121]. Assume that inequality (2.3) does not hold. Then there exists a sequence $(\varphi_n) \in H^m(D)$ with

$$\|\varphi_n\|^2_{H^m(D)} = 1$$

and

$$1 = \|\varphi_n\|^2_{H^m(D)} > n\big(\sum_{s=m} \|D^s\varphi_n\|^2_{L^2(D)} + p(\varphi_n)\big). \tag{2.4}$$

We have that

$$D^m\varphi_n \to 0 \text{ in } L^2(D). \tag{2.5}$$

On the other hand, the embedding theorems and Theorem 2.2.14 state, that $(\varphi_n)$ is also relatively compact in $H^{m-1}$, we have a subsequence $(\varphi_n)$, which has for simplicity the same notation, with

$$\varphi_n \to \varphi \text{ in } H^{m-1}(D).$$

We obtain by equation (2.5) that

$$\varphi_n \to \varphi \text{ in } H^m(D).$$

Equation (2.5) gives us that $\varphi \in P_{m-1}$ and inequality (2.4)

$$\lim_{n\to\infty} p(\varphi_n) = p(\varphi) = 0.$$

Due to the assumptions on $p$, we gain that $\varphi \equiv 0$, a contradiction to inequality (2.4). □

Examples of seminorms $p(u)$ can be found in [48, p.51].

### 2.3.3 Green's formula

In the context of integration by parts, we use the following Green's formula from [27, p.712].

**Theorem 2.3.10**
*Let $u, v \in C^2(\overline{D})$. Then*

$$\int_D \nabla v \nabla u \, dx = -\int_D u \Delta v \, dx + \int_{\partial D} \partial_\nu v u \, dS,$$

*where $dS$ is the integration w.r.t. the boundary.*

## 2.4 Concepts of solutions

In this section, we introduce the different types of solutions of nonlinear initial value problems. The notations and definitions are based on [41, p. 105]. Consider the following problem

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t), \quad t > 0 \\ u(0) &= u_0, \end{aligned} \tag{2.6}$$

where $f : [0, T) \to X$ and $A$ is the infinitesimal generator of an $C_0$-semigroup $T$ on $X$, where $X$ is a Banach– or Hilbert space. Consider $A$ as a (not necessarily positive) operator from $X$ to $X$.

### 2.4.1 Mild and classical solutions

We start to define mild solutions, which is the weakest concept of solution considered in this work.

**Definition 2.4.1 (mild solution)**
*Let $A$ be the generator of a $C_0$-semigroup $T$. Let $x \in X$ and $f \in L^1(0, T, X)$ Then*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\, ds \quad t \in [0, T] \tag{2.7}$$

*is called a **mild solution** of (2.6).*

We also know some other types of solutions:

**Definition 2.4.2 (Classical solution)**
*A function $u : [0, T) \to X$ is called* ***classical solution*** *of (2.6), if:*

- *$u$ fulfills (2.6)*
- *$u$ is continuous on $[0, T)$,*
- *$u$ is continuously differentiable on $(0, T)$,*
- *$u \in D(A)$ for $t \in (0, T)$.*

**Definition 2.4.3 (Strong solution)**
*A function $u : [0, T) \to X$ is called a* ***strong solution*** *of (2.6), if:*

- *$u$ is differentiable on $(0, T)$ almost everywhere,*
- *$u' \in L^1(0, T, X)$,*
- *$u'(t) = Au(t) + f(t)$ almost everywhere and $u(0) = u_0$.*

The following remark delivers a connection between mild and classical solutions.

**Remark 2.4.4 (Variation of constants)**
*If we have a classical solution, then it coincides with the mild solution. Consider the following inhomogeneuous initial value problem:*

$$\frac{du(t)}{dt} = Au(t) + f(t) \quad t > 0 \qquad u(0) = u_0, \tag{2.8}$$

*where $f \in L^1(0, T; X)$. Then we have*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\, ds.$$

*Proof.* We have to show that $u(t)$ solves in fact the equation and work out the details of the proof from [41, p. 105].
Let

$$g(s) := T(t-s)u(s).$$

This function is differentiable for $s \in (0, t)$ and we have due to Lemma 2.1.23 and the product rule

$$\begin{aligned} \frac{dg}{ds} &= -AT(t-s)u(s) + T(t-s)u'(s) \\ &= -AT(t-s)u(s) + T(t-s)Au(s) + T(t-s)f(s) \\ &= T(t-s)f(s). \end{aligned}$$

Since $f \in L^1(0, T; X)$, the right hand side of the equation is integrable, and thus

$$g(t) - g(0) = \int_0^t T(t-s)f(s)\, ds.$$

Additionally,

$$T(0)u(t) - T(t)u(0) = \int_0^t T(t-s)f(s)\,ds,$$

therefore, we have with $T(0) = \mathrm{Id}$ and $u(0) = u_0$:

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds$$

□

$u(t)$ is by definition a *mild solution* of the problem (2.8) and coincides with the classical solution.

### 2.4.2 Nonlinear case

Up to this moment, we have only considered problems, where the function $f$ does not depend on $u$. Now, we give up this restriction. If the conditions on $f$ are weak, i.e. if we assume less regularity on $f$, we need mild solutions. The variation of constants formula delivers us the solution.

**Theorem 2.4.5 (Variation of constants, nonlinear case)**
*Consider the following inhomogeneous initial value problem:*

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), \quad t > 0 \qquad u(0) = u_0, \tag{2.9}$$

*where $f : [0, T) \times X \to X$. $f$ is assumed as Lipschitz-continuous on $X$ and continuous on $[0, T)$.*
*Additionally, $f$ is measurable and the Lipschitz constant of $f$ is independent of $t$. $A$ is the generator of a $C_0$-semigroup $T(t)$. Then*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))\,ds$$

*is the unique mild solution of (2.9), see [41, p. 184].*

### 2.4.3 Weak solutions/Lax-Milgram theory

An alternative technique to develop a concept of solution is the usage of weak solutions. The Lax-Milgram theorem connects bilinear forms and linear operators. Suppose we have a bilinear continuous form $a$ on $V$. $V$ is a Hilbert space with scalar product $((u, v))$ and norm $\|u\|$. We can construct for every $a$ a linear continuous operator $A$ from $V$ to $V'$, $A \in L(V, V')$ and vice versa. Given a bilinear continuous form $a$ we construct a linear continuous operator $A \in L(V, V')$ by

$$A : u \mapsto \big(v \mapsto a(u, v)\big)$$

from $V$ to $V'$.
On the other hand, given a linear continuous operator $A \in L(V, V')$ we set

$$a(u, v) = \langle Au, v\rangle, \quad \forall u, v \in V,$$

where $\langle \cdot, \cdot \rangle$ is the duality map from $V' \times V \to \mathbb{R}$. If $a$ is coercive, i.e.

$$\exists \alpha > 0, \quad a(u, u) \geq \alpha \|u\|^2 \quad \forall u \in V.$$

we have the following Lax-Milgram theorem. $V'$ denotes the dual space of $V$ here.

**Theorem 2.4.6 (Lax-Milgram)**
*If $a$ is a bilinear continuous coercive form on $V$, then $A$ is an isomorphism from $V$ to $V'$.*

Thus, we can find $\forall f \in V'$ a $u \in V$, such that

$$\langle Au, v\rangle = (f, v) \quad \forall v \in V.$$

## 2.5 Evolution in time

We want to apply this theory to partial differential equations. In most settings, we have another Hilbert space $H$, with scalar product $(u, v)$ and norm $\|u\|$. Now we can construct an evolution triplet

$$V \subset H \equiv H' \subset V'$$

because of the Riesz representation theorem, this ensures $(h, v) = \langle h, v\rangle$ for $h \in H$. We define the domain of $A$ in $H$ as

$$D(A) = \{u \in V, Au \in H\}.$$

$A$ is also an isomorphism of $D(A)$ onto $H$. We assume again, that we have a bilinear coercive form $a$ on $V$. We consider the following initial value problem

$$\frac{du}{dt} + Au = f \text{ on } (0, T) \tag{2.10}$$

with $u(0) = u_0 \in H$ and $f \in L^2(0, T; H)$. To define a solution in the sense of distributions we need the following lemma, which includes a definition from [49, p. 69]. We need to point out, how $\frac{du}{dt}$ is understood in the distribution sense.

**Lemma 2.5.1**
*Let $X$ be a given Banach space with dual $X'$ and let $u$ and $g$ be two functions belonging to $L^1(a, b; X)$. Then the following three conditions are equivalent:*

- *$u$ is almost everywhere equal to a primitive function of $g$, i.e., there exists $\zeta \in X$ such that*

$$u(t) = \zeta + \int_a^t g(s)\, ds, \quad \text{for a.e. } t \in [a, b].$$

- *For every test function* $\varphi \in \mathcal{D}((a,b))$,

$$\int_a^b u(t)\varphi'(t)\,dt = -\int_a^b g(t)\varphi(t)\,dt.$$

- *For each* $\eta \in X'$,

$$\frac{d}{dt}\langle u,\eta\rangle = \langle g,\eta\rangle$$

*in the scalar distribution sense on* $(a,b)$.

*If all these conditions are satisfied, we say that* $g$ *is the* $X$*-valued distribution derivative of* $u$ *and* $u$ *is almost everywhere equal to a continuous function from* $[a,b]$ *into* $X$.

Now we can interprete (2.10) in the sense of distributions. If $u \in L^2(0,T;V)$, then $Au \in L^2(0,T;V')$.
Because of $f \in L^2(0,T;V')$ and $Au \in L^2(0,T;V')$ we have, that $u' = f - Au$ is in the distribution sense in $V'$ and we can interprete (2.10) as

$$\frac{d}{dt}\langle u,v\rangle + a(u,v) = \langle f,v\rangle, \quad \forall v \in V.$$

Note that by Lemma 2.5.1 $u \in C([0,T];V')$ and thus the initial condition of (2.10) is meaningful.
We state two basic theorems from [49, p.70] and [43] which are essential for existence and uniqueness. To assure existence and uniqueness, we have the following theorem:

**Theorem 2.5.2**
*Assume that* $u_0 \in H$ *and* $f \in L^2(0,T;V')$. *There exists a unique solution* $u$ *of (2.10) with*

$$u \in L^2(0,T;V) \cap C([0,T];H)$$

*and*

$$u' \in L^2(0,T;V').$$

*Proof.* The proof is based on the Faedo-Galerkin method, see [49, p.70]. The main argument is to show that

$$u_m \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{2.11}$$

where $u_m$ is the approximate solution of (2.10). A detailed demonstration of the techniques of this proof is given in Theorem 6.1.2. □

We take from [43, Theorem 7.2] the following useful theorem.

**Theorem 2.5.3**
*Suppose that*

$$u \in L^2(0,T;V) \text{ and } \frac{du}{dt} \in L^2(0,T;V').$$

*Then*

$$u \in C([0,T];H)$$

*after a modification on a set of measure zero, and*

$$\frac{d}{dt}\|u\|^2 = 2\langle\frac{du}{dt},u\rangle.$$

### 2.5.1 Examples

The following example to apply the Lax-Milgram theorem is taken from [49, p.63].

**Example 2.5.4** *We consider a homogeneous Dirichlet problem in $D$ associated with the Laplace operator.*

$$\begin{aligned} -\Delta u + \lambda u &= f \text{ in } D, \quad \lambda > 0. \\ u &= 0 \text{ in } \partial D. \end{aligned}$$

*By Green's formula 2.3.10, we obtain the following equivalent formulation with the Lax-Milgram theorem*

$$a(u,v) = (f,v)$$

*with*

$$a(u,v) = \int_D (\operatorname{grad} u \cdot \operatorname{grad} v + \lambda uv)\, dx.$$

*This leads to the following weak formulation of the problem.*
*For $f$ given in $H = L^2(D)$, find $u \in V = H_0^1(D)$, such that*

$$a(u,v) = (f,v) \quad \forall v \in V.$$

**Example 2.5.5 (Heat equation)**
*Let $D \subset \mathbb{R}^n$ be a bounded domain. We consider the following classical parabolic differential equation. Let $Q_T = D \times (0,T)$:*

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= f(t,x) \text{ on } Q_T \\ u(x,t) &= 0 \text{ on } \partial D \times [0,T], \\ u(x,0) &= u_0(x) \text{ on } D \end{aligned}$$

*This problem can also be formulated as evolution equation, and the operator $A$ generates a semigroup of linear operators. The details can be found in [49, p. 84]. $A := -\Delta$ together with the boundary condition generates a semigroup of linear operators*

$$T(t) : u_0 \in H \mapsto u(t) \in H.$$

**Example 2.5.6 (Wave equation in $\mathbb{R}^n$)**
*We consider the wave equation in $\mathbb{R}^n$:*

$$\begin{aligned} v_{tt} &= -\mathcal{A}v, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \\ v(0,x) &= v_0(x) \\ v_t(0,x) &= v_1(x) \end{aligned}$$

*With*

$$u = \begin{pmatrix} v \\ v_t \end{pmatrix}$$

*the problem is equivalent to*

$$u_t = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathcal{A} & 0 \end{pmatrix} u.$$

$$u(0,x) = u_0(x) = \begin{pmatrix} v_0(x) \\ v_1(x) \end{pmatrix}.$$

*On* $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathcal{A} & 0 \end{pmatrix}$$

*generates a semigroup of linear operators. In this context,* $\mathcal{A}$ *is a positive selfadjoint operator on* $D(A)$*, for example* $\mathcal{A} = -\Delta$.

We use the following example from [51, p. 166] of a parabolic problem with dynamical boundary conditions.

**Example 2.5.7**
*Let* $D$ *be a bounded domain in* $\mathbb{R}^n$ *with* $C^2$*-boundary* $\Gamma$*. Consider the problem*

$$\begin{aligned} u_t &= \Delta u \ \text{ on } [0,T] \times D \\ u_t + \partial_\nu u &= 0 \ \text{ on } [0,T] \times \Gamma \end{aligned}$$

*with initial condition*

$$u(x,0) = u_0(x) \in L^2(D) \times L^2(\Gamma).$$

We start with a result of regularity of solutions of a parabolic problem.

**Lemma 2.5.8**
*Let* $D$ *be a bounded domain in* $\mathbb{R}^n$ *with* $C^2$ *boundary* $\Gamma$ *and let* $\mu \geq 0, \lambda > 0$*. Then, for each* $f \in L^2(D)$ *and* $g \in L^2(\Gamma)$*, the elliptic problem*

$$\begin{aligned} \mu u - \Delta u &= f \\ \lambda v + \partial_\nu u &= g \\ u_{|\Gamma} &= v \end{aligned}$$

*has a unique solution* $u \in H^{3/2}(D)$ *with* $\Delta u \in L^2(D)$*,* $u_{|\Gamma}$ *and* $\partial_\nu u \in L^2(\Gamma)$*.*

The proof of this lemma is based on the Lax-Milgram theorem, see [51, p. 166]. The main idea is, that $u_\nu = g - \lambda v \in L^2(\Gamma)$ because the expression on the right hand side is in $L^2(\Gamma)$. The operator $A$ generated by the above problem has the following properties, the proof is based on the Hille-Yoshida theorem.

**Theorem 2.5.9**
*The operator*

$$A : D(A) \subset L^2(D) \times L^2(\Gamma) \to L^2(D) \times L^2(\Gamma),$$

*defined by*

$$D(A) = \{(u,v) \in L^2(D) \times L^2(\Gamma); \Delta u \in L^2(D), \partial_\nu u \in L^2(\Gamma), u_{|\Gamma} = v\},$$

$$A(u,v) = (\Delta u, -\partial_\nu u) \text{ for each } (u,v) \in D(A),$$

*is the infinitesimal generator of a compact and analytic* $C_0$*-semigroup of contractions.*

The theorem is proven in [51].
Finally, we have by Amann and Escher (see [2, Theorem 3.2]):

**Theorem 2.5.10** *The operator $-A$, coming from the bilinear form $a$, generates a strongly continuous positive contraction semigroup $S$ in the space*

$$\mathbb{C}(\bar{D}) := \left\{(u;\xi) \in C(\overline{D}) \times C(\partial D) : u = \xi \text{ on } \partial D\right\}. \tag{2.12}$$

*It has a unique extension to a strongly continuous positive contraction semigroup on $\mathbb{L}_p(D)$, $1 \leq p < \infty$, which is compact and analytic, if $1 < p < \infty$, where*

$$\mathbb{L}^p(D) := L^p(D) \times L^p(\Gamma).$$

# Chapter 3

# Linear partial differential equations with dynamical boundary conditions

## 3.1 Parabolic problems

In this chapter, we apply the $C_0$-semigroup theory to dynamical boundary operators. At the beginning, we start with first order in time differential equations and show, that these operators generate a $C_0$-semigroup. Later on, we extend this theory to second order in time partial differential equations, transform these equations to first order equations on appropriate function spaces and show the existence of a $C_0$–semigroup generated by the second order equation.
Let $D$ be a bounded smooth domain in $\mathbb{R}^d$. We consider the differential operator $\mathcal{A}$ of second order on the domain $D$

$$\mathcal{A}(x) := -\sum_{k,j} \partial_k(a_{kj}(x))\partial_j + a_0(x). \tag{3.1}$$

On the boundary $\Gamma := \partial D$ of $D$ we have the boundary operator

$$\mathcal{A}_\Gamma(x) = \sum_{k,j} \nu_k a_{kj}(x)\partial_j + b_0(x). \tag{3.2}$$

$\nu_k$ denote the $k$-th component of the outer normal on $\partial D$. The coefficients $a_{kj}$, $a_0$ and $b_0$ are smooth functions on $\overline{D}$. In addition $a_0(x) > 0$ and the symmetric matrix $(a_{kj}(x))_{k,j=1,\ldots d}$ is uniformly elliptic. That is there exists a $c > 0$ such that

$$\sum_{i,j} a_{ij}(x)\zeta_i\zeta_j \geq c\|\zeta\|^2 \text{ for all } x, \zeta \in \mathbb{R}^d.$$

The next theorem will show the existence of a $C_0$-semigroup generated by $(\mathcal{A}, \mathcal{A}_\Gamma)$ and we will specify the domain of the operator $A$.

**Theorem 3.1.1**
*$(\mathcal{A}, \mathcal{A}_\Gamma)$ generates a $C_0$-semigroup $S$ of contractions on $\mathbb{H} = L^2(D) \times L^2(\Gamma)$. The domain*

*of its generator $A$ is given by*

$$\begin{aligned} D(A) &= \{(u,w) \in L^2(D) \times L^2(\Gamma) : \mathcal{A}u \in L^2(D), \\ &\quad \mathcal{A}_\Gamma u \in L^2(\Gamma)\} \\ &\subset \{(u,w) : u \in H^{3/2}(D), u = w \ \textit{on}\ \Gamma\} \end{aligned}$$

*Proof.* First we show that the operator $A$ is symmetric. For $(u_1, w_1), (u_2, w_2) \in D(A)$ we have

$$\begin{aligned} &(A(u_1,w_1),(u_2,w_2))_{L^2(D)\times L^2(\Gamma)} \\ =\ & \int_D -\sum \partial_k a_{kj}\partial_j u_1 u_2\, dx + \int_\Gamma \sum \nu_k a_{kj}\partial_j w_1 w_2\, dx + (b_0 w_1, w_2)_{L^2(\Gamma)} \\ =\ & \int_D \sum a_{kj}\partial_j u_1 \partial_k u_2\, dx - \int_\Gamma \sum \nu_k a_{kj}\partial_j w_1 w_2\, dx + \int_\Gamma \sum \nu_k a_{kj}\partial_j w_1 w_2\, dx + (b_0 w_1, w_2)_{L^2(\Gamma)} \\ =\ & -\int_D \sum u_1 \partial_j a_{kj}\partial_k u_2\, dx + \int_\Gamma \sum \nu_j a_{kj}\partial_k w_2 w_1\, dx + (b_0 w_1, w_2)_{L^2(\Gamma)} \end{aligned}$$

Now, we are in the situation to apply Theorem 2.5.10. The properties of $D(A)$ will be discussed in Remark 3.1.3. The proof is complete. □

In addition, we state from [16] and [7] the following connection to Lax-Milgram theory. We consider the continuous symmetric positive bilinear form on the space $\mathbb{V} = H^1(D) \times H^{\frac{1}{2}}(\Gamma)$

$$\begin{aligned} a(U,V) = \sum_{k,j=1}^{d} \int_D a_{kj}(x)\partial_{x_k} u(x)\partial_{x_j} v(x)dx + \int_D a_0(x)u(x)v(x)dx \\ + \int_{\partial D} c(s)u_1(s)v_1(s)ds, \end{aligned} \tag{3.3}$$

where $U = (u, u_1)$, $V = (v, v_1) \in \mathbb{V}$. Following the Lax–Milgram theory, this bilinear form generates a positive selfadjoint operator $A$ in $\mathbb{H}$. Thus, $A$ is the generator of a semigroup of operators on $\mathbb{H}$. By Green's formula 2.3.10 $A$ is related to the pair $(\mathcal{A}(x,\partial), \mathcal{A}_\Gamma(x,\partial))$. In particular, we consider the elliptic problem

$$\begin{aligned} \mathcal{A}(x,\partial)u = f \quad \text{on } D, \\ \mathcal{A}_\Gamma(x,\partial)u = g \quad \text{on } \partial D. \end{aligned} \tag{3.4}$$

The domain of $A$ is considered in the following Lemma 3.1.2, which uses elliptic regularity theory.
We cite the following Lemma from [16].

**Lemma 3.1.2** *For $(f,g) \in \mathbb{H}$ (3.4) has a unique solution of the form $\Phi = u_* + u_{**}$ where $u_* \in H^2(D)$, $\mathcal{A}_\Gamma(x,\partial)u_* = 0$ on $\partial D$ and $u_{**} \in H^{3/2}(D)$, $\mathcal{A}(x,\partial)u_{**} = 0$.*

*Proof.* For a bounded set of right hand sides of (3.4) in $\mathbb{H}$ the set of solutions $\Phi$ for these right hand sides forms a bounded set in $H^{\frac{3}{2}}(D) \times H^1(\partial D)$. But this space is compactly embedded in $\mathbb{H}$. Hence, the positive symmetric operator $A$ has a compact inverse, so that we can apply spectral theory of selfadjoint operators. □

It follows that there exists an orthonormal basis $\{E_k\}_{k\in\mathrm{N}}$ in $\mathbb{H}$, such that

$$AE_k = \lambda_k E_k,\ k = 1,2,\dots, \quad 0 < \lambda_1 \le \lambda_2 \le \dots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

These eigenvalues have a finite multiplicity. By Lemma 3.1.2, we have obtained the domain $D(A)$ of the operator $A$ which is contained in $H^{\frac{3}{2}}(D) \times H^1(\partial D)$.
It is now straightforward that $-A$ is the generator of a $C_0$–semigroup $S(t)$.

**Remark 3.1.3**
*(1) We have by an interpolation argument*

$$D(A^s) \subset \begin{cases} H^{\frac{3}{2}s}(D) \times H^s(\partial D), & \text{for } 0 \leq s < 1, \\ H^{\frac{1}{2}+s}(D) \times H^s(\partial D), & \text{for } s \geq 1. \end{cases} \tag{3.5}$$

*For the first relation see Sell and You [46, Lemma 37.8].*
*(2) It follows from* (3.5) *that the eigenfunctions $E_k$ of the operator $A$ has the form*

$$E_k = (e_k; \gamma[e_k]), \quad k = 1, 2, \ldots,$$

*where $e_k \in C^\infty(\overline{D})$ and $\gamma$ is the trace operator on the boundary.*

The fractional spaces $D(A^s)$ are defined in Section 3.1.1 below. We can state the following theorem, which presents a connection between $D(A^{\frac{1}{2}})$ and $\mathbb{V}$.

**Theorem 3.1.4**
*Assume that $A$ is a positive selfadjoint operator and*

$$A : \mathbb{V} \mapsto \mathbb{V}'.$$

*Then*

$$\|V\|_{\mathbb{V}}^2 = (AV, V) = \|A^{\frac{1}{2}}V\|^2$$

*is a norm on $\mathbb{V}$.*

*Proof.* Due to the assumptions on $A$, the following spectral decomposition of $A$ is given by

$$AU = \sum_{i=1}^{\infty} \lambda_i (U, E_i) E_i,$$

where $\lambda_i$ are the eigenvalues and $E_i$ the eigenvectors of $A$. Then we obtain

$$\begin{aligned} a(U, U) &= \langle AU, U \rangle_{\mathbb{V}', \mathbb{V}} = (AU, U) \\ &= (\sum_{i=1}^{\infty} u_i A E_i, \sum_{i=1}^{\infty} u_i E_i) = (\sum_{i=1}^{\infty} u_i \lambda_i E_i, \sum_{i=1}^{\infty} u_i E_i) \\ &= \sum_{i=1}^{\infty} \lambda_i u_i^2. \end{aligned}$$

The last expression is a norm on $D(A^{\frac{1}{2}})$. Since $a(U, V)$ is the scalar product on $\mathbb{V}$, the assertion is proven. Note that we use for simplicity the same notation $A$ to denote the operator into $\mathbb{V}'$ as well as in $\mathbb{H}$, since we consider the relation only on

$$D(A) = \{u \in \mathbb{V}, AU \in \mathbb{H}\}.$$

We also use the Riesz representation theorem to identify the scalar product with the linear functional. □

It is also possible to prove the existence of a $C_0$–semigroup by applying the Hille-Yoshida Theorem 2.1.12. The example 2.5.7 is covered by this theorem.

**Theorem 3.1.5**
*$-A$ is the generator of a $C_0$–contraction semigroup on $\mathbb{H} = \mathbb{H}^0 = L^2(D) \times L^2(\Gamma)$.*

*Proof.* First, we note that $D(A)$ is dense in $\mathbb{H}$. In addition, $A$ is symmetric. Furthermore, we obtain that

$$AU + \lambda U = F \in \mathbb{H}$$

has a unique solution for every $\lambda > 0$. Hence, by the positivity of $A$

$$\lambda \|U\|^2 \leq \|F\| \|U\|,$$

which shows that $(A + \lambda \mathrm{id})^{-1} \leq \lambda^{-1}$ by calculating

$$\|(A + \lambda\, \mathrm{Id})^{-1}\| \leq \frac{1}{\|A + \lambda\, \mathrm{Id}\,\|}$$

and

$$\|A + \lambda\, \mathrm{Id}\,\| = \sup_{U \in \mathbb{H}} \frac{\|(A + \lambda\, \mathrm{Id})U\|}{\|U\|} = \sup_{U \in \mathbb{H}} \frac{\|F\|}{\|U\|} \geq \lambda.$$

In particular, there are no positive spectral values. The conclusion follows by Theorem 2.1.12.

□

Now, we will give two examples of parabolic operators with dynamical boundary conditions., the second example is taken from [56].

**Example 3.1.6 (1d-problem)**
*In the 1d-setting, the boundary is divided into two parts. We have only two simple points here. We consider the following problem on* $[0, 1]$

$$\begin{aligned} -u'' &= f \\ u(0) &= 0 \\ u'(1) + u(1) &= g \end{aligned} \tag{3.6}$$

*Multiplying by $v \in H^1$ with $v(0) = 0$ gives us the following weak formulation*

$$\begin{aligned} \int_0^1 f v \, dx = &- \int_0^1 u'' v \, dx \\ &= \int_0^1 v' u' \, dx + u(1)v(1) - g(1)v(1). \end{aligned}$$

*We have a problem of the following form*

$$L(v) = a(u, v),$$

*where a is a positive symmetric bilinear form. Exactly*

$$L(v) = \int_0^1 fv\,dx + g(1)v(1),$$
$$a(u,v) = \int_0^1 u'v'\,dx + u(1)v(1).$$

*The existence of a weak solution of (3.6) follows directly by Lemma 3.1.2.*

**Example 3.1.7 (Natural boundary conditions)**
*We formulate the third boundary value problem. Let $D$ be an open subset of $\mathbb{R}^n$ with boundary $\Gamma$ and $X = H^1(D)$. Consider the following equation*

$$\begin{aligned} -\Delta u &= f \text{ on } D \\ \partial_\nu u + hu &= g \text{ on } \Gamma, \end{aligned} \tag{3.7}$$

*where $\partial_\nu$ denotes the outer normal derivative on $\Gamma$. Again, we have a problem of the following form*

$$L(v) = a(u,v),$$

*where a is a positive symmetric bilinear form. Exactly*

$$L(v) = \int_D fv\,dx + \int_\Gamma gv\,d\sigma$$
$$a(u,v) = \int_D \nabla u \nabla v\,dx + \int_\Gamma huv\,d\sigma.$$

*Once more, we have the existence of a weak solution.*

### 3.1.1 Fractional spaces

We use the properties of $A$ to define fractional spaces. If the eigenvalues of $A$ are positive, increasing, countable and tend to infinity as $n \to \infty$, we can introduce the following function spaces for $s > 0$

$$D(A^s) = \{u = \sum_{i=1}^{\infty} \widehat{u_i} E_i : \quad \|u\|_{D(A^s)} = \sum_{i=1}^{\infty} \|\widehat{u_i}\|^2 \lambda_i^{2s} < \infty\}.$$

where $\lambda_i$ are the eigenvalues of $A$ and $\widehat{u_i} = (u, E_i)$. We refer to [48] for details. A connection between Sobolev spaces of fractional order and the $D(A^s)$–spaces is given in Theorem 3.1.4.

## 3.2 The Ritz method

We can also apply the Ritz method to find an approximate solution of (3.7), see [56, p. 28]. This is an alternative way to obtain the existence of weak solutions. The basic concept of this method is to vary over finite-dimensional subspaces, which satisfy the boundary conditions. We can reduce variational problems in function spaces to variational problems with finitely many variables. Let us consider again the problem (3.7)

$$\begin{aligned} -\Delta u =& f \text{ on } D \subset \mathbb{R}^n \\ \partial_\nu u + hu =& g \text{ on } \Gamma. \end{aligned}$$

This is by [56, p.28] equivalent with the variational problem

$$\int_D \big(\frac{1}{2}\sum_{i=1}^n (D_i u)^2 - uf\big)\, dx + \int_\Gamma (\frac{1}{2}u^2 - ug)\, dO = \min!, \quad u \in C^1(\overline{G})$$

and the generalized boundary value problem

$$\int_D \big(\sum_{i=1}^n D_i u D_i v - fv\big)\, dx + \int_\Gamma (hu - g)v\, dO = 0 \text{ for all } v \in C^1(\overline{G}).$$

We use the following ansatz for the solution $u$

$$u_m = \sum_{k=1}^m c_{km} w_k,$$

where $w_k$ are the eigenfunctions of the Laplacian, and replace our problem (3.7) by the approximated problem

$$\int_D \big(\frac{1}{2}\sum_{i=1}^n (D_i u_m)^2 - u_m f\big)\, dx + \int_\Gamma (\frac{1}{2}u_m^2 h - u_m g)\, dO = \min!, \quad c_{km} \in \mathbb{R}^n.$$

We now vary over all funtions $u_m$. Our goal is to determine the unknown coefficients $c_{1m}, \cdots, c_{mm}$. We have that

$$\int_D \big(\sum_{i=1}^n D_i u_m D_i w_j - f w_j\big)\, dx + \int_\Gamma (hu_m - g) w_j\, dO = 0, \quad j = 1, \cdots, m.$$

and get the equivalent Ritz equations

$$\sum_{k=1}^m c_{km}\Big(\int_D \big(\sum_{i=1}^n D_i w_k D_i w_j - f w_j\big)\, dx + \int_\Gamma (hw_k - g) w_j\, dO\Big) = 0, \quad j = 1, \cdots, m.$$

This is a linear system of equations, and we can determine the coefficients $c_{1m}, \cdots, c_{mm}$. The convergence of $u_m$ to $u$ is discussed in [56, chapter 22].

## 3.3 Hyperbolic problems

We have the same assumptions on the linear differential operators $\mathcal{A}(x, \partial)$ on $D$, see (3.1), and $\mathcal{A}_\Gamma(x, \partial)$ on $\Gamma$, see (3.2), as in the parabolic case. Now, we consider the hyperbolic problem

$$\begin{aligned} u'' + \mathcal{A}u &= 0 \text{ on } D \\ u'' + \mathcal{A}_\Gamma u &= 0 \text{ on } \Gamma \end{aligned}$$

with initial conditions $u(x,0) = u_0(x)$, $u'(x) = u_1(x)$ and respectively on the boundary. We transform this problem with $U = (u, u_\Gamma)$ and $V = (v, v_\Gamma)$ to

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = B \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & \text{id} \\ -A & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

where $A = (\mathcal{A}, \mathcal{A}_\Gamma)$ is the operator from the parabolic problem introduced in the section before. We can state the following Lemma, where the dissipativity of $B$ is computed directly.

**Lemma 3.3.1**
*We have that*

$$(\begin{pmatrix} U \\ V \end{pmatrix}, B \begin{pmatrix} U \\ V \end{pmatrix}) = 0,$$

*where the inner product is defined by*

$$(\begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} F \\ G \end{pmatrix}) = [\begin{pmatrix} u \\ u_\Gamma \\ v \\ v_\Gamma \end{pmatrix}, \begin{pmatrix} f \\ f_\Gamma \\ g \\ g_\Gamma \end{pmatrix}] := \int_D \nabla u \nabla f + vg\, dx + \int_\Gamma u_\Gamma f_\Gamma + v_\Gamma g_\Gamma \, d\sigma.$$

*Proof.* We have

$$\begin{aligned} [\begin{pmatrix} u \\ u_\Gamma \\ v \\ v_\Gamma \end{pmatrix}, B \begin{pmatrix} u \\ u_\Gamma \\ v \\ v_\Gamma \end{pmatrix}] &= \\ \int_D \nabla u \nabla u' + \Delta u u' \, dx + \int_\Gamma u_\Gamma v_\Gamma - \partial_\nu u_\Gamma v_\Gamma - u_\Gamma v_\Gamma \, d\sigma &= \\ \int_D \nabla u \nabla u' - \nabla u \nabla u' \, dx + \int_\Gamma \partial_\nu u_\Gamma v_\Gamma \, d\sigma - \int_\Gamma \partial_\nu u_\Gamma v_\Gamma \, d\sigma &= 0. \end{aligned}$$

□

This shows us the dissipativity of $B$, see Theorem 2.1.15.
To prove the semigroup property on $E_1 = D(A) \times \mathbb{V}$ we need the following more general lemma

**Lemma 3.3.2**
*We have that*

$$(\begin{pmatrix} U \\ V \end{pmatrix}, B \begin{pmatrix} U \\ V \end{pmatrix})_{E_j} = 0, \quad j = 1, 2$$

*with the following scalar product:*

$$(\begin{pmatrix} U_i \\ V_i \end{pmatrix}, \begin{pmatrix} F_i \\ G_i \end{pmatrix})_{E_j} = \lambda_i^{j+1} U_i F_i + \lambda_i^j V_i G_i.$$

*where $U_i, V_i, F_i, G_i$ are the coefficients of $U, V, F, G$ in the following basis of $E_j$*

$$E = (\hat{u}_i \begin{pmatrix} \lambda_i^{-\frac{1}{2}} \\ 0 \end{pmatrix} e_i, \hat{\hat{u}}_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_i)_{i=1,\cdots,\infty}.$$

*Here $\lambda_i$ are the eigenvalues and $e_i$ the eigenvectors of the operator $A$, see also [36, chapter 5].*

*Proof.* From now on we use the Einstein notation of sums and conclude

$$BE = \begin{pmatrix} \hat{\hat{u}}_i e_i \\ -\lambda_i \hat{u}_i \lambda_i^{-\frac{1}{2}} e_i \end{pmatrix}.$$

Thus

$$(BE, E)_{E_j} = (\begin{pmatrix} \hat{\hat{u}}_i e_i \\ -\hat{u}_i \lambda_i^{\frac{1}{2}} e_i \end{pmatrix}, \begin{pmatrix} \hat{u}_i \lambda_i^{-\frac{1}{2}} e_i \\ \hat{\hat{u}} e_i \end{pmatrix}_{E_j} = \lambda_i^{j+1-\frac{1}{2}} \hat{\hat{u}}_i \hat{u}_i - \lambda_i^{j+\frac{1}{2}} \hat{u}_i \hat{\hat{u}}_o = 0.$$

□

**Lemma 3.3.3**
*Define $E_0 = \mathbb{V} \times \mathbb{H}$. We have that*

$$D(B) = \mathrm{Dom}(B) \quad = \quad \{(U, V) \in E_0; (\mathcal{A}u, \mathcal{A}_\Gamma u_\Gamma) \in \mathbb{H}; \quad V \in \mathbb{V}\}$$

*Proof.* It holds, that

$$BW = B \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -A & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V \\ AU \end{pmatrix} \in \mathbb{V} \times \mathbb{H}.$$

□

**Theorem 3.3.4**
*$B$ is the generator of a $C_0$-contraction semigroup on $E_0$.*

*Proof.*
Consider as in [27, p.445] on

$$X := E_0 = H^1(D) \times L^2(D) \times H^{1/2}(\Gamma) \times L^2(\Gamma)$$

the operator equation

$$\lambda(U, V) - B(U, V) = (F, G),$$

with $(F, G) \in E_0$. This yields the following system of equations on $X$.

$$-v + \lambda u = f \tag{3.8}$$
$$-v_\Gamma + \lambda u_\Gamma = f_\Gamma \tag{3.9}$$
$$\mathcal{A}u + \lambda v = g \tag{3.10}$$
$$\mathcal{A}_\Gamma u_\Gamma + \lambda v_\Gamma = g_\Gamma \tag{3.11}$$

with

$$(f, f_\Gamma, g, g_\Gamma) \in H^1(D) \times H^{1/2}(\Gamma) \times L^2(D) \times L^2(\Gamma).$$

The existence of a solution is ensured by [27, Theorem 6.2.2]. We reformulate this system with $F = (f, f_\Gamma)$ and $G = (g, g_\Gamma)$ as

$$(U, V) = R_\lambda(F, G),$$

where $R_\lambda$ is the resolvent operator with

$$R_\lambda(F, G) := (\lambda I - B)^{-1}(F, G)$$

in this case. We insert all these equations (3.8)–(3.11) and obtain the equation

$$AU + \lambda^2 U = G + \lambda F.$$

The proof is now the same as in the parabolic case. As in Theorem 2.4.6, we have the following associated bilinear form to $A$

$$a(U, V) = (AU, V).$$

Starting with

$$\lambda\|V\|^2 + a(U, V) = (G, V)$$

we get by $V = (\lambda U - F)$

$$\begin{aligned} \lambda\big(\|V\|^2 + a(U, U)\big) &= (G, V) + a(U, F) \\ &\leq (\|G\|^2 + a(F, F))^{1/2}(\|V\|^2 + a(U, U))^{1/2} \end{aligned}$$

because

$$(G, V) + a(U, F)$$

is a scalar product and we can apply the Cauchy-Schwarz inequality.

$$\|(U, V)\|_0 = (a(U, U) + \|V\|^2)^{1/2}$$

is a norm on $X = E_0$ and thus, we have that

$$\|(U, V)\|_0 \leq \frac{1}{\lambda}\|(F, G)\|_0.$$

This gives us finally

$$\|R_\lambda\| \leq \frac{1}{\lambda}.$$

By applying the Hille-Yoshida theorem 2.1.12, the generation of a $C_0$-semigroup by $B$ is proven. □

**Remark 3.3.5**
*Due to Lemma 3.3.2, we have that $B$ is the generator of a $C_0$-contraction semigroup on $E_1$ with the same arguments as in Theorem 3.3.4.*

**Remark 3.3.6 (Damped equation)**
*We can apply Theorem 2.1.7 to a damped wave equation*

$$\begin{aligned} u'' + \alpha u' + \mathcal{A}u &= f \text{ on } D \\ u'' + \alpha u' + \frac{du}{d\nu} + u &= g \text{ on } \Gamma. \end{aligned} \tag{3.12}$$

*with initial condition $U_0 = (u_0, u_{0\Gamma}) \in \mathbb{H}$. We can conclude, that by this equation a $C_0$–semigroup is generated, see also Chapter 8.*

Finally, we can prove the existence of mild solutions of (3.12).

**Theorem 3.3.7 (Existence theorem)**
*There exists a mild solution to (3.12), under the assumptions, that $A$ is the generator of a $C_0$–semigroup $S(t)$ on $E_0$, and $Q = (f, g) \in E_0$.*

*Proof.* We reformulate (3.12),

$$BW = Q$$

and then follow the arguments as in [41, p.184]. The strategy is to show that

$$\mathcal{K}(W(t)) = S(t)W_0 + \int_0^t S(t-s)Q(s, W(s))\, ds$$

is a contraction on

$$Y = \{W \in X : \sup_{[0,T]} e^{-Lt} \|W(t)\|_{E_0} < \infty\}$$

where $L$ is the Lipschitz constant of $Q$. The fixpoint of this contraction is the mild solution of (3.12).

□

# Chapter 4

# Solutions of stochastic partial differential equations

## 4.1 Wiener process in a Hilbert space

The following definitions are from [19]. The Wiener process has to be defined on infinite-dimensional Hilbert spaces. The operator $Q$ plays the role of the covariance operator as in the finite-dimensional case.
At first, we have to generalize the theory of Gaussian measures to Hilbert spaces.

**Definition 4.1.1**
*Assume that $H$ is a Hilbert space with scalar product $(\cdot,\cdot)$, then a probability measure $\mu$ on $(H, B(H))$ is called Gaussian, if for arbitrary $h \in H$ there exists $m \in \mathbb{R}, q > 0$, such that*

$$\mu\{x \in H; \langle h, x\rangle \in A\} = N(m,q)(A), \quad A \in B(\mathbb{R}).$$

If $\mu$ is Gaussian, the following functionals

$$H \to \mathbb{R}, h \to \int_H \langle h, x\rangle \mu(dx)$$

and

$$H \times H \to \mathbb{R}, (h_1, h_2) \to \int_H \langle h_1, x\rangle\langle h_2, x\rangle \mu(dx)$$

are well defined. Due to [19, Lemma 2.14], they are also continuous. We have the following proposition and definition from [19, p. 54].

**Proposition 4.1.2**
*From [19, Lemma 2.14] it follows that if $\mu$ is Gaussian, then there exists an element $m \in H$ and a symmetric non negative continuous operator $Q$, such that*

$$\int_H \langle h, x\rangle \mu(dx) = \langle m, x\rangle, \quad \forall h \in H$$

*and*

$$\int_H \langle h_1, x\rangle\langle h_2, x\rangle \mu(dx) - \langle m, h_1\rangle\langle m, h_2\rangle = \langle Qh_1, h_2\rangle \quad \forall h_1, h_2 \in H.$$

*$m$ is called the mean and $Q$ the covariance operator of $\mu$. By a characteristic function argument, it follows, that $\mu$ is uniquely determined by $m$ and $Q$ and its distribution is denoted by $N(m, Q)$. We denote by $TrQ$ the trace of $Q$.*

Let $U$ and $H$ be two separable Hilbert spaces. Our goal is to construct the stochastic Ito integral

$$\int_0^t B(s)\, dW(s), t \in [0, T],$$

where $W(\cdot)$ is a Wiener process on $U$ and $B$ is a process with values in $L(U; H)$, the space of linear operators between $U$ and $H$. These operators are not necessarily bounded.

**Definition 4.1.3**
*Let $U$ be a separable Hilbert spaces, and $Q \in L(U)$ a symmetric nonnegative operator. A $U$-valued stochastic process $(W(t))_{t\geq 0}$ is called $Q$-Wiener-Prozess, if:*

- $W(0) = 0$,
- *$W$ has continuous trajectories,*
- *$W$ has independent increments,*
- *$(W(t) - W(s))$ is $N(0, (t - s)Q), t \geq s \geq 0$ distributed.*

*The exact interpretation of $N(0, (t - s)Q)$ can be found in [19, Chap. 2.2.].*

From [19, p. 37], we cite the following definition.

**Definition 4.1.4** *Note that a probability measure $\mu$ on $(H, B(H))$, where $H$ is a separable Banach space, is called a Gaussian measure, if and only if the law of an arbitrary linear function in $H^*$ considered as a random variable on $(H, B(H), \mu)$, is a Gaussian measure on $(\mathbb{R}^1, B(\mathbb{R}^1))$.*

This can be generalized to $H$ valued stochastic processes, see [31, Def. 2.4].

