Purity of G-zips

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Zusammenfassung. Sei k ein perfekter Körper der Characteristik $p > 0$, und S ein Schema über k. Ein F-zip ist ein lokal freier $\mathcal{O}_S$-Modul vom endlichen Rang versehen mit zwei Filtrierungen und einem Frobenius-linearen Isomorphismus zwischen deren graduierten Stücken. Eine natürliche Verallgemeinerung dieses Begriffs für eine reduktive algebraische Gruppe G/k ergibt einen "F-zip mit G-structure", so genannter G-zip, der zuerst in [PWZ12] eingeführt wurde. Ein G-zip $\underline{I}$ über S liefert die Zerlegung des Basisschemas $S = \bigcup_w S_{\underline{I}}^{[w]}$ in Strata, auf denen $\underline{I}$ lokal eine konstante Isomorphieklasse für fppf Topologie besitzt. Wir zeigen, dass $S_{\underline{I}}^{[w]} \hookrightarrow S$ affin sind, und geben eine Reihe geometrischer Anwendungen davon.

Abstract. Let k be a perfect field of characteristic $p > 0$, and S an scheme over k. An F-zip is basically a locally free $\mathcal{O}_S$-module of finite rank endowed with two filtration and an Frobenius-linear isomorphism between their graded pieces. The natural generalization of this notion for a reductive algebraic group G/k is an "F-zip with G-structure", a so-called G-zip introduced in [PWZ12]. A G-zip $\underline{I}$ over S yields the stratification of the base scheme $S = \bigcup_w S_{\underline{I}}^{[w]}$ in loci, where $\underline{I}$ has locally a constant isomorphism class for the fppf topology. We show that $S_{\underline{I}}^{[w]} \hookrightarrow S$ are affine and give a number of geometric applications of this purity result.

Introduction

Background and motivation. Let $k = \mathbb{F}_p$ for the sake of simplicity unless stated otherwise.

Giving a short historical account of purity problems in the algebraic and arithmetic geometry one should mention the Purity Theorem (2000) of de Jong-Oort [dJO00]:

Theorem 0.1. *Let S be an integral, excellent scheme in characteristic p. Let $\mathfrak{X} \to S$ be a Barsotti-Tate group over S. Further let $U \subset S$ be the largest (open dense) [1] set on which the Newton polygon is constant. Then, either $U = S$, or $S - U$ has codimension one in S.*

Let us remark that one could require some other regularity/finiteness conditions instead of "excellence" of S.

[1] these properties are automatically satisfied for a such set, see [Kat79, Thm. 2.3.1, p. 143]

This kind of result is referred by A. Vasiu in [Vas02] as “the weaker variant of purity”. In fact, he shows a stronger version of the above theorem which implies the de Jong-Oort’s result by applying the standard Hartogs-like yoga:

Theorem 0.2. *Let $\mathfrak{X}$, $U \subset S$ are as in Theorem 0.1. Then the open inclusion $U \hookrightarrow S$ is affine.*

Afterwards F. Oort gave an alternative proof of the above theorem in his conference talk (see [Oor02]) similar in flavor to that of A. Vasiu.

The authors of [NVW10] consider another purity problem for Barsotti-Tate groups: Pick $m \in \mathbb{N}$, and let S an arbitrary scheme over k. Let $\mathfrak{X}_m$ be an m-truncated Barsotti-Tate group over S. Further let $S^m_{X'}$ be the subscheme of S that describes the locus where the $\mathfrak{X}_m$ is locally for the fppf topology isomorphic to $X' \underset{k}{\times} S$, where X' is an m-truncated Barsotti-Tate group over k. As shown in *loc.cit.* the assertion $S^m_{X'} \hookrightarrow S$ affine holds for all primes $p \geq 5$, and under some strong conditions on X' it holds also for $p \in \{2, 3\}$. One should mention that the core of the proof is based on the case $m = 1$; this case readily implies the case $m > 1$. For $m = 1$ this purity result is equivalent to purity for a special class of F-zips, see below for an informal introduction to them.

Another motivation for this work comes from the fact that some data of geometrical origin, e.g., de Rham cohomology groups of certain projective varieties, has a structure of a so-called F-zip with maybe some additional structures. The notion of an F-zip was first introduced in [MW04]. Its authors B. Moonen and T. Wedhorn studied the de Rham cohomology $H^n_{\mathrm{dR}}(X/S)$ of a smooth proper scheme $f\colon X \to S$. They showed that under assumption of the so-called condition (D) which says: i) the higher direct images $R^a f_* \Omega^b_{X/S}$ for $a, b \in \mathbb{N}_0$ are locally free $\mathcal{O}_S$-modules of finite rank, and ii) the Hodge-de Rham spectral sequence degenerates at E_1, follows that $M := H^n_{\mathrm{dR}}(X/S)$ carries a structure of an F-zip, i.e. M is endowed with two filtrations (“Hodge” and “conjugate” filtration), and there is a Frobenius-linear morphism between their graded pieces induced by the Cartier isomorphism. For a general reductive algebraic group G, R. Pink, T. Wedhorn, P. Ziegler defined in [PWZ12] the notion of an F-zip with G-structure, called a *G-zip* (see Definition 1.5). These additional structures arise naturally: For instance assume that $f\colon X \to S$ is of pure dimension d with geometrically connected fibers satisfying condition (D). Then the cup pairing on the "middle" de Rham cohomology group $H^d_{\mathrm{dR}}(X/S)$ gives rise to a symplectic (resp. a symmetric) pairing for d odd (resp. even). In this case one obtains a G-zip, where $G = \mathrm{CSp}_{h,k}$ (resp. $G = \mathrm{CO}_{h,k}$), which is the group of symplectic simultudes (resp. of orthogonal simultudes) for $h = \operatorname{rank}_{\mathcal{O}_S} H^d_{\mathrm{dR}}(X/S)$, see [PWZ12, §8]. Another example are F-zips with additional structures associated to abelian varieties with certain extra data determined by a Shimura PEL-datum, see [VW12].

A G-zip over S yields a stratification of a base scheme S in similar fashion as explained above in case of an m-truncated Barsotti-Tate group over S. In its turn, giving an F-zip of rank n is equivalent to giving an $\mathrm{GL}_{n,k}$-zip.

In case of $S = \operatorname{Spec} k$ specifying these two filtrations for an F-zip is equivalent to giving two opposite parabolic subgroups of $\mathrm{GL}_{n,k}$, and a Frobenius-linear map between their Levi-factors. The generalization thereof leads to a concept of *algebraic zip datum* introduced in [PWZ11], which is a quadruple $\mathcal{Z} = (G, P, P', \varphi)$, where G is a reductive algebraic group over k, P and P' are parabolic subgroups with unipotent radicals R_uP resp. R_uP', and an isogeny $\varphi\colon P/R_uP \to P'/R_uP'$.

One could ask whether (G, P, P', φ) and (G, P, P', ψ) define the same algebraic zip datum up to a change of basis. To tackle this problem one defines an action of the associated zip group $E_{\mathcal{Z}} = \{(p', p) \in P' \times P : \varphi([p']) = [p]\}$ on G given by $((p', p), g) \mapsto p'gp^{-1}$♦.

The elements g and $g' \in G$ lie on the same orbit whenever they correspond the same φ up to a change of basis.

Let us remark that the notion of (non-connected) algebraic zip datum considered in [PWZ12] has a more general setting as above with G, P and P' playing a rôle of the neutral components of some, in general non-connected, algebraic groups. But the purity problem considered here can be reduced to the case of connected algebraic zip datum, see also Remark 2.1, hence we limit ourself to study the connected version.

A crucial rôle in [PWZ12] as well as in this paper plays the algebraic stack $[E_{\mathcal{Z}}\backslash G]$, which is the quotient stack with respect to the above action.

G-zips are the objects over S that look fibrewise like an algebraic zip datum. It turns out that their classifying stack is isomorphic to $[E_{\mathcal{Z}}\backslash G]$.

Results. Let k here be a perfect field containing $\mathbb{F}_p$, and S be a k-scheme.

Basically in this paper, we prove the following purity result and give several applications:

Theorem A. *Let k be an algebraically closed field. Suppose that G contains a finite number of $E_{\mathcal{Z}}$-orbits with respect to the action $\blacklozenge$ on it. Then $E_{\mathcal{Z}}$ acts on G with affine orbits.*

The above Theorem implies the following easy but important corollary:

Theorem B. *Let $\underline{I}$ be a G-zip of over S. $\underline{I}$ yields the finite decomposition $S = \bigcup_w S_{\underline{I}}^{[w]}$ in loci, where $\underline{I}$ has locally a constant isomorphism class for the fppf topology. Then $S_{\underline{I}}^{[w]} \overset{\jmath}{\hookrightarrow} S$ is affine.*

Check section 3 for more details about the index set of the above decomposition.

We also prove some variant of the lemma which allows to deduce a weak version of purity from the strong version, i.e. affineness of an inclusion.

Corollary C. *Let X be a scheme, Y be an locally-noetherian scheme, $X \hookrightarrow Y$ be an affine immersion. Denote by $\overline{X}$ the closure of X in Y and let Z be an irreducible component of $\overline{X} \backslash X \neq \varnothing$. Then* $\operatorname{codim}(Z, \overline{X}) = 1$.

In its turn, the above lemma implies the following weak purity result:

Theorem D. *Suppose that S is a locally noetherian k-scheme, Z a closed subscheme of S of codimension ≥ 2, which contains no embedded components of S that the restriction of $\underline{I}$ to $S \backslash Z$ is fppf locally constant, then $\underline{I}$ is fppf locally constant.*

Next we harvest results in the applications; first we reprove the result in [NVW10] about the purity of the stratification of a basis scheme S based on the local isomorphism class of $\mathfrak{X}_m$ discarding all restrictions in characteristics 2 and 3:

Theorem E. *In the notation of the previous section holds: The inclusion $S_{X'}^{m} \hookrightarrow S$ is affine.*

Let now X be a smooth proper scheme over S. At the very end of the paper we give some sufficient conditions and examples when de Rham cohomology $H^n_{\mathrm{dR}}(X/S)$ carries the structure of F-zip making in particular the purity result applicable in this case.

Proposition F. *Let $f\colon X \to S$ be a smooth proper morphism of schemes. Suppose that there is a lift of X in zero characteristic (see Definition 5.9), $\tilde{f}\colon \tilde{X} \to \tilde{S}$ such that $\tilde{X}$ and $\tilde{S}$ are locally noetherian schemes, $\tilde{f}$ is proper and smooth, and $\tilde{S}$ reduced.*

Further assume the Hodge numbers $\tilde{s} \mapsto \dim_{\kappa(\tilde{s})} H^b(\tilde{X}_{\tilde{s}}, \Omega^a_{\tilde{X}_{\tilde{s}}/\kappa(\tilde{s})})$ are locally constant on $\tilde{S}$ for all $a, b \in \mathbb{N}_0$.

Then f satisfies condition (D).

We also give examples of application of the last proposition.

Content. This paper is organized as follows. Section 1 contains a short recollection of basic facts about algebraic zip datum, the associated quotient stack, and F- and G-zips presented in [PWZ12] and [PWZ11].

Section 2 gives an insight in the geometry of the orbits in Theorem A, and culminates in its proof.

In section 3 will be explained how Theorem A implies purity results for the strata of Theorem B, and week purity results of Lemma C and Corollary D.

Section 4 outlines some applications of the purity results: In subsections 4.1 and 4.2 we concern us with the purity result of [NVW10], see Theorem E.

Section 5 focuses on the de Rham cohomology $H^n_{\mathrm{dR}}(X/S)$ of a proper smooth variety over S, and on conditions upon which it carries an F-zip structure. It discusses also some examples.

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1. Preliminaries: General notation and basic facts

Algebraic zip datum. Let k be a field extension of a finite field $\mathbb{F}_q$ of order q, which is a perfect field, and let S be a scheme over k. We denote by G a (connected) reductive quasi-split algebraic group over the field k, fix $T \subset G$ a maximal torus and $T \subset B \subset G$ a Borel subgroup. Further let $P, P' \subset G$ be parabolic subgroups such that $B \subset P$ and ${}^{g_0}B \subset P'$ for some fixed element $g_0 \in G$.

Denote by U and U' the unipotent radicals of P resp. P' and by L and L' their unique Levi-factors verifying $T \subset L$ and ${}^{g_0}T \subset L'$. In this way, we obtain two canonical projections $\pi_L \colon P \to L$, $\pi_{L'} \colon P' \to L'$.

Furthermore, we restrict our attention to such pairs (P', P), such there is an isogeny $\varphi \colon L' \to L$ satisfying the constraints $\varphi({}^{g_0}B \cap L') = B \cap L$ and $\varphi({}^{g_0}T) = T$.

We recall the following central definition introduced in [PWZ11]:

Definition 1.1. 1) A *connected algebraic zip datum* [3] $\mathcal{Z}$ is a quadruple $\mathcal{Z} = (G, P, P', \varphi)$ as above.

2) The linear algebraic group $E_{\mathcal{Z}}$ over k given by

$$E_{\mathcal{Z}} = \{(p', p) \in P' \times P : \varphi(\pi_{L'}(p')) = \pi_L(p)\} \tag{1.1}$$

is called *zip group associated to* $\mathcal{Z}$.