**Definition 4.1.5** *An $H$-valued stochastic process $\{X_t\}_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, P)$ is called Gaussian, if for any $n \in \mathbb{N}$ and $t_1, \ldots, t_n \geq 0$, $(X_{t_1}, \ldots, X_{t_n})$ is an $H^n$ valued Gaussion random variable.*

**Remark 4.1.6**
*Assume that $W_1$ and $W_2$ are two independent $Q$-Wiener processes. Then, we can also construct by*

$$W(t) = W_1(t), \quad W(-t) = W_2(t),$$

*a twosided Wiener process, which is zero in zero.*

The Wiener process has the following properties, [19, Prop. 4.1]:

**Proposition 4.1.7**
*Assume that $W$ is a $Q$-Wiener process, with $TrQ < \infty$. Then, the following statements hold.*

- *$W$ is a Gaussian process on $U$ and*

$$E(W(t)) = 0, \quad \mathrm{Cov}(W(t)) = tQ, \quad t \geq 0.$$

- *For arbitrary $t$, $W$ has the expansion*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \tag{4.1}$$

*where*

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}}\langle W(t), e_j\rangle$$

*with $e_j$ the eigenfunctions of $Q$.*

Now, we are able to construct the stochastic integral, see [19, p. 90]. Assume that, we have a $Q$-Wiener process in $(\Omega, \mathcal{F}, P)$ with values in $U$. This process is given by (4.1). Furthermore, we are also given a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in $\mathcal{F}$ and

- $W(t)$ is $\mathcal{F}_t$-measureable,
- $W(t+h) - W(t)$ is independent of $\mathcal{F}_t, \forall h \geq 0, \forall t \geq 0$.

At first, we assume that $B(t)$ is an elementary process, i.e. there exists a sequence $0 = t_0 < t_1 < \ldots < t_k = T$ and a sequence $B_0, B_1, \ldots B_{k-1}$ of $L(U;H)$-valued random variables, such that $B_m$ are $\mathcal{F}_{t_m}$ measureable and

$$B(t) = B_m \text{ for } t \in (t_m, t_{m+1}], m = 0, 1, \ldots, k-1.$$

Then, we can define the stochastic integral by, see [19, p. 90],

$$\begin{aligned}\int_0^t B(s)\, dW(s) &= \sum_{m=0}^{k-1} B_m(W(t_{m+1} \wedge t) - W(t_m \wedge t)) \\ &=: B \cdot W(t).\end{aligned}$$

As in the finite-dimensional case, see [39] we have to extend the class of elementary functions $B(t)$ to a proper class of integrands. These are in fact the predictable processes with values in $L_0^2 = L^2(U_0; H)$, see [19, Prop. 4.7(ii)] or [31, Prop. 2.2], where $U_0 = Q^{1/2}U$ and $L^2(U_0; H)$ is the space of Hilbert-Schmidt operators from $U_0$ into $H$. The Hilbert-Schmidt norm of an operator $L \in L^2(U_0; H)$ is by [31, (2.7)] given by

$$\begin{aligned}\|L\|^2_{L^2(U_0;H)} &= \sum_{j,i=1}^{\infty} (L(\lambda_j^{1/2} f_j), e_i)^2_H = \sum_{j,i=1}^{\infty} (LQ^{1/2} f_j), e_i)^2_H \\ &= \|LQ^{1/2}\|^2_{L^2(U;H)} = \mathrm{tr}((LQ^{1/2})(LQ^{1/2})^*).\end{aligned}$$

where $f_j$ is an orthonormal basis (ONB) in $U$ and $e_i$ in $H$.
Note that $L(U;H) \subset L^2(U_0;H)$, since for $k \in U_0$

$$Lk = \sum_{j=1}^{\infty} (k, \lambda_j^{1/2} f_j)_{U_0} \lambda_j^{1/2},$$

regarded as an operator from $U_0$ into $H$, has a finite Hilbert–Schmidt norm

$$\|L\|^2_{L^2(U_0;H)} = \sum_{j=1}^{\infty} \|L(\lambda_j^{1/2} f_j)\|^2_H = \sum_{j=1}^{\infty} \lambda_j \|Lf_j\|^2_H \leq \|L\|^2_{L(U,H)} \operatorname{tr}(Q).$$

The trace condition on $Q$ ensures, that the integrands are square integrable, such that one can generalize the methods of the finite-dimensional case [39, Chapter 3], and we conclude the following Lemma ([19, Prop. 4.5])

**Lemma 4.1.8** *If a process $B$ is elementary and*

$$\mathbb{E} \int_0^t \|B(s)\|^2_{L_0^2} \, ds < \infty$$

*then the process $B \cdot W(t)$ is a continuous, square integrable $H$–valued martingale and*

$$\mathbb{E}\|B \cdot W(t)\|^2 = \mathbb{E} \int_0^t \|B(s)\|^2_{L_0^2} \, ds < \infty.$$

## 4.2 Concepts of solutions

We can use the definition of a stochastic integral to introduce stochastic partial differential equations (spdes). We take the definitions from [19, Chapter 6]. Consider the following evolution equation on a separable Hilbert space $H$

$$\begin{aligned} dX(t) &= (AX + f(t))dt + B(X(t))dW(t) \\ X(0) &= \xi \end{aligned}$$

on a time interval $[0, T]$, where $A : D(A) \subset H \mapsto H$ is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$ and $W$ is a $Q$-Wiener process with $TrQ < \infty$. $\xi$ is a $\mathcal{F}_0$–measurable $H$-valued random variable, where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the Wiener process, where $f$ is a predictable process (see Definition 4.2.2) and $B : D(B) \subset H \to L_0^2$ is a linear operator. $\mathcal{F}_0$–measurability has to be understood in the sense, that the initial condition is independent of the Wiener process.

**Definition 4.2.1**
*Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a given standard filtration $\mathcal{F}_t$ generated by the Wiener process. $P_T$ denotes predictable $\sigma$-fields on $\Omega_T = [0, T] \times \Omega$. $\Omega_T$ the $\sigma$-field generated by the following sets, see also [19, p. 76],*

$$(s, t] \times F, \quad 0 \leq s < t < T, F \in \mathcal{F}_s \text{ and } \{0\} \times F, F \in \mathcal{F}_0.$$

**Definition 4.2.2**
*An arbitrary measurable mapping from $([0, T) \times \Omega, P_T)$ into $(H, B(H))$ is called a predictable process.*

**Definition 4.2.3 (Strong solution)**
*An $H$-valued predictable process $X(t)$ is called strong solution of the above equation, if*

$$P\Big(\int_0^T (\|X(s)\| + \|AX(s)\|)\, ds < \infty\Big) = 1,$$

$$P\Big(\int_0^T \|B(X(s))\|_{L^2(U;H)} < \infty\Big) = 1$$

*and*

$$X(t) = \xi + \int_0^t (AX(s) + f(s))\, ds + \int_0^t B(X(s))\, dW(s)$$

*holds for arbitrary $t \in [0, T]$.*

**Definition 4.2.4 (Weak solution)**
*An $H$-valued predictable process $X(t)$ is called weak solution of the above equation, if*

$$P\Big(\int_0^T \|X(s)\|\, ds < \infty\Big) = 1,$$

$$P\Big(\int_0^T \|B(X(s))\|_{L^2(U;H)} < \infty\Big) = 1$$

*and*

$$\langle X(t), \zeta\rangle = \langle \xi, \zeta\rangle + \int_0^t (\langle X(s), A^*\zeta\rangle + \langle f(s), \zeta\rangle)\, ds + \int_0^t \langle \zeta, B(X(s))\, dW(s)\rangle$$

*holds for arbitrary $t \in [0, T]$ and $\zeta \in D(A^*)$. $A^*$ denotes the adjoint operator of $A$.*

**Definition 4.2.5 (Mild solution)**
*An $H$-valued predictable process $X(t)$ is called mild solution of the above equation, if*

$$P\Big(\int_0^T \|X(s)\|\, ds < \infty\Big) = 1,$$

$$P\Big(\int_0^T \|B(X(s))\|_{L^2(U;H)} < \infty\Big) = 1$$

*and*

$$X(t) = T(t)\zeta + \int_0^t T(t-s)f(s)\, ds + \int_0^t T(t-s)B(X(s))\, dW(s)$$

*holds for arbitrary $t \in [0, T]$.*

There are the following connections between the different types of solutions.

**Remark 4.2.6**
*A strong solution is also a weak solution. A weak solution is also a mild solution. This means, that the concept of mild solutions is the weakest concept.*

## 4.3 Main theorem of existence and uniqueness of mild solutions

From now on, we consider the spde

$$\begin{aligned} dX(t) &= (AX + F(X(t),t))dt + B(X(t),t)dW(t) \\ X(0) &= \xi \end{aligned} \tag{4.2}$$

on a time interval $[0,T]$, where $A : D(A) \subset H \mapsto H$ is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$ and $W$ is a $Q$-Wiener process with $TrQ < \infty$. $\xi$ is a $\mathcal{F}_0$–measurable $H$-valued random variable, where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the Wiener process. Additionally, we have the Hypothesis 4.3.1 and 4.3.2 below. We cite Theorem 7.4 from [19]. This theorem gives us a survey on the existence of solutions of SPDEs. The key aspect is the introduction of the concept of mild solutions.

**Hypothesis 4.3.1**
*There exists a constant $C > 0$, such that*

$$\|F(t,\omega,x) - F(t,\omega,y)\| + \|B(t,\omega,x) - B(t,\omega,y)\|_{L^2(U_0;H)} \leq C\|x-y\| \tag{4.3}$$

*and*

$$\|F(t,\omega,x)\|^2 + \|B(t,\omega,x)\|^2_{L^2(U_0;H)} \leq C^2(1+\|x\|^2). \tag{4.4}$$

**Hypothesis 4.3.2**
*On $F$ and $B$ we have the following hypotheses:*

- *The mapping $F : [0,T] \times \Omega \times H \to H$, $(t,\omega,x) \to F(t,\omega,x)$ is measurable from $(\Omega_T \times H, P_T \times B(H))$ into $(H, B(H))$.*
- *The mapping $B : [0,T] \times \Omega \times H \to L^2_0$, $(t,\omega,x) \to F(t,\omega,x)$ is measurable from $(\Omega_T \times H, P_T \times B(H))$ into $(L^2_0, B(L^2_0))$.*

**Theorem 4.3.3 (Existence of Solutions of SPDE)**
*We assume $A$ is the infinitesimal generator of a strongly continuous semigroup and let hypotheses 4.3.1 and 4.3.2 be fulfilled. Then for any arbitrary $\mathcal{F}_0$ measurable initial condition with $E\|\xi\|^p < \infty$, $p \geq 1$, there exists a unique mild solution $X$ to (4.2) with*

$$X(t) = T(t)\zeta + \int_0^t T(t-s)F(s,X(s))\,ds + \int_0^t T(t-s)B(s,X(s))\,dW(s) \quad \forall t \in [0,T] a.s. \;.$$

# Chapter 5

# Random dynamics

## 5.1 Random dynamical systems

### 5.1.1 MDS

Before we study our randomly perturbed differential equations, we need some fundamental terms from the theory of random dynamical systems (RDS). The theory is introduced in the monograph by Arnold [3]. First, we define a metric dynamical system, which serves as a general model for a noise.

**Definition 5.1.1**
*Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A quadro-tuple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system, if $\theta$ is a measurable flow, i.e.*

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F})$$

*fulfills the flow property*

$$\theta_0 = \mathrm{id}, \quad \theta_t \circ \theta_\tau =: \theta_t\theta_\tau = \theta_{\tau+t} \text{ for } t, \tau \in \mathbb{R}.$$

*Here $\theta_t$ denotes the partial mappings of $\omega \mapsto \theta(t, \omega)$. In addition, the measure $\mathbb{P}$ is supposed to be ergodic with respect to the flow $\theta$ i.e. the $\theta_t$-invariant sets have either full or zero measure.*

As an example we study the *Brownian motion* metric dynamical system. We introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (C_0(\mathbb{R}; U), \mathcal{B}(C_0(\mathbb{R}; U)), \mathbb{P}_W)$ where $C_0(\mathbb{R}; U)$ consists of continuous functions on $\mathbb{R}$ with values on some separable Hilbert space $U$ , which are zero at zero equipped with the compact open topology, $\mathcal{B}(C_0(\mathbb{R}; U))$ is the Borel-$\sigma$-algebra of this space and $\mathbb{P} = \mathbb{P}_W$ the Wiener measure with respect to some covariance operator $K$ with finite trace on some appropriate function space. We introduce the flow defined by the shift operators

$$\theta_t\omega(\cdot) := \omega(\cdot + t) - \omega(t).$$

The Wiener measure is ergodic with respect to this flow. The associated probability space defines a canonical Wiener process. We also note, that such a Wiener process generates a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$,

$$\mathcal{F}_t = \{\omega(\tau) | \tau \le t\}.$$

There exist other metric dynamical systems, for instance generated by the *fractional Brownian motion*. For details see [4].
For the following, we suppose that $H$ is a separable Banach space with norm $\|\cdot\|$.

**Definition 5.1.2 (Random set)**
*A multivalued mapping $\omega \to M(\omega) \subset H$ with closed and non-empty values is called measurable, if the mapping*

$$\omega \to \inf_{y\in M(\omega)} \|x-y\|$$

*is a random variable for every $x \in H$.*

In addition we need:

**Definition 5.1.3 (Temperedness)**
*A random variable $X \in \mathbb{R}^+$ on $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called tempered, if there is a set of full measure such that*

$$\lim_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = 0.$$

This full set is $\theta_t$-invariant.
We note that in the ergodic case, there is only one alternative to this relation:

$$\limsup_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = \infty.$$

Later on, we will use the notation *tempered* if for *every* $\omega$ the above convergence property holds. In this sense, we introduce the set of tempered random sets $\mathcal{D}$. A multivalued mapping $\omega \to D(\omega)$ is contained in $\mathcal{D}$ if $D(\omega)$ is a bounded subset of $H$, $\bar{D}$ is a random set and

$$\omega \to \sup_{x\in\bar{D}(\omega)} \|x\|$$

is a tempered random variable. The measurability of this random variable is given by the representation

$$\overline{\bigcup_{i\in\mathbb{N}} x_i(\omega)} = D(\omega),$$

where $x_i(\omega)$ are measurable maps, see [9, Theorem III.9], and [44].
We can state some criteria, that a random variable is tempered. For the first one, see [3, p. 167].

**Remark 5.1.4**
*A sufficient condition that a positive random variable is tempered is that*

$$\mathbb{E} \sup_{t\in[0,1]} X(\theta_t\omega) < \infty.$$

**Remark 5.1.5**
*Let $h_1$ and $h_2$ be random variables, such that*

$$t \mapsto h_1(\theta_t\omega), \quad t \mapsto h_2(\theta_t\omega)$$

*are locally integrable with* $\mathbb{E}h_1 < 0$ *and* $h_2$ *is tempered. The random variable*

$$\int_{-\infty}^{0} e^{\int_t^0 h_1(\theta_s\omega)\,ds} h_2(\theta_t\omega)\,dt$$

*exists and is tempered.*

The proof can be found in [45] in the proof of Theorem 4.1($iii$).

### 5.1.2 Random Dynamical System

**Definition 5.1.6**
*Let* $H$ *be some separable Banach space. A measurable mapping*

$$\varphi : \mathbb{R}^+ \times \Omega \times H \to H$$

*having the cocycle property*

$$\varphi(0,\omega,\cdot) = \mathrm{id}_H, \quad \varphi(t,\theta_\tau\omega,\varphi(\tau,\omega,x)) = \varphi(t+\tau,\omega,x)$$

*for* $t,\tau \in \mathbb{R}^+, x \in H$ *and* $\omega \in \Omega$ *is called RDS.* $\varphi$ *is called continuous if* $H \ni x \to \varphi(t,\omega,x) \in H$ *is continuous for all* $t \geq 0$ *and* $\omega \in \Omega$.

### 5.1.3 Attractors

We now introduce the term random attractor, which allows to describe the long time behavior of a random dynamical system. Roughly speaking, the long time behavior can be described by a stationary set, which is compact. Stationarity has to be understood in the rds-sense, which is clarified in Definition 5.1.7 below. We will denote by $\mathcal{D}$ the family of random sets, which will be attracted by a random attractor. It is possible to choose different systems $\mathcal{D}$ of random sets, but we restrict ourselves on the family of tempered sets defined above in Definition 5.1.3.

**Definition 5.1.7**
*A random set* $\mathcal{A} \in \mathcal{D}$ *with* $\mathcal{A}(\omega)$ compact *is called a random (pullback) attractor in* $\mathcal{D}$*, if*

$$\varphi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \text{ for every } t \geq 0, \omega \in \Omega$$

*and*

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\omega, D(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0 \text{ for every } D \in \mathcal{D}, \omega \in \Omega$$

*The invariance of* $\mathbb{P}$ *also gives us the forward convergence in probability. This means*

$$\lim_{t\to\infty} \mathbb{P}(\mathrm{dist}(\overline{\varphi(t,\omega,\overline{D(\omega)})}, \mathcal{A}(\theta_t\omega)) > \delta) \to 0 \quad \forall\delta > 0.$$

*Here* dist *denotes the Hausdorff semidistance* $\mathrm{dist}(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$.

The measurability of $\mathcal{A}$ is given by [44, Theorem 2.4] .
The following theorem contains a sufficient condition for the existence and uniqueness of a random attractor [15].

**Theorem 5.1.8**
*Let $\varphi$ be a continuous RDS and let $B \in \mathcal{D}$ be, such that $B(\omega)$ is compact, and for every $D \in \mathcal{D}$ and $\omega \in \Omega$ there exists a $t(\omega, D) \geq 0$, such that for $t \geq t(\omega, D)$*

$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega).$$

*Then there exists a random attractor $\mathcal{A}$, which is* unique *in $\mathcal{D}$.*

*Proof.* Existence is given by [44, Theorem 2.4]. Assume, we have two attractors $A_i \in \mathcal{D}$, $i = 1, 2$. Then, it follows for any $\omega \in \Omega$ :

$$\operatorname{dist}(A_1(\omega), A_2(\omega)) = \lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega, A_1(\theta_{-t}\omega)), A_2(\omega)) = 0.$$

We conclude $A_1(\omega) \subset A_2(\omega)$ for any $\omega \in \Omega$. Changing the roles of $A_1$ and $A_2$ leads to $A_2(\omega) \subset A_1(\omega)$ for any $\omega \in \Omega$. Thus, the random $\mathcal{D}$-attractor is unique. □

Sometimes, it is not possible to prove compactness like in Theorem 5.1.8. But then one can use another concept introduced in the following theorem. The theorem is taken from [33, Theorem 2.2].

**Theorem 5.1.9**
*Suppose that for the random dynamical system $\varphi$ the mapping*

$$x \mapsto \varphi(t, \omega, x)$$

*is continuous and suppose, that there exists an attracting random compact set $C \in \mathcal{D}$. Recall that by definition of attracting random compact set $C$ has the following property*

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), C(\omega)) = 0.$$

*Then*

$$A(\omega) = \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, C(\theta_{-t}\omega))}$$

*is a random $\mathcal{D}$-attractor.*

### 5.1.4 Random invariant and inertial manifolds

In this section, we give the same definitions as in [7].

**Definition 5.1.10**
*Let $\varphi$ be an rds, and $H_1$, $H_2$ be linear subspaces of $H$, such that $H = H_1 \oplus H_2$ is a splitting of $H$ where $H_1$ is finite dimensional. The projections onto these spaces are denoted by $\pi_1$, $\pi_2$. A multi–function $\omega \mapsto M(\omega) \subset H$ is called random* invariant manifold *of the RDS $\varphi(t, \omega, x)$ if the following properties hold:*

- *$M$ is positively invariant:*

$$\varphi(t, \omega, M(\omega)) \subset M(\theta_t\omega) \quad \text{for all } t \in \mathbb{R}^+,\ \omega \in \Omega.$$

- *$M$ has a graph structure: There exists a mapping $m : H_1 \times \Omega \to H_2$, such that*

$$M(\omega) = \{x_1 + m(x_1, \omega) : x_1 \in H_1\},$$

*$m(\cdot, \omega) : H_1 \to H_2$ is Lipschitz-continuous and $C^k$-smooth for some $k \in \mathbb{N}$ for all $\omega \in \Omega$. $m(x_1, \cdot) : \Omega \to H_2$ is measurable for all $x_1 \in H_1$.*

**Definition 5.1.11** *An invariant manifold $M$ is called random* inertial manifold, *if $M$ is exponentially attracting:*

$$\text{For all } x \in H,\, \omega \in \Omega \text{ holds } \lim_{t\to\infty} \operatorname{dist}_H(\varphi(t, \omega, x), M(\theta_t \omega)) = 0$$

*with exponential convergence rate $\kappa$, i.e.*

$$\text{For all } x \in H,\, \omega \in \Omega \text{ holds } \operatorname{dist}_H(\varphi(t, \omega, x), M(\theta_t \omega)) \le Ce^{-\kappa t}.$$

*and some $C > 0$.*

We have the following remarks from [7]:

**Remark 5.1.12**
*Note that from the measurability properties of $m$, it follows that $m : H_1 \times \Omega \to H_2$ is measurable, see Castaing and Valadier [9, Lemma III.14]. Then straightforwardly, the multi–function $M$ is a random set. Since $H_1$ is finite dimensional, it is closed and hence $\pi_1$ is continuous. Then $\pi_2$ is continuous, such that $H_2 = (\mathrm{id} - \pi_1)H$ is a closed subspace, too.*

**Remark 5.1.13**
*The dynamics of $\varphi$ is finite dimensional on $M$. Let $x = x_1 + m(x_1, \omega) \in M(\omega)$, $x_1 \in H_1$. Then $\varphi$ is represented by*

$$\varphi(t, \omega, x) = \varphi_1(t, \omega, x_1) + m(\varphi_1(t, \omega, x_1), \theta_t \omega),$$

*where $\varphi_1(t, \omega, x_1) := \pi_1 \varphi(t, \omega, x_1 + m(x_1, \omega))$ which is an RDS on $H_1$.*

*Proof.* Assume, that $x \in M(\omega)$. Then, by the invariance property of $M$ $\varphi(t, \omega, x) \in M(\theta_t \omega)$. Therefore, there exists $x_2 \in H$ with

$$\varphi(t, \omega, x) = x_2 + m(x_2, \theta_t \omega). \tag{5.1}$$

Applying the projection on both sides of (5.1) we obtain

$$\pi_1 \varphi(t, \omega, x) = x_2.$$

This gives us the result.

□

### 5.1.5 Conjugated dynamical systems and their properties

We now describe conjugacy between random dynamical systems.

**Definition 5.1.14**
*Let $\varphi_1$ be a continuous RDS and let $T : \Omega \times H \to H$ a mapping with the following properties:*

- *For fixed $\omega$ this mapping $x \to T(\omega, x)$ is a homeomorphism on $H$. The inverse to this mapping is denoted by $T^{-1}(\omega, x)$.*
- *The mappings $\omega \to T(\omega, x)$ and $\omega \to T^{-1}(\omega, x)$ are measurable for $x \in H$.*

*Then the mapping*

$$(t, \omega, x) \to \varphi_2(t, \omega, x) := T(\theta_t\omega, \varphi_1(t, \omega, T^{-1}(\omega, x)))$$

*defines a continuous RDS which is called* conjugate *to $\varphi_1$.*

We have the following remark from [7].

**Remark 5.1.15**
*Suppose that $T(\cdot, \omega)$, $T^{-1}(\cdot, \omega)$ is continuous for all $\omega \in \Omega$ and that $T(x, \cdot)$, $T^{-1}(x, \cdot)$ is measurable for all $x \in H$. Applying Lemma III.14 in [9] again, we obtain that $T$, $T^{-1}$ is measurable. Then, if $\varphi$ is a continuous RDS so is*

$$(t, \omega, x) \mapsto \psi(t, \omega, x) := T(\theta_t\omega, \varphi(t, \omega, T^{-1}(\omega, x))). \tag{5.2}$$

The following lemma demonstrates the connection between attractors of conjugate RDS.

**Lemma 5.1.16**
*Let $\mathcal{A}_1$ be a random attractor of the RDS $\varphi_1$ and $T$ the transformation mapping. Suppose that the system $\{T(D)|D \in \mathcal{D}\}$ is contained in $\mathcal{D}$. Then $\mathcal{A}_2(\omega) = T(\omega, \mathcal{A}_1(\omega))$ generates a random attractor of the RDS $\varphi_2$.*

The proof can be found in [32, Theorem 2.1].
The following lemma from [7] gives the connection between inertial manifolds of the original and the transformated system.

**Lemma 5.1.17**
*(1) Let $\omega \mapsto M(\omega)$ be a random set, which is positively invariant with respect to the continuous RDS $\varphi$. In addition, let $T$ be a mapping introduced above. Then $\omega \to T(\omega, M(\omega)) =: M'(\omega)$ is a positively invariant random set for $\psi$.*

*(2) Let $M$ be an inertial manifold for $\varphi$ and suppose that $x \to T(x, \omega), T^{-1}(x_1, \omega)$ is Lipschitz continuous with a Lipschitz constant $L_T(\omega), L_{T^{-1}}(\omega)$ resp. which is tempered. In addition, suppose that $H_1 \ni x_1 \mapsto \pi_1 T(x_1 + m(\omega, x_1))$ is a homeomorphism on $H_1$ for every $\omega \in \Omega$. Let us denote the inverse mapping by $x_1 \mapsto (\pi_1 T)^{-1}(x_1, \omega)$. The mapping $\omega \mapsto (\pi_1 T)^{-1}(x_1, \omega)$ is supposed to be measurable. Then $M'$ is an inertial manifold for $\psi$ with the same exponential convergence rate and with the graph*

$$m'(\omega, y_1) := \pi_2 T((\pi_1 T)^{-1}(y_1, \omega) + m(\pi_1 T^{-1}(y_1, \omega), \omega)), \quad y_1 \in H_1. \tag{5.3}$$

*(3) In addition, if $T(\cdot, \omega)$, $T(\cdot, \omega)^{-1}$ is Lipschitz/$C^k$–smooth and $m(\cdot, \omega)$ is the graph of a $C^k$–manifold then $M'$ is a $C^k$–manifold.*

*Proof.* (1) follows directly by (5.2) and by the fact that $M$ can be represented by the closure of the union of countable many measurable selectors of the random set $M$.
The invariance of $M'$ is given by the relation

$$\begin{aligned}&\psi(t,\omega,T(\omega,M(\omega)))=T(\theta_t(\omega),\varphi(t,\omega,T^{-1}(\omega,T(\omega,M(\omega)))))\\ =&T(\theta_t\omega,\varphi(t,\omega,M(\omega)))\subset T(\theta_t\omega,M(\theta_t\omega)).\end{aligned}$$

(2) We have

$$\begin{aligned}&\mathrm{dist_H}(\psi(t,\omega,y),M'(\theta_t\omega))=\inf_{z\in M(\theta_t\omega)}\|T(\varphi(t,\omega,T^{-1}(y,\omega)))-T(z,\theta_t\omega)\|\\ &\le L_T(\theta_t\omega)\inf_{z\in M(\theta_t\omega)}\|\varphi(t,\omega,T^{-1}(y,\omega)))-z\|.\end{aligned}$$

The right hand side goes to zero and the temperedness of $L_T$ does not change the rate of exponential decay. On the other hand

$$\begin{aligned}M'(\omega)\ni y&=T(x_1+m(x_1,\omega),\omega)\\ y_1&=\pi_1 T(x_1+m(x_1,\omega),\omega)\end{aligned}$$

which gives the second conclusion. (3) follows from (5.3) and the regularity of $m$, $T$, $T^{-1}$. □

### 5.1.6 The Ornstein-Uhlenbeck process

We now introduce the method of Imkeller-Schmalfuß [32] and Flandoli [28] to transform a SPDE to a random partial differential equation. The main advantage of this method is that we can now solve our equation $\omega$-wise because of the absence of stochastic integrals. In general, we assume that $A$ is a positive self-adjoint operator on some Hilbert space $H$. In this way, we avoid the issue of exceptional sets of the solution depending on the initial condition.
We consider the following Langevin equation, whose solution is called an Ornstein-Uhlenbeck process

$$dZ+AZdt=dW. \tag{5.4}$$

$W$ is an appropriate Wiener process with covariance $K$, as in [19] and Definition 4.1.3. There exists a mild solution of this stochastic equation, because $A$ is positive and self-adjoint [19]. As we consider stochastic equations in infinite dimensions, we need the following lemma to describe the regularity of the system. We refer to [15].

**Lemma 5.1.18 (Regularity of the Ornstein-Uhlenbeck process)**
*Let $H$ be a Hilbert-space, $W$ a twosided Wiener-process (see Remark 4.1.6), $A$ a positive self-adjoint operator on $H$ and $K$ its covariance-operator so that*

$$\mathrm{tr}_H(KA^{2s-1+\epsilon})=\mathrm{tr}_H(A^{s-1/2+\epsilon/2}KA^{s-1/2+\epsilon/2})<\infty.$$

*Here $\mathrm{tr}_H$ denotes the trace of the covariance. Then a $\mathcal{F}_0$-measurable random variable $Z$ with values in $D(A^s)$ exists, and the process $(t,\omega)\to Z(\theta_t\omega)$ is a stationary solution of (5.4). Moreover, we have for $Z(0)=Z_0\in D(A^s)$ that (5.4) has a continuous stationary* *solution $\mathbb{R} \ni t \mapsto Z(t) \in D(A^s)$. In particular, this solution is given by an $\mathcal{F}_0$–measurable random variable $Z \in D(A^\alpha)$, such that*

$$(t,\omega) \mapsto Z(t,\omega) = Z(\theta_t\omega)$$

*is an $(\mathcal{F}_t)_{t\in\mathbf{R}}$–adapted stochastic process. Note that $(\mathcal{F}_t)_{t\in\mathbf{R}}$ is generated by the twosided Wiener process, see also Remark 4.1.6. The random variable $\|Z\|_{D(A^s)}$ is tempered and the mapping $t \mapsto Z(\theta_t\omega) \in D(A^s)$ is continuous, see Chueshov and Scheutzow [14] and the following condition holds:*

$$\mathbb{E}\|Z\|^2_{D(A^s)} = 1/2\,\mathrm{tr}_H(A^{s-1/2}KA^{s-1/2}) < \infty.$$

**Remark 5.1.19**

*(i) Because of the elliptic regularity theory the norms $\|\cdot\|_{D(A^s)}$ and $\|\cdot\|_{H^{2s}}$ are equivalent on $D(A^s)$; see also Lemma 5.1.18 and [48].*

*(ii) The properties of $Z$ formulated in the last lemma hold on a $\{\theta_t\}_{t\in\mathbb{R}}$–invariant set of full measure. We note that the solution of (5.4) satisfies*

$$\mathbb{E}\sup_{[0,T]}\|Z(\theta_t\omega)\|^2_{D(A^s)} < \infty.$$

*We then find the temperedness of $\|Z(\omega)\|^2_{D(A^s)}$. Hence, there exists a $\{\theta_t\}_{t\in\mathbb{R}}$–invariant set of full measure, where $\|Z(\theta_t\omega)\|^2_{D(A^s)}$ has a subexponential growth.*

*(iii) The process $t \to Z(\theta_t\omega)$ is a weak solution to (5.4) as seen in [19, Chapter 5]*

*(iv) We have a representation*

$$Z(\theta_t\omega) = \int_{-\infty}^{t} T(t-\tau)\,dW(\omega)$$

*almost surely for every $t \in \mathbb{R}$.*

*Proof.* (ii) follows by applying Burkholder's inequality and Birkhoff's ergodic theorem. The idea of (iv) is the same as in Lemma 5.1.20 below. □

In the following lemma we introduce as in [36, Lemma 2.3] the following Ornstein-Uhlenbeck process.

**Lemma 5.1.20**
*Consider the following stochastic differential equation*

$$dZ + \mu Z dt = dW, \tag{5.5}$$

*on some Hilbert space $U$ with $\mu > 0$ and $Tr_U Q < \infty$. This equation possesses the stationary solution $\hat{Z}$, later on, for simplicity also denoted by $Z$,*

$$\hat{Z}(\omega) := -\mu\int_{-\infty}^{0} e^{s\mu}W(s)\,ds.$$

*Proof.* We interprete equation (5.5) as integral equation and get

$$Z(t) + \mu \int_0^t Z(t)\, ds = Z(0) + W(t).$$

Applying Ito's formula to $e^{\mu t} Z(t)$ gives us

$$d(e^{\mu t} Z(t)) = \mu e^{\mu t} Z(t) dt + e^{\mu t}(dW - \mu Z(t) dt) = e^{\mu t} dW.$$

This means

$$Z(t,\omega) = e^{-\mu t} Z(0) + \int_0^t e^{\mu(s-t)}\, dW(s).$$

To find a candidate for a stationary solution we calculate

$$\begin{aligned} Z(t, \theta_{-t}\omega) &= e^{-\mu t} Z(0) + \int_0^t e^{\mu(s-t)}\, dW(\theta_{-t}\omega, \tau) \\ &= e^{-\mu t} Z(0) + \int_{-t}^0 e^{\mu s}\, dW(\tau) \\ t \overset{=}{\to} \infty & \int_{-\infty}^0 e^{\mu s}\, dW(\tau) := \hat{Z}(\omega). \end{aligned}$$

We conclude

$$\begin{aligned} \hat{Z}(\omega) e^{-\mu t} + \int_0^t e^{\mu(s-t)}\, dW &= \int_{-\infty}^0 e^{\mu s}\, dW(\tau) e^{-\mu t} + \int_0^t e^{\mu(s-t)}\, dW \\ &= \int_{-\infty}^t e^{\mu s}\, dW(\tau) = \int_{-\infty}^0 e^{\mu s}\, dW(\theta_t \omega, \tau) = \hat{Z}(\theta_t \omega) \text{ a.s.} \end{aligned}$$

Integration by parts and the subexponential growth of $W$ give us

$$\begin{aligned} \int_{-\infty}^0 e^{\mu s}\, dW(\tau) &= e^{0\mu} W(0) - \lim_{s \to -\infty} W(s) e^{\mu s} - \mu \int_{-\infty}^0 e^{s\mu} W(s)\, ds \\ &= -\mu \int_{-\infty}^0 e^{s\mu} W(s)\, ds := \hat{Z}(\omega). \end{aligned}$$

□

**Remark 5.1.21**
*Note that the set where $W$ grows subexponentially is a full $\theta$-invariant set. Therefore, we consider often spdes in the trace-$\sigma$ algebra $\Omega_L$. See also [23, Lemma 2.1].*

**Remark 5.1.22**
*We can apply the last Lemma 5.1.20 with*

$$Z = (z, z_\Gamma) \text{ and } \mu > 0$$

*and*

$$U = L^2(D) \times L^2(\Gamma).$$

We also have the following lemma from [8, Lemma 4.1].

**Lemma 5.1.23 (Asymptotic properties of $Z$)**
*Assume that $U=\mathbb{R}$. Then the stationary solution of (5.5) has the following properties on a $\theta_t$ invariant set*

$$\lim_{t\to\pm\infty}\frac{|Z(\theta_t\omega)|}{|t|}=0,$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t Z(\theta_\tau\omega)\,d\tau=0$$

*and*

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t |Z(\theta_\tau\omega)|\,d\tau=\mathbb{E}|Z|.$$

This was generalized by Keller and Schmalfuß [33, Lemma 2.5] to

**Lemma 5.1.24**
*Let $W$ be a twosided (see Remark 4.1.6) Wiener process with $\operatorname{Tr}_U Q<\infty$. Then, there exists a $\theta$-invariant measurable set $\Omega$ of full measure, such that*

- *(i) the mapping $t\mapsto Z(\theta_t\omega)$ is continuous on $D(A)$ for any $\omega\in\Omega$,*
- *(ii) for any $k>0$*
$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t \|Z(\theta_\tau\omega)\|_U^k\,d\tau=\mathbb{E}\|Z\|_U^k,$$
- *(iii) the mapping $t\mapsto\|Z(\theta_t\omega)\|_U$ grows sublinearly,*
- *(iv) for some $n>0,\alpha>0,c>0$ and some random variable $y$ such that $\mathbb{E}\|y\|^{2n}<\infty$ there exists a $\mu>0$ such that*
$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big(\int_0^\infty & e^{-\alpha s+c\int_0^s\|Z(\theta_{\nu+\tau}\omega)\|_U\,d\tau}y(\theta_{\nu+s}\omega)\,ds\Big)^n\,d\nu\\ &=\mathbb{E}\Big(\int_0^\infty e^{-\alpha s+c\int_0^s\|Z(\theta_\tau\omega)\|_U\,d\tau}y(\theta_s\omega)\,ds\Big)^n<\infty.\end{aligned}$$

*Proof.* The proof can be found in [33, Lemma 2.5]. (i) follows by [19, Lemma 5.13]. The finiteness of $E\|Z\|_U$ can be shown as follows: By Ito's Formula we obtain

$$2\mu\int_0^t\|Z(\theta_s\omega)\|_U^2+\|Z(\theta_t\omega)\|_U^2=\|Z(\omega)\|_U^2+2\int_0^t(Z(\theta_s\omega),dW)_U+\operatorname{tr}Qt.$$

We use the stationarity of $Z$ to conclude $E(\|Z(\theta_t\omega)\|_U^2)=E(\|Z(\theta_s\omega)\|_U^2)$ for $s\neq t$. Thus,

$$2\mu\int_0^t E\|Z(\omega)\|_U^2\,ds=\operatorname{tr}Qt$$

and

$$E\|Z(\theta_t\omega)\|_U^2=E\|Z(\omega)\|_U^2=\frac{\operatorname{tr}Q}{2\mu}.$$

□

# Chapter 6

# Attractors of stochastic parabolic equations

In this chapter, we consider random attractors of parabolic equations. We start with a simple reaction–diffusion equation with additive noise as motivating example. Here, a very simple Lipschitz non–linearity is chosen. Later on, we investigate a reaction–diffusion equation with multiplicative noise, with a more difficult non–linearity, which consists of a global– and a local–Lipschitz function. The assumptions on $F$ are taken from [16], but in our case, we consider multiplicative noise instead of additive noise in that article. Finally, we regard the Boussinesq problen, also known as the Bénard problem, see [48]. This section is based on [6], and we show the existence of a random attractor of the coupled system of equations.

## 6.1 Reaction–diffusion equation with additive noise

### 6.1.1 Introduction

At first, we study the system of a simple reaction-diffusion equation with dynamical boundary conditions and additive noise. This type of equation appears in chemistry and a popular example is the heat equation. We use classical energy estimates to prove the existence of a random attractor. We start with an estimate to obtain an absorbing set in $\mathbb{H} = L^2(D) \times L^2(\Gamma)$. The existence of an attractor is shown by a compactness argument. We also consider in this section the space $\mathbb{V} = H^1(D) \times H^{\frac{1}{2}}(\Gamma)$, introduced in Chapter 3. This space is compactly embedded in $\mathbb{H}$. We denote by $\|\cdot\|$ the norm $\|\cdot\|_{\mathbb{H}}$ on $\mathbb{H}$. The spirit of Remark 2.3.6 holds in this chapter.

### 6.1.2 General setting

Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\Gamma$. Consider the following reaction-diffusion equation with dynamical boundary conditions

$$\begin{aligned} u' + \mathcal{A}u &= f(u) + \frac{dw}{dt} \text{ on } D \\ u'_\Gamma + \mathcal{A}_\Gamma u_\Gamma &= f_\Gamma(u_\Gamma) + \frac{dw_\Gamma}{dt} \text{ on } \Gamma, \end{aligned} \tag{6.1}$$

or with $U = (u, u_\Gamma) \in \mathbb{R}$, $F = (f, f_\Gamma) \in \mathbb{R}$ and $AU = (\mathcal{A}u, \mathcal{A}_\Gamma u)$

$$dU + AUdt = F(U)dt + dW \tag{6.2}$$

with $U(0) = u_0 \in \mathbb{H}$ formulated as evolution equation and $W$ Wiener noise on $\mathbb{H}$. We set

$$\mathcal{A}u = -\Delta u \text{ and } \mathcal{A}_\Gamma u = u + \partial_\nu u. \tag{6.3}$$

**Hypothesis 6.1.1**
*We have the following assumption on $F$:*

$$\|F(U)\| \leq l\|U\| + C$$

*with growth constant $l$, and*

$$\|F(U_1) - F(U_2)\| \leq L\|U_1 - U_2\|. \tag{6.4}$$

*with Lipschitz constant $L$. Note, that in general $l < L$.*

### 6.1.3 Transformation by Ornstein-Uhlenbeck process

We use a transformation by the auxiliary Ornstein-Uhlenbeck equation

$$dz + \mathcal{A}zdt = dw$$
$$dz_\Gamma + \mathcal{A}_\Gamma z_\Gamma dt = dw_\Gamma,$$

or with $Z = (z, z_\Gamma)$, $W = (w, w_\Gamma)$

$$dZ + AZdt = dW \tag{6.5}$$

This equation has a stationary solution $Z(\theta_t\omega)$, see Lemma 5.1.18 and we introduce a new variable

$$V = U - Z(\theta_t\omega).$$

This formally gives us the random partial differential equation

$$V' + AV = F(V + Z(\theta_t\omega)). \tag{6.6}$$

Note that stochastic differentials does not appear in (6.6) and we can consider the solution path–wise for every $\omega$.

### 6.1.4 Existence and uniqueness

**Theorem 6.1.2**
*Assume that $V_0 \in \mathbb{H}$ and $F$ fullfills (6.4). Additionally $l < \frac{3}{8}\lambda_1$. Then there exists a unique weak solution $u$ of (6.6) with*

$$u \in L^2(0, T; \mathbb{V}) \cap C([0, T]; \mathbb{H})$$

*and*

$$u' \in L^2(0, T; \mathbb{V}').$$

The proof is divided into two Lemmas 6.1.3 and 6.1.4.

**Lemma 6.1.3 (Existence)**
*We have the same assumptions as in Theorem 6.1.2. Then there exists a weak solution of (6.6).*

*Proof.* First note that by operator $A$, given in (6.6), which is positive and self–adjoint, a coercive form $a$ associated to $A$ is generated, so that we are in the framework of Lax-Milgram theory. By operator $A$ we have given an orthonormal basis generated of $\mathbb{H}$ by the eigenvectors $\{E_i\}_{i\in\mathbb{N}}$. We denote by $\mathbb{H}_n$ span$\{E_1,\dots,E_n\}$ and by $P_n$ the corresponding projector on this subspace of $\mathbb{H}$. Therefore, we have the following finite dimensional equation

$$dV_n + AV_n dt = P_nF(V_n + Z(\theta_t\omega))dt, \quad V_n(0) = v_n^0 \in \mathbb{H}_n. \tag{6.7}$$

Like in [49, p.70] we define an approximate solution of equation (6.6)

$$V_n(t) := \sum_{i=1}^{n} g_{in}(t)E_i.$$

Consider the following system of ordinary differential equations (ODE) for $g_{in}$

$$\frac{d}{dt}(V_n, E_j) + a(V_n, E_j) = (P_nF(V_n + Z), E_j) \text{ for } i = 1, \cdots, n, \tag{6.8}$$

see also Theorem 2.4.6.
This equation possesses a unique, global and measurable solution with trajectories in $C([0,T];\mathbb{H}_n)$, see [49, p.70]. For existence and uniqueness of a weak solution, we have to prove that an approximate solution $V_n$ in the Galerkin approximation (6.7) of equation (6.6), see Theorem 2.5.2, is in $L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H})$. Summing up the systems of equations (6.8) leads to the following relation

$$\frac{1}{2}\frac{d}{dt}(V_n, V_n) + a(V_n, V_n) = (P_nF(V_n + Z), V_n) \text{ for } i = 1, \cdots, n. \tag{6.9}$$

This is equivalent to

$$(V_n', V_n) + (AV_n, V_n) = (P_nF(V_n + Z(\theta_t\omega)), V_n).$$

Note that

$$\|V_n\|^2 = \|v_n\|^2 + \|v_n\|_\Gamma^2 \text{ and } \|V_n\|_\mathbb{V}^2 = ((v_n, v_n)) + \|v_n\|_\Gamma^2,$$

and we obtain

$$\frac{1}{2}\frac{d}{dt}\|V_n\|^2 + \|V_n\|_\mathbb{V}^2 = (P_nF(V_n + Z(\theta_t\omega)), V_n). \tag{6.10}$$

We have the following relation between $\|\cdot\|_\mathbb{H}$ and $\|\cdot\|_\mathbb{V}$

$$\|V_n\|_\mathbb{V} \geq \lambda_1\|V_n\|_\mathbb{H} := \lambda_1\|V_n\|. \tag{6.11}$$

This gives us the following estimate in $\mathbb{H}$

$$\frac{d}{dt}\|V_n\|^2 + \|V_n\|_\mathbb{V}^2 + \lambda_1\|V_n\|^2 \leq 2(P_nF(V_n + Z(\theta_t\omega)), V_n).$$

Note that

$$\begin{aligned}(F(V_n+Z),V_n) &\leq (l\|V_n+Z\|+C)\|V_n\| \\ &\leq l(\|V_n\|+\|Z\|)\|V_n\|+C\|V_n\| \\ &\leq l\|V_n\|^2+K\|Z\|^2+\frac{\lambda_1}{4}\|V_n\|^2+K^C. \end{aligned} \tag{6.12}$$

We derive from this estimate, since $l<\frac{3}{8}\lambda_1$

$$\frac{d}{dt}\|V_n\|^2+\|V_n\|^2_{\mathbb{V}} \leq K^\beta+K\|Z\|^2.$$

This leads to

$$V_n \in L^2(0,T;\mathbb{V})\cap L^\infty(0,T;\mathbb{H}).$$

Inequality (6.12) provides us that $F$ is uniformly bounded in $\mathbb{H}$. Thus, we can conclude, following Theorem 8.4 in [43], that there exists a weak solution to (6.6).