The group $E_{\mathcal{Z}}$ acts on G by:

$$((p', p), g) \mapsto p' g p^{-1} \text{ for } (p, p') \in E_{\mathcal{Z}}, g \in G \tag{1.2}$$

Or, more explicitly, writing $P' = U' \rtimes L' = U' \cdot L'$ and $P = U \rtimes L = U \cdot L$, $p' = u'l'$, $p = ul$, this action becomes:

$$((p', p), g) \mapsto u'l'g\varphi(l')^{-1}u^{-1}.$$

Moreover, we impose the following additional condition:

(FC) 1.2. *For an algebraic closure $\bar{k}$ of k there is only a finite number of $E_{\mathcal{Z}}(\bar{k})$-orbits of $G(\bar{k})$.*

We will see in the section 2 that the condition (FC) is in particular fulfilled if Lie $\varphi = 0$, but in fact the latter condition is too strong.

Throughout this paper we consider the algebraic quotient stack $[E_{\mathcal{Z}} \backslash G]$.

The geometric situation described in [PWZ12] leads to some special kind of algebraic zip datum associated to a cocharacter $\chi \colon \mathbb{G}_{m,k} \to G$.

[3] this definition was originally made in the case of algebraically closed field k

We assume that the reductive algebraic group G is defined over $\mathbb{F}_q$ i.e. $G = G'_k$, where G' is a reductive algebraic group over $\mathbb{F}_q$. Let L be the centralizer of χ in G. Then, there are two opposite parabolic subgroups $P_\pm = L \ltimes U_\pm$ with the common Levi factor L and the unipotent radicals $U_\pm$, where the Lie algebras $\mathfrak{u}_\pm$ are directs sums of positive resp. negative weight spaces in the Lie algebra $\mathfrak{g}$ under $\mathrm{Ad} \circ \chi$.

We denote by $(.)^{(q)}$ a pullback of a scheme or a sheaf under q^{th}- power absolute Frobenius map $S \to S$ resp. $k \to k$.

Clearly, we have $G^{(q)} = G$.

Lemma 1.3. *Let G be a reductive algebraic group over k defined over $\mathbb{F}_q$. Furthermore, let P be a parabolic subgroup of G and $L \subset P$ be a Levi subgroup. There exist a maximal torus T, a Borel subgroup B of G both already defined over $\mathbb{F}_q$, and $\bar{g} \in G(k)$ such that $T \subset {}^{\bar{g}}L$ and $T \subset B \subset {}^{\bar{g}}P$.*

Proof. By the assumption, G is a quasi-split algebraic group, thus we can choose a torus T and a Borel subgroup $B \supset T$ defined over $\mathbb{F}_q$.

By [DG64, Exposé XXVI, Lemme 3.8.] there is the parabolic subgroup P' such that $B \subset P'$, and P' is of the same type as P. By Proposition 1.6 *loc.cit.* there is the unique Levi subgroup L' of P' such that $T \subset L'$. Then the assertion of the lemma is a direct consequence of Corollaire 5.5.(iv) *loc.cit.*. ☐

A new zip datum $(G, {}^{\bar{g}}P, {}^{\bar{g}}P', \mathrm{int}(\bar{g}) \circ \varphi \circ \mathrm{int}\left(\bar{g}^{-1}\right))$ for a $\bar{g} \in G(k)$ is obviously equivalent to the original one, so we assume by the previous lemma that there are a maximal torus $T \subset L$ and a Borel subgroup $P \supset B \supset T$ already defined over $\mathbb{F}_q$.

The relative Frobenius yields the isogeny $\mathrm{Frob}_q \colon L \to L^{(q)} \cong P_-^{(q)}/U_-^{(q)}$. In this way we obtain an algebraic zip datum:

Definition 1.4. The tuple $\mathcal{Z}_{G,\chi} = (G, P_-^{(q)}, P_+, \mathrm{Frob}_q)$ is called the *algebraic zip datum associated to G and χ.*

Note that due to the choice of an isogeny $\varphi = \mathrm{Frob}_q$ the condition (FC) is automatically fulfilled in this case.

The associated zip group to this zip datum is denoted by $E_{G,\chi}$, and the corresponding quotient stack by $[E_{G,\chi} \backslash G]$.

Quotient stack $[E_{G,\chi} \backslash G]$**.** Denote by $\mathrm{Transp}_{E_{G,\chi}}$ the k-scheme $(E_{G,\chi} \times G) \underset{\mu\ G\ \mathrm{id}}{\times} G$, where μ is given by the $E_{G,\chi}$ - group action 1.2.

We may think $[E_{G,\chi}\backslash G]$ as the stack associated to the k-groupoid $\{G/\mathrm{Transp}_{E_{G,\chi}}\}$(see [LMB91, (2.4.3)] for details), i.e.: For a k-scheme S the objects of the k-groupoid are the elements of $G(S)$ and morphisms between two objects g_1, g_2 are the S-valued points of the transporter $\mathrm{Transp}_{E_{G,\chi}}(g_1, g_2)(S)$ of $E_{G,\chi}$-action with the composition given by the multiplication map of $E_{G,\chi}$.

The underlying topological space of $[E_{G,\chi}\backslash G]$ has a following common description [Wed01].

If $k = \bar{k}$, the underlying set $\overline{\Xi}$ is a finite set of $E_{\mathcal{Z}}(\bar{k})$-orbits in $G(\bar{k})$, and the topology is induced by a partial order $\preceq$ on it: For two $E_{G,\chi}(\bar{k})$-orbits $\mathfrak{o}'$ and $\mathfrak{o}$ one sets $\mathfrak{o}' \preceq \mathfrak{o}$ if $\mathfrak{o}' \subset \overline{\mathfrak{o}}$, where $\overline{\mathfrak{o}}$ denotes the closure of $\mathfrak{o}$ in $G(\bar{k})$. The open sets in this topology are explicitly defined by the following property: U is open if and only if for some $\mathfrak{o} \in \overline{\Xi}$ such that $\mathfrak{o}' \preceq \mathfrak{o}$ for all $\mathfrak{o}' \in U$ follows that $\mathfrak{o} \in U$.

Let now k be an arbitrary field, and $\Gamma = \mathrm{Aut}\left(\bar{k}/k\right)$ be the profinite group of k-automorphisms of $\bar{k}$. Then Γ acts on $E_{G,\chi}(\bar{k})$-orbits of $G(\bar{k})$ preserving the order. Therefore, one obtains an induced order on the Γ-orbits of $\overline{\Xi}$, and the underlying topological space of $[E_{G,\chi}\backslash G]$ is isomorphic to $\Xi := \overline{\Xi}/\Gamma$ with the quotient topology.

More specifically, the topological space $\overline{\Xi}$ admits the following geometrical description [PWZ11].

Let $W := \mathrm{Norm}_G(T)(\bar{k})/T(\bar{k})$ be the Weyl group of G, w_0 be the element of maximal length in G, and R_s the corresponding set of simple reflections with respect to $T_{\bar{k}} \subset B_{\bar{k}}$.

Let $K \subset R_s$ be a subset. We denote by W_K the subgroup of the Weyl group W generated by K, and let (cf. [Car85, ch. 2.3])

$$^{K}W := \{w \in W : w \textit{ of the minimal length in the right coset } W_K w\}.$$

Note that the Frobenius isogeny $\varphi\colon G \to G$ induces an automorphism $\overline{\varphi}$ of the Weyl group W.

Let θ_0 be the element of minimal length in $W_J w_0 W_{\overline{\varphi}(I)}$.

Let further $I \subset R_s$ be the type of P_+ and let $J \subset R_s$ be the type of $(P_-)^{(q)}$. Then the restriction $\overline{\psi} := \mathrm{int}\,(\theta_0) \circ \overline{\varphi}\colon W \to W$ induces an isomorphism of Coxeter systems (W_I, I) and (W_J, J).

For $w', w \in {}^{I}W$ one sets $w' \preceq w$ if there is $u \in W_I$ such that $uw'\psi(u)^{-1} \leq w$ with respect to the Bruhat order on W.

As shown in [PWZ12, Subsection 3.5] $\overline{\Xi} \cong {}^{I}W$ with the topology induced by the partial order $\preceq$.

This quotient stack $[E_{G,\chi}\backslash G]$ is useful to describe the isomorphism classes of G-zips of type χ.

G-zips and F-zips. For an affine k-group scheme $\mathcal{G}$ we mean by $\mathcal{G}$-torsor a right $\mathcal{G}_S$-torsor over S for the fpqc-topology. In other words, $\mathcal{G}$-torsor I is a scheme I over S endowed with a right action of $I \times_S (\mathcal{G}_S) \to I$ written $(i,g) \mapsto ig$ such that the morphism $I \times_S \mathcal{G}_{\mathcal{S}} \to I \times_S I$ given by $(i,g) \mapsto (i, ig)$ is an isomorphism, and there is a scheme S' and an fpqc-morphism $S' \to S$ such that $I(S') \neq \varnothing$. Remark that the last condition can be omitted if the structure morphism $I \to S$ is fpqc.

Let $\mathcal{H}$ be a closed subgroup scheme of $\mathcal{G}$ over k . We say that a $\mathcal{H}$-torsor J is a subtorsor of $\mathcal{G}$-torsor J if there is an $\mathcal{H}$-equivariant inclusion $J \hookrightarrow I$, where $\mathcal{H}$ acts on I via restriction of the $\mathcal{G}$-action.

We recall the following definition introduced in [PWZ12].

Definition 1.5. 1) A G-zip of type χ over S is a tuple $\underline{I} = (I, I_+, I_-, \imath)$, where I is a G-torsor over S, $I_+ \subset I$ a P_+ subtorsor, $I_- \subset I$ a $P_-^{(q)}$ subtorsor, and $\imath\colon I_+^{(q)}/U_+^{(q)} \xrightarrow{\sim} I_-/U_-^{(q)}$ an isomorphism of $L^{(q)}$-torsors.

2) A morphism $(I, I_+, I_-, \imath) \to (I', I'_+, I'_-, \imath')$ of G-zips of type χ consists of G resp. $P_+, P_-^{(q)}$ equivariant morphisms $I \to I'$ resp. $I_\pm \to I'_\pm$ which are compatible with inclusions and the isomorphisms $\imath$ and $\imath'$.

The morphisms of G-zips of type χ over S are isomorphisms, hence such G-zips form a groupoid denoted by $G\text{-Zip}_k^\chi(S)$.

As shown in [PWZ12, Prop. 3.2], the groupoids $G\text{-Zip}_k^\chi(S)$ with the obvious pullback definition form a stack $G\text{-Zip}_k^\chi$ fibered over the category $(\mathbf{Sch}/k)$.

The stack $G\text{-Zip}_k^\chi$ is isomorphic to the stack $[E_{G,\chi}\backslash G]$ [PWZ12, Prop. 3.11].

The data coming from many interesting geometric objects in nonzero characteristic carries the structure of so-called F-zips (see [Wed08, Section 2]).

First, we recall the definitions.

Definition 1.6. Let $\mathcal{M}$ be a locally free $\mathcal{O}_S$-module of finite rank.

A *descending filtration* $C^\bullet$ of $\mathcal{M}$ is a family $\{C^i\mathcal{M}\}_{i\in\mathbb{Z}}$ of $\mathcal{O}_S$-submodules of $\mathcal{M}$ which are locally direct summands of $\mathcal{M}$ such that $C^{i+1}\mathcal{M} \subset C^i\mathcal{M}$ for all $i \in \mathbb{Z}$, and $C^i\mathcal{M} = \mathcal{M}$ for $i \ll 0$ and $C^i\mathcal{M} = 0$ for $i \gg 0$.

Similarly, an *ascending filtration* $D_\bullet$ of $\mathcal{M}$ is a family $\{D_i\}_{i\in\mathbb{Z}}$ of $\mathcal{O}_S$-submodules of $\mathcal{M}$ which are locally direct summands of $\mathcal{M}$ such that

$D_{i-1}\mathcal{M} \subset D_i\mathcal{M}$ for all $i \in \mathbb{Z}$, and $D_i\mathcal{M} = \mathcal{M}$ for $i \gg 0$ and $D_i\mathcal{M} = 0$ for $i \ll 0$.

If S has a finite number of connected components, the conditions of the previous definition imply that the subquotiens $\mathrm{gr}^i_C\mathcal{M} := C^i\mathcal{M}/C^{i+1}\mathcal{M}$ and $\mathrm{gr}^D_i\mathcal{M} := D_i\mathcal{M}/D_{i-1}\mathcal{M}$ are locally free $\mathcal{O}_S$-modules that vanish outside a finite index range. Note that locally free $\mathcal{O}_S$-modules endowed with a descending resp. an ascending filtration form a category. Objects of these categories are pairs $(\mathcal{M}, C^\bullet)$ resp. $(\mathcal{M}, D_\bullet)$ and maps are the morphisms of $\mathcal{O}_S$-modules that respect these filtrations.

Definition 1.7. [PWZ12, Definition 6.1.]
1) An F-zip over S is a tuple $\underline{\mathcal{M}} = (\mathcal{M}, C^\bullet, D_\bullet, \varphi_\bullet)$, where $\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of finite rank, $C^\bullet$ a descending filtration of $\mathcal{M}$, $D_\bullet$ an ascending filtration of $\mathcal{M}$, and $\varphi_i \colon \left(\mathrm{gr}^i_C\mathcal{M}\right)^{(q)} \xrightarrow{\sim} \mathrm{gr}^D_i\mathcal{M}$ are $\mathcal{O}_S$-linear isomorphisms.