□

**Lemma 6.1.4**
*If we assume that $V_1$ and $V_2$ are two solutions with the same initial conditions and setting $V=V_1-V_2$, we obtain uniqueness by (6.10), which yields that*

$$\frac{1}{2}\frac{d}{dt}\|V\|^2+\lambda_1\|V\|^2 \leq (F(V_1+Z)-F(V_2+Z),V) \leq L\|V\|^2.$$

**Remark 6.1.5**
*In the same way as in Lemma 6.1.4, we conclude that $V(t)\in\mathbb{H}$ depends continuously on the initial condition.*

**Lemma 6.1.6**
*The solution of (6.6) generates a continuous random dynamical system in $\mathbb{H}$. We use the standard arguments from [9] to prove the property, which are the same as in Lemma 6.2.5 below.*

We denote this random dynamical system in the following by $\varphi$.

**Remark 6.1.7**
*The solution is also contained in $C([0,T];\mathbb{H})$; see [49, Lemma II,3.2].*

### 6.1.5 Random attractor

We can state the following theorem of the existence of a random attractor.

**Theorem 6.1.8 (Random attractor)**
*Let $Z$ be the stationary solution of (6.5) and assume that $l<\frac{3}{8}\lambda$. Then the reaction-diffusion systems (6.2) and (6.6) have a unique random attractor.*

We divide the proof into two lemmas. The first Lemma 6.1.9 proves the existence of a random absorbing set and the second Lemma 6.1.10 the compactness of this set. Then Theorem 5.1.8 gives the existence of a random attractor. We consider the invariant set of $\omega$ that is defined in Remark 5.1.19 and such that $t \to \|Z(\theta_t\omega)\|_{D(A^s)}$ has a subexponential growth. We take the trace-$\sigma$-algebra of $\mathcal{F}$ for a set given by the intersection of this invariant set and the probability measure, which is the restriction of $\mathbb{P}$ to this new $\sigma$-algebra. Then, we obtain a new metric dynamical system. In particular, the flow $\theta$ is measurable with respect to this new $\sigma$–algebra [8]. For this new metric dynamical system we use the old notation $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

**Lemma 6.1.9**
*Let $\lambda := 2(\lambda_1 - l - \frac{\lambda_1}{4})$ with $\lambda$ defined in (6.11). Then the reaction-diffusion problem (6.6) has an absorbing set*

$$B(\omega) = B_{\mathbb{H}}(0, \rho(\omega)),$$

*where $B_{\mathbb{H}}(0, \rho(\omega))$ defines the closed ball in $\mathbb{H}$ with center 0 and tempered radius $\rho(\omega)$, which is defined as follows:*

$$\rho(\omega) = 2 \int_{-\infty}^{0} \exp(\lambda t) H(F, \theta_t\omega)\, dt,$$
$$H(F, \omega) = K^C + K\|Z(\omega)\|^2.$$

*Proof.* We multiply (6.6) with $V$ and get

$$(V', V) + (AV, V) = (F(V + Z(\theta_t\omega)), V).$$

This gives us the following estimate in $\mathbb{H}$

$$\frac{d}{dt}\|V\|^2 + \lambda\|V\|^2 \le (F(V + Z(\theta_t\omega)), V).$$

Note that

$$\begin{aligned}(F(V+Z), V) &\le (l\|V+Z\| + C)\|V\| \\ &\le l(\|V\| + \|Z\|)\|V\| + C\|V\| \\ &\le l\|V\|^2 + K\|Z\|^2 + \frac{\lambda_1}{4}\|V\|^2 + K^C.\end{aligned}$$

Thus we have

$$\frac{d}{dt}\|V\|^2 + 2\lambda_1\|V\|^2 \le K^\beta + 2l\|V\|^2 + K\|Z\|^2 + \frac{\lambda}{2}\|V\|^2$$

and thus with $\lambda = 2(\lambda_1 - l - \frac{\lambda_1}{4})$, and $H(F, \omega) = K^C + K\|Z(\omega)\|^2$

$$\|V(t)\|^2 \le \|V(0)\|^2 e^{-\lambda t} + \int_0^t H(F, \theta_\tau\omega) e^{-\lambda_1(t-\tau)}\, d\tau. \tag{6.13}$$

We replace $\omega$ by $\theta_{-t}\omega$, carry out an integral transform $\tau \to \tau - t$ and let $t \to \infty$ in the right hand side of (6.13) and conclude that the right hand side of (6.13) is bounded by

$$\int_{-\infty}^{0} e^{\lambda_1 \tau} H(F, \theta_\tau \omega)\, d\tau.$$

Applying Lemma 4.6 of [8] yields us the existence of an absorbing set in $\mathbb{H}$ because $\lambda > 0$ and $H$ is tempered. We can now conclude the existence of a random absorbing set [8, chapter 4].

□

We also have a similar estimate on $\|V\|_{\mathbb{V}}^2$

$$\frac{d}{dt}\|V\|^2 + \|V\|_{\mathbb{V}}^2 \le K_2^\beta + c_3^* \|Z(\theta_t\omega)\|^2 := H^*(F, \theta_t\omega). \tag{6.14}$$

**Lemma 6.1.10**
*Let $B$ be an absorbing set in $\mathbb{H}$. Then*

$$C(\omega) = \overline{\varphi(1, \theta_{-1}\omega, B(\theta_{-1}\omega))}$$

*is a compact and absorbing set of (6.6) in $\mathbb{H}$.*

*Proof.* Usually, we have to use for these estimates the Galerkin approximations, but we suppress the projections in these equations. The limit transition is well-defined because $V_m \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$ and $\frac{dV_m}{dt} \in L^2(0,T;\mathbb{V}')$. We use the notation from ([49]) and show some a priori estimates for $V$ in the $H^1(D) \times H^{\frac{1}{2}}(\Gamma)$-norm. By Theorem 2.3.9, this norm is equivalent to the norm generated by $\|V\|_{\mathbb{V}}$. To get the existence of an attractor, we need a regularization result. We will show that $\|V(1)\|_{\mathbb{V}}^2$ is bounded, if $\|V(0)\|^2$ is bounded.
We multiply (6.6) with $AV$ and get

$$(V', AV) + (AV, AV) = (F(V+Z), AV).$$

We obtain

$$\frac{1}{2}\frac{d}{dt}\|V\|_{\mathbb{V}}^2 + (\mathcal{A}v, \mathcal{A}v) + (\mathcal{A}_\Gamma v, \mathcal{A}_\Gamma v)_\Gamma = (F(V+Z), AV)$$

$F = (f, f_\Gamma)$ has the following property:

$$\begin{aligned}(F(V+Z), AV) &\le (l\|V+Z\| + C)\|AV\| \\ &\le l\|V\|\|AV\| + l\|Z\|\|AV\| + C\|AV\| \\ &\le K\|V\|^2 + \|AV\|^2 + K\|Z\|^2 + K^C.\end{aligned}$$

Hence, we obtain

$$\frac{d}{dt}\|V\|_{\mathbb{V}}^2 \le K^C + c_5\|V\|^2 + c_6\|Z(\theta_t\omega)\|^2 := G(F, V, \theta_t\omega).$$

Additionally, we have

$$\frac{d(t\|V(t)\|_{\mathbb{V}}^2)}{dt} = t\frac{d\|V(t)\|^2}{dt} + \|V(t)\|_{\mathbb{V}}^2 \le G(F, V, \theta_t\omega)t + \|V\|_{\mathbb{V}}^2.$$

Integration and (6.14) give us

$$\begin{aligned}\|V(1)\|_{\mathbb{V}}^2 &\leq \int_0^1 G(F,V,\theta_\tau\omega)\tau\,d\tau + \int_0^1 \|V(\tau)\|_{\mathbb{V}}^2\,d\tau\\ &\leq \int_0^1 G(F,V,\theta_\tau\omega)\,d\tau + \int_0^1 \big(\int_0^\tau H^*(F,\theta_s\omega)\,ds + \|V(0)\|^2\big)\,d\tau.\end{aligned}$$

$G(F,V,\theta_t\omega)$ is in $L^2(0,1)$ by Lemma 6.1.3 and $H^*(F,\theta_t\omega)$ is also in $L^2(0,1)$ by the assumptions on $F$ and $Z$, so that we can apply Theorem 5.1.8 and conclude the existence of a random attractor of the transformed equation (6.6). Lemma 5.1.16 gives us the existence of a random attractor of the original equation (6.1).

□

## 6.2 Reaction-diffusion equation with multiplicative noise

### 6.2.1 Introduction

We extend our theory of reaction-diffusion equation with additive noise to an equation with multiplicative noise. As in the chapter before, we are again looking for a random absorbing set in $\mathbb{H} = L^2(D)\times L^2(\Gamma)$. After that, we show the existence of an attractor by a compactness argument. We again consider in this section the space $\mathbb{V} = H^1(D)\times H^{\frac{1}{2}}(\Gamma)$. We again use the transformation method.

### 6.2.2 General setting

Again $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\Gamma$. Consider the following reaction-diffusion equation with dynamical boundary conditions

$$\begin{aligned}u' + \mathcal{A}u + f(u) &= bu\frac{dW}{dt} \text{ on } D\\ u' + \mathcal{A}_\Gamma u + h(u) &= bu\frac{dW}{dt} \text{ on } \Gamma.\end{aligned} \tag{6.15}$$

or

$$U' + AU + F(U) = bU\frac{dW}{dt}, \tag{6.16}$$

where $W$ is a one dimensional Wiener process, $A$ as in (6.3). On $F = (f,h)$ we assume Hypothesis 6.2.1, which are the slightly modified assumptions from [16]. Note that $f$ and $h$ do not need to be identical.

**Hypothesis 6.2.1**
*We suppose, that $f$ and $h$ have the following properties:*

- *the mapping $f : \mathbb{R}^d \to \mathbb{R}$ is continuous,*
- *$f$ has the representation*

$$f(u) = f_0(u) + f_1(u),$$

*where* $f_1 : \mathbb{R} \to \mathbb{R}$ *is globally Lipschitz with Lipschitz constant* $L$ *and* $f_0$ *is a locally Lipschitz function with the properties*

$$\alpha_1|s|^2 - \beta_1 \leq sf_0(s) \leq \alpha_2|s|^2 + \beta_2, \quad s \in \mathbb{R}, \tag{6.17}$$

*and*

$$(s_1 - s_2)(f_0(s_1) - f_0(s_2)) \geq -\alpha_3|s_1 - s_2|^2, \quad s_1, s_2 \in \mathbb{R}, \tag{6.18}$$

*where* $\alpha_i, \beta_i > 0$.

- *There exists a constant* $c > 0$, *such that*

$$f_1(u)u \geq -c, \quad u \in \mathbb{R}. \tag{6.19}$$

*On the boundary, we assume the similar properties*

- *the mapping* $h : \mathbb{R}^d \to \mathbb{R}$ *is continuous,*
- $h$ *has the representation*

$$h(u) = h_0(u) + h_1(u),$$

*where* $h_1 : \mathbb{R} \to \mathbb{R}$ *is globally Lipschitz with Lipschitz constant* $L$ *and* $h_0$ *is a locally Lipschitz function with the properties*

$$\alpha_1|s|^2 - \beta_1 \leq sh_0(s) \leq \alpha_2|s|^2 + \beta_2, \quad s \in \mathbb{R}, \tag{6.20}$$

*and*

$$(s_1 - s_2)(f_0(s_1) - f_0(s_2)) \geq -\alpha_3|s_1 - s_2|^2, \quad s_1, s_2 \in \mathbb{R}, \tag{6.21}$$

*where* $\alpha_i, \beta_i > 0$.

- *There exists a constant* $c > 0$, *such that*

$$h_1(u)u \geq -c, \quad u \in \mathbb{R}. \tag{6.22}$$

*Furthermore, we assume the following connection condition between* $f$ *and* $h$. *There exists a* $c_1 > 0$ *such that*

$$|h(u) - f(u)| \leq c_1(1 + |u|), \quad u \in \mathbb{R}. \tag{6.23}$$

The handling of the stochastic differential is not obvious and thus we use a transformation by a stationary Ornstein-Uhlenbeck process, which is introduced in the following.

### 6.2.3 Transformation by Ornstein-Uhlenbeck process

We use the transformation by a stationary Ornstein-Uhlenbeck process. Consider the auxiliary equation

$$dZ + \alpha Z dt = b dW, \quad \alpha > 0. \tag{6.24}$$

This equation has the stationary solution

$$Z(\theta_t \omega) = \int_{-\infty}^{t} e^{(t-\tau)\alpha} \, dW(\omega)$$

almost surely for every $t \in \mathbb{R}$. Note, that $W$ is a one-dimensional Wiener process. Then, we apply the transformation

$$V = T(U, \omega) := e^{-Z(\omega)} U,$$

where $Z$ is the stationary solution of (6.24). We define

$$\beta(\omega) := e^{-Z(\omega)}$$

and get by Ito‘s formula, omitting the $\omega$, when there are no confusions,

$$d\beta = -\beta b dW + \alpha Z \beta dt + \frac{1}{2}\beta b^2 dt.$$

This gives us the following equation

$$dV + AV dt + \frac{1}{2} b^2 V dt + \beta(\theta_t \omega) F(\beta(\theta_t \omega)^{-1} V) dt = \alpha Z(\theta_t \omega) V dt. \tag{6.25}$$

### 6.2.4 Properties of the non–linearity

We state two Lemmas, which give some basic properties of the non–linearity. The following Lemma is similar to [16, Lemma 3.2].

**Lemma 6.2.2**
*Assume that $U \in \mathbb{H}$. Then,*

$$(F(U), U) \geq \alpha_1 (\|u\|^2 + \|u_\Gamma\|_\Gamma^2) - c$$

*for some $c > 0$.*

*Proof.*

$$(f(u), u) = \int_D u f_0 \, dx + \int_D u f_1 \, dx \geq \alpha_1 \int_D |u|^2 \, dx - \int_D \beta_1 \, dx + (-c) \tag{6.26}$$

by (6.17) and (6.19). (6.20) and (6.22) give us

$$(h(u), u)_\Gamma \geq \alpha_1 \int_\Gamma |u|^2 \, dS - \int_\Gamma \beta_1 \, dx + (-c). \tag{6.27}$$

Adding (6.26) and (6.27) gives us the assertion. □

The next Lemma is similar to [16, Lemma 3.8].

**Lemma 6.2.3**
*Assume that $U \in D(A)$. Then,*

$$(F(U), AU) \geq -c - c\|U\|^2 - \frac{1}{2}\|AU\|^2$$

*for some $c > 0$.*

*Proof.* Integration by parts yields

$$\begin{aligned}(B(U), AU) &= \int_D f(u)\mathcal{A}u\,dx + \int_\Gamma h(u)\mathcal{A}_\Gamma u\,dS \\ &= \int_D \nabla f \nabla u\,dx + \int_\Gamma (h(u) - f(u))\mathcal{A}_\Gamma u\,dS + \int_\Gamma f(u)u\,dS. \qquad (6.28)\end{aligned}$$

We estimate the first term in (6.28) by

$$\begin{aligned}\int_D \nabla f \nabla u\,dx &= \int_D \sum_{i=1}^d \partial_{x_i} f \partial_{x_i} u\,dx \\ &= \int_D \sum_{i=1}^d \partial_{x_i} f_0 \partial_{x_i} u\,dx + \int_D \sum_{i=1}^d \partial_{x_i} f_1 \partial_{x_i} u\,dx \\ &\geq \int_D (\frac{\partial f_0(u)}{\partial u} + \frac{\partial f_1(u)}{\partial u})(\partial_{x_i} u)^2 \\ &\geq -c_1\|u\|_{H^1(D)} \geq -c_2\|U\|_{\mathbb{V}}^2 \geq -c_3\|U\|^2 - \frac{1}{4}\|AU\|^2.\end{aligned}$$

by an interpolation inequality, (6.18) and the global Lipschitz continuity of $f_1$. □

Furthermore, we estimate as in [16, Lemma 3.8].

$$\begin{aligned}\int_\Gamma |h(u) - f(u)|\mathcal{A}_\Gamma u\,dS &\leq c_4(1 + \|u\|_{H^1(D)})\|AU\| \\ &\leq c_5 + c_5\|U\|^2 + \frac{1}{4}\|AU\|^2.\end{aligned}$$

Finally, we obtain

$$\int_\Gamma f(u)u\,dS \geq -c_6 - c_6\|U\|^2$$

by the assumptions on $f$.

### 6.2.5 Existence and uniqueness

We use the same method as in the additive case. To show existence and uniqueness of a weak solution, we have again to prove that an approximate solution $V_m$ in the Galerkin approximation of equation (6.25), see Theorem 2.5.2, is in $L^2(0, T; \mathbb{V}) \cap L^\infty(0, T; \mathbb{H})$. This is given by Remark 6.2.10 and 6.2.11. Uniqueness is given by Remark 6.2.8.
We have again the following Remark:

**Remark 6.2.4**
*The solution is also contained in $C([0, T]; \mathbb{H})$; see [49, Lemma II,3.2] and Theorem 2.5.3*

We note that by the measurability $\omega \to V \in C([0, T]; \mathbb{V}')$ the mapping $\omega \to V(t, \omega) \in \mathbb{H}$ is measurable for $t \in [0, T]$; see [50, Bem. 4.1.3]. In addition, for fixed $t, \omega$ the mapping $V^0 \to V(t, \omega)$ is continuous by Remark 6.2.4. Applying [9, Lemma III.14] the mapping $(V^0, \omega, t) \to V(t, \omega) \in \mathbb{H}$ is measurable. Thus, we have again the following result.

**Lemma 6.2.5**
*The solution of (6.25) generates a continuous random dynamical system in $\mathbb{H}$.*

We generally denote this random dynamical system by $\varphi$.

### 6.2.6 The attractor of reaction-diffusion equation with multiplicative noise

In this section, we will apply the results of the preceding section to prove the existence of a random attractor for (6.25).

**Theorem 6.2.6 (Random attractor)**
*Let $W$ be a one dimensional–Wiener process and let $Z$ be the one-dimensional stationary solution of (6.24). We set*

$$Q(\omega) := 2(\lambda_1 + \alpha_1 - \alpha|Z(\omega)|) \tag{6.29}$$

*where the constant $\lambda_1$ is defined in (6.11). Assume that $Q(\omega)$ has the finite expectation*

$$\mathbb{E}Q > 0.$$

*Then, the reaction-diffusion systems (6.16) and (6.25) with multiplicative noise have a unique random attractor.*

We first note that from assumption (6.29) there exists a $\{\theta_t\}_{t\in\mathbb{R}}$–invariant set of full measure (see also Remark 5.1.19 (ii)) such that

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t Q(\theta_\tau\omega)d\tau = \mathbb{E}Q. \tag{6.30}$$

We consider the invariant set of $\omega$ such that $t \to |Z(\theta_t\omega)|$ has a subexponential growth and the invariant set of $\omega$ such that (6.29) holds. We take the trace-$\sigma$-algebra of $\mathcal{F}$ for a set given by the intersection of these invariant sets and the probability measure which is the restriction of $\mathbb{P}$ to this new $\sigma$-algebra. Then, we obtain a new metric dynamical system. In particular, the flow $\theta$ is measurable with respect to this new $\sigma$–algebra [8]. For this new metric dynamical system we use the old notation $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.
The proof is divided into two lemmas. After proving the existence of an absorbing set $B$ in $\mathbb{H}$, we show that this set can be modified into a compact absorbing set. We apply Theorem 5.1.8 together with Lemma 5.1.16 to conclude the existence of a random attractor.

**Lemma 6.2.7**
*Let $\mathbb{E}Q > 0$. Then the reaction-diffusion problem (6.25) has an absorbing set*

$$B(\omega) = B_{\mathbb{H}}(0, \rho(\omega)),$$

*where $B_{\mathbb{H}}(0, \rho(\omega))$ defines the closed ball in $\mathbb{H}$ with center 0 and radius $\rho(\omega)$ which is defined as follows:*

$$\rho(\omega) = 2\int_{-\infty}^{0} \exp\left(\int_t^0 Q(\theta_\tau\omega)\,d\tau\right) H(\theta_t\omega)\,dt$$

$$H(F,\omega) = 2c|\beta(\omega)|^2$$

Note that $H$ is a tempered random variable. Like in Lemma 6.1.5 we make some energy estimates. Multiplication of (6.25) with $V$ gives us

$$\frac{1}{2}\frac{d\|V\|^2}{dt} + \|V\|_{\mathbb{V}}^2 + \frac{1}{2}b^2\|V\|^2 + (\beta F(\beta^{-1}V), V) \leq \alpha|Z(\theta_t\omega)|\|V\|^2. \tag{6.31}$$

With Lemma 6.2.2 we achieve

$$\begin{aligned}(\beta F(\beta^{-1}V), V) &= \beta^2(F(\beta^{-1}V), \beta^{-1}V) \\ &\geq \beta^2(\alpha_1(|\beta^{-1}v|^2 + |\beta^{-1}v_\Gamma|_\Gamma^2) - c) \\ &= \alpha_1(|v|^2 + |v_\Gamma|^2) - c\beta^2.\end{aligned}$$

and conclude

$$\frac{d}{dt}\|V\|^2 + Q(\theta_t\omega)\|V\|^2 \leq H(F, \theta_t\omega)$$

with

$$Q(\omega) := 2(\lambda_1 + \alpha_1 - \alpha|Z(\omega)|).$$

By Gronwall's lemma we get

$$\|V(t)\|^2 \leq e^{\int_0^t -Q(\theta_s\omega)\,ds}\|V(0)\|^2 + \int_0^t H(\theta_s\omega)e^{\int_s^t -Q(\theta_\tau\omega)\,d\tau}\,ds. \tag{6.32}$$

The first expression on the right hand side behaves for $|t| \to \infty$ by Birkhoff's ergodic theorem like

$$e^{-\mathbb{E}Q|t|}\|V(0)\|^2$$

and the second tends by replacing $\omega \to \theta_t\omega$ and a simple integral transformation and $t \to \infty$ to

$$\int_{-\infty}^0 H(F, \theta_s\omega)e^{\int_s^0 -Q(\theta_\tau\omega)\,d\tau}\,ds.$$

The temperedness of $H$ and $\mathbb{E}Q > 0$ gives us by applying Lemma 4.6 of [8] the existence of a random absorbing set for (6.25).
Applying Lemma 4.6 of [8] provides us the existence of an absorbing set in $\mathbb{H}$ because $\mathbb{E}Q > 0$ and $H$ is tempered. We can conclude now the existence of a random absorbing set [8, chapter 4].
Similar as in Lemma 6.1.4 we obtain uniqueness.

**Remark 6.2.8**
*If we assume that $V_1$ and $V_2$ are two solutions with the same initial conditions and setting $V = V_1 - V_2$, we obtain uniqueness by (6.31).*

*Proof.*

$$\frac{1}{2}\frac{d}{dt}\|V\|^2 + \|V\|_{\mathbb{V}}^2 \leq \alpha|Z|\|V\|^2 + \beta(F(\beta^{-1}V_1) - F(\beta^{-1}V_2), V).$$

and

$$\beta(F(\beta^{-1}V_1) - F(\beta^{-1}V_2), V) \leq -(L + \alpha_3)\|V\|^2.$$

lead us to

$$\frac{d}{dt}\|V\|^2 \leq 2(\alpha|Z| + L + \alpha_3)\|V\|^2,$$

$V(0) = 0$ and applying Gronwall's Lemma gives us the uniqueness. □

**Remark 6.2.9**
*A similar calculation as in Lemma 6.2.8 provides us that $V(t)$ depends continuously on the initial condition.*

**Remark 6.2.10**
*If we replace $V$ in (6.32) by $V_m$, we get for the approximate solution that*

$$V_m \in L^\infty(0,T;\mathbb{H}).$$

**Remark 6.2.11**
*If we replace $V$ in (6.31) by $V_m$ and apply Remark (6.2.10), we get for the approximate solution that*

$$V_m \in L^2(0,T;\mathbb{V}).$$

*Proof.* From Inequality (6.31) we derive the following inequality

$$\frac{1}{2}\frac{d\|V_m\|^2}{dt} + \|V_m\|_{\mathbb{V}}^2 + \frac{1}{2}b^2\|V_m\|^2 + (\beta F(\beta^{-1}V_m), V_m) \leq \alpha|Z(\theta_t\omega)|\|V_m\|^2.$$

Then, we get by applying

$$(\beta F(\beta^{-1}V_m), V_m) \geq \alpha_1\|V_m\|^2 - c|\beta|^2$$

the following inequality

$$\frac{d\|V_m\|^2}{dt} + k\|V_m\|_{\mathbb{V}}^2 \leq H(F,\theta_t\omega) + 2\alpha|Z(\theta_t\omega)|\|V_m\|^2,$$

for some $k > 0$. Integrating from 0 to $t$ and Remark 6.2.10 gives us the result. □

**Lemma 6.2.12**
*Let $B$ be an absorbing set in $\mathbb{H}$. Then*

$$C(\omega) = \overline{\varphi(1,\theta_{-1}\omega, B(\theta_{-1}\omega))}$$

*is a compact and absorbing set of (6.16) in $\mathbb{H}$.*

*Proof.* Usually, we have to use for these estimates the Galerkin approximations, but we suppress again as in Lemma 6.1.10 the projections in these equations. The limit transition is well-defined because $V_m \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$ and $\frac{dV_m}{dt} \in L^2(0,T;\mathbb{V}')$. We use the notation from ([49]) and show some a priori estimates for $V$ in the $H^1$-norm. This norm is by Theorem 2.3.9 equivalent to the norm generated by $\|V\|_{\mathbb{V}}$. Multiplying of (6.25) by $AV$ gives us

$$(\frac{dV}{dt}, AV) + (AV, AV) + \frac{1}{2}(b^2V, AV) + (\beta(\theta_t\omega)F(\beta^{-1}V), AV) = \alpha(Z(\theta_t\omega)V, AV).$$

By

$$(V, AV) = (v, \mathcal{A}v) + (v, \mathcal{A}_\Gamma v) = ((v,v)) - (v, \partial_\nu v)_\Gamma + \|v\|_\Gamma^2 + (v, \partial_\nu v)_\Gamma = \|V\|_{\mathbb{V}}^2,$$

the estimate

$$\alpha(Z(\theta_t\omega)V, AV) \le |\alpha Z(\theta_t\omega)|(V, AV) \le K|Z(\theta_t\omega)|\|V\|^2 + \frac{1}{2}(AV, AV)$$

and by Lemma 6.2.3

$$(\beta(\theta_t\omega)F(\beta^{-1}V), AV) \ge -K\|V\|^2 - \frac{1}{2}(AV, AV) - K|\beta(\theta_t\omega)|^2$$

we get

$$\frac{1}{2}\frac{d}{dt}\|V\|_{\mathbb{V}}^2 \le K|\beta(\theta_t\omega)|^2 + K(|Z(\theta_t\omega)| + 1)\|V\|^2.$$

We calculate

$$\frac{d(t\|V(t)\|_{\mathbb{V}}^2)}{dt} = t\frac{d\|V(t)\|_{\mathbb{V}}^2}{dt} + \|V(t)\|_{\mathbb{V}}^2$$

and integrate from 0 to 1. □

Thus, we have

$$\|V(1)\|_{\mathbb{V}}^2 \le \int_0^1 K|\beta(\theta_s\omega)|^2\,ds + \sup_{t\in[0,1]}|Z(\theta_t\omega)|K\int_0^1 \|V(s)\|^2\,ds \tag{6.33}$$

$$+ K\int_0^1 \|V(s)\|^2\,ds + \int_0^1 \|V(s)\|_{\mathbb{V}}^2\,ds. \tag{6.34}$$

It is clear that $K|\beta(\theta_t\omega)|^2$ and $\|V\|^2$ are in $L^1(0,1)$, $\|V\|_{\mathbb{V}}^2$ is also in $L^1(0,1)$ by Remark (6.2.11) and depends on $\|V(0)\|^2$, therefore the right hand side of (6.34) is bounded for initial condition $V(0) \in \mathbb{H}$ and we can together with Lemma 6.2.7 apply Theorem 5.1.8 to get the existence of a random attractor. By Lemma 5.1.16 we obtain the existence of a random attractor of (6.16).

## 6.3 Boussinesq system

### 6.3.1 Introduction

The Navier-Stokes equations are often coupled with other equations, i.e. with the scalar transport equations for fluid density, salinity, or temperature. These coupled equations model a variety of phenomena in environmental, geophysical and climate systems [20, 42, 21, 40].

In this section, we consider the Boussinesq equations in which the scalar quantity is salinity, under dynamical (flux type) boundary conditions for the salinity. This models various phenomena in our climate system, for example, oceanic density currents and the thermohaline circulation.

We take random influences into account and formulate this problem as a system of stochastic partial differential equations (SPDEs). This is a coupled system of the stochastic Navier-Stokes equations and the stochastic transport equation.

The main differences to the standard Boussinesq model [20] are the dynamical boundary conditions. We emphasize that the noise also acts on the boundary.

In the first part of this section we give the general Boussinesq model with dynamical boundary conditions. The basic functional analytic setting and the appearing linear operators are analyzed. Later on, the properties of the trilinear form evolving for the Navier–Stokes equations are investigated. Again, we transform the SPDE to a random partial differential equation by an Ornstein Uhlenbeck process and show the existence of a random attractor.

### 6.3.2 General setting

Let $D \subset \mathbb{R}^2$ be a bounded $C^1$-smooth domain with the boundary $\partial D = \Gamma$, in the vertical plane. We consider a system of coupled partial differential equations (PDE's) with white noise and *random dynamical boundary conditions* of the form [40, 20]:

$$\begin{aligned}
\frac{du}{dt} &= (\frac{1}{Re}\triangle u - \nabla p - u\cdot\nabla u - \frac{1}{Fr^2}U\mathbf{k}) + \dot{W}_0 \text{ on } D\times\mathbb{R}_+ \\
\operatorname{div} u &= 0 \text{ on } D\times\mathbb{R}_+ \\
u &= 0 \text{ on } \Gamma\times\mathbb{R}_+ \\
u(0) &= u_0 \\
\frac{dU}{dt} &= (\frac{1}{RePr}\triangle U - u\cdot\nabla U) + \dot{W}_1 \text{ on } D\times\mathbb{R}_+ \\
\frac{dU_\Gamma}{dt} &= (\frac{-\partial_n U_\Gamma - cU_\Gamma + f(x)}{\epsilon_0}) + \dot{W}_2 \text{ on } \Gamma\times\mathbb{R}_+ \qquad (6.35)\\
\gamma U &= U_\Gamma \\
U(0) &= U_0, \qquad (6.36)
\end{aligned}$$

with velocity $u = u(t,x) \in \mathbb{R}^2$, salinity $U = U(t,x) \in \mathbb{R}$, pressure $p(t,x)$, where $x = (\xi,\eta) \in D \subset \mathbb{R}^2$ and $t > 0$. Here $\triangle$ is the Laplacian operator, $\gamma$ is the trace operator with respect to the boundary, $\nabla$ the gradient operator, div the divergence operator, $Fr$ is the Froude number, $Re$ is the Reynolds number and $Pr$ is the Prandtl number. Moreover, $\dot{W}_0$, $\dot{W}_1$ and $\dot{W}_2$ are white noise terms with values in appropriate function spaces; see also Lemma 5.1.18 below. The mathematical model for these noise terms are the generalized time derivatives of Brownian motions. Note also that $\epsilon_0$ and $c$ are some positive constants. When $\epsilon_0 \to 0$, the dynamical boundary condition (6.35) reduces to the usual Robin boundary condition. Finally, $f(x)$ is a given function describing the mean salinity flux through the boundary; $\mathbf{k} \in \mathbb{R}^2$ is a unit vector in the upward vertical direction (opposite to the gravity); $U_0$ and $u_0$ are the initial conditions; and $\partial_n U_\Gamma$ is the outer normal derivative, Without lost of generality for our stochastic analysis in this paper, we take $\epsilon_0$ to be 1.

In the following some function spaces are introduced to deal with this specific setting. General properties of Sobolev spaces can be found in Chapter 2.2.1. We set

$$L^2 := (L^2(D))^2 \times L^2(D) \times L^2(\Gamma)$$

The $L^2$-norm is denoted by $\|\cdot\|$. We define a function space, which incorporates the boundary and also the divergence-free condition

$$\mathcal{V} := \{(u,U,U_\Gamma) \in (C^\infty(D))^2 \times C^\infty(D) \times \gamma C^\infty(D) : \operatorname{div} u = 0\}.$$

Define

$$H_s^1 := \{(u, U, U_\Gamma) \in (H_0^1(D))^2 \times H^1(D) \times H^{\frac{1}{2}}(\Gamma)\},$$

where $H^1(D)$ is the usual Sobolev space and $H^{\frac{1}{2}}(\Gamma)$ is given by $\gamma(H^1(D))$. It can be endowed by a norm, for instance $\|\varphi\|_{H^{\frac{1}{2}}(\Gamma)} := \inf_{\gamma u = \varphi} \|u\|_{H^1(D)}$; see [49, p.48]

Related to the above functional setting, we define the Hilbert space $L^2$ given by the following inner product:

$$(\cdot,\cdot) = (\cdot,\cdot)_{(L^2(D))^2} + \frac{1}{\kappa\epsilon_0}(\cdot,\cdot)_{L^2(D)} + (\cdot,\cdot)_{L^2(\Gamma)},$$

with norm

$$\|\mathbf{U}\|^2 = (\mathbf{U},\mathbf{U})$$

$$\mathbf{U} = (u, U, U_\Gamma). \tag{6.37}$$

This norm is equivalent to the usual norm on $L^2(D)^2 \times L^2(D) \times L^2(\Gamma)$. Let $\mathbb{H} = \mathbb{H}_1 \times \mathbb{H}_2$ denote the closure of $\mathcal{V}$ with respect to the $L^2$-norm, $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$ denotes the closure of $\mathcal{V}$ with respect to the $H_s^1$-norm, and $\mathbb{V}'$ be the dual space of $\mathbb{V}$.

Using (6.37) we can reformulate our problem as an stochastic evolution equation:

$$d\mathbf{U} + A\mathbf{U}dt + B(\mathbf{U},\mathbf{U})dt = F(\mathbf{U})dt + dW, \tag{6.38}$$

with

$$A : \mathbb{V} \to \mathbb{V}',$$

$$A\mathbf{U} = \begin{pmatrix} A_1 u \\ A_2(U, U_\Gamma) \end{pmatrix}.$$

In particular,

$$A_1 u := -\nu \triangle u,$$

$$A_2 U := \begin{pmatrix} -\frac{1}{RePr}\triangle U \\ \frac{\partial_n U_\Gamma + cU_\Gamma}{\epsilon_0} \end{pmatrix},$$

$$B : \mathbb{V} \times \mathbb{V} \mapsto \mathbb{V}' :$$

$$B(\mathbf{U},\mathbf{V}) = \begin{pmatrix} B_1(u,v) \\ B_2(u,V) \\ 0 \end{pmatrix} := \begin{pmatrix} u\cdot\nabla v \\ u\cdot\nabla V \\ 0 \end{pmatrix},$$

and

$$F(\mathbf{U}) = \begin{pmatrix} -\frac{1}{Fr^2}U\mathbf{k} \\ 0 \\ \frac{f}{\epsilon_0} \end{pmatrix}, \quad f \in L^2(\Gamma).$$

Note that $A_1$ and $A_2$ are positive self-adjoint operators with domains $D(A_1)$ and $D(A_2)$. So $A$ has the same properties on $D(A) := D(A_1) \times D(A_2)$. Moreover, $B_1$ and $B_2$ are bilinear forms, which are well studied in the context of the Navier-Stokes equations [17].

**Definition 6.3.1**
*Let* $\langle\cdot,\cdot\rangle$ *be the duality mapping* $\mathbb{V}'\times\mathbb{V}\to\mathbb{R}$. *We denote by* $b(u,v,w)=\langle B(u,v),w\rangle$ *which can be represented by*

$$b_1(u,v,w)=\int_D\sum_{i,j=1}^{2}u_i\frac{\partial v_j}{\partial x_i}w_j\,dx,$$

$$b_2(u,V,W)=\int_D\sum_{i=1}^{2}u_i\frac{\partial V}{\partial x_i}W\,dx,$$

*and*

$$b(\mathbf{U},\mathbf{V},\mathbf{W})=b_1(u,v,w)+b_2(u,V,W)$$

*for*

$$\mathbf{U},\mathbf{V},\mathbf{W}\in\mathbb{V}.$$

We state that $B:\mathbb{V}'\to\mathbb{V}$ by [47, Lemma 3.4, p.92].

**Lemma 6.3.2 (Properties of** $A$**)**
*Both* $A_1$ *and* $A_2$ *are positive self-adjoint operators.*

*Proof.* Following a density argument, we now assume $U\in H^2(D)$ to obtain the following estimate, $n$ denotes the outer normal in this context. There exists a $\mu_2>0$, such that

$$\begin{aligned}
(A_2(U,U_\Gamma),(U,U_\Gamma)) &= \frac{1}{\kappa\epsilon_0}(-\kappa\triangle U,U)_{L^2(D)}+(\frac{1}{\epsilon_0}\partial_n U,U)_{L^2(\Gamma)}+(\frac{c}{\epsilon_0}U,U)_{L^2(\Gamma)}\\
&= \frac{1}{\kappa\epsilon_0}\int_D(-\kappa\triangle U)U\,dx+\int_\Gamma\frac{1}{\epsilon_0}(\partial_n U)U\,d\sigma+\int_\Gamma\frac{c}{\epsilon_0}U^2\,d\sigma\\
&= \frac{1}{\epsilon_0}(\int_D(\nabla U\cdot\nabla U)\,dx-\int_\Gamma(\partial_n U)U\,d\sigma)+\\
&\quad +\frac{1}{\epsilon_0}\int_\Gamma(\partial_n U)U\,d\sigma+\frac{c}{\epsilon_0}\int_\Gamma U^2\,d\sigma\\
&= \frac{1}{\epsilon_0}(\int_D(\nabla U\cdot\nabla U)\,dx+c\int_\Gamma U^2\,d\sigma)\\
&\geq \mu_2\|(U,U_\Gamma)\|^2.
\end{aligned}$$

The last inequality holds because of the generalized Poincaré inequality [49, p.51] or Theorem 2.3.9. Define

$$\lambda=\min(\mu_1,\mu_2), \tag{6.39}$$

with $(A_2(U,U_\Gamma),(U,U_\Gamma))\geq\mu_2\|(U,U_\Gamma)\|^2$ and $(A_1u,u)_{L^2}\geq\mu_1\|u\|^2_{(L^2(D))^2}$

such that

$$(A\mathbf{U},\mathbf{U})\geq\lambda\|\mathbf{U}\|^2.$$

□

Again, we can as in 3.1.1 also define the following function spaces with respect to the operator $A$, defined in (6.38), this is reasonable because $A^{-1}$ is compact and so the spectrum of $A$ is discrete with finite multiplicities. The spectrum of $A$ is denoted by $(\lambda_i)_{i\in\mathbb{N}}$ and the appropriate eigenfunctions are $(E_i)_{i\in\mathbb{N}}$ which form a complete orthonormal system in $\mathbb{H}$. The eigenvalues are positive, increasing and tend to infinity as $n\to\infty$. All these properties follow similar to [15]. This allows us to introduce the function spaces

$$D(A^s)=\{\mathbf{U}=\sum_{i=1}^{\infty}\widehat{u_i}E_i:\quad \|\mathbf{U}\|_{D(A^s)}=\sum_{i=1}^{\infty}\|\widehat{u_i}\|^2\lambda_i^{2s}<\infty\}$$

with $D(A^0)=\mathbb{H}$ and $D(A^{1/2})=\mathbb{V}$. We refer to [48] for details.

**Lemma 6.3.3** *[Properties of the trilinear form B]*
*The following properties hold for* $\mathbf{U},\mathbf{V},\mathbf{W}\in\mathbb{V}$*:*

$$b(\mathbf{U},\mathbf{V},\mathbf{W})=-b(\mathbf{U},\mathbf{W},\mathbf{V}) \tag{6.40}$$

*and*

$$\langle B(\mathbf{V},\mathbf{U}),\mathbf{U}\rangle=b(\mathbf{V},\mathbf{U},\mathbf{U})=0.$$

*Proof.* To prove (6.40) we conclude for $\mathbf{U},\mathbf{V}\in\mathbb{V}$:

$$\begin{aligned}\int_D(u\cdot\nabla V)W\,dx &= \int_D(u_1\partial_{x_1}VW+u_2\partial_{x_2}VW)\,dx\\ &= -\int_D V\partial_{x_1}(u_1W)+V\partial_{x_2}(u_2W)\,dx\\ &= -\int_D VW\partial_{x_1}u_1+VW\partial_{x_2}u_2+u_1V\partial_{x_1}W+u_2V\partial_{x_2}W\,dx\\ &= -\int_D u_1V\partial_{x_1}W+u_2V\partial_{x_2}W\,dx,\end{aligned}$$

by $\operatorname{div} u=0$. All these integrals are well-defined by the embedding $H^1\subset L^4$, see Theorem 2.2.19. Now, the second assertion is also clear, it is a direct consequence of (6.40). □

**Lemma 6.3.4 (Estimates for $b_1$ and $b_2$)**
*The trilinear forms $b_1$ and $b_2$ have the following properties: There exists a constant $c_B>0$, such that:*

$$\begin{aligned}|b_1(u,v,w)| &\le c_B\|u\|_{H^1}\|v\|_{D(A_1)}\|w\| \quad u\in\mathbb{V}_1,v\in D(A_1),w\in\mathbb{H}_1\\ |b_1(u,v,w)| &\le c_B\|u\|_{L^2}^{\frac12}\|u\|_{H^1}^{\frac12}\|v\|_{D(A_1)}\|w\|^{\frac12}\|w\|_{H^1}^{\frac12} \quad u\in\mathbb{V}_1,v\in D(A_1),w\in\mathbb{V}_1\\ |b_1(v,v_1,v)| &\le c_B\|v\|_{H^1}\|v_1\|_{H^1}\|v\| \quad v\in\mathbb{V}_1,v_1\in\mathbb{V}_1\\ |b_2(v,V_1,V)| &\le c_B\|v\|^{\frac12}\|v\|_{H^1}^{\frac12}\|V_1\|_{H^1}\|V\|^{\frac12}\|V\|_{H^1}^{\frac12} \quad v\in\mathbb{V}_1,V_1\in\mathbb{V}_2,V\in\mathbb{V}_1\\ |b_2(u,V,W)| &\le c_B\|u\|\|V\|_{D(A_2)}\|W\|_{H^1} \quad u\in\mathbb{H}_1,V\in D(A_2),W\in\mathbb{V}_2.\end{aligned}$$

*Proof.* The first inequality holds because

$$\begin{aligned}|b_1(u,v,w)| &\leq k_1\|u\|_{L^4}\|\nabla v\|_{L^4}\|w\|_{L^2}\\ &\leq c_{B_1}\|u\|_{H^1}\|v\|_{D(A_1)}\|w\|_{L^2}.\end{aligned}$$

The first step uses Hölder's inequality and the second step makes use of Sobolev's embedding theorem. The other inequalities are similar, but additionally Ladyzhenskaya's inequality is used:

$$\begin{aligned}|b_1(u,v,w)| &\leq k_2\|u\|_{L^4}\|\nabla v\|_{L^2}\|w\|_{L^4}\\ &\leq c_{B_2}\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}\|v\|_{D(A_1)}\|w\|_{L^2}^{\frac{1}{2}}\|w\|_{H^1}^{\frac{1}{2}}\end{aligned}$$

and it remains to show

$$\begin{aligned}|b_2(u,V,W)| &\leq k_3\|u\|\|\nabla V\|_{L^{\frac{8}{3}}}\|W\|_{L^8}\\ &\leq c_B\|u\|\|V\|_{D(A_2)}\|W\|_{H^1},\end{aligned}$$

for the details of the embeddings see Lemma 6.3.16 below. For a two-dimensional domain, Ladyzhenskaya's inequality does not depend on the boundary conditions [43, Lemma 5.27].

□

**Remark 6.3.5**
*$b$ is a continuous trilinear form on $\mathbb{V}$ into $\mathbb{R}$ which results by Lemma 6.3.4.*

We can write (6.36) as stochastic evolution equation

$$d\mathbf{U} + A\mathbf{U}dt + B(\mathbf{U},\mathbf{U})dt = F(\mathbf{U})dt + d\mathbf{W}, \quad \mathbf{U}(0) = u^0 \in \mathbb{H}. \tag{6.41}$$

In which sense we study solutions will be clarified later on. Formally, we transform this equation as follows. We apply the transformation in (5.4) where we chose $\mathbf{W}$ in the role of $W$ in (5.4) and $\mathbf{Z}$ as $Z$ respectively. Subtracting (5.4) from (6.41) with $\mathbf{V} = \mathbf{U} - \mathbf{Z}(\theta_t\omega)$ leads us to

$$\frac{d\mathbf{V}}{dt} + A\mathbf{V}dt + B(\mathbf{V} + \mathbf{Z}(\theta_t\omega), \mathbf{V} + \mathbf{Z}(\theta_t\omega)) = F(\mathbf{V} + \mathbf{Z}(\theta_t\omega)), \tag{6.42}$$

$$\mathbf{V}(0) = v^0 \in \mathbb{H}.$$

A solution of equation (6.42) will be defined for every $\omega$, here a solution is interpreted in the weak sense. From the existence of weak solutions of equation (6.42) we can derive the existence of weak solutions of equation (6.41).

To prove existence of a solution, we use the method of Galerkin approximations, but only the basic ideas are given here. We consider the orthonormal basis generated by the eigenvectors $\{E_i\}_{i\in\mathbf{N}}$ of the operator $A$. $\mathbb{H}_n$ denotes span$\{E_1,\dots E_n\}$ and $P_n$ the corresponding projector on this subspace of $\mathbb{H}$. The finite dimensional equation

$$\begin{aligned}d\mathbf{V}_n &+ A\mathbf{V}_n dt + P_nB(\mathbf{V}_n + \mathbf{Z}(\theta_t\omega), \mathbf{V}_n + \mathbf{Z}(\theta_t\omega))dt\\ &= P_nF(\mathbf{V}_n + P_n\mathbf{Z}(\theta_t\omega))dt, \quad \mathbf{V}_n(0) = v_n^0 \in \mathbb{H}\end{aligned}$$

possesses a unique, global and measurable solution with trajectories in $C([0,T];\in \mathbb{H}_n)$.
For the existence of solutions of equation (6.42), we note that the set of Galerkin approximations $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$ is relative compact in $L^2(0,T;\mathbb{H}) \cap C([0,T];\mathbb{V}')$, since $\frac{d\mathbf{V}_n}{dt}$ is in $L^2(0,T;\mathbb{V}')$. This gives us, that the $\mathbf{V}_n$ are contained in an equicontinuous set and hence we can apply Theorem 2.2.24. Limit points of this sets satisfy equation (6.42).
The existence of solutions in $L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$ can easily be shown, we refer to Remark 6.3.13 and 6.3.14. A straightforward computation shows us that we can apply the methods of [43, Chapter 9.4] and the compactness theorems referred therein. Due to those a priori estimates, the existence of the solution is proven. The solution is measurable as a mapping from $\Omega$ to $C([0,T];\mathbb{V}') \cap L^2(0,T;\mathbb{H})$ because the approximate solution is measurable as a solution of an ODE.

**Remark 6.3.6**
*The solution is also contained in* $C([0,T];\mathbb{H})$*; see [49, Lemma II,3.2].*

To avoid too much indices and for simplicity, constants are denoted by $K$.
We give some arguments to prove uniqueness of the solution. In the following, $\mathbf{V}_1$ and $\mathbf{V}_2$ are two solutions of (6.42) for the same initial condition. We set

$$\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2.$$

As a preparation for this purpose, we formulate the following both lemmas, which are a consequence of the properties formulated in Lemma 6.3.3.

**Lemma 6.3.7 (Properties of $b_i$)**
*Assume that* $\mathbf{V}_i \in \mathbb{V}$*. Then it is essential that*

$$b_1(v_1, v_1, v) - b_1(v_2, v_2, v) = b_1(v, v_1, v)$$

*and*

$$b_2(v_1, V_1, V) - b_2(v_2, V_2, V) = b_2(v, V_1, V).$$

*Proof.*

$$\begin{aligned} & b_2(v_1, V_1, V) - b_2(v_2, V_2, V) \\ = \; & b_2(v_1, V_1, V) + b_2(v_2, V, V) - b_2(v_2, V_1, V) \\ = \; & b_2(v_1 - v_2, V_1, V) = b_2(v, V_1, V) \end{aligned}$$

The proof of the properties of $b_1$ is analogeous, just replace $V$ and $V_i$ by $v$ and $v_i$. □

**Lemma 6.3.8 (Properties of $b_i$)**
*The following equalities hold for* $\mathbf{V}_i \in \mathbb{V}$ *and* $z, Z$ *sufficiently regular:*

$$\begin{aligned} & b_1(v_1 + z, v_1 + z, v) - b_1(v_2 + z, v_2 + z, v) \\ & = b_1(v, v_1, v) + b_1(v, z, v) \end{aligned}$$

*and*

$$\begin{aligned} & b_2(v_1 + z, V_1 + Z, V) - b_2(v_2 + Z, V_2 + Z, V) \\ & = b_2(v, V_1, V) + b_2(v, Z, V). \end{aligned}$$

*Proof.*

$$
\begin{aligned}
& b_1(v_1+z, v_1+z, v) - b_1(v_2+z, v_2+z, v) \\
= \; & b_1(v_1, v_1+z, v) + b_1(z, v_1+z, v) - b_1(v_2, v_2+z, v) - b_1(z, v_2+z, v) \\
= \; & b_1(v_1, v_1, v) + b_1(v_1, z, v) + b_1(z, v_1, v) + b_1(z, z, v) \\
& -b_1(v_2, v_2, v) - b_1(v_2, z, v) - b_1(z, v_2, v) - b_1(z, z, v) \\
= \; & b_1(v, v_1, v) + b_1(v, z, v) + b_1(z, v, v) \\
= \; & b_1(v, v_1, v) + b_1(v, z, v)
\end{aligned}
$$

and

$$
\begin{aligned}
& b_2(v_1+z, V_1+z, V) - b_2(v_2+z, V_2+Z, V) \\
= \; & b_2(v_1, V_1+Z, v) + b_2(z, V_1+Z, V) - b_2(v_2, V_2+Z, V) - b_2(z, V_2+Z, V) \\
= \; & b_2(v_1, V_1, V) + b_2(v_2, Z, V) + b_2(z, V_1, V) + b_2(z, Z, V) \\
& -b_2(v_2, V_2, V) - b_2(v_1, Z, V) - b_2(z, V_2, V) - b_2(z, Z, V) \\
= \; & b_2(v, V_1, V) + b_2(v, Z, V) + b_2(z, V, V) \\
= \; & b_2(v, V_1, V) + b_2(v, Z, V)
\end{aligned}
$$

□

Thus, we have

$$\frac{d}{dt}\|V\|^2 + \|V\|_{H^1}^2 \leq K\|\mathbf{V}\|^2(\|V_1\|_{H^1}^2 + \|Z(\theta_t\omega)\|_{H^1}^4) + \epsilon\|v\|_{H^1}^2$$

for some $\epsilon > 0$. This leads to

$$\frac{d}{dt}\|\mathbf{V}\|^2 \leq K(\|V_1\|_{H^1}^2 + \|v_1\|_{H^1}^2 + \|Z(\theta_t\omega)\|_{H^1}^4 + K)\|\mathbf{V}\|^2$$

and thus, we obtain uniqueness by applying the Gronwall lemma because of $\|v_1^0 - v_2^0\|^2 = 0$ and $\|V_1\|_{H^1}^2 + \|v_1\|_{H^1}^2 + \|Z(\theta_t\omega)\|_{H^1}^4 + K \in L^1(0,t)$ .

**Remark 6.3.9**
*A similar calculation gives us that $\mathbf{V}(t) \in \mathbb{H}$ depends continuously on the initial condition.*

We note that by the measurability $\omega \to \mathbf{V} \in C([0,T];\mathbb{V}')$ the mapping $\omega \to \mathbf{V}(t,\omega)$ is measurable for $t \in [0,T]$; see [50, Bem. 4.1.3]. In addition, for fixed $t, \omega$ the mapping $v^0 \to \mathbf{V}(t,\omega)$ is continuous by Remark 6.3.9. Applying [9, Lemma III.14] the mapping $(v^0, \omega, t) \to \mathbf{V}(t,\omega) \in \mathbb{H}$ is measurable. Thus, we have the following result.

**Lemma 6.3.10**
*The solution of (6.42) generates a continuous random dynamical system in $H$.*

We denote this random dynamical system by $\varphi = (\varphi_1, \varphi_2)$.

### 6.3.3 The attractor of the Boussinesq system

In this section, we will apply the results of the preceding section to prove the existence of a random attractor for (6.41).

**Theorem 6.3.11 (Random attractor)**
*Let $\mathbf{Z} = (z, Z, Z_\Gamma)$ be the stationary solution of (5.4). We set*

$$Q(\omega) := K^\epsilon \|Z(\omega)\|^2_{D(A_2)} + K^\epsilon \|z(\omega)\|^2_{H^1} - \lambda \in L^1, \tag{6.43}$$

*where the constant $K^\epsilon$ depends on the data of the problem and some $\epsilon$ that has to be chosen sufficiently small, and the constant $\lambda$ is defined in (6.39). Assume that $Q(\omega)$ has the finite expectation*

$$\mathbb{E}Q < 0.$$

*This can be ensured by Lemma 5.1.18. Then, the Boussinesq system (6.41) has a unique random attractor.*

We first note that from assumption (6.43) there exists a $\{\theta_t\}_{t\in\mathbb{R}}$–invariant set of full measure (see also Remark 5.1.19 (ii)), such that

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t Q(\theta_\tau \omega) d\tau = \mathbb{E}Q. \tag{6.44}$$

We again consider the invariant set of $\omega$ that is defined in Remark 5.1.19 and such that $t \to \|\mathbf{Z}(\theta_t\omega)\|_{D(A^s)}$ has a subexponential growth and the invariant set of $\omega$, such that (6.44) holds. We take the trace-$\sigma$-algebra of $\mathcal{F}$ for a set given by the intersection of these invariant sets and the probability measure which is the restriction of $\mathbb{P}$ to this new $\sigma$-algebra. Then, we obtain a new metric dynamical system. In particular, the flow $\theta$ is measurable with respect to this new $\sigma$–algebra [8]. For this new metric dynamical system we use the old notation $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.
The proof of the above theorem is divided into two lemmas. We first prove the existence of an absorbing set $B$ in $\mathbb{H}$ and then show that this set can be modified into a compact absorbing set. Then, together with a continuity result, we can apply Theorem 5.1.8 to conclude the existence of the random attractor.

**Lemma 6.3.12**
*Let $\mathbb{E}Q < 0$. Then the Boussinesq problem (6.42) has an absorbing set*

$$B(\omega) = B_{\mathbb{H}}(0, \rho(\omega)),$$

*where $B_{\mathbb{H}}(0, \rho(\omega))$ defines the closed ball in $\mathbb{H}$ with center 0 and radius $\rho(\omega)$ which is defined as follows:*

$$\begin{aligned}
\rho(\omega) &= 2 \int_{-\infty}^{0} \exp\Big( \int_{t}^{0} Q(\theta_\tau\omega)\, d\tau \Big) G(\theta_t\omega)\, dt \\
G(\omega) &= G_1(\omega) + G_2(\omega), \\
G_1(\omega) &= K^\epsilon + K^\epsilon(\|z(\omega)\| \|Z(\omega)\|_{D(A_2)})^2, \\
G_2(\omega) &= K^\epsilon(\|z(\omega)\|_{H^1} \|z(\omega)\|_{D(A_1)})^2 + K^\epsilon \|Z(\omega)\|^2.
\end{aligned}$$

*Proof.* Let

$$\mathbf{V} = (v, V, V_\Gamma)$$

be the solution of (6.42) and

$$\mathbf{Z} = (z, Z, Z_\Gamma)$$

be the stationary solution of (5.4). Multiplying (6.42) by $\mathbf{V}$ with respect to the $L^2$ inner product leads us to

$$\frac{d\|(V,V_\Gamma)\|^2}{dt} + 2(A_2(V,V_\Gamma),(V,V_\Gamma)) \leq 2(f,V_\Gamma) + 2\|b_2(v+z(\theta_t\omega), Z(\theta_t\omega), V)\| \tag{6.45}$$

and

$$\begin{aligned}\frac{d\|v\|^2}{dt} + 2(A_1 v, v) \leq &\frac{2}{Fr^2}\|V+Z\|\|v\| \\ &+ 2|b_1(z(\theta_t\omega), z(\theta_t\omega), v)| + 2|b_1(v, z(\theta_t\omega), v)|.\end{aligned} \tag{6.46}$$

We start to deal with (6.45). On account of Lemma 6.3.4 we obtain for some $\epsilon > 0$

$$\begin{aligned}\frac{d\|(V,V_\Gamma)\|^2}{dt} + 2(A_2V,V) &\leq 2|(f,V_\Gamma)| + 2|b_2(v+z(\theta_t\omega), Z(\theta_t\omega), V)| \\ &\leq 2|(f,V_\Gamma)| + K\|v+z(\theta_t\omega)\|\|Z(\theta_t\omega)\|_{D(A_2)}\|V\|_{H^1} \\ &\leq 2|(f,V)| + K\|v\|\|Z(\theta_t\omega)\|_{D(A_2)}\|V\|_{H^1} + \\ &+ K\|z(\theta_t\omega)\|\|Z(\theta_t\omega)\|_{D(A_2)}\|V\|_{H^1} \\ &\leq K^\epsilon + K^\epsilon(\|v\|\|Z(\theta_t\omega)\|_{D(A_2)})^2 + \\ &+ K^\epsilon(\|z(\theta_t\omega)\|\|Z(\theta_t\omega)\|_{D(A_2)})^2 + \epsilon\|V\|^2_{H^1},\end{aligned}$$

hence

$$\frac{d\|(V,V_\Gamma)\|^2}{dt} + \lambda\|(V,V_\Gamma)\|^2 + K\|V\|^2_{H^1} \leq K^\epsilon\|v\|^2\|Z(\theta_t\omega)\|^2_{D(A_2)} + G_1(\theta_t\omega). \tag{6.47}$$

Gronwall's lemma yields

$$\begin{aligned}\|(V,V_\Gamma)(t)\|^2 \quad + K\int_0^t e^{-\lambda(t-s)}\|V\|^2_{H_1}\,ds \quad &\leq K^\epsilon\int_0^t e^{-\lambda(t-s)}\|v\|^2\|Z(\theta_s\omega)\|^2_{D(A_2)}\,ds \\ &+ \int_0^t e^{-\lambda(t-s)}G_1(\theta_s\omega)\,ds + e^{-\lambda t}\|(V,V_\Gamma)(0)\|^2.\end{aligned}$$

Similar to the last calculation we conclude in (6.46); see also Lemma 6.3.4:

$$\begin{aligned}\frac{d\|v\|^2}{dt} + 2(A_1v,v) \le{}& \frac{2}{Fr^2}\|V+Z(\theta_t\omega)\|\|v\| + 2|b_1(z(\theta_t\omega),z(\theta_t\omega),v)| +\\ &+ 2|b_1(v,z(\theta_t\omega),v)|\\ \le{}& \frac{2}{Fr^2}(\|V\|\|v\| + \|Z(\theta_t\omega)\|\|v\|) + K\|z(\theta_t\omega)\|_{H^1}\|v\|\|v\|_{H^1} +\\ &+ K\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)}\|v\|\\ \le{}& K^\epsilon\|V\|^2 + \frac{\epsilon}{4}\|v\|^2 + K^\epsilon\|Z(\theta_t\omega)\|^2 + \frac{\epsilon}{4}\|v\|^2\\ &+ K^\epsilon\|z(\theta_t\omega)\|^2_{H^1}\|z(\theta_t\omega)\|^2_{D(A_1)}\\ &+ \frac{\epsilon}{4}\|v\|^2 + K^\epsilon\|z(\theta_t\omega)\|^2_{H^1}\|v\|^2 + \frac{\epsilon}{4}\|v\|^2_{H^1}.\end{aligned} \tag{6.48}$$

Thus, we obtain for small $\epsilon > 0$

$$\frac{d\|v\|^2}{dt} - Q_2(\theta_t\omega)\|v\|^2 \le K^\epsilon\|V\|^2 + G_2(\theta_t\omega),$$

where

$$Q_2(\omega) := K^\epsilon\|z(\omega)\|^2_{H^1} - \lambda.$$

Applying again Gronwall's lemma, we obtain

$$\begin{aligned}\|v(t)\|^2 \le{}& e^{\int_0^t Q_2(\theta_s\omega)\,ds}\|v(0)\|^2 + \int_0^t G_2(\theta_s\omega)e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\,ds +\\ &\int_0^t K^\epsilon\|V(s)\|^2 e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\,ds.\end{aligned}$$

We achieve

$$\begin{aligned}\|\mathbf{V}(t)\|^2 \le{}& e^{\int_0^t Q_2(\theta_s\omega)\,ds}\|\mathbf{V}(0)\|^2 + \int_0^t e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau} G(\theta_s\omega)\,ds\\ &+ \int_0^t K^\epsilon\|V\|^2 e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\,ds + K^\epsilon\int_0^t e^{-\lambda(t-s)}\|v\|^2\|Z(\theta_s\omega)\|^2_{D(A_2)}\,ds\end{aligned}$$

with

$$G(\omega) = G_1(\omega) + G_2(\omega).$$

We have by (6.47) and

$$Q_2(\omega) > -\lambda,$$

$$\frac{d\|V\|^2}{dt} - Q_2(\theta_t\omega)\|V\|^2 + K\|V\|^2_{H^1} \le K^\epsilon\|v\|^2\|Z(\theta_t\omega)\|^2_{D(A_2)} + G_1(\theta_t\omega)$$

such that the following estimate holds

$$\int_0^t K^\epsilon \|V(s)\|^2 e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\,ds \leq K^\epsilon \int_0^t \|V(s)\|_{H^1}^2 e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\,ds$$

$$\leq K^\epsilon e^{\int_0^t Q_2(\theta_\tau\omega)\,d\tau}\|V(0)\|^2 + K^\epsilon \int_0^t e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\|v(s)\|^2\|Z(\theta_s\omega)\|_{D(A_2)}^2\,ds$$

$$+K^\epsilon \int_0^t e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau} G_1(\theta_s\omega)\,ds.$$

We obtain the following integral inequality

$$\begin{aligned}\|\mathbf{V}(t)\|^2 \quad &\leq \quad K^\epsilon e^{\int_0^t Q_2(\theta_s\omega)\,ds}\|\mathbf{V}(0)\|^2 + K^\epsilon \int_0^t e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau} G(\theta_s\omega)\,ds \\ &\quad +K^\epsilon \int_0^t e^{\int_s^t Q_2(\theta_\tau\omega)\,d\tau}\|Z(\theta_s\omega)\|_{D(A_2)}^2\|\mathbf{V}(s)\|^2\,ds. \end{aligned} \tag{6.49}$$

In order to apply Gronwall's lemma, we multiply (6.49) by $e^{-\int_0^t Q_2(\theta_\tau\omega)\,d\tau}$ as an integrating factor:

$$\begin{aligned}\|\mathbf{V}(t)\|^2 e^{-\int_0^t Q_2(\theta_\tau\omega)\,d\tau} \quad &\leq \quad K^\epsilon\|\mathbf{V}(0)\|^2 + K^\epsilon \int_0^t e^{-\int_0^s Q_2(\theta_\tau\omega)\,d\tau} G(\theta_s\omega)\,ds \\ &\quad +K^\epsilon \int_0^t e^{-\int_0^s Q_2(\theta_\tau\omega)\,d\tau}\|Z(\theta_t\omega)\|_{D(A_2)}\|\mathbf{V}(s)\|^2\,ds.\end{aligned}$$

Applying Gronwall's lemma ([54, Lemma 29.2]) on $\|\mathbf{V}(t)\|^2 e^{-\int_0^t Q_2(\theta_\tau\omega)\,d\tau}$ provides us

$$\|\mathbf{V}(t)\|^2 \leq e^{\int_0^t Q(\theta_\tau\omega)\,d\tau}\Big(K^\epsilon\|\mathbf{V}(0)\|^2 + \int_0^t K^\epsilon e^{-\int_0^\tau Q(\theta_s\omega)\,ds} G(\theta_\tau\omega)\,d\tau\Big). \tag{6.50}$$

On account of Birkhoff's ergodic theorem, we obtain for $t \to \pm\infty$

$$e^{\int_0^t Q(\theta_\tau\omega)\,d\tau} \approx e^{\mathbb{E}Q|t|}.$$

For the second part of (6.50) we have

$$e^{\int_0^t Q(\theta_\tau\omega)\,d\tau}\int_0^t K^\epsilon e^{-\int_0^\tau Q(\theta_s\omega)\,ds} G(\theta_\tau\omega)\,d\tau$$

which yields by replacing $\omega \to \theta_{-t}\omega$ simple integral transformation $\tau \to \tau - t$ and $t \to \infty$ to

$$K^\epsilon \int_{-\infty}^{0} e^{\int_\tau^0 Q(\theta_s\omega)\,ds} G(\theta_\tau\omega)\,d\tau < \infty.$$

Applying Lemma 4.6 of [8] gives us the existence of an absorbing set in $\mathbb{H}$ because $\mathbb{E}Q < 0$ and $G$ is tempered. We can conclude now the existence of a random absorbing set [8, chapter 4]. □

**Remark 6.3.13**
*Similar calculations as in the last proof give us a bound of* $\mathbf{V}$ *in* $L^\infty(0,T;\mathbb{H})$. *This bound is uniform, if the initial conditions for (6.42) are in a bounded set in* $\mathbb{H}$.

**Remark 6.3.14**
*By Remark 6.3.13 and equations (6.47) and (6.48), we obtain directly, that* $\mathbf{V} \in L^2(0,T;\mathbb{V})$.

To prove the existence of a compact absorbing set, we need the following two lemmas. At first, we need the following modification of the uniform Gronwall lemma [49, p.91].

**Lemma 6.3.15**
*Suppose that* $t \in [0, \frac{1}{2}]$ *and*

$$\frac{dy}{dt} \le gy + h.$$

*with g,y and h are positive integrable functions. The the following estimate holds:*

$$y(\frac{1}{2}) \le 2\int_0^{\frac{1}{2}} y(s)\,ds \exp\Big(\int_0^{\frac{1}{2}} g(\tau)d\tau\Big) + \int_0^{\frac{1}{2}} h(s)\,ds \exp\Big(\int_0^{\frac{1}{2}} g(\tau)\,d\tau\Big). \tag{6.51}$$

In the following Lemma, we need estimates of the the trilinear form $b_1$. The proof is similar to Lemma 6.3.4.

**Lemma 6.3.16**
*Suppose that* $z$, $v \in D(A_1)$ *and* $V \in D(A_2)$. *Then, we have*

$$\begin{aligned}
|b_1(z,z,A_1v)| &\le K^\epsilon(\|z\|_{H^1}\|z\|_{D(A_1)})^2 + \epsilon\|A_1v\|^2,\\
|b_1(v,z,A_1v)| &\le K^\epsilon\|v\|_{H^1}^2\|z\|_{D(A_1)}^2 + \epsilon\|A_1v\|^2,\\
|b_1(z,v,A_1v)| &\le K^\epsilon\|v\|_{\mathbb{V}_1}^2\|z(\theta_t\omega)\|_{H_1}^4 + \epsilon\|A_1v\|^2,\\
|b_1(v,v,A_1v)| &\le K^\epsilon\|v\|^2\|v\|_{H^1}^4 + \epsilon\|A_1v\|^2
\end{aligned}$$

*and similarly*

$$\begin{aligned}
|b_2(z,Z,A_2V)| &\le K^\epsilon(\|z\|_{H^1}\|Z\|_{D(A_2)})^2 + \epsilon\|A_2V\|^2,\\
|b_2(v,Z,A_2V)| &\le K^\epsilon\|v\|_{H^1}^2\|Z\|_{D(A_2)}^2 + \epsilon\|A_2V\|^2,\\
|b_2(z,V,A_2V)| &\le K^\epsilon\|V\|_{\mathbb{V}_2}^2\|z(\theta_t\omega)\|_{H_1}^4 + \epsilon\|A_2v\|^2,\\
|b_2(v,V,A_2V)| &\le K^\epsilon\|A_1v\|^2\|V\|_{\mathbb{V}_2}^2 + \epsilon\|A_2V\|^2.
\end{aligned}$$

*Proof.* Applying generalized Hölder's inequality with parameters $(4,4,2)$ leads to:

$$\begin{aligned} b_1(z,z,A_1v) \le d_1\|z\|_{L^4}\|\nabla z\|_{L^4}\|A_1v\| &\le d_2\|z\|_{H^1}\|z\|_{D(A_1)}\|A_1v\| \\ &\le K^\epsilon(\|z\|_{H^1}\|z\|_{D(A_1)})^2+\epsilon\|A_1v\|^2 \end{aligned}$$

In the same way, we conclude

$$b_1(v,z(\theta_t\omega),A_1v) \le K^\epsilon\|v\|^2_{\mathbb{V}_1}\|z(\theta_t\omega)\|^2_{D(A_1)}+\epsilon\|A_1v\|^2,$$

Again with parameters $(4,4,2)$ we obtain:

$$b_1(z(\theta_t\omega),v,A_1v) \le K^\epsilon\|v\|^2_{\mathbb{V}_1}\|z(\theta_t\omega)\|^4_{H_1}+\epsilon\|A_1v\|^2,$$

In the following calculation again Hölder's inequality with parameter $(4,4,2)$, Young's inequality with $(4,\frac{4}{3})$ and Ladyzhenskaya's inequality are used:

$$\begin{aligned} b_1(v,v,A_1v) &\le d_6\|v\|_{L^4}\|\nabla v\|_{L^4}\|A_1v\| \\ &\le d_7\|v\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}_{\mathbb{V}_1}\|v\|^{\frac{1}{2}}_{\mathbb{V}_1}\|A_1v\|^{\frac{1}{2}}\|A_1v\| \\ &\le K^\epsilon\|v\|^2\|v\|^4_{\mathbb{V}_1}+\epsilon\|A_1v\|^2 \end{aligned}$$

We use Hölder's inequality with parameters $(8,\frac{8}{3},2)$ to obtain

$$b_2(z,Z,A_2V) \le K\|z\|_{L^8}\|\nabla Z\|_{L^{\frac{8}{3}}}\|A_2V\|.$$

Theorem 2.2.19 and the interpolation inequality [48, p. 49] combined with [38, Theorem B.8] leads to

$$\|\nabla Z\|_{L^{\frac{8}{3}}} \le K_1\|\nabla Z\|_{H^{\frac{1}{4}}} \le K_2\|\nabla Z\|^{\frac{1}{2}}_{L^2}\|\nabla Z\|^{\frac{1}{2}}_{H^{\frac{1}{2}}} \le K_3\|Z\|_{D(A_2)},$$

since $D(A_2)\subset H^{\frac{3}{2}}(D)$. Similarly we achieve $H^1(D)\subset L^8(D)$, thus $\|z\|_{L^8}\le K\|z\|_{H^1}$. Collecting all the last two estimates we obtain

$$b_2(z,Z,A_2V) \le K\|z\|_{H^1}\|\nabla Z\|_{D(A_2)}\|A_2V\| \le K^\epsilon(\|z\|_{H^1}\|\nabla Z\|_{D(A_2)})^2+\epsilon\|A_2V\|^2.$$

Analogeously, we find

$$b_2(v,Z,A_2V) \le K\|v\|_{H^1}\|\nabla Z\|_{D(A_2)}\|A_2V\| \le K^\epsilon(\|v\|_{H^1}\|\nabla Z\|_{D(A_2)})^2+\epsilon\|A_2V\|^2$$

and

$$b_2(z,V,A_2V) \le K\|z\|_{H^1}\|V\|^{\frac{1}{2}}_{\mathbb{V}_2}\|V\|^{\frac{1}{2}}_{D(A_2)}\|A_2V\| \le K^\epsilon(\|V\|^2_{\mathbb{V}_2}\|z\|^4_{H^1})+\epsilon\|A_2V\|^2.$$

The last inequality we achieve by

$$\begin{aligned} b_2(v,V,A_2V) &\le K\sup_{x\in D}\|v(x)\|\|\nabla V\|\|A_2V\| \\ &\le K\|v\|_{H^2}\|\nabla V\|\|A_2V\| \\ &\le K^\epsilon\|v\|^2_{D(A_1)}\|V\|^2_{\mathbb{V}_2}+\epsilon\|A_2V\|^2, \end{aligned}$$

by applying [43, Theorem 5.29] and the relation $D(A_1)\sim H^2(D)$.

□

**Lemma 6.3.17**
*Let $B$ be an absorbing set in $\mathbb{H}$. Then*

$$C(\omega) = \overline{\varphi(1, \theta_{-1}\omega, B(\theta_{-1}\omega))}$$

*is a compact and absorbing set in $\mathbb{H}$.*

*Proof.* This is a similar technique like in [15, Theorem 5.2]. To obtain this result, we show some a priori estimates in the $H^1$-norm for $\mathbf{V}$ in the same way like [49, p.111 and p.139]. In particular, we have to use for the estimates the Galerkin approximations but we again suppress the projections in these equations. For details we refer to [47, Theorem III.3.10] .

□

*Proof of Theorem 6.3.11.* Lemma 6.3.17 and Remark 6.3.9 allow us to apply Theorem 5.1.8 to the rds generated by (6.42). The transform $T(\omega, x) = x + \mathbf{Z}(\omega)$ has all properties assumed in Lemma 5.1.16 which gives us the existence of a random attractor for (6.41).
We have

$$\begin{aligned}\left(\frac{d\mathbf{V}}{dt}, A\mathbf{V}\right) + (A\mathbf{V}, A\mathbf{V}) &\leq |b_1(v + z(\theta_t\omega), v + z(\theta_t\omega), A_1 v)|) \\ &+ |b_2(v + z(\theta_t\omega), V + Z(\theta_t\omega), A_2 V)|) + |K((V + Z(\theta_t\omega))\mathbf{k}, A_1 v)| \\ &+ |(f, A_2 V_\Gamma)|.\end{aligned}$$

Multiplication of (6.42) with respect to the $L^2$ scalar-product respectively by $A = (A_1, A_2)$ gives us:

$$(\frac{dv}{dt}, A_1 v) + (A_1 v, A_1 v) + b_1(v + z(\theta_t\omega), v + z(\theta_t\omega), A_1 v) = K((V + Z(\theta_t\omega))\mathbf{k}, A_1 v) \quad (6.52)$$

$$(\frac{dV}{dt}, A_2 V) + (A_2 V, A_2 V) + b_2(v + z(\theta_t\omega), V + Z(\theta_t\omega), A_2 V) = (f, A_2 V). \quad (6.53)$$

At first, we deal with (6.52): The Cauchy-Schwarz-inequality and the chain rule provides us:

$$\frac{d\|v\|^2_{\mathbb{V}}}{dt} + \frac{5}{3}\|A_1 v\|^2 \leq |2b_1(v + z(\theta_t\omega), v + z(\theta_t\omega), A_1 v)| + K\|V\|^2 + K\|Z(\theta_t\omega)\|^2. \quad (6.54)$$

Inserting the estimates of Lemma 6.3.16 in (6.54) yields:

$$\begin{aligned}\frac{d\|v\|^2_{\mathbb{V}_1}}{dt} + \|A_1 v\|^2 \quad &\leq \quad K\|V\|^2 + K\|Z(\theta_t\omega)\|^2 + K^\epsilon(\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)})^2 + \\ &+K^\epsilon\|v\|^2_{\mathbb{V}_1}(\|z(\theta_t\omega)\|^2_{D(A_1)} + \|z(\theta_t\omega)\|^4_{H^1}) + K^\epsilon\|v\|^2\|v\|^4_{\mathbb{V}_1}. \quad (6.55)\end{aligned}$$

We estimate $V$ in a similar way and obtain

$$\begin{aligned}\frac{d\|V\|^2_{\mathbb{V}_2}}{dt} + \|A_2 V\|^2 \quad &\leq \quad K^\epsilon(\|v\|_{\mathbb{V}_1}\|Z(\theta_t\omega)\|_{D(A_2)})^2 + K^\epsilon(\|z(\theta_t\omega)\|_{H^1}\|Z(\theta_t\omega)\|_{D(A_2)})^2 \\ &+K^\epsilon\|A_1 v\|^2\|V\|^2_{\mathbb{V}_2} + K^\epsilon\|z(\theta_t\omega)\|^4\|V\|^2_{\mathbb{V}_2} + K^\epsilon.\end{aligned} \quad (6.56)$$

by the estimates of $b_2$ in Lemma 6.3.16.
(6.55) and (6.56) are of uniform Gronwall type. Adding them with

$$y := \|v\|^2_{\mathbb{V}_1} + \|V\|^2_{\mathbb{V}_2} = \|\mathbf{V}\|^2_{\mathbb{V}}$$

leads to

$$\frac{d\|\mathbf{V}\|^2_{\mathbb{V}}}{dt} \leq g\|\mathbf{V}\|^2_{\mathbb{V}} + h, \tag{6.57}$$

with

$$h(\omega) := K\|V\|^2 + H_1(\omega),$$

and

$$g(\omega) := H_2(\omega) + K_2\|v\|^2\|v\|^2_{\mathbb{V}} + \|A_1 v\|^2.$$

$H_i$ are defined as follows

$$H_1(\omega) = K^\epsilon + K^\epsilon(\|z(\omega)\|_{H^1}\|Z(\omega)\|_{D(A_2)})^2 + K\|Z(\omega)\|^2 + K^\epsilon(\|z(\omega)\|_{H^1}\|z(\omega)\|_{D(A_1)})^2$$

and

$$H_2(\omega) = K^\epsilon\|z(\omega)\|^2_{D(A_1)} + K^\epsilon\|Z(\omega)\|^2_{D(A_2)} + K^\epsilon\|z\|^4_{H^1}.$$

After applying (6.51) on (6.57), we want to show that $\|\mathbf{V}(1)\|^2_{\mathbb{V}}$ is bounded, if $\mathbf{V}_0$ is contained in a bounded set in $L^2$. We can write

$$\varphi(1, \theta_{-1}\omega, \mathbf{V}_0) = \varphi(\frac{1}{2}, \theta_{-\frac{1}{2}}\omega, \varphi(\frac{1}{2}, \theta_{-1}\omega, \mathbf{V}_0)),$$

for $\mathbf{V}(0) \in B(\theta_{-1}\omega)$ and define

$$\overline{\mathbf{V}_0} := \varphi(\frac{1}{2}, \theta_{-1}\omega, \mathbf{V}_0).$$

We have to demonstrate that $\int_0^{\frac{1}{2}} \|\mathbf{V}(s)\|^2_{\mathbb{V}}\, ds$, $\int_0^{\frac{1}{2}} g(s)\, d\tau$ and $\int_0^{\frac{1}{2}} h(s)\, ds$ are respectively finite, if $\|\overline{\mathbf{V}_0}\|^2$ is contained in a bounded set.
Therefore, we have to make some energy estimates on the $\|\cdot\|_{\mathbb{V}}$-norm to show that $\int_0^{\frac{1}{2}} \|\mathbf{V}(s)\|^2_{\mathbb{V}}\, ds$ is bounded. We assume for this moment that $\int_0^{\frac{1}{2}} \|A_1 v\|^2\, ds$ is finite. This will be proven in Lemma 6.3.20 below, if we additionally assume that $\overline{v_0} = \varphi_1(\frac{1}{2}, \theta_{-1}\omega, v_0)$ is bounded in $\mathbb{V}_1$, which is shown in Lemma 6.3.18 below. Multiplying (6.42) by $V$ gives us

$$\frac{d\|V\|^2}{dt} + \|V\|^2_{\mathbb{V}_2} \leq K_1 + K_2(\|v\|\|Z(\theta_t\omega)\|_{D(A_2)})^2 + K_3(\|z(\theta_t\omega)\|\|Z(\theta_t\omega)\|_{D(A_2)})^2.$$

Summarizing

$$\int_0^t \|V\|^2_{\mathbb{V}_2}\, ds \leq \|\overline{V(0)}\|^2_{L^2} + K_1 + K_2\int_0^t (\|v\|_{L^2}\|Z(\theta_s\omega)\|_{D(A_2)})^2\, ds + \tag{6.58}$$

$$+K_3\int_0^t (\|z(\theta_s\omega)\|_{L^2}\|Z(\theta_s\omega)\|_{D(A_2)})^2\, ds.$$

Similar we obtain for $v$: Multiplication of (6.42) by $v$ with respect to the $(L^2)^2$-scalar product implies:

$$\begin{aligned}\frac{d\|v\|^2}{dt}+2\|v\|^2_{\mathbb{V}_1} &\leq \frac{2}{Fr^2}\|V+Z\|\|v\|+K\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)}\|v\|+\\ &\quad \tilde{K}\|z(\theta_t\omega)\|_{D(A_1)}\|v\|\|v\|_{\mathbb{V}_1}\\ &\leq \frac{2}{Fr^2}(\|V\|\|v\|+\|Z(\theta_t\omega)\|\|v\|)+K^\epsilon(\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)})^2\\ &\quad +\epsilon\|v\|^2+\tilde{K}\|z(\theta_t\omega)\|^2_{D(A_1)}\|v\|^2+\epsilon\|v\|^2_{\mathbb{V}_1}\\ &\leq K^\epsilon\|V\|^2_{L^2}+K^\epsilon\|Z(\theta_t\omega)\|^2+K^\epsilon(\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)})^2\\ &\quad +K\|z(\theta_t\omega)\|^2_{D(A_1)}\|v\|^2+3\epsilon\|v\|^2_{\mathbb{V}_1}.\end{aligned}$$

Now observe that

$$\begin{aligned}\frac{d\|v\|^2}{dt}+\|v\|^2_{\mathbb{V}_1} &\leq K^\epsilon\|V\|^2_{L^2}+K^\epsilon\|Z(\theta_t\omega)\|^2+K^\epsilon(\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)})^2\\ &\quad +K\|z(\theta_t\omega)\|^2_{D(A_1)}\|v\|^2.\end{aligned}$$

Integrating from 0 to $t$ leads us to

$$\begin{aligned}\int_0^t \|v\|^2_{\mathbb{V}_1}\,ds &\leq \|\overline{v(0)}\|^2+\int_0^t (K^\epsilon\|V\|^2_{L^2}+K^\epsilon\|Z(\theta_s\omega)\|^2+ \\ &\quad +K^\epsilon(\|z(\theta_s\omega)\|_{H^1}\|z(\theta_s\omega)\|_{D(A_1)})^2+K\|z(\theta_s\omega)\|^2_{D(A_1)}\|v\|^2)\,ds.\end{aligned} \tag{6.59}$$

We now conclude

$$\int_0^{\frac{1}{2}} \|\mathbf{V}(t)\|^2_{\mathbb{V}}\,dt \leq M(\omega,\|\overline{\mathbf{V}_0}\|).$$

$M$ is bounded, if $\overline{\mathbf{V}_0}$ is contained in a bounded set.

To obtain that $\int_0^{\frac{1}{2}} g(s)\,ds$ is bounded, it is sufficient to show that $\int_0^{\frac{1}{2}}(\|v\|^2\|v\|^2_{\mathbb{V}_1}+\|A_1v\|^2)\,dt$ is bounded. But this is a direct consequence of Remark (6.3.14) and (6.3.13). The boundedness of $\int_0^{\frac{1}{2}}\|A_1v\|^2\,dt$ is shown in Lemma 6.3.20 below.

It is clear by the existence of an absorbing set in $L^2$ that $\int_0^1 h(s)\,ds$ is bounded. By the uniform Gronwall lemma [49, p.91] we obtain that

$$\|\varphi(1)\|_{\mathbb{V}}=\|\varphi(\frac{1}{2},\theta_{-\frac{1}{2}}\omega,\overline{\mathbf{V}_0})\|_{\mathbb{V}}$$

is bounded if $\|\overline{\mathbf{V}_0}\|^2$ are contained in a bounded set, in particular if

$$\overline{\mathbf{V}_0}\in B(\theta_{-\frac{1}{2}}\omega),$$

defined in Lemma 6.3.17. The last statement is clear by

$$\varphi(\frac{1}{2}, \theta_{-1}\omega, \mathbf{V}_0) \in B(\theta_{-\frac{1}{2}}\omega) \text{ if } \mathbf{V}_0 \in B(\theta_{-1}\omega)$$

and the invariance property of $B$.
Lemma 6.3.12, Lemma 6.3.17 and Theorem 5.1.8 give the existence of a random attractor.

**Lemma 6.3.18**
*Assume that $v_0 \in B(\theta_{-1}\omega)$. Then*

$$\sup_{v_0 \in B(\theta_{-1}\omega)} \|\varphi_1(\frac{1}{2}, \theta_{-1}\omega, v_0)\|^2_{\mathbb{V}_1} < \infty.$$

*Proof.* We start with inequality (6.55) and obtain inequality (6.57) with $\|\mathbf{V}\|_{\mathbb{V}}$ replaced by $\|v\|_{\mathbb{V}_1}$ and follow then the arguments of the proof of Theorem 6.3.11 with

$$h(\omega) = K\|V\|^2 + K\|Z(\theta_t\omega)\|^2 + K^\epsilon(\|z(\theta_t\omega)\|_{H^1}\|z(\theta_t\omega)\|_{D(A_1)})^2$$

and

$$g(\omega) = K^\epsilon(\|z(\theta_t\omega)\|^2_{D(A_1)} + \|z(\theta_t\omega)\|^4_{H^1}) + K^\epsilon\|v\|^2\|v\|^2_{\mathbb{V}_1}$$

The integrals from 0 to $\frac{1}{2}$ of $g$ and $h$ are respectively finite and the assertion is proven. □

The following Lemma is needed as preparation for Lemma 6.3.20.

**Lemma 6.3.19**
*Assume that $v_0$ is bounded in $\mathbb{V}_1$. Then*

$$v \in L^\infty(0, \frac{1}{2}; \mathbb{V}_1).$$

*Proof.* We obtain by inequality (6.55):

$$\frac{dy}{dt} \leq h + yg, \tag{6.60}$$

with $y(t) = \|v\|^2_{\mathbb{V}_1}$, $h$ and $g$ from Lemma 6.3.18. Integrating (6.60) leads to

$$y(t) \leq H(t) + \int_0^t g(\tau)y(\tau)\, d\tau,$$

with

$$H(t) = \int_0^t h(\tau)\, d\tau + y(0).$$

We observe by Gronwall's Lemma [54, Lemma 29.2]

$$y(t) \leq e^{\int_0^t g(\tau)\, d\tau} y(0) + \int_0^t h(\tau) e^{\int_\tau^t g(s)\, ds}\, d\tau. \tag{6.61}$$

The expressions on the right hand side of (6.61) are bounded by the properties of $h$ and $g$ and that the initial condition $y(0)$ is bounded in $\mathbb{V}$. Therefore, $y(t)$ is bounded and we achieve the proof. □

**Lemma 6.3.20**
*Assume that*

$$v \in L^{\infty}(0, \frac{1}{2}; \mathbb{V}_1).$$

*Then*

$$\sup_{v_0 \in B(\theta_{-\frac{1}{2}}\omega)} \int_0^{\frac{1}{2}} \|A_1 v\|^2 \, dt < \infty$$

*Proof.* We obtain by integrating inequality (6.55) fom 0 to $\frac{1}{2}$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \|A_1 v\|^2 \, dt \quad \leq \quad & \|v(0)\|_{\mathbb{V}_1}^2 + K \int_0^{\frac{1}{2}} \|V\|^2 \, dt + K \int_0^{\frac{1}{2}} \|Z(\theta_t\omega)\|^2 \, dt \\
& + K^{\epsilon} \int_0^{\frac{1}{2}} (\|z(\theta_t\omega)\|_{H^1} \|z(\theta_t\omega)\|_{D(A_1)})^2 \, dt \\
& + K^{\epsilon} \int_0^{\frac{1}{2}} \|v\|_{\mathbb{V}_1}^2 (\|z(\theta_t\omega)\|_{D(A_1)}^2 \|z(\theta_t\omega)\|_{H^1}^4) \, dt + \int_0^{\frac{1}{2}} K^{\epsilon} \|v\|^2 \|v\|_{\mathbb{V}_1}^4 \, dt,
\end{aligned}$$

which is bounded, since $v \in L^{\infty}(0, \frac{1}{2}; \mathbb{V}_1)$. □

# Chapter 7

# Invariant and inertial manifolds for parabolic equations

## 7.1 Introduction

We use the Lyapunov–Perron–transform, which gives us a fixed point, which represents the graph of an inertial manifold. Again, we transform the original stochastic partial differential equation into a non–autonomous partial differential equation, which generates a random dynamical system, to get the random fix point. For this, we have to show in addition that the non–linear dynamical boundary value problem has a mild solution. The proof of existence and uniqueness does not use the Galerkin method from Chapter 6.1.4, now a fix–point method is used. Note that the focus on the dynamical boundary condition is at the example at the end of this chapter, where the spectral properties of an operator with dynamical boundary conditions are analyzed. Finally, we collect special properties for the linear part and the non–linear part of our equation with dynamical boundary conditions that allows us to find an inertial manifold.
We consider similar to Chapter 6 the following problem.