2) A *homomorphism* $f \colon \underline{\mathcal{M}} = (\mathcal{M}, C^\bullet, D_\bullet, \varphi_\bullet) \to \underline{\mathcal{N}} = (\mathcal{N}, C^\bullet, D_\bullet, \varphi'_\bullet)$ of F-zips over S is a homomorphism of the underlying $\mathcal{O}_S$-modules $\underline{\mathcal{M}} \to \underline{\mathcal{N}}$ satisfying for all $i \in \mathbb{Z}$ the constraints $f(C^i\mathcal{M}) \subset C^i\mathcal{N}$ and $f(D_i\mathcal{M}) \subset D_i\mathcal{N}$ and making the following diagram commute:

$$\begin{array}{ccc} \left(\mathrm{gr}^i_C\mathcal{M}\right)^{(q)} & \xrightarrow[\sim]{\varphi_i} & \mathrm{gr}^D_i\mathcal{M} \\ \Big\downarrow{\scriptstyle \left(\mathrm{gr}^i_C f\right)^{(q)}} & & \Big\downarrow{\scriptstyle \mathrm{gr}^D_i f} \\ \left(\mathrm{gr}^i_C\mathcal{N}\right)^{(q)} & \xrightarrow[\sim]{\varphi_i'} & \mathrm{gr}^D_i\mathcal{N} \end{array}$$

The resulting category of F-zips over S is denoted by F-Zip(S). Its simplest objects are so-called Tate F-zips.

Example 1.8. The Tate F-zip of weight $d \in \mathbb{Z}$ is the F-zip $\underline{\mathbb{1}}(d) = (\mathcal{O}_S, C^\bullet, D_\bullet, \varphi_\bullet)$, where

$$C^i = \begin{cases} \mathcal{O}_S & \text{for } i \leq d, \\ 0 & \text{for } i > d \end{cases} \text{ and } D_i = \begin{cases} 0 & \text{for } i < d, \\ \mathcal{O}_S & \text{for } i \geq d \end{cases} \tag{1.3}$$

with φ_d is the identity on $\mathcal{O}_S = (\mathcal{O}_S)^{(q)}$.

With the natural definition of the tensor product and the duals [PWZ12, Section 6] the $\mathbb{F}_q$-linear category F-Zip(S) becomes a rigid tensor category with the unit object $\underline{\mathbb{1}}(0)$.

Definition 1.9. An F-zip is called of *rank* n, or *height* n if its underlying $\mathcal{O}_S$-module has constant rank n.

Let $\underline{n}\colon \mathbb{Z} \to \mathbb{N}_0$ be a function with finite support. An F-zip $\underline{\mathcal{M}}$ is called of type $\underline{n}$ if the graded pieces $\mathrm{gr}^i_C\mathcal{M}$, or equivalently $\mathrm{gr}^D_i\mathcal{M}$, are locally free $\mathcal{O}_S$-modules of constant rank $n_i := \underline{n}(i)$ for all $i \in \mathbb{Z}$.

Let $k = \mathbb{F}_q$, and S be a $\mathbb{F}_q$-scheme. Denote by $F\text{-Zip}^{\underline{n}}_k(S)$ be a subcategory of $F\text{-Zip}(S)$ whose objects are F-zips of type $\underline{n}$ and morphisms are isomorphisms. Since F-zips consist of quasi-coherent sheaves and the morphisms thereof, they satisfy the effective descent with respect to any fpqc-morphism of $\mathbb{F}_q$-schemes $S' \to S$. Therefore, one obtains the category $F\text{-Zip}^{\underline{n}}_k$ fibered in groupoids which is a stack.

Let $\underline{\mathcal{M}} = (\mathcal{M}, C^\bullet, D_\bullet, \varphi_\bullet)$ be an F-zip of type $\underline{n}$ over S.

It is immediate from the definition that $\mathcal{M}$ is Zariski locally isomorphic to the free $\mathcal{O}_S$-module $(k^n)_S = \mathcal{O}_S^n$, and the filtered $\mathcal{O}_S$-modules $(\mathcal{M}, C^\bullet)$ and $(\mathcal{M}, D_\bullet)$ to $(k^n, C^\bullet)_S$ resp. $(k^n, D_\bullet)_S$. Moreover, by a change of basis, we can assume that $\mathrm{gr}^i_C k^n = \mathrm{gr}^D_i k^n = k^{n_i}$.

Let $P_+ := \underline{\mathrm{Aut}}\left((k^n, C^\bullet)\right)$ and $P_- := \underline{\mathrm{Aut}}\left((k^n, D_\bullet)\right)$. As result, we get two opposite parabolic subgroups of GL_k defined over $k = \mathbb{F}_q$ together with the isogeny of their common Levi factor $L = P_+ \cap P_-$ induced by the Frobenius. As usual denote by U_+ and U_- their unipotent radicals.

Now on obtains the corresponding $\mathrm{GL}_{n,k}$-zip $\underline{I} = (I, I_+, I_-, \varphi)$ by taking

$$\begin{aligned} I = {} & \underline{\mathrm{Iso}}\left((k^n)_S, \mathcal{M}\right), \\ I_+ = {} & \underline{\mathrm{Iso}}\left((k^n, C^\bullet)_S, (\mathcal{M}, C^\bullet)\right), \\ I_- = {} & \underline{\mathrm{Iso}}\left((k^n, D_\bullet)_S, (\mathcal{M}, D_\bullet)\right) \end{aligned}$$

Forgetting filtrations gives the $P_\pm$ equivariant embeddings $I_\pm \hookrightarrow I$. Moreover, the isomorphism $\varphi_\bullet\colon (\mathrm{gr}^\bullet_C\mathcal{M})^{(q)} \xrightarrow{\sim} \mathrm{gr}^D_\bullet\mathcal{M}$ induces an isomorphism of L-torsors:

$$\begin{aligned} &(I_+)^{(q)}/U_+ \cong \underline{\mathrm{Iso}}\left((\mathrm{gr}^\bullet_C k^n)_S, (\mathrm{gr}^\bullet_C\mathcal{M})^{(q)}\right) \xrightarrow{\sim} \underline{\mathrm{Iso}}\left((\mathrm{gr}^D_\bullet k^n)_S, \mathrm{gr}^D_\bullet\mathcal{M}\right) \\ &\cong I_-/U_-. \end{aligned}$$

As shown in [PWZ12, Subsection 8.1], the assignment of an F-zip to $\mathrm{GL}_{n,k}$-zip, which is $\mathbb{F}_q$-linearly functorial and compatible with the pullback, gives rise to an isomorphism of stacks $F\text{-Zip}^{\underline{n}}_k$ and $\mathrm{GL}_n\text{-Zip}^\chi_k$. Recall that $\mathrm{GL}_n\text{-Zip}^\chi_k \cong \left[E_{\mathrm{GL}_{n,k},\chi}\backslash\mathrm{GL}_{n,k}\right]$.

The F-zips with additional structures can also be translated to G-zips for an appropriate reductive group (see *loc. cit.*).

The practical upshot from the above discussion is that the study of isomorphism classes of F-zips or G-zips of the fixed type reduces to the study of the stack $[E_{G,\chi}\backslash G]$.

2. Affineness of the orbits

Let $k = \bar{k}$ throughout this subsection. Its aim is to show that under assumption of the condition (FC) the orbits of the action 1.2 are affine. As we will see further on, it implies purity of G-zips.