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}u &= f(x,u) + \eta_0 \text{ on } D \times \mathbb{R}^+ \\ \frac{\partial u}{\partial t} + \mathcal{A}_\Gamma u &= g(x,u) + \eta_1 \text{ on } \partial D \times \mathbb{R}^+ \\ u(0,x) &= u_0(x), \quad x \in D, \\ u(0,\xi) &= u_0(\xi), \quad \xi \in \partial D \end{aligned} \tag{7.1}$$

with a smooth bounded domain $D$ with boundary $\partial D$ and

$$u : \mathbb{R}^+ \times D \to \mathbb{R}$$

The properties of the differential operations $\mathcal{A}(x, \partial)$ and $\mathcal{A}_\Gamma(x, \partial)$

$$\mathcal{A}(x, \partial) := -\sum_{k,j=1}^{n} \partial_{x_k}(a_{kj}(x)\partial_{x_j}) + a_0(x) \tag{7.2}$$

and

$$\mathcal{A}_\Gamma(x,\partial) = \sum_{k,j=1}^{n} \nu_k a_{kj}(x)\partial_{x_j} + c(x), \tag{7.3}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal to $\partial D$, are considered in Chapter 3.1. Note that $Au = (\mathcal{A}u, \mathcal{A}_\Gamma u) = (-\Delta u, \partial u + c(x)u)$ as in (1.2) and Chapter (6) are covered in this section.

## 7.2 Transformation

We consider (7.1) as an RDS. As before, see Chapter 6, we need again the same transformation by an Ornstein-Uhlenbeck process.
At first, we reformulate (7.1) as the following spde

$$du + Au\,dt = F(u)\,dt + dW, \quad u(0) = u_0 \in \mathbb{H}, \tag{7.4}$$

where $-A$ is is the generator of the semigroup $S(t)$ and $\mathbb{H} = L^2(D) \times L^2(\Gamma)$. For the properties of the non–linearity $F$ and the Wiener process see below. The existence of a semigroup for $A$ allows to interpret the solution to this spde as a mild solution. For details see Da Prato and Zabczyk [19].
As in Chapter 6, we consider the following random evolution equation

$$\frac{dv}{dt} + Av = F(v + Z(\theta_t\omega)), \qquad v(0) = v_0. \tag{7.5}$$

We abbreviate $F(v + Z(\omega)) =: F(v,\omega)$.

## 7.3 Inertial manifolds for random dynamical systems

In this section, we apply the Lyapunov–Perron transform to show the existence of an inertial manifold for the random pde introduced in (7.5) above. Let $\mathcal{H}_1$ be the space spanned by the eigenelements related to the first $N$ eigenvalues of the positive symmetric operator $A$. Then $\pi_1$ is an orthogonal projection from $\mathbb{H}$ on $\mathcal{H}_1$. Similarly, we can describe the infinite dimensional space $\mathcal{H}_2$ given by the span of the eigenelements of $\lambda_{N+1}, \cdots$, where the related projection is denoted by $\pi_2$. We consider the semigroup $S$ defined on $\mathbb{H}$, see Theorem 2.5.10.

**Lemma 7.3.1**
*Let $\alpha \in [0,1)$ be a constant.*
*(1) We have*

$$\pi_i S(t) = S(t)\pi_i, \quad \text{for } t \geq 0,\ i = 1,\ 2.$$

*(2) We have for $t > 0$*

$$\|\pi_2 S(t)\|_{L(\mathbb{H},D(A^\alpha))} \leq \left(\frac{\alpha^\alpha}{t^\alpha} + \lambda_{N+1}^\alpha\right) e^{-\lambda_{N+1}t}$$

*(for $\alpha = 0$ we have the convention $0^0 = 0$) and for $t \leq 0$*

$$\|\pi_1 S(t)\|_{L(\mathbb{H},D(A^\alpha))} \leq \lambda_N^\alpha e^{-\lambda_N t} = \lambda_N^\alpha e^{\lambda_N |t|}.$$

Our intention is now to interpret (7.5) as an RDS. However, in the classical theory of stochastic differential equations solutions of these equations are only defined *almost surely* where the exceptional set depends on the initial condition. Such a dependence contradicts the definition of an RDS. Therefore, we have to go another way to prove that (7.5) generates an RDS.
We have

**Lemma 7.3.2**
*(1) Suppose that the mapping*

$$F : D(A^\alpha) \to \mathbb{H}, \quad \alpha \in [0,1)^1$$

*is Lipschitz continuous with Lipschitz constant $L_F$. Then for $v_0 \in D(A^\alpha), \omega \in \Omega$ (7.5) has a unique mild solution in $C([0,T]; D(A^\alpha))$ for $T > 0$.*
*(2) Suppose now that $v_0 \in \mathbb{H}$ and $\alpha \in [0,1/2)$, then there exists a unique mild solution which is contained in $C([0,T];\mathbb{H})$. This solution depends continuously on $v_0$. In particular, this solution generates an RDS on $\mathbb{H}$ denoted by $\varphi$.*

*Proof.* (1) For the first part, we use the regularization property of $S$ similar to Lemma 7.3.1 which follows because $S$ is analytic. We need that

$$\|S(t)\|_{L(\mathbb{H},D(A^\alpha))} \le \frac{\alpha^\alpha e^{-\alpha}}{t^\alpha} \quad \text{for } t > 0,\ \alpha \in (0,1), \tag{7.6}$$

where this operator norm is bounded uniformly in $t \ge 0$ if $\alpha = 0$, see Chueshov [13]. For a given $T$ we consider the norm

$$\|u\|_{\sigma,D(A^\alpha)} = \sup_{t\in[0,T]} e^{-\sigma t}\|u(t)\|_{D(A^\alpha)},$$

which is equivalent to the standard norm of $C([0,T]; D(A^\alpha))$. $\sigma = \sigma(T)$ is chosen sufficiently large, such that

$$T_{v_0}(\cdot,\omega) : v(\cdot) \mapsto S(\cdot)v_0 + \int_0^\cdot S(\cdot - \tau)F(v(\tau), \theta_\tau\omega)d\tau$$

is a contraction on $C([0,T] : D(A^\alpha))$ for every $\omega \in \Omega$ with contraction constant

$$L_F \alpha^\alpha e^{-\alpha} \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-\tau)^\alpha} e^{-\sigma(t-\tau)} d\tau.$$

The Banach fixed point theorem then gives the conclusion.
(2) Let $T > 0$. The size will be determined later. Consider the separable Banach space $\mathcal{W}$ of measurable and square integrable mappings $[0,T] \mapsto D(A^\alpha)$ with the norm

$$|||v||| = \left(\int_0^T \|v(\tau)\|^2_{D(A^\alpha)} d\tau)\right)^{\frac{1}{2}}.$$

[1] For $\alpha = 0$ we replace $D(A^\alpha)$ by $\mathbb{H}$.

We show that $T_{v_0}(\cdot,\omega)$ satisfies the conditions of the Banach fixed point theorem for sufficiently small $T$. Suppose for a while $T_{v_0}(\cdot,\omega):\mathcal{W}\to\mathcal{W}$. Let us consider the contracting property of this mapping.

$$\begin{aligned}|||T_{v_0}(v_1,\omega)-T_{v_0}(v_2,\omega)|||^2 \le L_F^2\int_0^T\bigg(\int_0^t\|S(t-\tau)\|^2_{L(\mathbb{H},D(A^\alpha))}d\tau\times\\ \times\int_0^t\|v_1(\tau)-v_2(\tau)\|^2_{D(A^\alpha)}d\tau\bigg)dt\\ \le \alpha^{2\alpha}e^{-2\alpha}L_F^2\int_0^T\int_0^t\frac{1}{(t-\tau)^{2\alpha}}d\tau dt|||v_1-v_2|||^2\le k^2|||v_1-v_2|||^2,\end{aligned}$$

where $k<1$ for small $T$. Similar estimates ensure that $T_{v_0}(\cdot,\omega)$ maps $\mathcal{W}$ into itself. We also note that $S(\cdot)v_0\in\mathcal{W}$ and the mapping $v_0\mapsto S(\cdot)v_0$ is continuous from $\mathbb{H}$ to $\mathcal{W}$ by the estimate on the operator–norm (7.6). The fixed points $v=v(\omega)$ can be presented as a point-wise limit of the sequence with elements given by the iteration of the mapping $T_{v_0}$. Hence, the fixed point measurably depends on $\omega$.
Since the contraction constant is independent of $v_0$, a standard argument shows that the unique fixed point of $T_{v_0}(\omega)$ continuously depends on $v_0$.
Now, we show that the fixed point $v$ has a continuous version, i.e. it is contained in $C([0,T],D(A^\alpha))$. We have for $0\le t_1<t_2\le T$

$$\begin{aligned}&\|\int_0^{t_2}S(t_2-\tau)F(v(\tau),\theta_\tau\omega)d\tau-\int_0^{t_1}S(t_1-\tau)F(v(\tau),\theta_\tau\omega)d\tau\|\\ &\le\|\int_0^{t_2}S(t_2-\tau)F(v(\tau),\theta_\tau\omega)d\tau-\int_0^{t_1}S(t_2-\tau)F(v(\tau),\theta_\tau\omega)d\tau\|\\ &+\|\int_0^{t_1}S(t_2-\tau)F(v(\tau),\theta_\tau\omega)d\tau-\int_0^{t_1}S(t_1-\tau)F(v(\tau),\theta_\tau\omega)d\tau\|\\ &\le\int_{t_1}^{t_2}L_F(\|F(0,\theta_\tau\omega)\|+\|v(\tau)\|_{D(A^\alpha)})d\tau\\ &+\int_0^{t_1}\|S(t_2-t_1)-I)\|_{L(D(A^\alpha),\mathbb{H})}\|S(t_1-\tau)\|_{L(\mathbb{H},D(A^\alpha))}\times\\ &\times L_F(\|F(0,\theta_\tau\omega)\|+\|v(\tau)\|_{D(A^\alpha)})d\tau.\end{aligned}$$

We note that the first integral on the right hand side is arbitrarily small, if $t_2-t_1$ is sufficiently small. We have for the second integral

$$\begin{aligned}&C|t_2-t_1|^\alpha\int_0^{t_1}\frac{1}{(t_1-\tau)^\alpha}L_F(\|F(0,\theta_\tau\omega)\|+\|v(\tau)\|_{D(A^\alpha)})d\tau\\ &\le C'|t_2-t_1|^\alpha\bigg(\int_0^{t_1}\frac{d\tau}{(t_1-\tau)^{2\alpha}}\bigg)^{\frac12}\bigg(\int_0^{t_1}L_F^2(\|F(0,\theta_\tau\omega)\|+\|v(\tau)\|_{D(A^\alpha)})^2d\tau\bigg)^{\frac12}\end{aligned}$$

which can be made small in the above sense. Straightforwardly $v_0\mapsto S(\cdot)v_0\in C([0,T];\mathbb{H})$ is continuous. In addition, by the definition of $F(v,\omega)$ the mapping $t\mapsto F(0,\theta_t\omega)\in\mathbb{H}$ is continuous. We consider from now on the continuous version of this fixed point $v$. Let us denote this version by $v(t,\omega,v_0)$. Then

$$\begin{aligned} t \mapsto & v(t,\omega,v_0),\ v_0 \mapsto v(t,\omega,v_0) \quad \text{are continuous}\\ \omega \mapsto & v(t,\omega,v_0) \quad \text{is measurable,} \end{aligned}$$

such that by Castaing and Valadier [9, Lemma III.14 ] the mapping $(t,\omega,v_0)\mapsto v(t,\omega,v_0)\in \mathbb{H}$ is measurable.
Let now $T$ be any positive number. Let $[0,T]=\cup_{i=1}^n[T_i,T_{i+1}]$ where the intervals $[T_i,T_{i+1}]$ are so small that the above fixed point principle can be applied for every of these intervals. Since the Lipschitz constant of $F$ is global and independent of $\omega\in\Omega$ we can divide $[0,T]$ into finitely many of these intervals independently of $\omega$. By the continuity property we can *concatenate* these pieces of solutions to a measurable and continuous solution on $[0,T]$ which generates an RDS on $\mathbb{H}$. □

Now, we are in a position to prove the existence of an inertial manifold for the RDS related to (7.5). In a first step we show that the *Lyapunov–Perron–transform* has a unique fixed point in a particular space of trajectories. We are going to apply some standard technique that one can find in Chow et al. [12], [11] to our random problem.
We consider the following mapping

$$\begin{aligned} v\mapsto \mathcal{T}_{x_1}(v,\omega)(t) = \pi_1 S(t)x_1 - \int_t^0 S(t-\tau)\pi_1 F(v(\tau),\theta_\tau\omega)\,d\tau \\ + \int_{-\infty}^t S(t-\tau)\pi_2 F(v(\tau),\theta_\tau\omega)\,d\tau, \quad x_1\in\mathcal{H}_1,\omega\in\Omega,\ t\in\mathbb{R}^- \end{aligned} \tag{7.7}$$

where

$$v\in\mathcal{E}^\beta := \{u\in C((-\infty,0];\mathbb{H}): \quad \|u\|_\beta := \sup_{t\le 0} e^{\beta t}\|u(t)\|_{D(A^\alpha)}<\infty\}.$$

The constant $\beta$ will be determined below. The following theorem is a direct preparation for the main result of this section. We prove for general random evolution equation of the type of (7.5) the existence of an inertial manifold. The techniques are similar to Chow et al. [12] where deterministic evolution equations are considered. However, the fact that we have to deal with *random* dynamical systems generates some differences in the proof.

**Theorem 7.3.3**
*(1) Suppose that $F:\mathbb{H}\to\mathbb{H}$ is Lipschitz continuous with Lipschitz constant $L_F$. Let the gap condition*

$$\lambda_{N+1}-\lambda_N > 4L_F$$

*be satisfied. Then there exists a Lipschitz continuous inertial manifold $M$ for the RDS generated by (7.5).*
*(2) Suppose that $F$ satisfies the assumptions of Lemma 7.3.2 satisfying the gap condition*

$$\lambda_{N+1}-\lambda_N > \frac{2L_F}{k}((1+c_\alpha)\lambda_{N+1}^\alpha+\lambda_N^\alpha)),\ \textit{for } 0<k<1, \tag{7.8}$$

*with*

$$c_\alpha = \alpha^\alpha\int_0^\infty \tau^{-\alpha}e^{-\tau}\,d\tau.$$

*Let $\beta=\lambda_N+\frac{2L_F}{k}\lambda_N^\alpha$ which is in $(\lambda_N,\lambda_{N+1})$. Then the RDS has an inertial manifold of dimension $N$.*

The proof is divided into three lemmas.

**Lemma 7.3.4**
*Suppose that the assumptions of Theorem 7.3.3 are satisfied. Then (7.7) has a unique fixed point in $\mathcal{E}^\beta$ which depends Lipschitz–continuously on $x_1 \in \mathcal{H}_1$.*

*Proof.* (a) Applying the Banach fixed point theorem we have to show that $\mathcal{T}_{x_1}$ maps $\mathcal{E}^\beta$ to $\mathcal{E}^\beta$. From the definition of $\beta$ we can see that $\mathbb{R}^- \ni t \mapsto \|S(t)x_1\|_{D(A^\alpha)} \in \mathcal{E}^\beta$. To see that the integrals in (7.7) are in $\mathcal{E}^\beta$, we note that from the Lipschitz continuity of $F$ we have

$$\|F(u,\omega)\| \le L_F\|u\|_{D(A^\alpha)} + \|F(0,\omega)\|.$$

Then we can use the same facts that we need to prove that $\mathcal{T}_{x_1}$ is a contraction:
(b) Let $v_1$, $v_2 \in \mathcal{E}^\beta$ and let us denote $F(v_1,\omega) - F(v_2,\omega)$ by $\Delta F$. We have:

$$\begin{aligned}
\|\mathcal{T}_{x_1}(v_1,\omega) - \mathcal{T}_{x_1}(v_2,\omega)\|_\beta \le \sup_{t\le 0} e^{\beta t}\Big(\|\int_t^0 \pi_1 S(t-\tau)\Delta F d\tau\|_{D(A^\alpha)}\\
+ \|\int_{-\infty}^t \pi_2 S(t-\tau)\Delta F d\tau\|_{D(A^\alpha)}\Big)\\
\le \sup_{t\le 0} e^{\beta t}\Big(\int_t^0 \|A^\alpha \pi_1 S(t-\tau)\Delta F\|d\tau + \int_{-\infty}^t \|A^\alpha \pi_2 S(t-\tau)\Delta F\|d\tau\Big)\\
\le \sup_{t\le 0} e^{\beta t}\Big(\int_t^0 \lambda_N^\alpha e^{-\lambda_N(t-\tau)}e^{-\beta\tau}e^{\beta\tau}\|\Delta F\|\,d\tau + \int_{-\infty}^t \lambda_{N+1}^\alpha e^{-\lambda_{N+1}(t-\tau)}e^{-\beta\tau}\times\\
\times e^{\beta\tau}\|\Delta F\|\,d\tau + \alpha^\alpha\int_{-\infty}^t \frac{1}{(t-\tau)^\alpha}e^{(-\lambda_{N+1}+\beta)(t-\tau)}\|\Delta F\|d\tau\Big)\\
\le L_F \sup_{t\le 0}\Big(\lambda_N^\alpha\int_t^0 e^{(-\lambda_N+\beta)(t-\tau)}\,d\tau + \alpha^\alpha\int_{-\infty}^t\frac{1}{(t-\tau)^\alpha}e^{(-\lambda_{N+1}+\beta)(t-\tau)}d\tau\\
+ \lambda_{N+1}^\alpha\int_{-\infty}^t e^{(-\lambda_{N+1}+\beta)(t-\tau)}d\tau\Big)\|v_1-v_2\|_\beta
\end{aligned}$$

We obtain for the supremum in the last expression the bound

$$k = L_F\Big(\frac{\lambda_{N+1}^\alpha}{\lambda_{N+1}-\beta} + \frac{c_\alpha}{(\lambda_{N+1}-\beta)^{1-\alpha}} + \frac{\lambda_N^\alpha}{\beta-\lambda_N}\Big) < 1.$$

(c) Let us denote by $\Gamma(.,\omega,x_1)$ the fixed point of (7.7).

$$\begin{aligned}
\|\Gamma(\cdot,\omega,x_1) - \Gamma(\cdot,\omega,x_1')\|_\beta = \|\mathcal{T}_{x_1}(\Gamma(\cdot,\omega,x_1),\omega) - \mathcal{T}_{x_1'}(\Gamma(\cdot,\omega,x_1'),\omega)\|_\beta\\
\le \|\mathcal{T}_{x_1}(\Gamma(\cdot,\omega,x_1),\omega) - \mathcal{T}_{x_1'}(\Gamma(\cdot,\omega,x_1),\omega)\|_\beta\\
+ \|\mathcal{T}_{x_1'}(\Gamma(\cdot,\omega,x_1),\omega) - \mathcal{T}_{x_1'}(\Gamma(\cdot,\omega,x_1'),\omega)\|_\beta\\
\le \|\pi_1 S(\cdot)(x_1-x_1')\|_\beta + k\|\Gamma(\cdot,\omega,x_1) - \Gamma(\cdot,\omega,x_1')\|_\beta
\end{aligned}$$

which is the Lipschitz continuous dependence. □

We now show that

$$m(x_1,\omega) := \pi_2\Gamma(0,\omega,x_1) \tag{7.9}$$

defines the graph of the invariant manifold $M(\omega)$. Indeed, by (c) of the last proof, this graph is Lipschitz.

**Lemma 7.3.5**
*Let $M(\omega)=\{x_1+m(x_1,\omega): x_1\in\mathcal{H}_1\}$. Then $M$ is positively invariant.*

*Proof.* Let

$$X_T(\sigma,\omega) := \begin{cases} \varphi(\sigma+T,\omega,\Gamma(0,\omega,x_1)) & : \quad \sigma\in[-T,0] \\ \Gamma(\sigma+T,\omega,x_1) & : \quad \sigma<-T \end{cases}.$$

To see the global invariance

$$\varphi(T,\omega,M(\omega))\subset M(\theta_T\omega), \quad T\ge 0$$

we show that

$$\Gamma(\cdot,\theta_T\omega,\pi_1\varphi(T,\omega,x_1+m(x_1,\omega))) = X_T(\cdot,\omega).$$

To see this equality, we consider the $\pi_2$-part. Let $\sigma=t-T, \quad \sigma\ge -T$. Then

$$\begin{aligned}\pi_2X_T(\sigma,\omega) =& \pi_2\varphi(\sigma+T,\omega,\Gamma(0,\omega,x_1))\\ =& \int_{-\infty}^0 \pi_2S(\sigma+T-\tau)F(\Gamma(\tau,\omega,x_1),\theta_\tau\omega)\,d\tau\\ &+\int_0^{\sigma+T}\pi_2S(\sigma+T-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_\tau\omega)\,d\tau\\ =&\int_{-\infty}^{\sigma}\pi_2S(\sigma-\tau)F(X_T(\tau,\omega),\theta_{\tau+T}\omega)\,d\tau.\end{aligned}$$

Let $\sigma\le -T$. Then

$$\begin{aligned}&\int_{-\infty}^{\sigma}\pi_2S(\sigma-\tau)F(X_T(\tau,\omega),\theta_{\tau+T}\omega)d\tau\\ =&\int_{-\infty}^{\sigma+T}\pi_2S(t-\tau)F(X_T(\tau-T,\omega),\theta_\tau\omega)\,d\tau\\ =&\int_{-\infty}^{\sigma+T}\pi_2S(T+\sigma-\tau)F(\Gamma(\tau,\omega,x_1),\theta_\tau\omega)d\tau\\ =&\pi_2\Gamma(\sigma+T,\omega,x_1)\\ =&\pi_2X_T(\sigma,\omega).\end{aligned}$$

Now, we consider the $\pi_1$-part. Let $\sigma = t - T, \quad \sigma \geq -T$. Then

$$
\begin{aligned}
&\pi_1 X_T(\sigma,\omega)\\
=&\pi_1\varphi(\sigma+T,\omega,\Gamma(0,\omega,x_1))\\
=&\pi_1\varphi(t,\omega,\Gamma(0,\omega,x_1))\\
=&\pi_1 S(t)x_1 + \int_0^t \pi_1 S(t-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_\tau\omega)\,d\tau\\
=&\pi_1 S(t-T)\Big(S(T)x_1 + \int_0^T \pi_1 S(T-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_\tau\omega)\,d\tau\\
&+\int_T^t \pi_1 S(T-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_\tau\omega)\,d\tau\Big)\\
=&\pi_1 S(\sigma)\varphi(T,\omega,\Gamma(0,\omega,x_1))\\
&+\int_0^\sigma \pi_1 S(\sigma-\tau)F(\varphi(\tau+T,\omega,\Gamma(0,\omega,x_1)),\theta_{\tau+T}\omega)\,d\tau\\
=&\pi_1 S(\sigma)\varphi(T,\omega,\Gamma(0,\omega,x_1)) + \int_0^\sigma \pi_1 S(\sigma-\tau)F(X_T(\tau,\omega),\theta_{\tau+T}\omega)\,d\tau
\end{aligned}
$$

and for $\sigma < -T$

$$
\begin{aligned}
&\pi_1 X_T(\sigma,\omega) = \pi_1\Gamma(\sigma+T,\omega,x_1)\\
=&\pi_1 S(t)x_1 - \pi_1\int_t^0 S(t-\tau)F(\Gamma(\tau,\omega,x_1),\theta_\tau\omega)\,d\tau\\
=&\pi_1 S(t-T)\Big(S(T)x_1 + \int_0^T S(T-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_\tau\omega)\,d\tau\Big)\\
&-\int_{\sigma+T}^0 \pi_1 S(\sigma+T-\tau)F(\Gamma(\tau,\omega,x_1),\theta_\tau\omega)\,d\tau-\\
&-\int_0^T \pi_1 S(t-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_\tau\omega)\,d\tau\\
=&\pi_1 S(\sigma)\varphi(T,\omega,\Gamma(0,\omega,x_1)) - \int_\sigma^{-T} \pi_1 S(\sigma-\tau)F(\Gamma(\tau+T,\omega,x_1),\theta_{\tau+T}\omega)\,d\tau\\
&-\int_{-T}^0 \pi_1 S(\sigma-\tau)F(\varphi(\tau,\omega,\Gamma(0,\omega,x_1)),\theta_{\tau+T}\omega)\,d\tau\\
=&\pi_1 S(\sigma)\varphi(T,\omega,\Gamma(0,\omega,x_1)) - \pi_1\int_\sigma^0 S(\sigma-\tau)F(X_T(\tau,\omega),\theta_{\tau+T}\omega)\,d\tau.
\end{aligned}
$$

□

**Lemma 7.3.6** *Suppose that assumptions of Theorem 7.3.3 hold. Then the manifold $M$ is exponentially attracting.*

We skip the proof because the idea for this proof is quite similar to Lemma 7.3.4. The proof can be found in [14, Theorem 4.2].

**Remark 7.3.7** *We note that we can prove that the inertial manifold is $C^1$ following the methods in Chow et at. [11], [12]. In particular, suppose that $F : D(A^\alpha) \mapsto H$ is $C^1$ where a bound for a derivative gives the constant $L_F$ in Theorem 7.3.3. Then the graph of the manifold is $C^1$–smooth. Conditions for $\alpha$, such that the Nemytzkii operator has these properties, are derived in the next section.*

## 7.4 An example

We have to consider (7.5) where the linear differential operator $A$ is generated by the pair $(\mathcal{A}, \mathcal{A}_\Gamma)$. Let us start to describe the asymptotic behavior of the eigenvalues of $A$. The asymptotic behavior can be used to get some impressions about the size of the Lipschitz constant so that the assumptions of Theorem 7.3.3 are fulfilled. Recall that $A$ is a positive symmetric operator so that there exists a sequence of positive eigenvalues of finite multiplicity tending to $+\infty$.

**Lemma 7.4.1** *The eigenvalues $(\lambda_N)_{N\in\mathbf{N}}$ of the operator $A$ considered in (7.5) have the following asymptotic properties: There exist positive constants $K_1$, $K_2$, such that*

$$K_1 N^{1/n-1} \leq \lambda_N \leq K_2 N^{2/n}.$$

*Proof.* Consider the three eigenvalue problems:

$$\begin{aligned}
\mathcal{A}u &= \lambda_{1,N} u \text{ on } D \\
\mathcal{A}_\Gamma u - c(x)u &= \lambda_{1,N} u \text{ on } \partial D \\
\mathcal{A}u &= \lambda_N u \text{ on } D \\
\mathcal{A}_\Gamma u &= \lambda_N u \text{ on } \partial D \\
\mathcal{A}u &= (\lambda_{2,N} - \hat{c})u \text{ on } D \\
\mathcal{A}_\Gamma u - c(x)u &= (\lambda_{2,N} - \hat{c})u \text{ on } \partial D.
\end{aligned} \tag{7.10}$$

We show that the eigenvalue expansion of (7.10) is similar to the others by application of the Courant-Fischer principle and using the same techniques as in [29]. This is clear by the following inequality, with $\hat{c} = \sup_{x\in\partial D} c(x)$ and $\mathbb{H}_N$ the class of $N$ dimensional subspaces of $\mathbb{H}$

$$\begin{aligned}
\lambda_{1,N} &= \min_{E\in\mathbb{H}_{N+1}} \max_{(u,\gamma u)\in E\setminus\{0\}} \frac{a(U,U) - \int_{\partial D} cu^2 ds}{\int_D u^2\,dx + \int_{\partial D} u^2\,ds} \\
&\leq \min_{E\in\mathbb{H}_{N+1}} \max_{(u,\gamma u)\in E\setminus\{0\}} \frac{a(U,U)}{\int_D u^2\,dx + \int_{\partial D} u^2\,ds} = \lambda_N \\
&\leq \min_{E\in\mathbb{H}_{N+1}} \max_{(u,\gamma u)\in E\setminus\{0\}} \frac{a(U,U) + \int_{\partial D}(\hat{c} - c)u^2 ds + \int_D \hat{c}u^2\,dx}{\int_D u^2\,dx + \int_{\partial D} u^2\,ds} = \lambda_{2,N}
\end{aligned}$$

where $U = (u, \gamma u)$. By [29] Theorem 3.1 we get

$$\lambda_{2,N} \leq K_2 N^{2/n}$$

and so

$$\lambda_N \leq K_2 N^{2/n}.$$

Similarly [29] Theorem 3.3 gives us

$$\lambda_N \geq \lambda_{N,1} \geq K_1 N^{1/(n-1)}.$$

□

We consider for the non–linearity $F$ defined by a Nemytzki operator $(f, g)$ where

$$f : \bar{D} \times \mathbb{R} \to \mathbb{R}, \qquad g : \partial D \times \mathbb{R} \to \mathbb{R}$$

**Theorem 7.4.2**
*(1) Let $f(x,u)$, $g(\xi,u)$ be Lipschitz continuous with respect to the second variable uniformly in $x \in \bar{D}$ and $\xi \in \partial D$ with Lipschitz constant $L_f$, $L_g$. If (7.8) is satisfied with*

$$L_F = L_f + L_g \tag{7.11}$$

*for some $N$, then the RDS generated by (7.5) has an inertial manifold which is Lipschitz.*
*(2) Suppose that $f$, $g$ are twice continuously differentiable with respect to the second variable with bounded first derivative $D_2 f$, $D_2 g$ and second derivative $D_2^2 f$, $D_2^2 g$, uniformly in $x \in \bar{D}$ and $\xi \in \partial D$, such that (7.8) is satisfied for some $\alpha < 1/2$ and let $n = 2$. Suppose that the uniform bound of $D_2 f$, $D_2 g$ is denoted by $L_f$, $L_g$, such that for $L_F$ given by (7.11) the inequality (7.8) is satisfied. Then, there exists a $C^1$ smooth inertial manifold.*

*Proof.* The first part of the theorem follows straighforwardly by the first part of Theorem 7.3.3. We now consider the case that the manifold is smooth, see Remark 7.3.7. Let $F(U)[x,\xi] = (f(x,u), g(\xi,u_1))$, $U = (u, u_1)$. We show that

$$\|F(U+H) - F(U) - F'(U) \cdot H\| \leq o(\|H\|_{D(A^\alpha)}) \tag{7.12}$$

where $F'(U) \cdot H[\cdot] = (D_2 f(\cdot, u(\cdot))h(\cdot), D_2 g(\cdot, u_1(\cdot))h_1(\cdot))$, and $U = (u, u_1)$, $H = (h, h_1)$. By Taylors formula

$$\|f(\cdot, u(\cdot) + h(\cdot)) - f(\cdot, u(\cdot)) - f'(\cdot, u(\cdot))h(\cdot)\|^2_{L^2(D)} \leq c \int_D h(x)^4 dx$$

for an appropriate constant $c$, which is related to the second derivative of $f$. For the Sobolev–Slobodecki spaces we have the continuous embedding

$$L^4(D) \supset H^{\frac{3}{2}\alpha}(D),$$

because of

$$\frac{3}{2}\alpha \geq n\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{n}{4} \tag{7.13}$$

for $\alpha < 1$, see Remark 3.1.3 (1). Similarly we can conclude for $g$

$$\|g(\cdot, u_1(\cdot) + h_1(\cdot)) - g(\cdot, u_1(\cdot)) - g'(\cdot, u_1(\cdot))h_1(\cdot)\|^2_{L^2(\partial D)} \leq c \int_{\partial D} h_1(\xi)^4 d\xi$$

and $L^4(\partial D) \supset H^\alpha(\partial D)$, if

$$\alpha \geq (n-1)\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{n-1}{4}.$$

On account of Remark 3.1.3(1), the right side of (7.12) can be estimated by $C\|H\|^4_{D(A^\alpha)}$, such that $F : D(A^\alpha) \mapsto \mathbb{H}$ is differentiable. In addition, the derivative $DF$ as a mapping from $D(A^\alpha) \to L(D(A^\alpha), \mathbb{H})$ is continuous. This follows because the second derivative of $f$ is uniformly bounded:

$$
\begin{aligned}
\|F'(u)\cdot H - F'(\bar u)\cdot H\|^2 &= \int_D |f'(x,u(x))h(x) - f'(x,\bar u(x))h(x)|^2 dx \\
&\quad + \int_{\partial D} |g'(\xi,u_1(\xi))h_1(\xi) - g'(\xi,\bar u_1(\xi))h_1(\xi)|^2 d\xi \\
&\le c\int_D |u(x)-\bar u(x)|^2 h(x)^2 dx + c\int_{\partial D} |u_1(\xi)-\bar u_1(\xi)|^2 h_1(\xi)^2 d\xi \\
&\le c\|u-\bar u\|^2_{L^4(D)}\|h\|^2_{L^4(D)} + c\|u_1-\bar u\|^2_{L^4(\partial D)}\|h_1\|^2_{L^4(\partial D)} \\
&\le c'\|U-\bar U\|^2_{D(A^\alpha)}\|H\|^2_{D(A^\alpha)}.
\end{aligned}
$$

The assumptions that the first derivatives of $f$, $g$ are *small* in the sense that

$$|D_2 f(x,u)| \le L_f, \quad |D_2 g(\xi,u)| \le L_g$$

uniformly for $u \in \mathbb{R}$, $x \in \bar D$, $\xi \in \partial D$ then $F$ is Lipschitz with a Lipschitz constant given in (7.11). According to Lemma 7.3.2, this Lipschitz continuity and the assumption $\alpha \in (0,1/2)$ ensures that (7.5) generates an RDS $\varphi$ on $\mathbb{H}$. But if $\alpha < 1/2$ then (7.13) is only possible for $n = 2$. □

Now, we are ready to investigate the dynamics (7.5). We define the mappings

$$(x,\omega) \mapsto T(x,\omega) := x + Z(\omega), \qquad (x,\omega) \mapsto T^{-1}(x,\omega) := x - Z(\omega).$$

Considering

$$(t,\omega) \mapsto T(\varphi(t,\omega,T^{-1}(x,\omega)),\theta_t\omega) = \psi(t,\omega,x)$$

gives us a solution version to (7.4). Since $t \mapsto \varphi(t,\omega,T^{-1}(x,\omega))$ is $(\mathcal{F}_t)_{t\in\mathbf{R}^+}$–adapted so is $t \mapsto \psi(t,\omega,x)$. Then we have

**Theorem 7.4.3**
*Let the assumptions of Theorem 7.4.2 are satisfied. Then the RDS $\psi$ generated by (7.4) has an inertial manifold with the same smoothness as the inertial manifold for $\varphi$.*

*Proof.* We can apply for this random dynamical system Lemma 5.1.17. We only need to remark that

$$\pi_1 T(x_1 + m(x_1,\omega),\omega) = x_1 + \pi_1 Z(\omega)$$

satisfies all assumptions of this lemma. □

# Chapter 8

# Attractors of hyperbolic equations

## 8.1 Introduction

In this section, we consider attractors of hyperbolic partial differential equations. At first, we consider hyperbolic equations by the theory of $C_0$-semigroups, where we use methods from Vrabie [51]. Starting with the simple wave equation, we later introduce dynamical boundary conditions (DyBC) and apply semigroup theory to these equations to get results for existence and uniqueness, see also Chapter 3.3. Later on, we apply the classical theory from Temam and Ghidaglia [49], but we extend the theory of Dirichlet, Neumann and periodic boundary conditions to dynamical boundary conditions. In the final section of this chapter, we derive a theory for hyperbolic SPDE with DyBC, which was introduced by Keller [34], but has multiplicative instead of additive noise. The stochastic theory uses a method to transform the stochastic equation to get a random equation, which can be considered pathwise as a deterministic equation.

## 8.2 Wave equation

At first, we consider the wave equation with homogeneous Dirichlet boundary conditions to point out the differences and similarities to dynamical boundary conditions and show that this equation generates a semigroup. We use a method based on the Lumer-Philips theorem 2.1.17 in contrast to the methods used in Chapter 3.3. We write formally

$$\begin{aligned} u''(x,t) &= \Delta u \quad \text{on } D \times \mathbb{R} \\ u(x,t) &= 0 \quad \text{on } \Gamma \times \mathbb{R} \end{aligned}$$

with initial conditions

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in D.$$

We rewrite this problem as

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \text{id} \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Now, we introduce the following inner product on $H_0^1(D) \times L^2(D)$

$$[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}] := \int_D \nabla u \nabla f + vg\, dx.$$

Note that by definition $u' = v$. We have

$$[\begin{pmatrix} u \\ v \end{pmatrix}, B\begin{pmatrix} u \\ v \end{pmatrix}] = \int_D \nabla u \nabla u' + \Delta u u'\, dx = 0. \tag{8.1}$$

We set $\mathcal{A}u = -\Delta u$. Equation (8.1) gives us Condition 2.1 of the Lumer-Philips theorem 2.1.17. We have to prove Condition 2.2 of the Lumer-Philips theorem 2.1.17. Then we obtain by the theorem, that $B$ is a generator of a $C_0$-semigroup because every positive $\lambda$ is contained in the resolvent set of $B$ by considering the problem

$$\lambda u - v = f, \lambda v + \mathcal{A}u = g, f \in H_0^1(D), g \in L^2(D).$$

This leads to the equation

$$\mathcal{A}u + \lambda^2 u = g + \lambda f.$$

By the elliptic-boundary value theory, see Chapter 3.1, we have that this equation has a unique solution for every $\lambda$ and hence that by this operator a $C_0$-semigroup is generated. We have also from [51, p.94] the following result by a different proof by the Stone theorem.

**Theorem 8.2.1**
*The operator $B$ is the generator of a $C_0$-group of unitary operators on $H_0^1(D) \times L^2(D)$.*

*Proof.* Equation (8.1) give us that $B$ is skew symmetric and because of the calculations in Chapter 3.3 with $\lambda = \pm 1$, $B$ is also skew-adjoint (see [51, Lemma 1.6.1]). Now, we can apply the Stone Theorem 2.1.20. This gives us the proof. □

## 8.3 Wave equation with dynamical boundary conditions

We modify the homogeneous Dirichlet boundary conditions to dynamical boundary conditions. We get the same result, this equation generates also a semigroup. Consider

$$\begin{aligned} u'' &= \Delta u \quad \text{on } D \times \mathbb{R}^+ \\ u''_\Gamma &= -\partial_\nu u - u_\Gamma \quad \text{on } \Gamma \times \mathbb{R}^+ \end{aligned}$$

with initial conditions

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x)$$

and

$$u_\Gamma(x,0) = u_{\Gamma 0}(x), \quad u_\Gamma{}'(x,0) = u_{\Gamma 1}(x)$$

We rewrite this problem to a first–order equation

$$\begin{pmatrix} u' \\ v' \\ u'_\Gamma \\ v'_\Gamma \end{pmatrix} = B \begin{pmatrix} u \\ v \\ u_\Gamma \\ v_\Gamma \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} \\ -\partial_\nu & 0 & -\mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ u_\Gamma \\ v_\Gamma \end{pmatrix}.$$

Note that this problem is considered in Chapter 3.3.

## 8.4 Wave equation with damping and dynamical boundary conditions

To prove the existence of attractors, it is necessary to add some damping in the equation. In our case, we use for sake of simplicity the same damping in the inner domain and also on the boundary, but it is also possible to use a different damping coefficient on the boundary.

### 8.4.1 Dissipativity–Generation of a semigroup

We consider the following hyperbolic PDE with DyBc and damping $\alpha > 0$.

$$\begin{aligned} u'' + \alpha u' + (-\Delta)u &= f \text{ on } D \\ u_\Gamma'' + \alpha u_\Gamma' + \partial_\nu u + u_\Gamma &= g \text{ on } \Gamma. \end{aligned}$$

We rewrite the equation as a first-order evolution equation for some $\epsilon > 0$

$$\psi' + B_\epsilon \psi = (0, f, 0, g)$$

with

$$\psi = \begin{pmatrix} u \\ v = u' + \epsilon u \\ u_\Gamma \\ v_\Gamma = u_\Gamma' + \epsilon u_\Gamma \end{pmatrix}$$

and

$$B_\epsilon = \begin{pmatrix} \epsilon \,\mathrm{id} & -\mathrm{id} & 0 & 0 \\ -\Delta - \epsilon(\alpha-\epsilon)\,\mathrm{id} & (\alpha-\epsilon)\,\mathrm{id} & 0 & 0 \\ 0 & 0 & \epsilon\,\mathrm{id} & -\mathrm{id} \\ \partial_\nu & 0 & \mathrm{id} - \epsilon(\alpha-\epsilon)\,\mathrm{id} & (\alpha-\epsilon)\,\mathrm{id} \end{pmatrix}.$$

We show that the operator $B_\epsilon$, now on the left hand side, is a positive operator which yields dissipativity. Therefore, we introduce the following scalar product on $E_0 = D(A^{\frac{1}{2}}) \times \mathbb{H}$, where $\mathbb{H} = L^2(D) \times L^2(\Gamma)$ and $A$ is defined in (8.4) below.

$$[ \begin{pmatrix} u \\ v \\ u_\Gamma \\ v_\Gamma \end{pmatrix}, \begin{pmatrix} f \\ g \\ f_\Gamma \\ g_\Gamma \end{pmatrix} ]_{E_0} := \int_D \nabla u \nabla f + vg \, dx + \int_\Gamma u_\Gamma f_\Gamma + v_\Gamma g_\Gamma \, d\sigma.$$

As in [48, p.183], we may assume that $u$ is sufficiently regular

$$u \in D(A) \cap C^2(\overline{D}).$$

The abstract arguments of [48, Lemma II.4.1] show us that the following integrals are well–defined. Then, we can conclude

$$\begin{aligned}
&[\begin{pmatrix} u \\ v \\ u_\Gamma \\ v_\Gamma \end{pmatrix}, B_\epsilon \begin{pmatrix} u \\ v \\ u_\Gamma \\ v_\Gamma \end{pmatrix}] \\
=&\int_D \nabla u \nabla(\epsilon u - v)\,dx + \int_D v(-\Delta u) - \epsilon(\alpha-\epsilon)uv + (\alpha-\epsilon)v^2\,dx \\
&+ \int_\Gamma u_\Gamma(\epsilon u_\Gamma - v_\Gamma)\,d\sigma + \int_\Gamma v_\Gamma \partial_\nu u + v_\Gamma u_\Gamma - v_\Gamma \epsilon(\alpha-\epsilon)u_\Gamma + (\alpha-\epsilon)v_\Gamma^2\,d\sigma \\
=&\epsilon \int_D \nabla u \nabla u\,dx - \int_D \nabla u \nabla v\,dx + \int_D \nabla u \nabla v\,dx - \int_\Gamma \partial_\nu u v_\Gamma\,d\sigma - \\
&- \int_D \epsilon(\alpha-\epsilon)uv + (\alpha-\epsilon)v^2\,dx + \epsilon \int_\Gamma u_\Gamma u_\Gamma\,d\sigma - \int_\Gamma u_\Gamma v_\Gamma\,d\sigma \\
&+ \int_\Gamma \partial_\nu u v_\Gamma\,d\sigma + \int_\Gamma u_\Gamma v_\Gamma\,d\sigma - \int_\Gamma v_\Gamma \epsilon(\alpha-\epsilon)u_\Gamma\,d\sigma + \int_\Gamma (\alpha-\epsilon)v_\Gamma v_\Gamma\,d\sigma \\
=&\int_D \epsilon \nabla u \nabla u + \epsilon(\alpha-\epsilon)uv + (\alpha-\epsilon)v^2\,dx + \int_D \epsilon u_\Gamma u_\Gamma + \epsilon(\alpha-\epsilon)u_\Gamma v_\Gamma + (\alpha-\epsilon)v_\Gamma^2\,d\sigma \\
\geq&\frac{\epsilon}{2}((u,u)) + \frac{\epsilon}{2}\|v\|^2 + \frac{\alpha}{2}\|v\|^2 + \frac{\epsilon}{2}\|u_\Gamma\|^2 + \frac{\epsilon}{2}\|v_\Gamma\|^2 + \frac{\alpha}{2}\|v_\Gamma\|^2 \\
\geq&\alpha_1 \|(u,v,u_\Gamma,v_\Gamma)\|_{E_0}^2,
\end{aligned}$$

for all $\epsilon$ where Inequality (8.6) holds and $\alpha_1 = \min(\frac{\epsilon}{2}, \frac{\alpha}{2})$. The last inequality is given by Lemma 8.4.2 below. Thus, this operator generates a positive $C_0$-semigroup on $E_0 = D(A^{\frac{1}{2}}) \times \mathbb{H}$ by applying the Lumer-Phillips Theorem 2.1.17, Condition 2.2 is fulfilled by the arguments in Chapter 3.1, where only a positive operator is needed. Alternatively, one can apply Remark 2.1.7, which directly yields the result. These calculations can also be found in [30], where the space $\hat{\mathcal{H}}_{en}$ is exactly our space $E_0$.

### 8.4.2 Exponential decay of the solution

Now, we consider again the following hyperbolic PDE with DyBC and damping. Our goal is at first to prove the existence of an absorbing set.

$$\begin{aligned} u'' + \alpha u' + (-\Delta)u &= f \text{ on } D \\ u_\Gamma'' + \alpha u_\Gamma' + \partial_\nu u + u_\Gamma &= g \text{ on } \Gamma. \end{aligned} \tag{8.2}$$

To prove the existence of an absorbing set, we need the following transformation

$$V = U' + \epsilon U$$

to get the transformed equation

$$V' + (\alpha-\epsilon)V + (A - \epsilon(\alpha-\epsilon))U = (f,g), \tag{8.3}$$

where in our special case

$$AU = (\mathcal{A}u, \mathcal{A}_\Gamma u) = (-\Delta u, \partial_\nu u + u_\Gamma) = \begin{pmatrix} -\Delta & 0 \\ \partial_\nu & \mathrm{Id} \end{pmatrix} U. \tag{8.4}$$

We set $U = (u, u_\Gamma)$ and $V = (v, v_\Gamma)$ respectively. As in [49, p.181], we assume that $A$ is the associated operator to a bilinear coercive and symmetric continuous form $a(U, V)$ from a given Hilbert space $\mathbb{V}$ into its dual $\mathbb{V}'$. As in Chapter 3.1.1, we can define fractional powers of the operator $A$ and function spaces

$$\mathbb{V}_{2s} = D(A^s), \quad s \in \mathbb{R}.$$

Note that $\mathbb{V}_1 = \mathbb{V}$. These spaces are Hilbert spaces for the following scalar product and norm

$$(U, V)_{2s} = (A^s U, A^s V), \quad \|U\|_{2s}^2 = (U, U)_{2s}, \quad \forall U, V \in D(A).$$

**Lemma 8.4.1**
*Assume that $\alpha > 0$ and*

$$\epsilon \le \epsilon_0 := \min(\frac{\alpha}{4}, \frac{\lambda_1}{2\alpha}). \tag{8.5}$$

*Assume that $(U, V) \in \mathbb{V}_{s+1} \times \mathbb{V}_s$ Then*

$$\epsilon\|U\|_{s+1}^2 + (\alpha - \epsilon)\|V\|_s^2 - \epsilon(\alpha - \epsilon)(U, V)_s \ge \frac{\epsilon}{2}\|U\|_{s+1}^2 + \frac{\alpha}{2}\|V\|_s^2 \quad s = 0, 1.$$

*Proof.* We may assume that

$$\|U\|_s^2 \le \frac{1}{\lambda_1}\|U\|_{s+1}^2 \text{ for } U \in \mathbb{V}_{s+1},$$

where $\lambda_1 > 0$ is the first eigenvalue of $A$. Then we conclude

$$\begin{aligned}
&\epsilon\|U\|_{s+1}^2 + (\alpha - \epsilon)\|V\|_s^2 - \epsilon(\alpha - \epsilon)(U, V)_s \\
\ge&\epsilon\|U\|_{s+1}^2 + (\alpha - \epsilon)\|V\|_s^2 - \frac{\epsilon(\alpha - \epsilon)}{\sqrt{\lambda_1}}\|U\|_{s+1}\|V\|_s \\
\ge&\epsilon\|U\|_{s+1}^2 + \frac{3}{4}\alpha\|V\|_s^2 - \frac{\alpha\epsilon}{\sqrt{\lambda_1}}\|U\|_{s+1}\|V\|_s \\
\ge&\frac{\epsilon}{2}\|U\|_{s+1}^2 + \frac{\alpha}{2}\|V\|_s^2.
\end{aligned}$$

The last inequality is given by

$$\frac{\alpha\epsilon}{\sqrt{\lambda_1}}ab = \sqrt{\epsilon}a\frac{\alpha\sqrt{\epsilon}}{\sqrt{\lambda_1}}b \le \frac{1}{2}\epsilon a^2 + \frac{1}{2}\frac{\epsilon\alpha^2}{\lambda_1}b^2 \le \frac{1}{2}\epsilon a^2 + \frac{1}{4}\alpha b^2.$$

□

More details of the functional analytic setting can be found in Definition 8.5.1.
We can also prove the general result given by Lemma 8.4.1 in our special case directly by Lemma 8.4.2 and Lemma 8.5.20, where $A$ has the form of (8.4) . To prove Lemma 8.4.2 directly, we need a version the generalized Poincaré inequality, which can be found in [49] and Theorem 2.3.9. We also use the notation

$$((u, u)) = \|\nabla u\|_{L^2(D)}^2 = \|u\|^2.$$

Note that $((u,v))$ is not an inner product on $H^1(D)$. We continue with this non–standard lemma, which is the analogon to the standard estimate for Dirichlet boundary conditions

$$\epsilon((u,u)) + (\alpha-\epsilon)\|v\|^2 - \epsilon(\alpha-\epsilon)(u,v) \geq \frac{\epsilon}{2}((u,u)) + \frac{\alpha}{2}\|v\|^2,$$

which cannot be used in the case of DyBC because of the additionally appearing terms on the boundary. The main tools in this lemma are Poincaré's and Young's inequalities.

**Lemma 8.4.2**
*Assume that* $\alpha > 0$,

$$\epsilon \leq \epsilon_0 := \min\left(\frac{\alpha}{4}, \frac{1}{2C(D)^2\alpha}, \frac{1}{\alpha}\right) \tag{8.6}$$

*and*

$$(U,V) \in E_0.$$

*Then there exists an* $\alpha_1 > 0$, *such that*

$$\begin{aligned} & \epsilon\|\nabla u\|^2 + (\alpha-\epsilon)\|v\|^2 - \epsilon(\alpha-\epsilon)(u,v) + \epsilon\|u\|_\Gamma^2 + (\alpha-\epsilon)\|v\|_\Gamma^2 - \epsilon(\alpha-\epsilon)(u,v)_\Gamma \\ \geq\ & \frac{\epsilon}{2}\|\nabla u\|^2 + \frac{\alpha}{6}\|v\|^2 + \frac{\epsilon}{8}\|u\|_\Gamma^2 + \frac{\alpha}{2}\|v\|_\Gamma^2 \\ \geq\ & \alpha_1\Big(\|\nabla u\|^2 + \|v\|^2 + \|u\|_\Gamma^2 + \|v\|_\Gamma^2\Big), \end{aligned}$$

*where*

$$\Big(\|\nabla u\|^2 + \|v\|^2 + \|u\|_\Gamma^2 + \|v\|_\Gamma^2\Big) =: \|(u,v,u_\Gamma,v_\Gamma)\|_{E_0}^2$$

*is a norm on* $E_0$.

*Proof.* At first, we consider the part in the interior of the domain. Applying Poincaré's and Young's inequalities yields us by the assumptions on $\epsilon$

$$\begin{aligned} & \epsilon\|\nabla u\|^2 + (\alpha-\epsilon)\|v\|^2 - \epsilon(\alpha-\epsilon)(u,v) \\ \geq\ & \epsilon\|\nabla u\|^2 + \frac{3}{4}\alpha\|v\|^2 - \epsilon(\alpha-\epsilon)(u,v) \\ \geq\ & \epsilon\|\nabla u\|^2 + \frac{3}{4}\alpha\|v\|^2 - \epsilon\alpha\big(C(D)(\|\!|u|\!\| + \|u\|_\Gamma)\big)\|v\| \\ \geq\ & \frac{\epsilon}{2}\|\nabla u\|^2 + \frac{\alpha}{2}\|v\|^2 - \epsilon\alpha C(D)\|u\|_\Gamma\|v\| \\ \geq\ & \frac{\epsilon}{2}\|\nabla u\|^2 + \frac{\alpha}{2}\|v\|^2 - \frac{3}{8}\epsilon\|u\|_\Gamma^2 - \frac{1}{3}\alpha\|v\|^2 \end{aligned} \tag{8.7}$$

because of

$$\begin{aligned} \alpha\epsilon C(D)\|\!|u|\!\|\|v\| &= \sqrt{\epsilon}\|\!|u|\!\|\frac{C(D)\alpha\epsilon}{\sqrt{\epsilon}}\|v\| \\ &\leq \frac{\epsilon}{2}\|\!|u|\!\|^2 + \frac{1}{2}\epsilon\alpha^2C(D)^2\|v\|^2 \leq \frac{\epsilon}{2}\|\!|u|\!\|^2 + \frac{1}{4}\alpha\|v\|^2 \end{aligned}$$

and analogically

$$\begin{aligned} \alpha\epsilon C(D)\|u\|_\Gamma\|v\| &= \sqrt{\frac{3}{4}\epsilon}\|u\|_\Gamma\frac{C(D)\alpha\epsilon}{\sqrt{\frac{3}{4}\epsilon}}\|v\| \\ &\leq \frac{3}{8}\epsilon\|u\|_\Gamma^2 + \frac{2}{3}\epsilon\alpha^2C(D)^2\|v\|^2 \leq \frac{3}{8}\epsilon\|u\|_\Gamma^2 + \frac{1}{3}\alpha\|v\|^2. \end{aligned}$$

We have similarly on the boundary, here we need that $\epsilon < \frac{1}{\alpha}$,

$$\begin{aligned} & \epsilon\|u\|_\Gamma^2 + (\alpha - \epsilon)\|v\|_\Gamma^2 - \epsilon(\alpha-\epsilon)(u,v)_\Gamma \\ \geq \; & \frac{\epsilon}{2}\|u\|_\Gamma^2 + \frac{\alpha}{2}\|v\|_\Gamma^2. \end{aligned} \tag{8.8}$$

Collecting the terms of both equations (8.7) and (8.8) and setting $\alpha_1 = \min(\frac{\epsilon}{8}, \frac{\alpha}{6})$ gives us the result. □

In the following, we use the latter Lemma 8.4.2 to get some energy estimates. In fact, the calculations are very similar in comparison to the damped wave equation with Dirichlet boundary conditions but we have to respect the additional terms appearing by integration by parts.

Taking the inner product of Equation (8.3) by $v$ on $D$ provides us:

$$\begin{aligned} (v', v) + (\alpha-\epsilon)(v,v) + ((\mathcal{A} - \epsilon(\alpha-\epsilon))u, v) &= (f, v) \\ \Rightarrow \frac{1}{2}\frac{d}{dt}\|v\|^2 + (\alpha-\epsilon)\|v\|^2 + (\mathcal{A}u, v) - \epsilon(\alpha-\epsilon)(u,v) &= (f,v) \\ \Rightarrow \frac{1}{2}\frac{d}{dt}(\|v\|^2 + \|\nabla u\|^2) + (\alpha-\epsilon)\|v\|^2 + \epsilon\|\nabla u\|^2 - \epsilon(\alpha-\epsilon)(u,v) - & \\ -\epsilon\int_\Gamma \partial_\nu u u d\sigma - \int_\Gamma \partial_\nu u u' d\sigma &= (f,v) \end{aligned} \tag{8.9}$$

because of

$$\begin{aligned} (\mathcal{A}u, u' + \epsilon u) &= \epsilon(\mathcal{A}u, u) + (\mathcal{A}u, u') \\ &= \epsilon\|\nabla u\|^2 - \epsilon\int_\Gamma \partial_\nu u u d\sigma + (\mathcal{A}u, u'). \end{aligned}$$

Furthermore, we have by Green's Formula 2.3.10, which we can apply by the arguments in [48, p. 193],

$$(\mathcal{A}u, u') = (\nabla u, \nabla u') - (\partial_\nu u, u') = \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \int_\Gamma \partial_\nu u u' d\sigma.$$

Multiplying the equation on the boundary $\Gamma$ by $v_\Gamma = u'_\Gamma + \epsilon u_\Gamma$ yields, omitting the $|_\Gamma$ in this calculations

$$\begin{aligned} (v', v) + (\alpha-\epsilon)(v,v) + ((\partial_\nu - \epsilon(\alpha-\epsilon))u, v) + (u,v) &= (g, v) \\ \Rightarrow \frac{1}{2}\frac{d}{dt}\|v\|^2 + (\alpha-\epsilon)\|v\|^2 - \epsilon(\alpha-\epsilon)(u,v) + (u,v) & \\ +(\partial_\nu u, u') + \epsilon(\partial_\nu u, u) &= (g,v). \end{aligned} \tag{8.10}$$

Adding both Equations (8.9) and (8.10) gives us for the boundary terms on the left hand side

$$\frac{1}{2}\frac{d}{dt}\|v\|_\Gamma^2 + (\alpha-\epsilon)\|v\|_\Gamma^2 - \epsilon(\alpha-\epsilon)(u,v)_\Gamma + (u,u')_\Gamma + \epsilon(u,u)_\Gamma$$

Lemma 8.4.2 leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|v\|_\Gamma^2+\|u\|_\Gamma^2+\|v\|^2+\|\nabla u\|^2\Big)+\alpha_1\Big(\|v\|_\Gamma^2+\|u\|_\Gamma^2+\|v\|^2+\|\nabla u\|^2\Big)\\
\leq\ & (f,v)+(g,v)_\Gamma\leq C(\epsilon_2,\alpha_2)\Big(\|f\|^2+\|g\|_\Gamma^2\Big)+\epsilon_2(\|v\|^2+\|v\|_\Gamma^2)
\end{aligned}$$

for some $\epsilon_2<\alpha_1$. By

$$h=\|v\|_\Gamma^2+\|u\|_\Gamma^2+\|v\|^2+\|\nabla u\|^2$$

and

$$\alpha_2=\alpha_1-\epsilon_2$$

we conclude

$$\frac{d}{dt}h+\alpha_2 h\leq C(\epsilon_2,\alpha_2)\Big(\|f\|^2+\|g\|_\Gamma^2\Big).$$

Now Gronwall's Lemma can be applied like in [49, p.184] to get the following estimate

$$h(t)\leq Ch(0)e^{-\alpha_2 t}+C(\epsilon_2,\alpha_2)\Big(\|f\|^2+\|g\|_\Gamma^2\Big)(1-e^{-\alpha_2 t}). \tag{8.11}$$

Thus, we have an estimate on the exponential decay of the first term.

## 8.5 Wave equation with damping and dynamical boundary conditions and multiplicative noise

This section deals with the most general assumptions on the noise. Instead of simple additive noise, we have more general multiplicative noise. In fact, we need that the drift term has to fullfill some trace condition, see Hypothesis 8.5.6[vii]. Otherwise, it is not possible to use integration by parts successfully. There are two different approaches included in this section. Both the approach of mild and weak solutions is considered. Weak solutions were analyzed in the work by Keller [33]. His ansatz is generalized to dynamical boundary conditions in this chapter. Following the theory of weak solutions, we can show the existence of a random attractor on (8.12). Additionally, there exists to each lemma considering weak solutions an associated remark, which proves the statement of the corresponding lemma, if the reader only considers mild solutions.

### 8.5.1 General setting

We consider the following hyperbolic SPDE with DyBC and damping with coefficient $\alpha>0$.We formally write

$$\begin{aligned}
u''+\alpha u'+(-\Delta)u &= f(u)+c(u)\frac{dw}{dt}\text{ on }D \\
u_\Gamma''+\alpha u_\Gamma'+\partial_\nu u+u_\Gamma &= f_\Gamma(u_\Gamma)+c_\Gamma(u)\frac{dw}{dt}\text{ on }\Gamma.
\end{aligned}\tag{8.12}$$

or

$$U''+\alpha U'+AU=F(U)+C(U)\frac{dw}{dt}.$$

The operator $A$ is given by (8.4), the nonlinearity is an operator

$$F:\mathbb{V}\mapsto\mathbb{H}$$

on which we have several assumptions, see Hypothesis 8.5.5. As in [33], the operator $C$ maps $\mathbb{V}$ into the space of Hilbert Schmidt operators with respect to the covariance operator $Q$ of the Wiener process $W$

$$C:\mathbb{V}\mapsto L^2(U_0;\mathbb{H}).$$

More details of Hilbert Schmidt operators and the covariance operator $Q$ of the Wiener process $W$ can be found in Chapter 4.1 and in the monograph [19]. Furthermore, we suppose that the operator $C$ possesses a Fréchet derivative

$$C':\mathbb{V}\mapsto L(U\times\mathbb{V};\mathbb{H}),$$

which is uniformly bounded.
Later, we transform this equation into a random evolution equation by a version of a stationary Ornstein-Uhlenbeck process. We can rewrite this problem by the following coordinate change

$$V=U'+\epsilon U$$

in the following way where $\epsilon>0$, re–ordering the equations

$$\begin{aligned}
du &= (v-\epsilon u)dt \text{ on } D\\
du_\Gamma &= (v_\Gamma-\epsilon u_\Gamma)dt \text{ on } \Gamma\\
dv &= ((\epsilon-\alpha)v+\epsilon(\alpha-\epsilon)u-(-\Delta)u-f(u))dt+c(u)dw \text{ on } D\\
dv_\Gamma &= ((\epsilon-\alpha)v_\Gamma+\epsilon(\alpha-\epsilon)u_\Gamma-\partial_\nu u-u_\Gamma-f_\Gamma(u_\Gamma))dt+c_\Gamma(u)dw \text{ on } \Gamma,
\end{aligned}\tag{8.13}$$

so that a second derivative does not appear.
Now, we can rewrite the equation as a first-order evolution equation

$$\varphi'+(B+L_\epsilon)\varphi=(0,0,f(u),f_\Gamma(u_\Gamma))+(0,0,C(U))[dW]$$

with

$$\varphi=\begin{pmatrix} u\\ u_\Gamma\\ v=u'+\epsilon u\\ v_\Gamma=u'_\Gamma+\epsilon u_\Gamma\end{pmatrix},$$

$$B=\begin{pmatrix} 0 & 0 & -\operatorname{id} & 0\\ 0 & 0 & 0 & -\operatorname{id}\\ -\Delta & 0 & 0 & 0\\ \partial_\nu & \operatorname{id} & 0 & 0\end{pmatrix},\tag{8.14}$$

$$C(U)=(c(u),c_\Gamma(u)),$$

and

$$L_\epsilon=\begin{pmatrix} \epsilon\operatorname{id} & 0 & 0 & 0\\ 0 & \epsilon\operatorname{id} & 0 & 0\\ -\epsilon(\alpha-\epsilon)\operatorname{id} & 0 & (\alpha-\epsilon)\operatorname{id} & 0\\ 0 & -\epsilon(\alpha-\epsilon)\operatorname{id} & 0 & (\alpha-\epsilon)\operatorname{id}\end{pmatrix}.\tag{8.15}$$

**Definition 8.5.1 (Function spaces)**
*In this setting, $w$ is a twosided Wiener process with values in some separable Hilbert space $U$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The covariance $Q$ of this Wiener process is finite: $\operatorname{Tr}_U Q < \infty$, see also Chapter 4.1. Then, this process has continuous trajectories in $U$, see [19]. The operator $A$ is defined in (8.4).*

In the following we denote

$$\mathbb{H} = L^2(D) \times L^2(\Gamma),$$

$$\mathbb{V} = D(A^{\frac{1}{2}}).$$

$$E_0 = \mathbb{V} \times \mathbb{H}$$

and

$$E_1 = D(A) \times \mathbb{V}.$$

**Remark 8.5.2**
*We also have that*

$$D(B) = E_1,$$

*see [36, Remark 3.1.] and Lemma 3.3.3. This could also be found in [51, p. 93].*

**Remark 8.5.3**
*We can write*

$$\mathbb{V} = D(A^{\frac{1}{2}}).$$

*since to the operator $A$ is associated to a positive bilinear form $a(U, V)$ defined in Equation 3.3 for suitable $a_{ik}$, $a_i$ and $c_i$, and thus we can apply Theorem 3.1.4.*

**Definition 8.5.4 (Norms)**
*We define by*

$$\|V\|^2 := \|v\|^2 + \|v\|_\Gamma^2$$

*a norm on $\mathbb{H}$. Note that by the generalized Poincaré inequality 2.3.9 and the definition of the $H^{1/2}(\Gamma)$-norm*

$$\|U\|_{\mathbb{V}}^2 := \|\nabla u\|^2 + \|u\|_\Gamma^2$$

*is a norm on $\mathbb{V}$ by the relation*

$$(A^{\frac{1}{2}}U, A^{\frac{1}{2}}U) = (AU, U) = (\nabla u, \nabla u) + (u, u)_\Gamma,$$

*which is equivalent to the $H^1$-norm for $U \in H^1$.*
*We also define*

$$\|\varphi\|_{E_0}^2 = \|v\|^2 + \|\nabla u\|^2 + \|v\|_\Gamma^2 + \|u\|_\Gamma^2,$$

*and*

$$\|\varphi\|_{E_1}^2 := \|v\|^2 + \|\mathcal{A}u\|^2 + \|v\|_\Gamma^2 + \|\mathcal{A}_\Gamma u\|_\Gamma^2$$

*for*

$$\varphi = (u, u_\Gamma, v, v_\Gamma).$$

On the nonlinearity

$$F(\varphi) := F(U) := (f(u), f_\Gamma(u_\Gamma)),$$

we have the following assumptions like in [33].

**Hypothesis 8.5.5**

*(i)* $F : \mathbb{V} \mapsto \mathbb{H}, \|F(U)\| \le C_{G,H}$ *for* $U \in \mathbb{V}$,

*(ii)* $F : D(A) \to \mathbb{V}$, $\|F(U)\|_{\mathbb{V}} \le C_{G,V}(1 + \|U\|_{\mathbb{V}})$ *for* $U \in D(A)$,

*(iii)* $F$ *is Lipschitz continuous on* $\mathbb{V}$ *to* $\mathbb{H}$.

On the operator $C(U) := (c(u), c_\Gamma(u))$, we have the following assumptions like in [36]:

**Hypothesis 8.5.6**

*(i)* $C : \mathbb{V} \mapsto L(U, \mathbb{H})$, *the space of all linear bounded operators from* $U$ *to* $\mathbb{H}$.

*(ii)* $C$ *is a continuously differentiable nonlinear operator with* $C'$ *from* $\mathbb{V}$ *into* $\mathcal{L}(U \times \mathbb{V}, \mathbb{V})$.

*(iii)* $C(\cdot)[z] : L^2(0, T; \mathbb{H}) \to L^2(0, T; \mathbb{H})$ *and*
$C'(\cdot)[z] : L^2(0, T; \mathbb{H}) \to L^2(0, T; L(U \times \mathbb{H}, \mathbb{H}))$ *are continuous for* $T > 0$ *and* $z \in U$.

*(iv)* $\|C(U)[z]\| \le c_H \|z\|_U$, *for* $U \in \mathbb{V}$,

*(v)* $\|c(u_1)[z] - c(u_2)[z]\| \le c_V \|z\|_U (1 + \min(\|u_1\|, \|u_2\|))\|u_1 - u_2\|$,
$\|c_\Gamma(u_1)[z] - c_\Gamma(u_2)[z]\|_\Gamma \le c_V \|z\|_U (1 + \min(\|u_1\|, \|u_2\|))\|u_1 - u_2\|$ *for* $U_1, U_2 \in \mathbb{V}$

*(v*)* $\|c(u_1)[z] - c(u_2)[z]\| \le c_V \|z\|_U \|u_1 - u_2\|$,
$\|c_\Gamma(u_1)[z] - c_\Gamma(u_2)[z]\|_\Gamma \le c_V \|z\|_U \|u_1 - u_2\|$ *for* $U_1, U_2 \in \mathbb{V}$

*(vi)* $\|C(U)[z]\|_{D(A)} \le c_{D(A)} \|z\|_U$ *for* $U \in D(A)$.

*(vii)* $c(u)[z]_{|\Gamma} = c_\Gamma(u)[z]$ *for* $U \in \mathbb{V}$.

*(viii) We also have an estimate on the operator norm of* $C'$:

$$\|C'(U)[z]\|_{L(\mathbb{H},\mathbb{H})} \le c'_H \|z\|_U \text{ for } U \in \mathbb{V}.$$

*(ix)* $c'(u)$ *and* $c'_\Gamma(u_\Gamma)$ *can be extended to a bounded linear operator from* $\mathbb{H}$ *to* $L(U, \mathbb{H})$ *and there exists* $c'_C > 0$ *such that*

$$\|c'(u_1)[\cdot, \cdot] - c'(u_2)[\cdot, \cdot]\|_{L(U,\mathbb{H})} \le c'_C \|u_1 - u_2\|, \text{ for } U_1, U_2 \in \mathbb{V}.$$

*and respectively on the boundary.*

$L(U, \mathbb{V})$ *denotes the space of bounded linear operators between* $U$ *and* $\mathbb{V}$. *Note that condition* (v) *can be replaced by simple Lipschitz continuity, but this condition gives us more generality, though it is not used in the example at the end of this chapter. Furthermore, we assume like in [33] that the operators* $C$ *and* $C'$ *can be extended as uniformly bounded continuous functions*

$$C : \mathbb{V} \mapsto L(U, \mathbb{V}), \quad C' : \mathbb{V} \mapsto L(U \times \mathbb{H}, \mathbb{V}).$$

First, we state a theorem of existence and uniqueness of the stochastic equation.

**Theorem 8.5.7 (Existence and uniqueness)**
*Let $(U_0, V_0)$ be a $\mathcal{F}_0$–measurable random variable with values in $E_0$ and that the assumptions of Hypothesis 8.5.5 and 8.5.6 hold. Then on every interval $[0, T]$, $T > 0$ Equation (8.12) has a unique measurable mild solution which is continuous in $E_0$ for almost all $\omega \in \Omega$. $\mathcal{F}_0$–measurability has to be understood in the sense, that the initial condition is independent of the Wiener process.*

*Proof.* At first, we may assume that our operator $C$ is Lipschitz continuous form $E_0$ to $E_0$ by 8.5.6[v*]. We have to apply Theorem 7.4 from Da Prato and Zabczyk, see also Theorem 4.3.3. The proof is based on the fact that $B_\epsilon := B + L_\epsilon$ generates a group, which is proven in Theorem 2.1.7 and that $F$ and $C$ are Lipschitz–continuous. But we only have local Lipschitz–continuity on $C$, if we assume Hypothesis 8.5.6[v]. Thus, we have to apply Theorem 3.6.5 from [10] to get a unique mild solution. In the proof of this theorem a stopping time argument is used, which yields global Lipschitz continuity. The solution cannot explode, since we can use the result of Lemma 8.5.13 below as a priori estimates. Thus, $F$ and $G$ are also linear bounded. □

### 8.5.2 Transformation

We now transform our stochastic partial differential equation into a random differential equation, where the stochastic white noise term dissappears. This equation has random coefficients and the tranformation is stationary, so that the cocycle property is ensured to the transformed equation. We consider the following equation

$$
\begin{aligned}
\frac{d\hat{u}}{dt} &= \hat{v} - \epsilon\hat{u} + c(\hat{u})[z(\theta_t\omega)] \text{ on } D \\
\frac{d\hat{u}_\Gamma}{dt} &= \hat{v}_\Gamma - \epsilon\hat{u}_\Gamma + c_\Gamma(\hat{u})[z(\theta_t\omega)] \text{ on } \Gamma \\
\frac{d\hat{v}}{dt} &= (\epsilon-\alpha)\hat{v} + \epsilon(\alpha-\epsilon)\hat{u} - \mathcal{A}\hat{u} - f(\hat{u}) - (\alpha-\mu-\epsilon)c(\hat{u})[z(\theta_t\omega)] \\
&\quad + c'(\hat{u})[z(\theta_t\omega), \hat{v} - \epsilon\hat{u} + c(\hat{u})[z(\theta_t\omega)]] \text{ on } D \\
\frac{d\hat{v}_\Gamma}{dt} &= (\epsilon-\alpha)\hat{v}_\Gamma + \epsilon(\alpha-\epsilon)\hat{u}_\Gamma - \mathcal{A}_\Gamma\hat{u} - f_\Gamma(\hat{u}_\Gamma) - (\alpha-\mu-\epsilon)c_\Gamma(\hat{u})[z(\theta_t\omega)] \\
&\quad + c'_\Gamma(\hat{u})[z(\theta_t\omega), \hat{v} - \epsilon\hat{u} + c(\hat{u})[z(\theta_t\omega)]] \text{ on } \Gamma, \qquad (8.16)
\end{aligned}
$$

and show that it is equivalent to the origin equation. $z$ is the $U$-valued Ornstein-Uhlenbeck–process introduced in Equation (5.5).

$$dz + \mu z dt = dW, \quad \mu > 0.$$

For simplicity, we use the same notation

$$\psi := (\hat{U}, \hat{V}) = (\hat{u}, \hat{u}_\Gamma, \hat{v}, \hat{v}_\Gamma).$$

We rewrite (8.16) as first order evolution equation as

$$\frac{d\psi}{dt} = (B + L_\epsilon)\psi + H(\theta_t\omega, \psi), \qquad (8.17)$$

with $L_\epsilon$ defined in (8.15) and

$$H(\omega,\psi)=\begin{pmatrix} c(\hat u)[z(\omega)] \\ c_\Gamma(\hat u)[z(\omega)] \\ -f(\hat u)-(\alpha-\mu-\epsilon)c(\hat u)[z(\omega)]-c'(\hat u)[z(\omega),\hat v-\epsilon\hat u+c(\hat u)[z(\omega)]] \\ -f_\Gamma(\hat u_\Gamma)-(\alpha-\mu-\epsilon)c_\Gamma(\hat u)[z(\omega)]-c'_\Gamma(\hat u)[z(\omega),\hat v-\epsilon\hat u+c(\hat u)[z(\omega)]] \end{pmatrix}$$

We abbreviate

$$B_\epsilon = B + L_\epsilon.$$

**Remark 8.5.8**
*As in [33] and [36], we redefine $z$ on $(\Omega_L\cup\Omega_C)^c$ by zero. $\Omega_L\cup\Omega_C$ is a full set, where $\Omega_C$ is the full set of Remark 5.1.19(ii). $\Omega_L$ is the set of all $\omega\in\Omega$, where $w$ grows subexponentially. This is a full $\theta$–invariant set in $\Omega$. Thus, the properties of $z$ hold for all $\omega\in\Omega$.*

At first, we will prove existence and uniqueness of a mild solution of Equation (8.16).

**Theorem 8.5.9 (Existence and Uniqueness)**
*We assume, that the assumptions of Hypothesis 8.5.5 and 8.5.6 hold. Then, for every $\omega\in\Omega$ and $\varphi_0 := (U_0,V_0)\in E_0$ equation (8.16) has a global mild solution $\psi(\cdot,\omega,\varphi_0)$ with values in $E_0$, for all $\omega\in\Omega$, see also Remark 5.1.21 and 8.5.8. Additionally, $\psi_0\mapsto\psi(t,\omega,\psi_0)$ is Lipschitz continuous in $E_0$ and generates an RDS.*

*Proof.* The proof is based on the techniques of Chapter 6 in [41] and uses the fact, that $F$ is Lipschitz continuous and $C$ is local Lipschitz continuous and linear bounded together with the continuity of $z$, which is given by the subexponential growth. These facts give us that the operator $H$ is linear bounded and local Lipschitz continuous in $E_0$. We have, using Hypothesis 8.5.6[v*], that $H$ is Lipschitz-continuous from $E_0$ to $E_0$, if we only assume Hypothesis 8.5.6[v], we have only local Lipschitz continuity. Hypothesis 8.5.6[ix] gives us, that the last two components of $H$ are local-Lipschitz continuous from $\mathbb{V}\to\mathbb{H}$, and thus, $H$ is (local)-Lipschitz continuous from $E_0$ to $E_0$, and we can apply Theorem 1.4 from Chapter 6 in [41]. We achieve the local-Lipschitz continuity by the a priori estimates in in Lemma 8.5.13 and the following estimates using Hypothesis 8.5.6[v]

$$\begin{aligned}\|C(U_1)[z]-C(U_2)[z]\|_{\mathbb{H}} &\le k_1\|C(U_1)[z]-C(U_2)[z]\|_{\mathbb{V}}\\ &\le k_2\|z\|_U(1+\min(\|U_1\|_{\mathbb{V}}+\|U_2\|_{\mathbb{V}}))\|U_1-U_2\|\|U_1\|_{\mathbb{V}}\end{aligned}$$

or assuming Hypothesis 8.5.6[v*] instead of Hypothesis 8.5.6[v]

$$\|C(U_1)[z]-C(U_2)[z]\|_{\mathbb{H}}\le k_1\|C(U_1)[z]-C(U_2)[z]\|_{\mathbb{V}}\le k_2\|z\|_U\|U_1-U_2\|\|U_1\|_{\mathbb{V}}.$$

The last expression in $H$ can be estimated by

$$\begin{aligned}&\|C'(U_1)[z,V_1-\epsilon U_1+C(U_1)[z]]-C'(U_2)[z,V_2-\epsilon U_2+C(U_2)[z]]\|_{\mathbb{H}}\\ \le&\|C'(U_1)[z,U_1']-C'(U_2)[z,U_2']-C'(U_2)[z,U_1']+C'(U_2)[z,U_1']\|_{\mathbb{H}}\\ \le&c_H'\|U_1'\|_{\mathbb{H}}\|U_1-U_2\|_{\mathbb{V}}+\|C'(U_2)[z,U_1']-C(U_2)[z,U_2']\|_{\mathbb{H}}\\ \le&c_H'\|U_1'\|_{\mathbb{H}}\|U_1-U_2\|_{\mathbb{V}}+\|C'(U_2)[z,U_1'-U_2']\|_{\mathbb{H}}\\ \le&c_H'\|U_1'\|_{\mathbb{H}}\|U_1-U_2\|_{\mathbb{V}}+k\|z\|_U\|U_1'-U_2'\|_{\mathbb{H}}\\ \le&c_H'\|U_1'\|\|U_1-U_2\|_{\mathbb{V}}+k\|z\|_U(\|V_1-V_2\|+\epsilon\|U_1-U_2\|_{\mathbb{V}}+k\|z\|_U\|U_1-U_2\|_{\mathbb{V}}),\end{aligned}\tag{8.18}$$

by inserting

$$U_i' = V_i - \epsilon U_i + C(U_i)[z],$$

see also the first two equations in (8.16).
Thanks to the bound of $\psi \in E_0$ derived in the a priori estimates in Lemma 8.5.13, we can find local solutions and extend them into global solutions by iterating [41, Theorem 6.1.4]. □

Solutions in $E_1$ are considered in Theorem 8.5.19.
We can also prove the existence and uniqueness of a weak solution of (8.16).

**Theorem 8.5.10 (Existence and Uniqueness)**
*We assume, that the assumptions of Hypothesis 8.5.5 and 8.5.6 hold. Then, for every $\omega \in \Omega$ and $\varphi_0 := (U_0, V_0) \in E_0$ (8.16) has a global weak solution $\psi(\cdot, \omega, \varphi_0)$ with values in $E_0$, for all $\omega \in \Omega$, see also Remark 5.1.21 and 8.5.8. Additionally, $\psi_0 \mapsto \psi(t, \omega, \psi_0)$ is Lipschitz continuous in $E_0$ and generates an RDS.*

*Proof.* The proof is based on the Galerkin method introduced in [49, Theorem IV.4.1]. The main idea of the proof is to obtain some a priori estimates. We achieve these estimates by replacing $U$ by

$$U_m = \sum_{i=1}^{m} g_{im}(t) E_i$$

where $E_i$ are the eigenfunctions of $A$ in the following Theorem 8.5.12. The weak–star convergence of $\psi_m = (U_m, U_m') \to \psi$ follows by the estimate (8.25). Hence, we can follow the proof in [49, Theorem IV.4.1], use the compactness Theorem 2.2.25 and conclude that

$$U_m \to U \text{ in } L^2(0, T; \mathbb{H}) \text{ strongly.}$$

Due to the properties of $H$, $H(U_m)$ converges to $H(U)$ weakly in $L^2(0, T; \mathbb{V})$: We have to estimate $\|P_m H(U_m) - H(U)\|_{\mathbb{V}}$:

$$\|P_m H(U_m) - H(U)\|_{\mathbb{V}} \leq \|P_m H(U_m) - P_m H(U)\|_{\mathbb{V}} + \|P_m H(U) - H(U)\|_{\mathbb{V}}. \tag{8.19}$$

The convergence of the second expression to zero of the right–hand side of (8.19) is clear and the convergence of the first expression follows by the Local-Lipschitz continuity of $H$, replace $U_1$ by $U_m$ and $U_2$ by $U$ in (8.18), together with the a priori estimates of $U_m$ and Bessel's inequality.
Thus, we can find a solution $U$ of (8.16), such that

$$U \in L^\infty(0, T; \mathbb{V}), \quad U' \in L^\infty(0, T; \mathbb{H}).$$

We obtain that

$$U \in C([0, T]; \mathbb{V}), \quad U' \in C([0, T]; \mathbb{H}).$$

by [49, Theorem II.4.1]. □

Existence and uniqueness of solutions in $E_1$ are proven quite similar, see Temam [49, p.214].

**Remark 8.5.11**
*Mild solutions and weak solutions coincide in our setting.*

*Proof.* We consider the Galerkin approximations of the mild solution. By Remark 8.5.15 we obtain an a priori estimate similar to Lemma 8.5.13. Then, we can follow the proof in Theorem 8.5.10. □

The next theorem transforms the stochastic partial differential equation (8.13) into a random partial differential equation (8.16) and is adapted from [36, Theorem 3.6].

**Theorem 8.5.12 (Transformation)**
*Let $z(\omega)$ be the random variable defined in equation (5.5) with the modification of Remark 8.5.8 and let $T : E_0 \times \Omega \mapsto E_0$ be given by*

$$\psi := (u, u_\Gamma, v, v_\Gamma) \mapsto (u, u_\Gamma, v + c(u)[z(\omega)], v_\Gamma + c_\Gamma(u_)[z(\omega)]) \text{ for } \omega \in \Omega.$$

*Then $T$ is a random homeomorphism with the inverse*

$$T^{-1}(\cdot, \omega) : \varphi := (u, u_\Gamma, v, v_\Gamma) \mapsto (u, u_\Gamma, v - c(u)[z(\omega)], v_\Gamma - c_\Gamma(u)[z(\omega)]) \text{ for } \omega \in \Omega.$$

*Additionally, the mapping*

$$t \mapsto T(\psi(t, \omega, T^{-1}(\varphi_0, \omega)), \theta_t \omega) =: \varphi(t, \omega, \varphi_0) \in E_0$$

*defines a random dynamical system on $\mathbb{R}$. For $\varphi_0 \in E_0$ and $z$ defined by (5.5), the process*

$$(t, \omega) \mapsto \varphi(t, \omega, \varphi_0)$$

*is a $\{\mathcal{F}_t\}_{t \geq 0}$-measurable version of the mild solution of the spde (8.13).*

*Proof.* It is easy to show, that $T(\cdot, \omega)$ is a homeomorphism on $E_0$ with inverse $T^{-1}$. $T(\psi, \cdot)$ and $T^{-1}(\psi, \cdot)$ are measurable.

$$(\varphi, \omega) \mapsto T^{-1}(\psi, \cdot), (\psi, \omega) \mapsto T(\psi, \omega)$$

are measurable by Castaing and Valadier [9, Chapter 3] because of the local Lipschitz continuity of $C$. The main part is now to show that $T$ transforms a solution of (8.16) to a version of the mild solution of (8.13).
We have that $\psi(t)$ is the mild solution of (8.17) with initial condition $\psi(0) = \psi_0 \in E_0$. Furthermore, $G(t)$ is the $C_0$–group generated by the linear operator $B_\epsilon$, see also Theorem 2.1.7. Then, by the definition of a mild solution, we have

$$\psi(t) = G(t)\psi_0 + \int_0^t G(t - \tau) H(\theta_\tau \omega, \psi(\tau))\, d\tau.$$

Additionally, $\varphi(t)$ fulfills

$$
\begin{aligned}
&\varphi(t)\\
= \;& T(\psi(t),\theta_t\omega)) = \begin{pmatrix} \hat{u}(t) \\ \hat{u}_\Gamma(t) \\ \hat{v}(t) + c(\hat{u}(t))[z(\theta_t\omega)] \\ \hat{v}_\Gamma(t) + c_\Gamma(\hat{u}(t))[z(\theta_t\omega)] \end{pmatrix}\\
= \;& G(t)\psi_0 + \int_0^t G(t-\tau)H(\theta_\tau\omega,\psi(\tau))\,d\tau + \begin{pmatrix} 0 \\ 0 \\ c(\hat{u}(t))[z(\theta_t\omega)] \\ c_\Gamma(\hat{u}(t))[z(\theta_t\omega)] \end{pmatrix}\\
= \;& G(t)\begin{pmatrix} u_0 \\ u_{\Gamma 0} \\ v_0 - c(u_0)[z(\omega)] \\ v_{\Gamma 0} - c_\Gamma(u_0)[z(\omega)] \end{pmatrix} + \int_0^t G(t-\tau)H(\theta_\tau\omega,\psi(\tau))\,d\tau + \begin{pmatrix} 0 \\ 0 \\ c(\hat{u}(t))[z(\theta_t\omega)] \\ c_\Gamma(\hat{u}(t))[z(\theta_t\omega)] \end{pmatrix}\\
= \;& G(t)\varphi_0 + \int_0^t G(t-\tau)\begin{pmatrix} 0 \\ 0 \\ -f(u(\tau)) \\ -f_\Gamma(u_\Gamma(\tau)) \end{pmatrix} d\tau +\\
& + G(t)\begin{pmatrix} 0 \\ 0 \\ -c(u_0)[z(\omega)] \\ -c_\Gamma(u_0)[z(\omega)] \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c(\hat{u}(t))[z(\theta_t\omega)] \\ c_\Gamma(\hat{u}(t))[z(\theta_t\omega)] \end{pmatrix}\\
& + \int_0^t G(t-\tau)\begin{pmatrix} c(u(\tau))[z(\theta_\tau\omega)] \\ c_\Gamma(u(\tau))[z(\theta_\tau\omega)] \\ -(\alpha-\mu-\epsilon)c(u(\tau))[z(\theta_\tau\omega)] - c'(u(\tau))[z(\theta_\tau\omega), v-\epsilon u] \\ -(\alpha-\mu-\epsilon)c_\Gamma(u(\tau))[z(\theta_\tau\omega)] - c'_\Gamma(u(\tau))[z(\theta_\tau\omega), v-\epsilon u] \end{pmatrix} d\tau.
\end{aligned}
\tag{8.20}
$$

We have to show that the mild solution $\varphi(t)$ of the spde (8.13) coincides with the $\varphi(t)$ in (8.20). We start with

$$
\begin{aligned}
\varphi(t) \;=\;& G(t)\begin{pmatrix} u_0 \\ u_{\Gamma 0} \\ v_0 \\ v_{\Gamma 0} \end{pmatrix} + \int_0^t G(t-\tau)\begin{pmatrix} 0 \\ 0 \\ -f(u(\tau)) \\ -f_\Gamma(u_\Gamma(\tau)) \end{pmatrix} d\tau\\
& + \int_0^t G(t-\tau)\begin{pmatrix} 0 \\ 0 \\ c(u(\tau))dw \\ c_\Gamma(u(\tau))dw \end{pmatrix} d\tau
\end{aligned}
$$

We define

$$
U^m(t) = \sum_{i=1}^m U_m(t)E_m \text{ and } V^m(t) = \sum_{i=1}^m V_m(t)E_m
$$

with

$$U_m(t) = (U(t), E_m) \text{ and } V_m(t) = (V(t), E_m).$$

$(E_i)_{i\in\mathbb{N}}$ are the eigenfunctions of the operator $A$. By Hypothesis 8.5.6[i], the existence Theorem 8.5.7 and the continuous embedding from $\mathbb{V}$ into $\mathbb{H}$, we have that

$$E\int_0^t \|C(U(s))\|^2_{L(U;\mathbb{H})}\, ds < \infty.$$

This guarantees, by Theorem 6.5 from [19], that the mild of the spde solution is also a weak solution. Therefore, we have

$$\begin{pmatrix} u^m(t) \\ u^m_\Gamma(t) \end{pmatrix} = \begin{pmatrix} u^m_0 \\ u_{\Gamma 0}^m \end{pmatrix} + \int_0^t \begin{pmatrix} v^m(\tau) - \epsilon u^m(\tau) \\ v^m_\Gamma(\tau) - \epsilon u^m_\Gamma(\tau) \end{pmatrix} d\tau.$$

$u(t)$ and $v(t)$ are continuous, and therefore $u^m(t)$ is continuously differentiable in $t$ on $D(A)$ with

$$\frac{du^m}{dt} = v^m(t) - \epsilon u^m(t).$$

Using $u^m(t) \to u(t)$ and again Hypothesis 8.5.6[i], we get for $t > 0$ together with the trace condition Hypothesis 8.5.6[vii]

$$L^2 - \lim_{m\to\infty} \int_0^t G(t-\tau) \begin{pmatrix} 0 \\ 0 \\ c(u^m)dw \\ c_\Gamma(u^m(\tau))dw \end{pmatrix} d\tau = \int_0^t G(t-\tau) \begin{pmatrix} 0 \\ 0 \\ c(u(\tau))dw \\ c_\Gamma(u(\tau))dw \end{pmatrix} d\tau. \quad (8.21)$$