Remark 2.1. An orbit of the action 1.2 on $\hat{G}$ in the non-connected setting (see [PWZ12, Definition 3.6.]) is a finite scheme theoretically disjoint union of locally closed subsets of $\hat{G}$ which are isomorphic to the orbits of the related connected zip data given by the neutral components G of $\hat{G}$, and two parabolic subgroups. Hence, as already noted in the introduction, we can restrict us to the study of the connected case.

We first recall some basic facts.

Theorem 2.2. [Car85, Lang-Steinberg thm., section 1.17] *Let $\mathcal{G}$ be an affine connected algebraic group over k, $F\colon \mathcal{G} \to \mathcal{G}$ an isogeny, such that $\mathcal{G}^F = \{g \in \mathcal{G} : F(g) = g\}$ is finite. Then the morphism of k-varieties $\mathcal{L}\colon \mathcal{G} \to \mathcal{G}, g \mapsto g^{-1}F(g)$ is surjective. In particular, taking for F a Frobenius map satisfies the condition of the theorem.*

Remark 2.3. Going through the proof one can see the theorem holds if the above finiteness condition is replaced by the condition that Lie F is nilpotent.

We suppose now that the conditions of the previous theorem hold. Then:

(i) By composing with the map $g \mapsto g^{-1}$ we conclude $\mathcal{L}'\colon \mathcal{G} \to \mathcal{G}, g \mapsto gF(g)^{-1}$ is also surjective.
(ii) Let $\mathcal{G}$ act on itself by F-conjugation, i.e. $(g, x) \mapsto gxF(g)^{-1}$, $g, x \in G$. Then $\mathcal{G}$ coincides with the orbit of 1, hence this action is transitive.

The following two easy corollaries will be further useful.

Corollary 2.4. *Let $F\colon \mathcal{G} \to \mathcal{G}$ be an isogeny, and x_0 be a point of $\mathcal{G}$. The following statements are equivalent:*

(i) *$\mathcal{L}$ is surjective*
(ii) *$\mathcal{G}^F$ is finite*
(iii) *$\mathcal{L}_{x_0}\colon \mathcal{G} \to \mathcal{G},\ g \mapsto gx_0F(g)^{-1}$ is surjective*

(iv) $\mathcal{G}^{x_0,F} := \{g \in \mathcal{G} : gx_0F(g)^{-1} = x_0\}$ *is finite*
(v) $\mathcal{L}_{x_0}$ *is a finite morphism*

Proof. (ii) $\Rightarrow$ (i) is exactly the statement of Lang-Steinberg theorem.

(i) $\Rightarrow$ (ii): By the previous remark the F-conjugation is transitive. Then, for the dimension reason, the stabilizers of all points of $\mathcal{G}$, in particular $1 \in \mathcal{G}$ which is $\mathcal{G}^F$, are finite.

(i) $\Leftrightarrow$ (iii): Both statements are equivalent to the transitivity of F-conjugation.

(iii) $\Rightarrow$ (iv): The F-conjugation is transitive by the previous remark. Then, for the dimensions reasons, the stabilizers of all points of $\mathcal{G}$, in particular $\mathcal{G}^{x_0,F}$, are finite.

(iv) $\Rightarrow$ (iii): Consider $F'(g) := x_0\ F(g)x_0^{-1}$. Then by Lang-Steinberg theorem the map $\mathcal{L}'' : g \mapsto gF'(g)^{-1}$ is surjective. Hence $\mathcal{L}_{x_0} = \mathcal{L}''x_0$ is so as well.

(v) $\Rightarrow$ (iv): $\mathcal{G}^{x_0,F}$ is finite as being a fiber of a finite (in particular a quasi-finite) morphism.

(iv) $\Rightarrow$ (v): Note that $\mathcal{L}_{x_0} \colon \mathcal{G} \to \mathcal{G}$ is a (right) $\mathcal{G}^{x_0,F}$-torsor.

By passing to $(G)_{\mathrm{red}}$ we can assume G is smooth. Thus we have the following Cartesian diagram:

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G}^{x_0,F} & \xrightarrow{\pi_1} & \mathcal{G} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \mathcal{L}_{x_0}} \\ \mathcal{G} & \xrightarrow{\mathcal{L}_{x_0}} & \mathcal{G} \end{array}$$

Here π_1 is a projection onto the first factor. The map $\mathcal{L}_{x_0}$ is quasi-finite and surjective by the foregoing part of the proof, hence it is faithfully flat as a morphism of smooth varieties. The projection π_1 is clearly finite, hence by faithfully flat descent $\mathcal{L}_{x_0}$ is so as well.

□

Corollary 2.5. *Let $F \colon \mathcal{G} \to \mathcal{G}$ be an isogeny. The $\mathcal{G}$-action on $\mathcal{G}$ by F-conjugation (cf.2.3(ii)) is transitive if and only if the stabilizers of all points of $\mathcal{G}$ are finite. Otherwise there exist infinitely many orbits of $\mathcal{G}$-action by F-conjugation in $\mathcal{G}$.*

Proof. The first statement is immediate by Corollary 2.4. Suppose now there are finitely many orbits of the $\mathcal{G}$-action by F-conjugation. Since $\mathcal{G}$ is connected, one of them lies dense in $\mathcal{G}$. Therefore, for the dimension reason it has a finite stabilizer. Now by 2.4 follows that the $\mathcal{G}$-action by F-conjugation is transitive. □

Now let $I \in R_s$ resp. $J \in R_s$ be the types of the parabolic subgroups ${}^{g_0^{-1}}P'$ and P both containing Borel subgroup B. Recall that W_I and W_J are subgroup of Weyl group W generated by sets of simple reflections I resp. J.

As is well known, we have the following Bruhat decomposition of G:

$$G = \bigcup_{w \in {}^I W^J} P' w P,$$

where ${}^I W^J$ is a system of representatives for $W_I \backslash W / W_J$ in the normalizer of T. By the left translation with g_0 it yields the decomposition: $G = \bigcup_{w \in {}^I W^J} P' g_0 w P$.

We now fix some arbitrary w as above. The set $P' g_0 w P$ being $P' \times P$ orbit in G is locally closed, and obviously stable under the $E_{\mathcal{Z}}$-action 1.2, hence each orbit is contained in exactly one of such pairwise disjoint pieces. Moreover, it is clear by the definition of $E_{\mathcal{Z}}$-action that each orbit in $P' g_0 w P$ contains an element of the form $g = g_0 w l$ for some $l \in L$.

Now consider the right homogeneous space G/U with the action of P' on it given by

$$(p', [g]) \mapsto [p' g \varphi(\pi_{L'}(p'))^{-1}].$$

Note that the restriction of the projection $P' \times P \to P'$ gives rise to the surjective morphism $E_{\mathcal{Z}} \to P'$. Via this morphism we obtain the action of $E_{\mathcal{Z}}$ on G/U making the quotient map $G \to G/U$ $E_{\mathcal{Z}}$-equivariant. Thus, we get the faithfully flat morphism bijectively mapping the $E_{\mathcal{Z}}$-orbits of G onto the P'-orbits of G/U. Moreover, this map is affine by [GW10, ch. 12, Prop.12.3.(3)] since G is affine, and, as is well known, the homogeneous space G/U is a quasi-projective variety [4] (see [SR05, Ch. 7, Thm. 4.2]), in particular separated.