We now rewrite the stochastic integral. Let $0 = \tau^n_0 < \tau^n_1 < \cdots < \tau^n_n = t$ be a sequence of partitions of $[0, t]$, such that the maximal mesh size tends to zero for $n \to \infty$. Equation (5.5) gives us the following relation

$$z(t) - z(s) + \mu \int_s^t z(\tau)\, d\tau = w(t) - w(s),$$

so that we get by multiplying with $C(U^m)$ the following equation (8.22). This is the limit in probability of the left–hand side of (8.21). We have by the definition of the Ito-Integral

$$\begin{aligned}
&\sum_{i=1}^{n-1} G(t-\tau^n_i) \begin{pmatrix} 0 \\ 0 \\ c(u^m(\tau^n_i))[w(\tau^n_{i+1}) - w(\tau^n_i)] \\ c_\Gamma(u^m(\tau^n_i))[w(\tau^n_{i+1}) - w(\tau^n_i)] \end{pmatrix} \qquad (8.22)\\
= &\sum_{i=1}^{n-1} G(t-\tau^n_i) \begin{pmatrix} 0 \\ 0 \\ \mu c(u^m(\tau^n_i))[\int_{\tau^n_i}^{\tau^n_{i+1}} z(\theta_s\omega)\, ds] \\ \mu c_\Gamma(u^m(\tau^n_i))[\int_{\tau^n_i}^{\tau^n_{i+1}} z(\theta_s\omega)\, ds] \end{pmatrix} \\
&+ \sum_{i=1}^{n-1} G(t-\tau^n_i) \begin{pmatrix} 0 \\ 0 \\ c(u^m(\tau^n_i))[z(\tau^n_{i+1}) - z(\tau^n_i)] \\ c_\Gamma(u^m(\tau^n_i))[z(\tau^n_{i+1}) - z(\tau^n_i)] \end{pmatrix}.
\end{aligned}$$

Additionally, we have that

$$\sum_{i=1}^{n-1} G(t-\tau_i^n) \begin{pmatrix} 0 \\ 0 \\ \mu c(u^m(\tau_i^n))[\int_{\tau_i^n}^{\tau_{i+1}^n} z(\theta_s\omega)\,ds] \\ \mu c_\Gamma(u^m(\tau_i^n))[\int_{\tau_i^n}^{\tau_{i+1}^n} z(\theta_s\omega)\,ds] \end{pmatrix}$$

tends in probability to

$$\int_0^t G(t-\tau) \begin{pmatrix} 0 \\ 0 \\ \mu c(u^m(\tau))[z(\theta_s\omega)] \\ \mu c_\Gamma(u^m(\tau))[z(\theta_s\omega)] \end{pmatrix} d\tau. \tag{8.23}$$

By defining

$$h(\tau_i^n) = G(t-\tau_i^n)c(u^m(\tau_i^n))$$

and applying an intermediate value theorem, we achieve

$$\begin{aligned} & -h(0)[z(\omega)] + h(\tau_{n-1}^n)[z(\theta_{\tau_n^n}\omega)] - \sum_{i=1}^{n-1}(h(\tau_i^n)[z(\theta_{\tau_i^n}\omega)] - h(\tau_{i-1}^n)[z(\theta_{\tau_i^n}\omega)]) \\ = \;& -h(0)[z(\omega)] + h(\tau_{n-1}^n)[z(\theta_{\tau_n^n}\omega)] - \sum_{i=1}^{n-1}\int_{\tau_{i-1}^n}^{\tau_i^n} D_\tau h(\tau)[z(\theta_{\tau_i^n}\omega)]\,d\tau \\ \to \;& -h(0)[z(\omega)] + h(t)[z(\theta_t\omega)] - \int_0^t D_\tau h(\tau)[z(\theta_\tau\omega)]\,d\tau \end{aligned}$$

because $\tau \to h(\tau)z$ is continuously differentiable and $\tau \to z(\theta_\tau\omega)$ is continuous. On the boundary we have a similar result. Applying the chain rule, we get

$$\begin{aligned} & \begin{pmatrix} 0 \\ 0 \\ c(u^m(t))[z(\theta_t\omega)] \\ c_\Gamma(u^m(t))[z(\theta_t\omega)] \end{pmatrix} + \int_0^t G(t-\tau)B_\epsilon \begin{pmatrix} 0 \\ 0 \\ c(u^m(\tau))[z(\theta_\tau\omega)] \\ c_\Gamma(u^m(\tau))[z(\theta_\tau\omega)] \end{pmatrix} d\tau \\ & + \int_0^t G(t-\tau) \begin{pmatrix} 0 \\ 0 \\ -c'(u^m(\tau))[z(\theta_\tau\omega), \frac{d}{d\tau}u^m(\tau)] \\ -c_\Gamma'(u^m(\tau))[z(\theta_\tau\omega), \frac{d}{d\tau}u^m(\tau)] \end{pmatrix} d\tau - G(t) \begin{pmatrix} 0 \\ 0 \\ c(u_0^m)[z(\omega)] \\ c_\Gamma(u_0^m)[z(\omega)] \end{pmatrix} d\tau \end{aligned}$$

which is equal to

$$\begin{pmatrix} 0 \\ 0 \\ c(u^m(t))[z(\theta_t\omega)] \\ c_\Gamma(u^m(t))[z(\theta_t\omega)] \end{pmatrix} - G(t) \begin{pmatrix} 0 \\ 0 \\ c(u_0^m)[z(\omega)] \\ c_\Gamma(u_0^m)[z(\omega)] \end{pmatrix} d\tau$$
$$+ \int_0^t G(t-\tau) \begin{pmatrix} c(u^m(\tau))[z(\theta_\tau\omega)] \\ c_\Gamma(u^m(\tau))[z(\theta_\tau\omega)] \\ -(\alpha-\epsilon)c(u^m(\tau))[z(\theta_\tau\omega)] \\ -(\alpha-\epsilon)c_\Gamma(u^m(\tau))[z(\theta_\tau\omega)] \end{pmatrix} d\tau$$
$$+ \int_0^t G(t-\tau) \begin{pmatrix} 0 \\ 0 \\ -c'(u^m(\tau))[z(\theta_\tau\omega), \frac{d}{d\tau}u^m(\tau)] \\ -c'_\Gamma(u^m(\tau))[z(\theta_\tau\omega), \frac{d}{d\tau}u^m(\tau)] \end{pmatrix} d\tau,$$

by using the explicit form of $B_\epsilon$. Letting $m \to \infty$ delivers us the equivalence of (8.13) and (8.16), in the sense that the the transformed mild solution of (8.16) is a version of the mild solution of (8.13). The terms containing $F$ are considered analogeously. □

### 8.5.3 Absorbing set

In this section, the existence of a random attractor is shown. Due to [18] and [15] the proof is divided in two parts. At first, we show the existence of a random absorbing set, then the attraction property of this set by a compactness argument. Then, we can apply Theorem 5.1.9. The main tools in this section are some kind of energy estimates, but because of the DyBC we cannot use the standard methods of integration by parts. In this section we set

$$\begin{aligned} ((u,v)) &:= (\nabla u, \nabla v) \\ \mathcal{A} &:= -\Delta \\ \mathcal{A}_\Gamma &:= \partial_\nu + \mathrm{Id} \\ (u,v)_D &:= (u,v) \end{aligned}$$

and write $u$ instead of $\hat{u}$ and $v$ instead of $\hat{v}$ respectively. Note that $((\cdot,\cdot))$ is not an inner product on $H^1(D)$. The proof of a random absorbing set is divided into several lemmas, which use the techniques of weak solutions. After each lemma, there is also a remark, how to achieve the precedent result by techniques of mild solutions.

**Lemma 8.5.13 (Random absorbing set)**
*The RDS (8.16) possesses a random absorbing set.*

*Proof.* We estimate

$$\|\psi\|_{E_0}^2 = \|v\|^2 + \|\nabla u\|^2 + \|v\|_\Gamma^2 + \|u\|_\Gamma^2$$

by the chain rule. The scalar products are well-defined, since we can apply the abstract result from [49, Lemma II.4.1], and the arguments of [49, Chapter $IV$], so that

$$\begin{aligned} (\frac{dv}{dt}, v) =& (\epsilon-\alpha)(v,v) + \epsilon(\alpha-\epsilon)(u,v) - (\mathcal{A}u,v) - (\alpha-\mu-\epsilon)(c(u)[z(\theta_t\omega)],v) \\ &- (f(u),v) - (c'(u)[z(\theta_t\omega), v-\epsilon u + c(u)[z(\theta_t\omega)]],v). \end{aligned}$$

This leads to

$$\begin{aligned}\frac{1}{2}\frac{d\|v\|^2}{dt} =&(\epsilon-\alpha)\|v\|^2+\epsilon(\alpha-\epsilon)(u,v)-(\mathcal{A}u,v)-(\alpha-\mu-\epsilon)(c(u)[z(\theta_t\omega)],v)\\ &-(f(u),v)-(c'(u)[z(\theta_t\omega),v-\epsilon u+c(u)[z(\theta_t\omega)]],v)\end{aligned}$$

with

$$\begin{aligned}-(\mathcal{A}u,v)&=-(\mathcal{A}u,u'+\epsilon u-c(u)[z(\theta_t\omega)])\\ &=-(\nabla u,\nabla u')+(\partial_\nu u,u'_\Gamma)_\Gamma-\epsilon(\nabla u,\nabla u)+\epsilon(\partial_\nu u,u_\Gamma)_\Gamma\\ &\quad+(\nabla u,\nabla c(u)[z(\theta_t\omega)])-(\partial_\nu u,c_\Gamma(u)[z(\theta_t\omega)])_\Gamma\\ &\le-\frac{1}{2}\frac{d\|\nabla u\|^2}{dt}-\epsilon\|\nabla u\|^2+(\nabla u,\nabla c(u)[z(\theta_t\omega)])+h_\Gamma(u,\theta_t\omega),\end{aligned}$$

where $h_\Gamma$ is defined by

$$h_\Gamma(u,\omega)=(\partial_\nu u,u'_\Gamma)_\Gamma+\epsilon(\partial_\nu u,u_\Gamma)_\Gamma-(\partial_\nu u,c_\Gamma(u)[z(\omega)])_\Gamma.$$

As in [34] we have by Hypothesis 8.5.6[v] the estimate

$$\begin{aligned}(\nabla u,\nabla c(u)[z(\theta_t\omega)])&\le c_V\|z(\theta_t\omega)\|_U\|\nabla u\|^2+\|c(0)[z(\theta_t\omega)]\|\|\nabla u\|\\ &\le c_0\|z(\theta_t\omega)\|_U+c_0\|z(\theta_t\omega)\|_U\|\nabla u\|^2\end{aligned}$$

and similarly

$$\begin{aligned}2(\mu+\epsilon-\alpha)(c(u)[z(\theta_t\omega)],v)&\le 2\|\mu+\epsilon-\alpha\|c_H\|z(\theta_t\omega)\|_U\|v\|\\ &\le c_H^2\|\mu+\epsilon-\alpha\|^2\|z(\theta_t\omega)\|_U+k\|z(\theta_t\omega)\|_U\|v\|^2\\ 2(c'(u)[z(\theta_t\omega),v-\epsilon u+c(u)[z(\theta_t\omega)]],v)&\le 2\|c'(u)[z(\theta_t\omega),v-\epsilon u+c(u)[z(\theta_t\omega)]]\|\|v\|\\ &\le c'_H\|z(\theta_t\omega)\|_U(\|v\|+\|u\|+\|z(\theta_t\omega)\|_U)\|v\|\\ &\le c_2\|z(\theta_t\omega)\|_U(\|v\|+\|\nabla u\|+\|u\|_\Gamma+\|z(\theta_t\omega)\|_U)\|v\|\\ &\le c_3\|z(\theta_t\omega)\|_U(\|v\|+\|\nabla u\|+\|z(\theta_t\omega)\|_U)\|v\|\\ &\quad+c_4\|z(\theta_t\omega)\|_U\|u\|_\Gamma\|v\|\\ &\le c_5\|z(\theta_t\omega)\|_U\|\psi\|_{E_0}^2+c_6\|z(\theta_t\omega)\|_U^3,\end{aligned}$$

where $c_5$ and $c_6$ depend on $c'_H$ defined in Hypothesis 8.5.6[viii]. We analogically estimate on the boundary

$$\begin{aligned}\frac{1}{2}\frac{d\|v_\Gamma\|_\Gamma^2}{dt}\;=\;&(\epsilon-\alpha)\|v_\Gamma\|_\Gamma^2+\epsilon(\alpha-\epsilon)(u_\Gamma,v_\Gamma)_\Gamma-(u_\Gamma+\partial_\nu u,v_\Gamma)_\Gamma\\ &-((\alpha-\mu-\epsilon)c_\Gamma(u)[z(\theta_t\omega)],v_\Gamma)_\Gamma-(f_\Gamma(u_\Gamma),v_\Gamma)_\Gamma\\ &-(c'_\Gamma(u)[z(\theta_t\omega),v-\epsilon u+c(u)[z(\theta_t\omega)]],v_\Gamma)_\Gamma\end{aligned}$$

We have that

$$-(u_\Gamma+\partial_\nu u,v_\Gamma)_\Gamma=-\frac{1}{2}\frac{d\|u_\Gamma\|_\Gamma^2}{dt}-\epsilon\|u_\Gamma\|^2+(u_\Gamma,c_\Gamma(u)[z(\theta_t\omega)])_\Gamma-h_\Gamma(u,\theta_t\omega)$$

by inserting

$$\frac{du_\Gamma}{dt} = v_\Gamma - \epsilon u_\Gamma + c_\Gamma(u)[z(\theta_t\omega)].$$

By Hypothesis 8.5.6[iv], we have for any $\epsilon_2 > 0$ and a $k_{\epsilon_2} > 0$

$$(u, c_\Gamma(u)[z(\theta_t\omega)])_\Gamma \le \epsilon_2 \|u\|_\Gamma^2 + k_{\epsilon_2}\|z(\theta_t\omega)\|_U^2.$$

We obtain on the boundary as well as in the inner domain by Hypothesis 8.5.6[iv] and 8.5.6[vii]

$$\begin{aligned} 2(\mu+\epsilon-\alpha)(c_\Gamma(u)[z(\theta_t\omega)], v)_\Gamma &\le 2|\mu+\epsilon-\alpha| c_H \|z(\theta_t\omega)\|_U \|v_\Gamma\|_\Gamma \\ &\le c_H^2|\mu+\epsilon-\alpha|^2 \|z(\theta_t\omega)\|_U + \|z(\theta_t\omega)\|_U\|v\|_\Gamma^2 \\ 2(c'_\Gamma(u)[z(\theta_t\omega), v-\epsilon u + c(u)[z(\theta_t\omega)]], v)_\Gamma &\le k\|z(\theta_t\omega)\|_U\|\psi\|_{E_0}^2 + c\|z(\theta_t\omega)\|_U^3. \end{aligned}$$

Again, we could estimate the nonlinearity by Hypothesis 8.5.5[i], that for every $\epsilon > 0$ there is a $K_\epsilon$ such that

$$\|(f_\Gamma(u_\Gamma), v_\Gamma)_\Gamma\| \le K_\epsilon + \frac{\epsilon}{4}\|v\|_{E_0}^2.$$

Lemma 8.4.2 delivers

$$\epsilon|\!|\!|u|\!|\!|^2 + (\alpha-\epsilon)\|v\|^2 - \epsilon(\alpha-\epsilon)(u,v) + \epsilon\|u_\Gamma\|_\Gamma^2 + (\alpha-\epsilon)\|v_\Gamma\|_\Gamma^2 - \epsilon(\alpha-\epsilon)(u_\Gamma, v_\Gamma)_\Gamma \ge \alpha_1\|\psi\|^2.$$

This leads to the following estimate with some $0 < \alpha_2 < \alpha_1$ and some constant $c_1 > 0$

$$\frac{d}{dt}\|\psi\|_{E_0}^2 + \alpha_2\|\psi\|_{E_0}^2 \le c_1\|z(\theta_t\omega)\|_U\|\psi\|_{E_0}^2 + c_2(\|z(\theta_t\omega)\|_U^3 + 1). \tag{8.24}$$

Integration from 0 to $t$ gives us

$$\|\psi(t)\|_{E_0}^2 \le \|\psi_0\|_{E_0}^2 + \int_0^t (c_1\|z(\theta_\tau\omega)\|_U - \alpha_2)\|\psi(\tau)\|_{E_0}^2\,d\tau + \int_0^\tau c_2(\|z(\theta_\tau\omega)\|_U^3 + 1)\,d\tau.$$

The inequality has the form

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)\,d\tau$$

with

$$v(t) = \|\psi(t)\|_{E_0}^2, \quad h(\tau) = c_1\|z(\theta_\tau\omega)\|_U - \alpha_2, \quad g(t) = \|\psi_0\|_{E_0} + \int_0^t c_2(\|z(\theta_\tau\omega)\|_U^3 + 1)\,d\tau.$$

Now we can apply the Gronwall Lemma [54, Lemma 29.2] and conclude

$$\begin{aligned} \|\psi(t,\omega,\psi_0)\|_{E_0}^2 \le & e^{\int_0^t (c_1\|z(\theta_\tau\omega)\|_U - \alpha_2)\,d\tau}\|\psi_0\|_{E_0}^2 \\ & + e^{\int_0^t (c_1\|z(\theta_\tau\omega)\|_U - \alpha_2)\,d\tau}\int_0^t c_2(\|z(\theta_\tau\omega)\|_U^3 + 1)e^{-\int_0^\tau (c_1\|z(\theta_s\omega)\|_U - \alpha_2)\,ds}\,d\tau. \end{aligned}$$

We replace $\omega$ by $\theta_{-t}\omega$ and achieve for

$$\|\psi\|_{E_0} := \|\psi(t,\theta_{-t}\omega,\psi_0)\|_{E_0}$$

the following inequality

$$\begin{aligned}\|\psi(t,\theta_{-t}\omega,\psi_0)\|_{E_0}^2 \leq & \|\psi_0\|_{E_0}^2 e^{-\alpha_2 t + c_1\int_{-t}^0 \|z(\theta_\tau\omega)\|_U\,d\tau} \\ & + \int_{-t}^0 c_2 e^{\alpha_2 s + c_1 \int_s^0 \|z(\theta_\tau\omega)\|_U\,d\tau}(\|z(\theta_s\omega)\|_U^3 + 1)\,ds.\end{aligned}$$

Therefore, we obtain that the closed ball $B(\omega)$ in $E_0$ with center zero and square radius

$$\rho_0(\omega)^2 := 2\int_{-\infty}^0 c_2 e^{\alpha_2 s + c_1\int_s^0 \|z(\theta_\tau\omega)\|_U\,d\tau}(\|z(\theta_s\omega)\|_U^3 + 1)\,ds \tag{8.25}$$

is an absorbing set for the random dynamical system $\psi$, if we choose $\mathrm{Tr}_U\, Q$ small enough, such that

$$\mathbb{E}c_1\|z\|_U - \alpha_2 < 0.$$

□

On the properties of the random ball $B(\omega)$, we can state as in [33] the following remarks.

**Remark 8.5.14**
*$\rho_0$ fullfills the assumptions of Remark 5.1.5 and thus the radius $\rho_0^2$ defined in (8.25) is tempered.*

**Remark 8.5.15**
*We can also prove the existence of a random absorbing set of $\psi$ in $E_0$ by semigroup methods considering the mild solution of (8.16). We use Hypothesis 8.5.6[v*] in these calculations.*

*Proof.* The mild solution

$$\psi(t) = G(t)\psi_0 + \int_0^t G(t-\tau)H(\theta_\tau\omega,\psi(\tau)))\,d\tau$$

provides us, that

$$\|\psi(t)\|_{E_0} \leq \|G(t)\psi_0\|_{E_0} + \int_0^t \|G(t-\tau)H(\theta_\tau\omega,\psi(\tau))\|_{E_0}\,d\tau.$$

The exponential decay of the semigroup, see Lemma 8.4.1 or explicitly Lemma 8.4.2 leads to the estimate

$$\|\psi(t)\|_{E_0} \leq e^{-\lambda t}\|\psi_0\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)}\|H(\theta_\tau\omega,\psi(\tau)\|_{E_0}\,d\tau$$

for some $\lambda > 0$ given by Lemma 8.4.1.
We estimate

$$\begin{aligned}\|H(\omega,\psi)\|_{E_0} \leq & k_1\|z(\omega)\|_U + c_V\|z(\omega)\|_U\|U\| + C_{G,H} + C_H\|z(\omega)\|_U \\ & + k_2\|z(\omega)\|_U\|V\| + k_3\|z(\omega)\|_U\|U\| + k_4\|z(\omega)\|_U^2.\end{aligned}$$

We set

$$\Theta(\omega) = k_1\|z(\omega)\|_U + C_{G,H} + C_H\|z(\omega)\|_U + k_4\|z(\omega)\|_U^2.$$

Note that $\Theta$ is a tempered random variable. We obtain

$$\|\psi(t)\|_{E_0} \le e^{-\lambda t}\|\psi_0\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)}(k_2+k_3+c_V)\|z(\theta_\tau\omega)\|_U\|\psi(\tau)\|_{E_0} + e^{-\lambda(t-\tau)}\Theta(\theta_\tau\omega)\,d\tau$$

or

$$\|\psi(t)\|_{E_0}e^{\lambda t} \le \|\psi_0\|_{E_0} + \int_0^t e^{\lambda\tau}(k_2+k_3+c_V)\|z(\theta_\tau\omega)\|_U\|\psi(\tau)\|_{E_0} + e^{\lambda\tau}\Theta(\theta_\tau\omega)\,d\tau.$$

The inequality has the form

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)\,d\tau$$

with

$$v(t) = \|\psi(t)\|_{E_0}e^{\lambda t}, \quad h(\tau) = (k_2+k_3+C_V)\|z(\theta_\tau\omega)\|_U, \quad g(t) = \|\psi_0\|_{E_0} + \int_0^t e^{\lambda\tau}\Theta(\theta_\tau\omega)\,d\tau.$$

and set $k_5 = k_2 + k_3 + C_V$. We obtain

$$v(t) \le e^{k_5\int_0^t \|z(\theta_\tau\omega)\|_U\,d\tau}(g(0) + \int_0^t e^{\lambda\tau}\Theta(\theta_\tau\omega)e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\,ds}\,d\tau).$$

Thus, we achieve

$$\|\psi(t)\|_{E_0} \le e^{k_5\int_0^t \|z(\theta_\tau\omega)\|_U\,d\tau - \lambda t}(g(0) + \int_0^t e^{\lambda\tau}\Theta(\theta_\tau\omega)e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\,ds}\,d\tau).$$

and finally

$$\|\psi(t)\|_{E_0} \le e^{k_5\int_0^t \|z(\theta_\tau\omega)\|_U\,d\tau - \lambda t}\|\psi_0\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)+\int_\tau^t \|z(\theta_s\omega)\|_U\,ds}\Theta(\theta_\tau\omega)\,d\tau. \qquad (8.26)$$

Replacing $\omega$ by $\theta_{-t}\omega$, we get the following inequality

$$\|\psi(t,\theta_{-t}\omega,\psi_o)\|_{E_0} \le \|\psi_0\|_{E_0}e^{k_5\int_{-t}^0 \|z(\theta_\tau\omega)\|_U\,d\tau - \lambda t} + \int_{-t}^0 e^{k_5\int_\tau^0 \|z(\theta_s\omega)\|_U\,ds + \lambda\tau}\Theta(\theta_\tau\omega)\,d\tau.$$

This delivers us again that the closed ball $\hat{B}(\omega)$ in $E_0$ with center zero and radius

$$\rho_0^m(\omega) := 2\int_{-\infty}^0 e^{k_5\int_\tau^0 \|z(\theta_s\omega)\|_U\,ds + \lambda\tau}\Theta(\theta_\tau\omega)\,d\tau \qquad (8.27)$$

is an absorbing set for the random dynamical system $\psi$, if we choose $\mathrm{Tr}_U\, Q$ small enough, such that

$$\mathbb{E}k_5\|z\|_U - \lambda < 0.$$

The right hand side of (8.27) defines a tempered random variable and we can conclude the existence of a random absorbing set.

□

### 8.5.4 Splitting the solution operator

Now, we want to construct a compact random attracting set for the RDS $\psi$. In the deterministic theory, the solution is split up into two parts. In our setting, we need a third part because of the $C$-term arising from the spde. This term has to be sufficiently regular in the equation of $\psi_c$, and thus we get by the residual the third equation. We again apply the methods from Keller [33]. We use the following splitting method

$$\psi = \psi_l + \psi_r = \psi_l + \psi_c + (\psi_r - \psi_c),$$

where $\psi_l$ denotes the cocycle generated by the linear homogeneous problem and solves

$$\begin{aligned}
\frac{du_l}{dt} &= v_l - \epsilon u_l \text{ on } D \\
\frac{du_{\Gamma l}}{dt} &= v_{\Gamma l} - \epsilon u_{\Gamma l} \text{ on } \Gamma \\
\frac{dv_l}{dt} &= \epsilon(\alpha - \epsilon)u_l - \mathcal{A}u_l - (\alpha - \epsilon)v_l \text{ on } D \\
\frac{dv_{\Gamma l}}{dt} &= \epsilon(\alpha - \epsilon)u_{\Gamma l} - \mathcal{A}_\Gamma u_l - (\alpha - \epsilon)v_{\Gamma l} \text{ on } \Gamma
\end{aligned}$$

with initial condition $\psi_l(0) = \psi_0 \in E_0$.
Then, we define the compact part analogeously as in [49]

$$\begin{aligned}
\frac{du_c}{dt} &= v_c - \epsilon u_c + c(u_c)[z] \text{ on } D && (8.28)\\
\frac{dv_c}{dt} &= \epsilon(\alpha - \epsilon)u_c - \mathcal{A}u_c - (\alpha - \epsilon)v_c + f(u) \\
&\quad +(\mu + \epsilon - \alpha)c(u)[z] + c'(u)[z, v_c - \epsilon u + c(u)[z]] \text{ on } D \\
\frac{du_{\Gamma c}}{dt} &= v_{\Gamma c} - \epsilon u_c + c_\Gamma(u_c)[z] \text{ on } \Gamma \\
\frac{dv_{\Gamma c}}{dt} &= \epsilon(\alpha - \epsilon)u_{\Gamma c} - \mathcal{A}_\Gamma u_c - (\alpha - \epsilon)v_{\Gamma c} + f_\Gamma(u) \\
&\quad +(\mu + \epsilon - \alpha)c_\Gamma(u)[z] + c'_\Gamma(u)[z, v_c - \epsilon u + c(u)[z]] \text{ on } \Gamma
\end{aligned}$$

with initial condition $\psi_c(0) = (0, 0, 0, 0)$. Note that by the estimates in Lemma 8.5.21 $(U_c, V_c) \in E_1 = D(A) \times \mathbb{V}$. We set

$$\psi_d := \psi_r - \psi_c$$

and obtain

$$\begin{aligned}
\frac{du_d}{dt} &= v_d - \epsilon u_d + c(u)[z] - c(u_c)[z] \text{ on } D \\
\frac{dv_d}{dt} &= \epsilon(\alpha - \epsilon)u_d - \mathcal{A}u_d - (\alpha - \epsilon)v_d && (8.29)\\
&\quad +c'(u)[z, v - v_c] \text{ on } D \\
\frac{du_{\Gamma d}}{dt} &= v_{\Gamma d} - \epsilon u_{\Gamma d} + c_\Gamma(u)[z] - c_\Gamma(u_c)[z] \text{ on } \Gamma \\
\frac{dv_{\Gamma d}}{dt} &= \epsilon(\alpha - \epsilon)u_{\Gamma d} - \mathcal{A}_\Gamma u_d - (\alpha - \epsilon)v_{\Gamma d} && (8.30)\\
&\quad +c'_\Gamma(u)[z, v - v_c] \text{ on } \Gamma
\end{aligned}$$

with initial condition $\psi_d(0) = (0, 0, 0, 0)$.
We can state the same Lemma as in [33, Lemma 4.8].

**Lemma 8.5.16 (Estimate of $\psi_l$)**
*We have for any $D$ in $\mathcal{D}$*

$$\sup_{\psi_0 \in D(\theta_{-t}\omega)} \|\psi_l(s, \theta_{-t}\omega, \psi_0)\|_{E_0} \le d(\theta_{-t}\omega)e^{-\gamma s} \textit{ for } t, s \ge 0 \textit{ and } \gamma < \lambda,$$

*where $\lambda$ is defined in Lemma 8.4.1. $d(\omega)$ depends on the intial condition in the following way:*

$$d(\omega) := \sup_{x \in D(\omega)} \|x\|_{E_0}.$$

*This is the classical deterministic theory developed for equations with dynamical boundary conditions at the beginning of this chapter.*

**Lemma 8.5.17 (Estimate of $\psi_c$ in $E_0$)**
*The compact part $\psi_c$ has the absorption property in $E_0$.*

*Proof.* The proof is the same as in Lemma 8.5.13, except the estimate of

$$2(c'(u)[z(\omega), v_c - \epsilon u + c(u)[z(\omega)]], v_c).$$

By

$$\begin{aligned} k_0\|z(\omega)\|_U\|u\|\|v_c\| &\le k_1\|z(\omega)\|_U\|v_c\|^2 + k_2\|z(\omega)\|_U\|u\|^2 \\ &\le k_1\|z(\omega)\|_U\|v_c\|^2 + k_3\|z(\omega)\|_U^2 + k_4\|u\|^4 \end{aligned}$$

and similarly on the boundary we conclude similarly as in Lemma 8.5.13 that

$$\frac{d}{dt}\|\psi_c\|^2 + \alpha_2\|\psi_c\|^2 \le \hat{c}_1\|z(\theta_t\omega)\|_U\|\psi_c\|^2 + c_2(\|z\|_U^3 + 1) + c_3\|\psi\|^4.$$

We may assume that $2c_1 \le \hat{c}_1$, see Inequality (8.24), and obtain

$$\begin{aligned} \|\psi(s, \omega, \psi_0)\|^4 &\le (\|\psi(s, \omega, \psi_0)\|^2)^2 \\ &\le 2(\|\psi_0\|^2 e^{-\alpha_2 s + c_1 \int_{-s}^0 \|z(\theta_{\tau+s}\omega)\|_U \, d\tau})^2 \\ &\quad + 2(2\int_{-\infty}^0 c_2 e^{-\alpha_2 r + c_1 \int_r^0 \|z(\theta_{\tau+s}\omega)\|_U \, d\tau}(\|z(\theta_{r+s}\omega)\|_U^3 + 1)\, dr)^2 \\ &\le \|\psi_0\|^4 2(e^{-\alpha_2 s + c_1 \int_{-s}^0 \|z(\theta_{\tau+s}\omega)\|_U \, d\tau})^2 + 2(\rho_0(\theta_s))^4. \end{aligned}$$

Thus, we have since $\psi_0 \in B(\theta_{-t}\omega)$ and

$$\|\psi(s, \theta_{-t}\omega, \psi_0)\|^4 \le \rho_0(\theta_{-t}\omega)^4 2(e^{-\alpha_2 s + c_1 \int_{-s}^0 \|z(\theta_{\tau+s}\omega)\|_U \, d\tau})^2 + 2(\rho_0(\theta_{s-t}))^4.$$

We follow the proof in [33] and get that

$$\begin{aligned}
&\sup_{\psi_0\in B(\theta_{-t}\omega)} \|\psi_c(t,\theta_{-t}\omega,\psi_0)\|_{E_0}^2\\
&\le \int_0^t c_2 e^{\int_s^t -\alpha_2+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau}(\|z(\theta_{s-t}\omega)\|_U^3+1)\,ds\\
&+\int_0^t c_2 e^{\int_s^t -\alpha_2+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau} 2\rho_0(\theta_{-t}\omega)^4 e^{2\int_0^s -\alpha_2+c_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau}\,ds\\
&+\int_0^t c_2 e^{\int_s^t -\alpha_2+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau} 2\rho_0(\theta_{s-t}\omega)^4\,ds.
\end{aligned}$$

The main difficulty is to estimate the second integral. We estimate

$$\begin{aligned}
&\int_0^t c_2 e^{\int_s^t -\alpha_2+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau} 2\rho_0(\theta_{-t}\omega)^4 e^{2\int_0^s -\alpha_2+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau}\,ds\\
&\qquad\le 2c_2 t\rho_0(\theta_{-t}\omega)^4 e^{\int_0^t -\alpha_2+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau}.
\end{aligned}\tag{8.31}$$

There exists a $c_{\alpha_2}>0$, such that

$$2c_2 t\le c_\alpha e^{\frac{\alpha_2}{2}t} \text{ for } t\ge 0.$$

Therefore, the last expression in (8.31) can be estimated by

$$c_{\alpha_2}(\rho_0(\theta_{-t}\omega)^2 e^{\int_0^t -\frac{\alpha_2}{4}+\hat{c}_1\|z(\theta_{\tau-t}\omega)\|_U\,d\tau})^2$$

This is a tempered random variable for $\mu$ large enough, see Lemma 5.1.24.
We obtain with the same arguments as in [33] the following radius of the absorbing set:

$$\begin{aligned}
\rho_0^c(\omega) &:= \int_{-\infty}^0 c_2 e^{\int_s^0 -\alpha_2+c_1\|z(\theta_\tau\omega)\|\,d\tau}(2\rho_0(\theta_s\omega)^4+\|z(\theta_s\omega)\|_U^3+1)\,ds\\
&\quad +c_{\alpha_2}\Big(\int_{-\infty}^0 c_2 e^{\int_s^0 -\frac{\alpha_2}{4}+c_1\|z(\theta_r\omega)\|\,dr}(\|z(\theta_s\omega)\|_U^3+1)\,ds\Big)^2
\end{aligned}$$

□

We can also consider (8.28) in the context of mild solutions. Therefore, we rewrite (8.28) as first order evolution equation as

$$\frac{d\psi_c}{dt}=B_\epsilon\psi_c+H_c(\theta_t\omega,\psi_c),\tag{8.32}$$

with $B_\epsilon$ defined in (8.15) and

$$H_c(\omega,\psi_c)=\begin{pmatrix} c(u_c)[z(\omega)]\\ c_\Gamma(u_c)[z(\omega)]\\ -f(u)-(\alpha-\mu-\epsilon)c(u)[z(\omega)]-c'(u)[z(\omega),v_c-\epsilon u+c(u)[z(\omega)]]\\ -f_\Gamma(u_\Gamma)-(\alpha-\mu-\epsilon)c_\Gamma(u)[z(\omega)]-c'_\Gamma(u)[z(\omega),v_c-\epsilon u+c(u)[z(\omega)]]\end{pmatrix}$$

**Remark 8.5.18**
*We can also prove the existence of a random absorbing set of $\psi_c$ in $E_0$ by semigroup methods considering the mild solution of (8.28). We use Hypothesis 8.5.6[v*] in these calculations.*

*Proof.* By the mild solution

$$\psi_c(t) = G(t)\psi_{c0} + \int_0^t G(t-\tau)H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\,d\tau.$$

we conclude, that

$$\|\psi_c(t)\|_{E_0} \le \|G(t)\psi_{c0}\|_{E_0} + \int_0^t \|G(t-\tau)H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\|_{E_0}\,d\tau.$$

The exponential decay of the semigroup, see Lemma 8.5.20 leads us to the estimate

$$\|\psi_c(t)\|_{E_0} \le e^{-\lambda t}\|\psi_{c0}\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)}\|H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\|_{E_0}\,d\tau$$

for some $\lambda > 0$ defined in Lemma 8.4.1.
We estimate

$$\begin{aligned}&\|H_c(\omega, \psi_c(\tau), \psi(\tau))\|_{E_0}\\ \le&k_1\|z(\omega)\|_U + c_V|\!|\!|U_c|\!|\!| + C_{G,H} + c_H'\|z(\omega)\|_U(\|V_c\| + \|U\| + \|z(\omega)\|_U)\\ \le&k_1\|z(\omega)\|_U + k_2\|z(\omega)\|_U^2 + k_3\|z(\omega)\|_U\rho_0^m(\omega) + C_{G,H} + k_4\|\psi_c(\tau)\|_{E_0}.\end{aligned}$$

We set

$$\Theta_c^0(\omega) = k_1\|z(\omega)\|_U + k_2\|z(\omega)\|_U^2 + k_3\|z(\omega)\|_U\rho_0^m(\omega) + C_{G,H}$$

Note that $\Theta_c^0$ is a tempered random variable. We obtain

$$\|\psi_c(t)\|_{E_0} \le e^{-\lambda t}\|\psi_{c0}\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)}k_4\|z(\theta_\tau\omega)\|_U\|\psi_c(\tau)\|_{E_0} + e^{-\lambda(t-\tau)}\Theta_c^0(\theta_\tau\omega)\,d\tau$$

or

$$\|\psi_c(t)\|_{E_0}e^{\lambda t} \le \|\psi_{c0}\|_{E_0} + \int_0^t e^{\lambda\tau}k_4\|z(\theta_\tau\omega)\|_U\|\psi_c(\tau)\|_{E_0} + e^{\lambda\tau}\Theta_c^0(\theta_\tau\omega)\,d\tau.$$

The inequality has the form

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)\,d\tau.$$

with

$$v(t) = \|\psi_c(t)\|_{E_1}e^{\lambda t}, \quad h(\tau) = k_4\|z(\theta_\tau\omega)\|_U, \quad g(t) = \|\psi_{c0}\|_{E_1} + \int_0^t e^{\lambda\tau}\Theta_c(\theta_\tau\omega, U)\,d\tau.$$

Gronwall's Lemma yields us

$$v(t) \le e^{k_4\int_0^t\|z(\theta_\tau\omega)\|_U\,d\tau}(g(0) + \int_0^t e^{\lambda\tau}\Theta_c^0(\theta_\tau\omega)e^{-\int_0^\tau\|z(\theta_s\omega)\|_U\,ds}\,d\tau).$$

Thus, we achieve

$$\|\psi_c(t)\|_{E_0} \leq e^{k_4 \int_0^t \|z(\theta_\tau\omega)\|_U\, d\tau - \lambda t}(g(0) + \int_0^t e^{\lambda\tau}\Theta_c^0(\theta_\tau\omega)e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\, ds}\, d\tau).$$

and finally

$$\|\psi_c(t)\|_{E_0} \leq e^{k_4 \int_0^t \|z(\theta_\tau\omega)\|_U\, d\tau - \lambda t}\|\psi_{c0}\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)+\int_\tau^t \|z(\theta_s\omega)\|_U\, ds}\Theta_c^0(\theta_\tau\omega)\, d\tau. \quad (8.33)$$

Now, we can follow the arguments in Remark 8.5.15 and conclude that there exists closed ball $\hat{B}_c(\omega)$ in $E_0$ with center zero and radius

$$\rho_{c,0}^m(\omega) := 2\int_{-\infty}^0 e^{k_2 \int_\tau^0 \|z(\theta_s\omega)\|_U\, ds + \lambda\tau}\Theta_c(\theta_\tau\omega)\, d\tau \quad (8.34)$$

which is an absorbing set, if we choose $\mathrm{Tr}_U\, Q$ small enough, such that

$$\mathbb{E}k_4\|z\|_U - \lambda < 0.$$

The right hand side of Inequality (8.34) defines a tempered random variable.

□

To obtain estimates in $E_1$, we have to show existence und uniqueness in $E_1$ at first. Again, we can directly prove the existence and uniqueness of a mild solution of Equation (8.32).

**Theorem 8.5.19 (Existence and Uniqueness of in $E_1$)**
*For every $\omega \in \Omega$ and $\varphi_{c0} := (U_{c0}, V_{c0}) \in E_1$ equation (8.32) has a global mild solution $\psi_c(\cdot, \omega, \varphi_{c0})$ with values in $E_1$, for all $\omega \in \Omega$, see also Remark 5.1.21 and8.5.8.*

*Proof.* The proof uses the techniques of Chapter 6 in [41] and uses the fact, that $F(U)$ is bounded in $\mathbb{V}$ and $C(U)$ is bounded in $\mathbb{V}$. Furthermore, $C(U_c)$ is bounded in $D(A)$ by Hypothesis 8.5.6[vi]. The linearity of $C'$ in $V_c$ gives us, that the last two components of $H_c$ are Lipschitz continuous from $\mathbb{V} \to \mathbb{V}$. We achieve this if we assume Hypothesis 8.5.6[v*] very obviously by the Inequalities (8.37) and (8.38). If we assume Hypothesis 8.5.6[v], we could only use the Inequalities (8.35) and (8.36), but then we could use that $U$ is in set bounded by the tempered radius $\rho_0$. Thus, $H_c$ is (Local)-Lipschitz continuous from $E_1$ to $E_1$, and we can apply Theorem 1.4 from Chapter 6 in [41]. Thanks to the bound of $\psi_c \in E_1$ derived in the a priori estimates in Lemma 8.5.21, we can find local solutions and extend them into global solutions by iterating [41, Theorem 6.1.4]. The semigroup property of $B_\epsilon$ on $E_1$ is given by Lemma 3.3.2. □

For the compact part, we can state a lemma similar to Lemma 8.4.2. This result is also given by Lemma 8.4.1.

**Lemma 8.5.20**
*Assume that*

$$\epsilon \leq \epsilon_0 := \min(\frac{\alpha}{4}, \frac{1}{2C(D)^2\alpha}, \frac{1}{\alpha}).$$

*Then, we have the following estimate*

$$\begin{aligned} & \epsilon\|\mathcal{A}u\|^2 + (\alpha-\epsilon)\|\!|v|\!\|^2 - \epsilon(\alpha-\epsilon)(\nabla u, \nabla v) + \epsilon\|\mathcal{A}_\Gamma u\|_\Gamma^2 \\ & +(\alpha-\epsilon)\|v\|_\Gamma^2 - \epsilon(\alpha-\epsilon)(u,v)_\Gamma \\ \geq \; & \frac{\epsilon}{2}\|\mathcal{A}u\|^2 + \frac{\alpha}{6}\|\!|v|\!\|^2 + \frac{\epsilon}{8}\|\mathcal{A}_\Gamma u\|_\Gamma^2 + \frac{\alpha}{2}\|v\|_\Gamma^2 \\ \geq \; & \alpha_1\Big(\|\mathcal{A}u\|^2 + \|\!|v|\!\|^2 + \|\mathcal{A}_\Gamma u\|_\Gamma^2 + \|v\|_\Gamma^2\Big) \end{aligned}$$

*for*

$$\alpha_1 = \min(\frac{\alpha}{6}, \frac{\epsilon}{8})$$

*and* $(U,V) \in D(A) \times \mathbb{V}$.