Note that the above morphism $E_{\mathcal{Z}} \to P'$ induces an isomorphism $\iota \colon \mathrm{Stab}_{E_{\mathcal{Z}}}(g_0 w l) \to \mathrm{Stab}_{P'}([g_0 w l])$ with respect to the actions of $E_{\mathcal{Z}}$ resp. P'. The inverse map $\iota^{-1} \colon \mathrm{Stab}_{P'}([g_0 w l]) \to \mathrm{Stab}_{E_{\mathcal{Z}}}(g_0 w l)$ is given by $p' \mapsto (p', {}^{(g_0 w l)^{-1}} p')$.

Clearly, $\mathrm{Stab}_{P'}([g_0 w l]) \subset P' \cap {}^{g_0 w} P$, so we have $\mathrm{Stab}_{P'}([g_0 w l]) = \mathrm{Stab}_{P' \cap {}^{g_0 w} P}([g_0 w l])$.

[4] G/U is even quasi-affine since U is observable in G (cf. [SR05, Ch. 10, Observation 2.4.,Thm. 5.4]).

Remark 2.6. There is a possibility to pass from an original zip datum $\mathcal{Z} = (G, P, P', \varphi)$ to some new zip datum $\mathcal{Z}_n$ containing the algebraic groups of lower dimensions.

This reduction process is introduced in [PWZ11, Section 4], and it yields a one-to-one closure preserving correspondence between the orbits inside a $E_{\mathcal{Z}}$-stable piece $P' g_0 w P$ and the orbits in L with respect to the $E_{\mathcal{Z}_n}$-action. The new zip datum is given by $\mathcal{Z}_n := (L, Q, Q', \psi)$, where $Q := \varphi(L' \cap {}^{g_0 w}P)$, $Q' := L \cap {}^{w^{-1}g_0^{-1}}P'$ are two parabolic subgroups of L together with the isogeny

$$\psi := \varphi \circ \operatorname{int}(g_0 w)\,|_{L \cap {}^{w^{-1}g_0^{-1}}L'} \colon L \cap {}^{w^{-1}g_0^{-1}}L' \to \varphi(L' \cap {}^{g_0 w}L)$$

between their Levi factors. Note that Lie $\varphi = 0$ implies Lie $\psi = 0$.

As the dimensions of the algebraic groups get smaller, the reduction process terminates in a finite number of steps. For a terminating zip datum holds $G = L$. Thus, it must be of the form (G, G, G, φ), and $E_{\mathcal{Z}} \cong G$ acts on G by: $(g, x) \mapsto g x \varphi(g)^{-1}$, $g, x \in G$.

Since we assume the condition (FC) 1.2 it follows that there is the only finite number of orbits with respect of φ-conjugate G-action. Then by Corollary 2.5 we conclude that G acts transitively with finite stabilizers.

This reduction process makes it possible to give an inductive description of the stabilizers of the point $g_0 w l \in P' g_0 w P$.

Lemma 2.7. *There is an exact sequence of algebraic groups:*
(2.1)

$$1 \longrightarrow \ker e_n \longrightarrow \operatorname{Stab}_{P' \cap {}^{g_0 w}P}([g_0 w l]) \xrightarrow{\ e_n\ } \operatorname{Stab}_{E_{\mathcal{Z}_n}}(l) \longrightarrow 1$$

where e_n denotes the restriction of the morphism $e_n \colon P' \cap {}^{g_0 w}P \to E_{\mathcal{Z}_n}$ given by $e_n(p') := (\pi_L({}^{w^{-1}g_0^{-1}}p'), \varphi(\pi_{L'}(p')))$.

The reduced group scheme $(\ker e_n)_{\mathrm{red}}$ is isomorphic to $U' \cap {}^{g_0 w}U$.

Proof. It's immediate from the definition of e_n that $U' \cap {}^{g_0 w}U \subset (\ker e_n)_{\mathrm{red}} \subset (\ker \varphi \cdot U') \cap {}^{g_0 w}U$.

Since φ is an isogeny between connected algebraic groups, it follows that $(\ker \varphi)_{\mathrm{red}}$ lies in the center of L' (see [Spr98, 5.3.5]). Thus, $(\ker \varphi)_{\mathrm{red}}$ lies in some torus of L', hence $(\ker \varphi)_{\mathrm{red}} \cap {}^{g_0 w}U = 1$. As $U' \cap {}^{g_0 w}U$ is smooth, it implies $(\ker e_n)_{\mathrm{red}} = U' \cap {}^{g_0 w}U$.

And finally, the map e_n is faithfully flat (we will see below in the proof of Theorem 2.9 that it is split). $\square$

Remark 2.8. Let $\mathcal{G}$ be an affine connected algebraic group over algebraically closed field k and $\mathcal{H} \subset \mathcal{G}$ be a smooth closed connected subgroup. Furthermore let ${}_{\mathcal{H}}\mathcal{M}$ and ${}_{\mathcal{G}}\mathcal{M}$ be the categories of rational $\mathcal{H}$- resp. $\mathcal{G}$-modules.

A closed subgroup $\mathcal{H}$ is by definition *exact* in $\mathcal{G}$ if the induction functor $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \colon {}_{\mathcal{H}}\mathcal{M} \to {}_{\mathcal{G}}\mathcal{M}$ is exact. It is a well known fact that $\mathcal{G}/\mathcal{H}$ is affine if and only if $\mathcal{H}$ is exact in $\mathcal{G}$ [SR05, ch. 11, Theorems 4.5.and 6.7.]. As the induction functor $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \colon {}_{\mathcal{H}}\mathcal{M} \to {}_{\mathcal{G}}\mathcal{M}$ is an adjoint functor to the restriction functor $\mathrm{Res}_{\mathcal{G}}^{\mathcal{H}} \colon {}_{\mathcal{G}}\mathcal{M} \to {}_{\mathcal{H}}\mathcal{M}$, and since restriction is transitive, induction is so as well.

Specifying the exact group theoretical conditions such that $\mathcal{H}$ is exact in $\mathcal{G}$ turns out to be a hard problem [CPS77]. Nevertheless, there are some easy situations, e.g. $\mathcal{H}$ is a closed subgroup of the unipotent radical $\mathcal{R}_u\mathcal{G}$ of $\mathcal{G}$, the case we will study next.

Assume now: $\mathcal{H} \subset \mathcal{R}_u\mathcal{G}$. Thus, proving that $\mathcal{H}$ is exact in $\mathcal{G}$ amounts to showing that $\mathcal{H}$ is exact in $\mathcal{R}_u\mathcal{G}$, and $\mathcal{R}_u\mathcal{G}$ is exact in $\mathcal{G}$. The latter is obvious as the quotient is an affine (reductive) algebraic group.