**Lemma 8.5.21 (Estimate of $\psi_c$ in $E_1$)**
*The compact part $\psi_c$ has the absorption property in $E_1$.*

*Proof.* We calculate $\|v_c\|_{E_1}^2$ by the chain rule and consider at first the interior part:

$$\begin{aligned} \frac{dv_c}{dt} = v_c' \; = \; & \epsilon(\alpha-\epsilon)u_c - \mathcal{A}u_c + (\epsilon-\alpha)v_c + f(u) \\ & +(\mu+\epsilon-\alpha)c(u)[z] - c'(u)[z, v_c - \epsilon u + c(u)[z]]. \end{aligned}$$

Multiplying by $\mathcal{A}v_c$ gives us

$$\begin{aligned} (v_c', \mathcal{A}v_c) \; = \; & \epsilon(\alpha-\epsilon)(u_c, \mathcal{A}v_c) - (\mathcal{A}u_c, \mathcal{A}v_c) + (\epsilon-\alpha)(v_c, \mathcal{A}v_c) \\ & +\Big((\mu+\epsilon-\alpha)c(u)[z] - c'(u)[z, v_c - \epsilon u + c(u)[z]] + f(u)\Big), \mathcal{A}v_c). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\!|v_c|\!\|^2 - (v_{\Gamma}{}_c', \partial_\nu u_c)_\Gamma \\ =& \epsilon(\alpha-\epsilon)(\nabla u_c, \nabla v_c) - \epsilon(\alpha-\epsilon)(u_{\Gamma c}, \partial_\nu v_c)_\Gamma + (\epsilon-\alpha)\|\!|v_c|\!\|^2 - (\epsilon-\alpha)(v_{\Gamma c}, \partial_\nu v_c)_\Gamma \\ & - (\mathcal{A}u_c, \mathcal{A}u_c') - \epsilon(\mathcal{A}u_c, \mathcal{A}u_c) + (\mathcal{A}u_c, \mathcal{A}c(u)[z]) \\ & + (\Big((\mu+\epsilon-\alpha)c(u)z - c'(u)[z, v_c - \epsilon u + c(u)[z]] + f(u)\Big), \mathcal{A}v_c), \end{aligned}$$

and on the boundary by multiplying the boundary part with $\mathcal{A}_\Gamma v$, we omit the index $|_\Gamma$ at the scalar product as before in this calculations because we only consider the scalar product on the boundary

$$\begin{aligned} & (v_{\Gamma}{}_c', \partial_\nu v_c) + (v_{\Gamma}{}_c', v_{\Gamma c}) \\ =& \epsilon(\alpha-\epsilon)(u_{\Gamma c}, v_{\Gamma c}) + \epsilon(\alpha-\epsilon)(u_{\Gamma c}, \partial_\nu v_c) \\ & + (\epsilon-\alpha)(v_{\Gamma c}, \partial_\nu v_c) + (\epsilon-\alpha)(v_{\Gamma c}, v_{\Gamma c}) \\ & - (\mathcal{A}_\Gamma u_c, \mathcal{A}_\Gamma u_c') - \epsilon\|\mathcal{A}_\Gamma u_c\|^2 + (\mathcal{A}_\Gamma u_c, \mathcal{A}_\Gamma(c(u)[z])) \\ & + (\Big((\mu+\epsilon-\alpha)c_\Gamma(u)[z] - c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]] + f_\Gamma(u_\Gamma)\Big), \mathcal{A}_\Gamma v_c). \end{aligned}$$

We define $k = \mu + \epsilon - \alpha$ and derive the following estimate

$$\begin{aligned}
((\mu+\epsilon-\alpha)c(u)[z], \mathcal{A}v_c) &= k((c(u)[z], v_c)) - k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma \\
&\leq k\|c(u)[z]\|\|v_c\| - k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma \\
&\leq k_1\Big(\|c(0)[z]\| + c_V\|z\|_U\|u\|\Big)\|v_c\| - k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma \\
&\leq k_2\|z\|_U\|v_c\|^2 + k_3(1+\|z\|_U^2 + \|u\|^4) - k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma
\end{aligned}$$

and on the boundary,

$$\begin{aligned}
((\mu+\epsilon-\alpha)c_\Gamma(u)[z], \mathcal{A}_\Gamma v_c)_\Gamma &= k(c_\Gamma(u)[z], v_{\Gamma c})_\Gamma + k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma \\
&\leq k\|c_\Gamma(u)[z]\|\|v_{\Gamma c}\|_\Gamma + k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma \\
&\leq k_4\|z\|_U\|v_{\Gamma c}\|_\Gamma + k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma \\
&\leq k_5\|z\|_U\|v_{\Gamma c}\|_\Gamma^2 + k_6\|z\|_U^2 + k(c_\Gamma(u)[z], \partial_\nu v_c)_\Gamma.
\end{aligned}$$

We have for $c'$ the following estimate by Hypothesis 8.5.6[v]

$$\|c'(u)[z,h]\| \leq K\|z\|_U(\|h\|\|u\| + \|h\|), H \in \mathbb{V}, \tag{8.35}$$

and on the boundary

$$\|c'_\Gamma(u)[z,h]\| \leq K\|z\|_U(\|h\|\|u\| + \|h\|), H \in \mathbb{V}, \tag{8.36}$$

where $H = (h, h_\Gamma)$ by calculating

$$\begin{aligned}
\|P_m c'(u)[z,h]\| &\leq \lim_{s\to 0} \sup_{\|q\|\leq\|h\|} \frac{\|P_m c(u+sq)[z] - P_m c(u)[z]\|}{s} \\
&\leq K\|z\|_U(\|h\|\|u\| + \|h\|),
\end{aligned}$$

and similarly on the boundary. With the same arguments as in [33], these are the Fréchet differentiability of $C : \mathbb{V} \to L(U, \mathbb{H})$ and the extension of $C'$ as a continuous operator from $\mathbb{V} \to L(U \times \mathbb{V}, \mathbb{V})$ we obtain that the last inequality also follows for $m \to \infty$. $\mathbf{P}_m = (P_m, \gamma P_m)$ denotes the projection on the first $m$ eigenvectors of $A$. Note that the eigenfuntions of $A$ are contained in $D(A)$.
If we replace Hypothesis 8.5.6[v] by Hypothesis 8.5.6[v*], we achieve the simplier estimate

$$\|c'(u)[z,h]\| \leq K\|z\|_U\|h\|, H \in \mathbb{V}, \tag{8.37}$$

and on the boundary

$$\|c'_\Gamma(u)[z,h]\|_\Gamma \leq K\|z\|_U\|h\|, H \in \mathbb{V}. \tag{8.38}$$

We can thus derive by (8.35)

$$\begin{aligned}
& (c'(u)[z, v_c - \epsilon u + c(u)[z]], \mathcal{A}v_c) \\
= \;& ((c'(u)[z, v_c - \epsilon u + c(u)[z]], v_c)) - (c'_\Gamma(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \;& 2c\|z\|_U\big(\|v_c\| + \epsilon\|u\| + \|c(u)[z]\|\big)(\|u\|+1)\|v_c\| \\
& -(c'_\Gamma(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \;& (\epsilon_2\rho_0^2 + k_{\epsilon_2}\|z\|_U^2 + k_7\|z\|_U)\|v_c\|^2 + k_8(1 + \|z\|_U^{\gamma_1} + \|u\|^{\gamma_2}) \\
& -(c'_\Gamma(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c)
\end{aligned}$$

and on the boundary

$$
\begin{aligned}
& (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \mathcal{A}_\Gamma v_c)_\Gamma \\
= \; & (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], v_\Gamma) + (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \; & 2c\|z\|_U \big(\|v_c\| + \epsilon\|u\| + \|c(u)[z]\|\big)(\|u\| + 1)\|v_{\Gamma c}\| \\
& + (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \; & (\epsilon_2 \rho_0^2 + k_{\epsilon_2}\|z\|_U^2 + k_7\|z\|_U)\|\psi_c\|^2 + k_8(1 + \|z\|_U^{\gamma_1} + \|u\|^{\gamma_2}) \\
& + (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c).
\end{aligned}
$$

for an $\epsilon_2 > 0$ and some $\gamma_1, \gamma_2 > 0$. We have use in all these calculation the estimate

$$
\|c(u)[z]\| \leq \|c(0)[z]\| + c_V\|z\|_U\|u\| \leq k\|z\|_U + c_V\|z\|_U\|u\|,
$$

arising from Hypothesis 8.5.6[v]. If we use the estimate (8.37) derived by Hypothesis 8.5.6[v*] instead of Hypothesis 8.5.6[v], we obtain

$$
\begin{aligned}
& (c'(u)[z, v_c - \epsilon u + c(u)[z]], \mathcal{A} v_c) \\
= \; & ((c'(u)[z, v_c - \epsilon u + c(u)[z]], v_c)) - (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \; & 2c\|z\|_U \big(\|v_c\| + \epsilon\|u\| + \|c(u)[z]\|\big)\|v_c\| \\
& - (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \; & (\epsilon_2 \rho_0^2 + k_{\epsilon_2}\|z\|_U^2 + k_7\|z\|_U)\|v_c\|^2 + k_8(1 + \|z\|_U^4 + \|u\|^4) \\
& - (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c)
\end{aligned}
$$

and on the boundary

$$
\begin{aligned}
& (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \mathcal{A}_\Gamma v_{\Gamma c})_\Gamma \\
= \; & (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], v_\Gamma) + (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \; & 2c\|z\|_U \big(\|v_c\| + \epsilon\|u\| + \|c(u)[z]\|\big)\|v_{\Gamma c}\| \\
& + (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c) \\
\leq \; & (\epsilon_2 \rho_0^2 + k_{\epsilon_2}\|z\|_U^2 + k_7\|z\|_U)\|\psi_c\|^2 + k_8(1 + \|z\|_U^4 + \|u\|^4) \\
& + (c_\Gamma'(u)[z, v_c - \epsilon u + c(u)[z]], \partial_\nu v_c).
\end{aligned}
$$

for an $\epsilon_2 > 0$.
We estimate the nonlinearity by integration by parts and the trace condition of the nonlinearity, see Hypothesis 8.5.5,

$$
\begin{aligned}
& (f(u), \mathcal{A} v_c) + (f_\Gamma(u_\Gamma), \mathcal{A}_\Gamma v_c) \\
\leq \; & \hat{k}_1(1 + \|\psi\|)\|\psi_c\| \\
\leq \; & \hat{k}_2 + \hat{k}_3\|\psi\|^2 + \frac{\epsilon}{8}\|\psi_c\|^2
\end{aligned}
$$

for an appropriate constant $k$. Furthermore, we have by Hypothesis 8.5.6[vi]

$$
(AC(U_c)[z], AU_c) \leq c_{D(A)}\|z\|_U\|AU_c\| \leq \epsilon_3\|AU_c\|^2 + K_{\epsilon_3}\|z\|_U^2.
$$

Summing up all these estimates we get with

$$\|\psi\|_{E^1}^2 := \|v\|^2 + \|Au\|^2 + \|v_\Gamma\|_\Gamma^2 + \|\mathcal{A}_\Gamma u\|_\Gamma^2$$

the final estimate for some $k_1, k_2, k_3, k_4 > 0$

$$\frac{d}{dt}\|\psi_c\|_{E^1}^2 \leq -\alpha_2\|\psi_c\|_{E^1}^2 + 2(k_1\|z\|_U + \epsilon_2\rho_0^2 + k_{\epsilon_2}\|z\|_U^2)\|\psi_c\|_{E^1}^2 + k_3(k_4 + \|z\|_U^{\gamma_1} + \|u\|^{\gamma_2} + \|u\|^{\gamma_2}).$$

Assuming Hypothesis 8.5.6[v*] instead of Hypothesis 8.5.6[v] we have $\gamma_1 = 4$ and $\gamma_2 = 4$. If we choose now $Tr_U Q$ sufficiently small such that

$$\mathbb{E}(k_1\|z\|_U + \epsilon_2\rho_0^2 + k_{\epsilon_2}\|z\|_U^2) < \frac{\alpha_2}{4} \tag{8.39}$$

we get by ergodic theory like in Lemma 8.5.13 a ball with tempered radius $\rho_c$ and center zero. The existence of the expectation (8.39) is ensured by Lemma 5.1.24 and Lemma 5.1.5. Lemma 5.1.24 gives us that $\mathbb{E}\epsilon_2\rho_0^2$ is finite. Then, we choose $\epsilon_2$ small enough, such that $\mathbb{E}\epsilon_2\rho_0^2 < \frac{\alpha_2}{8}$. This is possible by Lemma 5.1.24 for a fixed $\mu$ and $\mathrm{Tr}_U Q$ small enough. Then we chose $\mathrm{Tr}_U Q$ small enough, such that

$$\mathbb{E}(k_1\|z\|_U + k_{\epsilon_2}\|z\|_U^2) < \frac{\alpha_2}{8}.$$

to obtain inequality (8.39) and then follow the arguments in Lemma 8.5.13. □

**Remark 8.5.22**
*We can also prove the existence of a random absorbing set of $\psi_c$ in $E_1$ by semigroup methods considering the mild solution of (8.28). We use Hypothesis 8.5.6[v*] in these calculations.*

*Proof.* The mild solution

$$\psi_c(t) = G(t)\psi_{c0} + \int_0^t G(t-\tau)H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\,d\tau.$$

provides us that

$$\|\psi_c(t)\|_{E_1} \leq \|G(t)\psi_{c0}\|_{E_1} + \int_0^t \|G(t-\tau)H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\|_{E_1}\,d\tau.$$

The exponential decay of the semigroup, see Lemma 8.4.1 or Lemma 8.5.20 leads us to the estimate

$$\|\psi_c(t)\|_{E_1} \leq e^{-\lambda t}\|\psi_{c0}\|_{E_1} + \int_0^t e^{-\lambda(t-\tau)}\|H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\|_{E_1}\,d\tau$$

for some $\lambda > 0$.
We estimate

$$\begin{aligned}
&\|H_c(\theta_\tau\omega, \psi_c(\tau), \psi(\tau))\|_{E_1}\\
\leq &2c_{D(A)}\|z\|_U + C_{G,V}(1 + \|U\|_{\mathbb{V}}) + \|C(U)[z, V_c - \epsilon U + C(U)[z]]\|_{\mathbb{V}}\\
\leq &2c_{D(A)}\|z\|_U + C_{G,V}(1 + \|U\|_{\mathbb{V}}) + k_1\|z\|_U\|V_c - \epsilon U + C(U)[z]]\|_{\mathbb{V}}\\
\leq &2c_{D(A)}\|z\|_U + C_{G,V}(1 + \|U\|_{\mathbb{V}}) + k_2\|z\|_U\|\psi_c(t)\|_{E_1}\\
&+ k_3\|z\|_U\rho_0^m(\theta_\tau\omega) + k_4\|z\|_U^2\rho_0^m(\theta_\tau\omega) + k_5\|z\|_U^2.
\end{aligned}$$

We set

$$\Theta_c^1(\omega) = c_{D(A)}\|z(\omega)\|_U + C_{G,V} + (k_3 + C_{G,V})\|z(\omega)\|_U \rho_0^m(\omega) + k_4\|z\|_U^2 \rho_0^m(\omega) + k_5\|z(\omega)\|_U^2.$$

Note that $\Theta_c^1$ is a tempered random variable. We obtain

$$\|\psi_c(t)\|_{E_1} \le e^{-\lambda t}\|\psi_{c0}\|_{E_1} + \int_0^t e^{-\lambda(t-\tau)} k_2\|z(\theta_\tau\omega)\|_U \|\psi_c(\tau)\|_{E_1} + e^{-\lambda(t-\tau)}\Theta_c^1(\theta_\tau\omega)\, d\tau$$

or

$$\|\psi_c(t)\|_{E_1} e^{\lambda t} \le \|\psi_{c0}\|_{E_1} + \int_0^t e^{\lambda\tau} k_2\|z(\theta_\tau\omega)\|_U \|\psi_c(\tau)\|_{E_1} + e^{\lambda\tau}\Theta_c^1(\theta_\tau\omega)\, d\tau.$$

The inequality has the form

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)\, d\tau.$$

with

$$v(t) = \|\psi_c(t)\|_{E_1} e^{\lambda t}, \quad h(\tau) = k_2\|z(\theta_\tau\omega)\|_U, \quad g(t) = \|\psi_{c0}\|_{E_1} + \int_0^t e^{\lambda\tau}\Theta_c^1(\theta_\tau\omega, U)\, d\tau.$$

Gronwall's Lemma yields us

$$v(t) \le e^{k_2 \int_0^t \|z(\theta_\tau\omega)\|_U\, d\tau}(g(0) + \int_0^t e^{\lambda\tau}\Theta_c^1(\theta_\tau\omega) e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\, ds}\, d\tau).$$

Thus, we achieve

$$\|\psi_c(t)\|_{E_1} \le e^{k_2 \int_0^t \|z(\theta_\tau\omega)\|_U\, d\tau - \lambda t}(g(0) + \int_0^t e^{\lambda\tau}\Theta_c^1(\theta_\tau\omega) e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\, ds}\, d\tau).$$

and finally

$$\|\psi_c(t)\|_{E_1} \le e^{k_2 \int_0^t \|z(\theta_\tau\omega)\|_U\, d\tau - \lambda t}\|\psi_{c0}\|_{E_1} + \int_0^t e^{-\lambda(t-\tau) + \int_\tau^t \|z(\theta_s\omega)\|_U\, ds}\Theta_c^1(\theta_\tau\omega)\, d\tau. \quad (8.40)$$

We can now follow the arguments in Remark 8.5.15 and conclude that there exists closed ball $\hat{C}(\omega)$ in $E_1$ with center zero and radius

$$\rho_{c1}^m(\omega) := 2\int_{-\infty}^0 e^{k_2 \int_\tau^0 \|z(\theta_s\omega)\|_U\, ds + \lambda\tau}\Theta_c^1(\theta_\tau\omega)\, d\tau \quad (8.41)$$

which attracts the set $\hat{B}$ defined in Remark 8.5.15, if we choose $\mathrm{Tr}_U\, Q$ small enough, such that

$$\mathbb{E}k_2\|z\|_U - \lambda < 0.$$

The right hand side of Equality (8.41) defines a tempered random variable.

□

**Definition 8.5.23**

*We denote the random ball with radius $\rho_c(\omega)$ in $E_1$ by $C(\omega)$ from Lemma 8.5.21.*

**Lemma 8.5.24 (Estimate of $\psi_d$)**
*We have that*

$$\lim_{t\to\infty} \sup_{x\in B(\theta_{-t}\omega)} \|\psi_d(t,\theta_{-t}\omega,x)\|_{E_0} = 0. \tag{8.42}$$

*B is the ball from Lemma (8.5.13).*

*Proof.* We have to estimate $\|\psi_d\|^2_{E_0}$. Multiplying of (8.29) by $v_d$ gives us the following equation in the interior domain

$$(v_d', v_d) = \epsilon(\alpha-\epsilon)(u_d, v_d) - (\mathcal{A}u_d, v_d) + (\epsilon-\alpha)(v_d, v_d) + (c'(u)[z, v-v_c], v_d)$$

and on the boundary by multiplying (8.30) by $v_d$,

$$(v_d', v_d) = \epsilon(\alpha-\epsilon)(u_d, v_d) - (\mathcal{A}_\Gamma u_d, v_d) + (\epsilon-\alpha)(v_d, v_d) + (c_\Gamma'(u)[z, v-v_c], v_d).$$

By the same arguments as for $\|\psi\|^2$, see Lemma 8.5.13, we achieve that

$$\begin{aligned}\frac{d}{dt}\|\psi_d\|^2 + \alpha_2\|\psi_d\|^2 \le{}& 2(c(u)[z] - c(u_c)[z], \mathcal{A}u_d) + (c(u)[z] - c(u_c)[z], \mathcal{A}_\Gamma u_d)_\Gamma \\ &+ 2(c'(u)[z, v-v_c], v_d) + 2(c_\Gamma'(u)[z, v-v_c], v_d)_\Gamma.\end{aligned}$$

The terms on the right hand side appear from the relation

$$v_d = u_d' + \epsilon u_d - (c(u)[z] - c(u_c)[z])$$

and respectively on the boundary. We obtain

$$\begin{aligned}&(c(u)[z] - c(u_c)[z], \mathcal{A}u_d) + (c_\Gamma(u)[z] - c_\Gamma(u_c)[z], \mathcal{A}_\Gamma u_d)_\Gamma \\ =&2((c(u)[z] - c(u_c)[z], u_d)) + 2(c_\Gamma(u)[z] - c_\Gamma(u_c)[z], u_{\Gamma d})_\Gamma.\end{aligned}$$

Now, we apply Hypothesis 8.5.6(v) and get in the inner domain

$$\begin{aligned}((c(u)[z] - c(u_c)[z], u_d)) &\le c_V\|z\|_U\big(|\!|\!|u|\!|\!||\!|\!|u-u_c|\!|\!| + |\!|\!|u-u_c|\!|\!|\big)|\!|\!|u_d|\!|\!| \\ &\le c_V\|z\|_U\big(|\!|\!|u|\!|\!||\!|\!|u_l+u_d|\!|\!| + |\!|\!|u_l+u_d|\!|\!|\big)|\!|\!|u_d|\!|\!| \\ &\le c_V\|z\|_U\big(|\!|\!|u|\!|\!||\!|\!|u_l|\!|\!||\!|\!|u_d|\!|\!| + |\!|\!|u|\!|\!||\!|\!|u_d|\!|\!|^2 + |\!|\!|u_d|\!|\!|^2 + |\!|\!|u_l|\!|\!||\!|\!|u_d|\!|\!|\big).\end{aligned}$$

On the boundary, we have the similar estimate

$$\begin{aligned}&(c_\Gamma(u)[z] - c_\Gamma(u_c)[z], u_{\Gamma d})_\Gamma \\ \le& c_V\|z\|_U\big(\|u\|\|u_l\|\|u_d\|_\Gamma + \|u\|\|u_d\|\|u_d\|_\Gamma + \|u_d\|\|u_d\|_\Gamma + \|u_l\|\|u_d\|_\Gamma\big).\end{aligned}$$

If we weaken Hypothesis 8.5.6[v] by Hypothesis 8.5.6[v*], we again achieve the simplier estimate

$$((c(u)[z] - c(u_c)[z], u_d)) \le c_V\|z\|_U\big(|\!|\!|u_l|\!|\!| + |\!|\!|u_d|\!|\!|\big)|\!|\!|u_d|\!|\!|$$

and

$$(c_\Gamma(u)[z] - c_\Gamma(u_c)[z], u_{\Gamma d}) \le c_V\|z\|_U\big(\|u_l\| + \|u_d\|\big)\|u_d\|_\Gamma$$

which is only a simple improvement. We can go further on with the same calculations. Note that $\|\!|u|\!\|$ and $\|u\|$ can be estimated by the radius $\rho_0$ and thus we have

$$\begin{aligned} c_V\|z\|_U\|\!|u|\!\|\|\!|u_l|\!\|\|\!|u_d|\!\| &\leq k\rho_0\|z\|_U\|\!|u_l|\!\|^2 + k\rho_0\|z\|_U\|\!|u_d|\!\|^2 \\ c_V\|z\|_U\|\!|u|\!\|\|\!|u_d|\!\|^2 &\leq k\rho_0\|z\|_U\|\!|u_d|\!\|^2 \leq (\epsilon_3\rho_0^2 + k_{\epsilon_3}\|z\|_U^2)\|\!|u_d|\!\|^2 \\ c_V\|z\|_U\|u\|\|u_l\|\|u_d\|_\Gamma &\leq k\rho_0\|z\|_U\|u_l\|^2 + k\rho_0\|z\|_U\|u_d\|_\Gamma^2 \\ c_V\|z\|_U\|u\|\|u_d\|\|u_d\|_\Gamma &\leq k\rho_0\|z\|_U\|u_d\|\|u_d\|_\Gamma \leq (\epsilon_3\rho_0^2 + k_{\epsilon_3}\|z\|_U^2)\frac{1}{2}\|\psi_d\|^2. \end{aligned}$$

We can also estimate in the inner domain and on the boundary by Hypothesis 8.5.6(viii)

$$\begin{aligned} (c'(u)[z, v - v_c], v_d) &\leq c'_H\|z\|_U\|v_l + v_d\|\|v_d\| \\ &\leq k\|z\|_U\big(\|v_d\|^2 + \|v_l\|^2\big). \end{aligned}$$

and on the boundary

$$\begin{aligned} (c'_\Gamma(u)[z, v - v_c], v_d) &\leq c'_H\|z\|_U\|v_l + v_d\|_{\|}v_d\|_\Gamma \\ &\leq k\|z\|_U\big(\|\psi_d\|^2 + \|v_l\|^2\big). \end{aligned}$$

Thus, we can derive the estimate for some $k_1, k_2, k_3 > 0$

$$\frac{d}{dt}\|\psi_d\|^2 + \frac{\alpha_2}{2}\|\psi_d\|^2 \leq h_1(\theta_t\omega)\|\psi_d\|^2 + h_2(\theta_t\omega)\|\psi_l\|^2,$$

where

$$h_1 = \frac{3}{2}(\epsilon_3\rho_0(\omega)^2 + k_{\epsilon_3}\|z(\omega)\|_U^2) + k_1\|z(\omega)\|_U$$

and

$$h_2 = k_2(\|z(\omega)\|_U^2 + \rho_0^2(\omega) + 1) + k_3\|z(\omega)\|_U.$$

We can conclude by the variation of constants formula the result, if we assume that $\mathbb{E}h_1 < \frac{\alpha_2}{4}$ and we can follow the arguments as in Lemma 8.5.21. Then, we can conclude as in [33], using the exponential decay of the linear part, proven in Lemma 8.5.16

$$\begin{aligned} &\sup_{x\in B(\theta_t\omega)} \|\psi_d(t, \theta_{-t}\omega, x)\|_{E_0}^2 \\ \leq &\int_0^t e^{\int_s^t -\frac{\alpha_2}{2} + h_1(\theta_{-t+\tau}\omega)\,d\tau}\rho_0(\theta_{-t}\omega)^2 e^{\frac{-\alpha_2}{4}s}h_2(\theta_{-t+s}\omega)\,ds \\ \leq &\rho_0(\theta_{-t}\omega)^2 e^{-\frac{\alpha_2}{4}t}\int_{-t}^0 e^{\int_s^0 -\frac{\alpha_2}{4} + h_1(\theta_\tau\omega)\,d\tau}h_2(\theta_s\omega)\,ds \\ \leq &\rho_0(\theta_{-t}\omega)^2 e^{-\frac{\alpha_2}{4}t}\int_{-\infty}^0 e^{\int_s^0 -\frac{\alpha_2}{4} + h_1(\theta_\tau\omega)\,d\tau}h_2(\theta_s\omega)\,ds. \end{aligned}$$

Since the integral on the right hand side defines a tempered random variable, we conclude that the right hand side tends to zero.

□

Alternatively, we can also prove the property (8.42) by the techniques of mild solutions. Therefore, we rewrite (8.29) and (8.30) as first order evolution equation as

$$\frac{d\psi_d}{dt} = B_\epsilon \psi_d + H_d(\theta_t\omega, \psi_d), \tag{8.43}$$

with $B_\epsilon$ defined in (8.15) and

$$H_d(\omega, \psi_c) = \begin{pmatrix} c(u)[z(\omega)] - c(u_c)[z(\omega)] \\ c_\Gamma(u)[z(\omega)] - c_\Gamma(u_c)[z(\omega)] \\ -c'(u)[z(\omega), v - v_c] \\ -c'_\Gamma(u)[z(\omega), v - v_c] \end{pmatrix}$$

We can state a remark similar to Lemma 8.5.21.

**Remark 8.5.25**
*We can prove property of $\psi_d$ (8.42) in $E_0$ by semigroup methods considering the mild solution of (8.43). We use Hypothesis 8.5.6[v*] in these calculations.*

*Proof.* The mild solution

$$\psi_d(t) = G(t)\psi_{d0} + \int_0^t G(t-\tau)H_d(\theta_\tau\omega, \psi_d(\tau), \psi_l(\tau))\, d\tau.$$

provides us that

$$\|\psi_d(t)\|_{E_0} \le \|G(t)\psi_{d0}\|_{E_0} + \int_0^t \|G(t-\tau)H_d(\theta_\tau\omega, \psi_d(\tau), \psi_l(\tau))\|_{E_0}\, d\tau.$$

The exponential decay of the semigroup, see Lemma 8.4.1 or Lemma 8.5.20, leads us to the estimate

$$\|\psi_d(t)\|_{E_0} \le e^{-\lambda t}\|\psi_{d0}\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)}\|H_d(\theta_\tau\omega, \psi_d(\tau), \psi_l(\tau))\|_{E_0}\, d\tau$$

for some $\lambda > 0$.
We estimate

$$\begin{aligned} &\|H_d(\theta_\tau\omega, \psi_c(\tau), \psi_l(\tau))\|_{E_0} \\ \le& c_V\|U - U_C\|_V\|z(\theta_\tau\omega)\|_U + c'_H\|V - V_C\|\|z(\theta_\tau\omega)\|_U \\ \le& k_1\|z(\theta_\tau\omega)\|_U\|\psi_d\| + k_2\|z(\theta_\tau\omega)\|_U\|\psi_l\|. \end{aligned}$$

We set

$$\Theta_d(\omega) = k_2\|z(\omega)\|_U\|\psi_l\|.$$

Note that

$$\Theta_d(\omega) \le k_2\|z(\omega)\|_U\rho_0^m(\omega)e^{-\frac{\lambda}{2}s}.$$

by Lemma 8.5.16.
We obtain

$$\|\psi_d(t)\|_{E_0} \le e^{-\lambda t}\|\psi_{d0}\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)}k_1\|z(\theta_\tau\omega)\|_U\|\psi_d(\tau)\|_{E_0} + e^{-\lambda(t-\tau)}\Theta_d(\theta_\tau\omega)\, d\tau$$

or

$$\|\psi_d(t)\|_{E_0} e^{\lambda t} \le \|\psi_{d0}\|_{E_0} + \int_0^t e^{\lambda\tau} k_1 \|z(\theta_\tau\omega)\|_U \|\psi_d(\tau)\|_{E_0} + e^{\lambda\tau}\Theta_d(\theta_\tau\omega)\,d\tau.$$

The inequality has the form

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)\,d\tau.$$

with

$$v(t) = \|\psi_d(t)\|_{E_1} e^{\lambda t}, \quad h(\tau) = k_1\|z(\theta_\tau\omega)\|_U, \quad g(t) = \|\psi_{d0}\|_{E_0} + \int_0^t e^{\lambda\tau}\Theta_d(\theta_\tau\omega)\,d\tau.$$

Gronwall's Lemma gives us

$$v(t) \le e^{k_1\int_0^t \|z(\theta_\tau\omega)\|_U\,d\tau}(g(0) + \int_0^t e^{\lambda\tau}\Theta_d(\theta_\tau\omega)e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\,ds}\,d\tau).$$

Thus, we achieve

$$\|\psi_d(t)\|_{E_0} \le e^{k_1\int_0^t \|z(\theta_\tau\omega)\|_U\,d\tau - \lambda t}(g(0) + \int_0^t e^{\lambda\tau}\Theta_d(\theta_\tau\omega)e^{-\int_0^\tau \|z(\theta_s\omega)\|_U\,ds}\,d\tau).$$

and finally, replacing $\omega$ by $\theta_{-t}\omega$,

$$\|\psi_d(t)\|_{E_0} \le e^{k_1\int_0^t \|z(\theta_{\tau-t}\omega)\|_U\,d\tau - \lambda t}\|\psi_{d0}\|_{E_0} + \int_0^t e^{-\lambda(t-\tau)+\int_\tau^t \|z(\theta_{s-t}\omega)\|_U\,ds}\Theta_d(\theta_{\tau-t}\omega)\,d\tau. \tag{8.44}$$

We can now follow the arguments in Remark 8.5.15 and conclude that

$$\begin{aligned}
&\sup_{x\in\hat{B}(\theta_{-t}\omega)} \|\psi_d(t,\theta_{-t}\omega,x)\|_{E_0} \\
\le& e^{k_1\int_0^t \|z(\theta_{\tau-t}\omega)\|_U\,d\tau-\lambda t}\int_0^t e^{-\lambda(t-\tau)+\int_\tau^t \|z(\theta_{s-t}\omega)\|_U\,ds}\Theta_d(\theta_{\tau-t}\omega)\,d\tau \\
\le& \int_0^t e^{\int_\tau^t -\lambda+k_1\|z(\theta_{-t+s}\omega)\|_U\,ds}\rho_0^m e^{-\frac{\lambda}{2}\tau}k_2\|z(\theta_{-t+\tau}\omega)\|_U\,d\tau \\
\le& \rho_0^m(\theta_{-t}\omega)e^{-\frac{\lambda}{2}t}\int_{-\infty}^0 e^{\int_\tau^0 \lambda+k_1\|z(\theta_s\omega)\|_U\,ds}k_2\|z(\theta_\tau\omega)\|_U\,d\tau.
\end{aligned} \tag{8.45}$$

The right hand side of Inequality (8.45) tends to zero because the integal is a tempered random variable.

□

Now, we can follow again the arguments in [33, 4.5]. Collecting Lemmas 8.5.24, 8.5.17, 8.5.21 and 8.5.16 shows us the existence of a compact random set, which attracts the random set $B$ in the space $E_0$. We have also shown that the random set $B$ is absorbing for our RDS. Thus, $C$ defined in Definition 8.5.23 attracts any set of $\mathcal{D}$ by the cocycle property. The compactness of $C$ is proven in Lemma 8.5.21. Applying Theorem 5.1.9 yields the following theorem:

**Theorem 8.5.26 (Existence of Random Attractor)**
*The hyperbolic stochastic equation (8.12) possesses a random attractor, if we choose $\mu$ big enough and $\mathrm{Tr}_U Q$ sufficiently small.*

**Example 8.5.27 (An example)**
*We consider the following non-linear kernel operator*

$$C(U)[z](x) = \big(\int_D k(x,u(y))z(y)\,dy, \int_D k(x,u(y))z(y)\,dy|_\Gamma\big).$$

*Note that this operator is defined on $\overline{D} = D \cup \Gamma$, so that the restriction of $c$ on $\Gamma$ is well-defined. We also set $U = D(A)$. We assume that $k(x,u)$ is three times continuously differentiable in $x$, two times continuously differentiable in $u$ and the derivatives are bounded, we set*

$$M(x) = \max_{u \in \mathbf{R}} k(x,u)$$

*To show that Hypothesis 8.5.6 is fulfilled we start with*

$$\begin{aligned}\|c(u)[z](x)\|^2 &= \int_D \big(\int_D k(x,u(y))z(y)\,dy\big)^2\,dy \\ &\le \int_D \|M(x)\|^2\|z\|^2\,dx = \|z\|^2\int_D \|M(x)\|^2\,dx \\ &\le K_1\|z\|^2_{D(A)},\end{aligned}$$

*for an appropriate constant $K_1$, which depends on $D$. This delivers us condition 8.5.6[iv]. The condition on the boundary follows by replacing the first $D$ by $\Gamma$.*
*To obtain condition 8.5.6[v] we calculate at first on the boundary*

$$\begin{aligned}&\int_\Gamma \|\big(\int_D (k(x,\varphi(y)) - k(x,\psi(y)))z(y)\,dy\big)\|^2_\Gamma\,dx \\ &\quad\le \int_\Gamma L(x)^2\big(\int_D \|\varphi(y)-\psi(y)\|\|z(y)\|\,dy\big)^2\,dx \\ &\le \int_\Gamma L(x)^2\|\varphi-\psi\|^2_D\|z\|^2_D\,dx \le K_2\|z\|^2_U\|\varphi-\psi\|^2_D.\end{aligned}$$

*We have used the Lipschitz continuity of $k$ in $u(y)$, exactly*

$$L(x) = \max_{u\in\mathbf{R}}\|D_x k(x,u)\|.$$

*$K_2$ again depends on $D$. In the inner domain we have by applying a mean value theorem that*

$$\begin{aligned}\|c(\varphi)[z] - c(\psi)[z]\|^2 &= \int_D \big(\nabla_x \int_D (k(x,\varphi(y)) - k(x,\psi(y)))z(y)\,dy\big)^2\,dx \\ &\le K_3\|z\|^2_U\|\varphi-\psi\|^2_{E_1}.\end{aligned}$$

*To prove condition 8.5.6[vi], we use that*

$$\|D_x^2 k(x,u)\| \le C_1.$$

*Thus, we obtain*

$$\begin{aligned}\|c(u)[z](x), c_\Gamma(u)[z](x)\|^2_{D(A)} &\le K_4 \|c(u)[z](x)\|^2_{H^2} \\ &= K_4 \int_D \sum_{\|\alpha\|\le 2} D_x^\alpha \big( \int_D k(x,u(y))z(y)\,dy \big)^2\,dy \\ &\le K_4 \int_D \| \sum_{\|\alpha\|\le 2} D_x^\alpha k(x)\|^2 \|z\|_D^2\,dx \\ &\le K_4 \|z\|_U^2 \int_D \| \sum_{\|\alpha\|\le 2} D_x^\alpha k(x)\|^2\,dx \\ &\le K_5 \|z\|_U^2,\end{aligned}$$

*By Taylor's formula we achieve with a constant*

$$\int_D \big(k(x,u(y)+h(y)) - k(x,u(y)) - D_u k(x,u(y))\big) z(y)\,dy \le \int_D D_u^2 k(x,u)|h(y)|^2 |z(y)|\,dy.$$

*The boundedness of $D_u^2 k$ gives us that*

$$\int_D \big( \int_D (k(x,u(y)+h(y)) - k(x,u(y)) - D_u k(x,u(y))) z(y)\,dy \big)^2\,dx \le K_5 \|h\|^4_{L^2} \|z\|^2_{L^\infty}.$$

*By Sobolev's embedding theory for dimension $n = 1,2$ we obtain for some $\epsilon > 0$*

$$\|z\|_{L^\infty} \le K_6 \|z\|_{H^{1+\epsilon}} \le K_7 \|z\|_{D(A)},$$

*so that the upper integral can be estimated by*

$$K_8 \|h\|^4_{L^2} \|z\|^2_{D(A)}$$

*and the existence of the Fréchet derivative is ensured. Note that the arguments of $C'$ are a priori in $\mathbb{V}$ and then in particular in $\mathbb{H}$ and so the extension property of $C'$ is also ensured. The last condition 8.5.6[viii], this is to estimate the derivatives with respect to $u$, we get by*

$$\begin{aligned}\|c'(u)[z]\|^2 &= \int_D \big( \int_D D_u k(x,u(y))z(y)\,dy \big)^2\,dx \\ &\le K_9 \|z\|^2_{D(A)},\end{aligned}$$

*since $k$ is one-time continuously differentiable with respect to $u$. The extension property of $C'$ Hypothesis 8.5.6[ix] is given because the arguments of $K$ are a priori in $\mathbb{H}$.*

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# Bibliography

[1] Robert A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] Herbert Amann and Joachim Escher. Strongly continuous dual semigroups. *Ann. Mat. Pura Appl. (4)*, 171:41–62, 1996.

[3] Ludwig Arnold. *Random dynamical systems.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[4] M.J. Garrido Atienza, K. Lu, and B. Schmalfuß. Unstable manifolds for a stochastic partial differential equation driven by a fractional Brownian motion. Manuscript.

[5] A. F. Bennett and P. E. Kloeden. Viscous motion in an oceanic circulation model. *Bull. Austral. Math. Soc.*, 23(3):443–460, 1981.

[6] Peter Brune, Jinqiao Duan, and Björn Schmalfuß. Random dynamics of the Boussinesq system with dynamical boundary conditions. *Stoch. Anal. Appl.*, 27(5):1096–1116, 2009.

[7] Peter Brune and Björn Schmalfuss. Inertial manifolds for stochastic PDE with dynamical boundary conditions. *Commun. Pure Appl. Anal.*, 10(3):831–846, 2011.

[8] Tomás Caraballo, Peter E. Kloeden, and Björn Schmalfuß. Exponentially stable stationary solutions for stochastic evolution equations and their perturbation. *Appl. Math. Optim.*, 50(3):183–207, 2004.

[9] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions.* Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 580.

[10] Pao-Liu Chow. *Stochastic partial differential equations.* Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[11] Shui-Nee Chow and Kening Lu. Invariant manifolds for flows in Banach spaces. *J. Differential Equations*, 74(2):285–317, 1988.

[12] Shui-Nee Chow, Kening Lu, and George R. Sell. Smoothness of inertial manifolds. *J. Math. Anal. Appl.*, 169(1):283–312, 1992.

[13] I. D. Chueshov. *Introduction to the Theory of Infinite-Dimensional Dissipative Systems.* AKTA, Kharkiv, 2002.

[14] I. D. Chueshov and M. Scheutzow. Inertial manifolds and forms for stochastically perturbed retarded semilinear parabolic equations. *J. Dynam. Differential Equations*, 13(2):355–380, 2001.

[15] Igor Chueshov and Björn Schmalfuss. Parabolic stochastic partial differential equations with dynamical boundary conditions. *Differential Integral Equations*, 17(7-8):751–780, 2004.

[16] Igor Chueshov and Björn Schmalfuss. Qualitative behavior of a class of stochastic parabolic PDEs with dynamical boundary conditions. *Discrete Contin. Dyn. Syst.*, 18(2-3):315–338, 2007.

[17] Peter Constantin and Ciprian Foias. *Navier-Stokes equations.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[18] Hans Crauel, Arnaud Debussche, and Franco Flandoli. Random attractors. *J. Dynam. Differential Equations*, 9(2):307–341, 1997.

[19] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1992.

[20] Henk A. Dijkstra. *Nonlinear physical oceanography.* Kluwer Academic Publishers, Boston, 2000.

[21] Jinqiao Duan, Hongjung Gao, and Björn Schmalfuß. Stochastic dynamics of a coupled atmosphere-ocean model. *Stoch. Dyn.*, 2(3):357–380, 2002. Special issue on stochastic climate models.

[22] Jinqiao Duan, Kening Lu, and Björn Schmalfuss. Invariant manifolds for stochastic partial differential equations. *Ann. Probab.*, 31(4):2109–2135, 2003.

[23] Jinqiao Duan, Kening Lu, and Björn Schmalfuss. Smooth stable and unstable manifolds for stochastic evolutionary equations. *J. Dynam. Differential Equations*, 16(4):949–972, 2004.

[24] Yu. V. Egorov and M. A. Shubin. Linear partial differential equations. Foundations of the classical theory. In *Partial differential equations, I*, volume 30 of *Encyclopaedia Math. Sci.*, pages 1–259. Springer, Berlin, 1992.

[25] J. Escher. A note on quasilinear parabolic systems with dynamical boundary conditions. In *Progress in partial differential equations: the Metz surveys, 2 (1992)*, volume 296 of *Pitman Res. Notes Math. Ser.*, pages 138–148. Longman Sci. Tech., Harlow, 1993.

[26] Joachim Escher. Quasilinear parabolic systems with dynamical boundary conditions. *Comm. Partial Differential Equations*, 18(7-8):1309–1364, 1993.

[27] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[28] Franco Flandoli and Björn Schmalfuß. Weak solutions and attractors for three-dimensional Navier-Stokes equations with nonregular force. *J. Dynam. Differential Equations*, 11(2):355–398, 1999.

[29] Gilles François. Spectral asymptotics stemming from parabolic equations under dynamical boundary conditions. *Asymptot. Anal.*, 46(1):43–52, 2006.

[30] Ciprian G. Gal, Gisèle Ruiz Goldstein, and Jerome A. Goldstein. Oscillatory boundary conditions for acoustic wave equations. *J. Evol. Equ.*, 3(4):623–635, 2003. Dedicated to Philippe Bénilan.

[31] Leszek Gawarecki and Vidyadhar Mandrekar. *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[32] Peter Imkeller and Björn Schmalfuss. The conjugacy of stochastic and random differential equations and the existence of global attractors. *J. Dynam. Differential Equations*, 13(2):215–249, 2001.

[33] H. Keller. Attractors for stochastic sine gordon equations with nontrivial noise. *Preprint Universität Bremen*, 1999.

[34] H. Keller and B. Schmalfuss. Attractors for stochastic differential equations with nontrivial noise. *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, (1):43–54, 150, 153–154, 1998.

[35] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[36] Kening Lu and Björn Schmalfuß. Invariant manifolds for stochastic wave equations. *J. Differential Equations*, 236(2):460–492, 2007.

[37] A. C. McBride. *Semigroups of linear operators: an introduction*, volume 156 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1987.

[38] William McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[39] Bernt Øksendal. *Stochastic differential equations.* Universitext. Springer-Verlag, Berlin, sixth edition, 2003. An introduction with applications.

[40] T. Ozgokmen, T. Iliescu, P. Fischer, A. Srinivasan, and J. Duan. Large eddy simulation of stratified mixing in two-dimensional dam-break problem in a rectangular enclosed domain. 16:106–140, 2007.

[41] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[42] J. P. Peixoto and A. H. Oort. *Physics of Climate.* Springer, New York, 1992.

[43] James C. Robinson. *Infinite-dimensional dynamical systems.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001. An introduction to dissipative parabolic PDEs and the theory of global attractors.

[44] Björn Schmalfuß. The random attractor of the stochastic Lorenz system. *Z. Angew. Math. Phys.*, 48(6):951–975, 1997.

[45] Björn Schmalfuss. A random fixed point theorem and the random graph transformation. *J. Math. Anal. Appl.*, 225(1):91–113, 1998.

[46] George R. Sell and Yuncheng You. *Dynamics of evolutionary equations*, volume 143 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 2002.

[47] Roger Temam. *Navier-Stokes equations*, volume 2 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, revised edition, 1979. Theory and numerical analysis, With an appendix by F. Thomasset.

[48] Roger Temam. *Navier-Stokes equations and nonlinear functional analysis*, volume 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1995.

[49] Roger Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences.* Springer-Verlag, New York, second edition, 1997.

[50] M. J. Višik and A. V. Fursikov. *Mathematische Probleme der statistischen Hydromechanik*, volume 41 of *Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology].* Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1986. Translated from the Russian by Bernd Krause.

[51] Ioan I. Vrabie. *$C_0$-semigroups and applications*, volume 191 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 2003.

[52] T. Wanner. Linearization of random dynamical systems. In *Dynamics reported*, volume 4 of *Dynam. Report. Expositions Dynam. Systems (N.S.)*, pages 203–269. Springer, Berlin, 1995.

[53] Dirk Werner. *Funktionalanalysis.* Springer-Verlag, Berlin, extended edition, 2000.

[54] Joseph Wloka. *Partielle Differentialgleichungen.* B. G. Teubner, Stuttgart, 1982. Sobolevräume und Randwertaufgaben. [Sobolev spaces and boundary value problems], Mathematische Leitfäden. [Mathematical Textbooks].

[55] Desheng Yang and Jinqiao Duan. An impact of stochastic dynamic boundary conditions on the evolution of the Cahn-Hilliard system. *Stoch. Anal. Appl.*, 25(3):613–639, 2007.

[56] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York, 1990. Linear monotone operators, Translated from the German by the author and Leo F. Boron.