The exactness of $\mathcal{H}$ in $\mathcal{R}_u\mathcal{G}$ is also clear [5]: Since unipotent groups have only trivial characters, there exist a rational module M and a point $x \in M$ such that $\mathcal{R}_u\mathcal{G}/\mathcal{H}$ is an $\mathcal{R}_u\mathcal{G}$-orbit of x (cf.[SR05, Ch.7, Corollary 3.6.]). Therefore $\mathcal{R}_u\mathcal{G}/\mathcal{H}$ being an orbit of a unipotent group in the affine variety M is closed in M, and hence $\mathcal{G}/\mathcal{H}$ is affine.

Theorem 2.9. *Suppose the condition (FC) 1.2 is verified. Then the $E_{\mathcal{Z}}$- orbits in G are affine.*

Proof. Let $\mathfrak{O}$ be an $E_{\mathcal{Z}}$-orbit of some element $g_0wl \in G$. As the quotient morphism $G \to G/U$ is affine, it suffices to show that the corresponding P'-orbit of $[g_0wl] \in G/U$ is affine.

Consider the inclusion $\mathrm{Stab}_{P' \cap {}^{g_0w}P}([g_0wl])^0_{\mathrm{red}} \hookrightarrow \mathrm{Stab}_{P' \cap {}^{g_0w}P}([g_0wl])$ whose cokernel is a finite algebraic group over k, say γ. Then $\mathfrak{O}$ is isomorphic to the quotient of $P'/\mathrm{Stab}_{P' \cap {}^{g_0w}P}([g_0wl])^0_{\mathrm{red}}$ by γ.

Thus, we get a finite surjective map $P'/\mathrm{Stab}_{P' \cap {}^{g_0w}P}([g_0wl])^0_{\mathrm{red}} \to \mathfrak{O}$. Hence, by Chevalley's theorem [GW10, Theorem 12.39.] it suffices to investigate the homogeneous space $P'/\mathrm{Stab}_{P' \cap {}^{g_0w}P}([g_0wl])^0_{\mathrm{red}}$ for affineness.

Claim: $\mathrm{Stab}_{P' \cap {}^{g_0w}P}([g_0wl])^0_{\mathrm{red}}$ is a closed subgroup of U'.

---

[5] see also [CPS77, Corollary 2.2] for another proof

Proof of the claim: We proceed inductively. The reduction process mentioned in Remark 2.6 terminates if $P = P' = G$. In this case the stabilizer is finite by *loc. cit.*.

Therefore the claim obviously holds in this case.

Due to the inductive assumption we have $\mathrm{Stab}_{Q'}([l])^0_{\mathrm{red}} \subset \mathcal{R}_u Q' = L \cap {}^{w^{-1}g_0^{-1}}U'$. Note that exact sequence in 2.1 splits, the splitting morphism is given by the restriction of the map $f_n \colon E_{\mathcal{Z}_n} \to P' \cap {}^{g_0 w}P$ given by $(q', q) \mapsto {}^{g_0 w}q'$.

Thus we have $f_n(\mathcal{R}_u Q') \subset U'$, and $\mathrm{Stab}_{P' \cap {}^n P}([g_0 w l])^0_{\mathrm{red}}$ is the product of two closed subgroups of U'.

Therefore, $\mathrm{Stab}_{P' \cap {}^{g_0 w}P}([g_0 w l])^0_{\mathrm{red}}$ is a closed connected subgroup of $\mathcal{R}_u P'$, and it is exact in P' due to the remark 2.8, hence their quotient is affine. □

If we abandon the condition (FC) 1.2 the claim of the previous theorem fails already in simplest cases:

Counterexample 2.10. Consider the zip datum $\mathcal{Z} = (\mathrm{GL}_2, \mathrm{GL}_2, \mathrm{GL}_2, \mathrm{Id})$. Hence $E_{\mathcal{Z}}$-action amounts to the conjugation in GL_2. Observe that the morphism of k-varieties $\lambda \colon \mathrm{GL}_2 \to \mathbb{A}_2$ given on k-valued points by $\lambda \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + d, ad - bc)$ is constant on the GL_2-orbits.

Denote by $\mathfrak{O}_1$ the orbit of the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2$ of dimension 2, and by $\mathfrak{O}_2$ the orbit of $\mathrm{Id} \in \mathrm{GL}_2$, which is just a point. Note that $\mathfrak{O}_1 \cup \mathfrak{O}_2 = \lambda^{-1}(2, 1)$ is closed in GL_2, and hence affine.

But $\mathfrak{O}_1$ is not closed in GL_2, by conjugating by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \in \mathrm{GL}_2$ $(t \neq 0)$ we conclude that $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \mathfrak{O}_1$ for all $t \neq 0$, therefore we have $\mathrm{Id} \in \overline{\mathfrak{O}_1} \setminus \mathfrak{O}_1$.

It follows that $\mathfrak{O}_1$ has codimension 2 in its closure $\overline{\mathfrak{O}_1} = \lambda^{-1}(2, 1)$. Thus, $\mathfrak{O}_1$ is not affine, otherwise it clearly contradicts the algebraic version of Hartogs' theorem [GW10, Theorem 6.45.], see also Lemma 3.3.

3. Purity of G-zip stratification

Let $\underline{I}$ be a G-zip of type χ over S. Recall that the stack $G\text{-}\mathrm{Zip}^{\chi}_k$ is isomorphic to $[E_{G,\chi} \backslash G]$ [PWZ12, Prop. 3.11.] with underlying set $\Gamma \backslash {}^I W$.

$\underline{I}$ defines for all Γ - orbits $[w]$ locally closed subschemes $S_{\underline{I}}^{[w]} \overset{\jmath}{\hookrightarrow} S$ which are loci, where $\underline{I}$ has locally the constant isomorphism class $[w]$. We will recall an exact definition of $S_{\underline{I}}^{[w]}$ below in this section.

There is the following set theoretically disjoint decomposition

$$S = \bigcup_{[w]\in {}^I W/\Gamma} S_{\underline{I}}^{[w]}. \tag{3.1}$$

First we explain how the previous section implies that the immersion $\jmath$ is affine.

We recall construction of $S_{\underline{I}}^{[w]}$ [PWZ12, Subsection 3.6]:

A G-zip $\underline{I}$ over S defines by Yoneda lemma the 1-morphism $\zeta\colon h_S \to G\text{-Zip}_k^\chi$, where h_S is the stack associated to the k-scheme S i.e. to its functor of points.

Due to [PWZ12, Prop. 2.2.] we can consider $[w]$ as a smooth, locally closed algebraic substack of $G\text{-Zip}_k^\chi$, let $S_{\underline{I}}^{[w]}$ be the schematic inverse image $\zeta^{-1}([w])$.

Theorem 3.1. *The immersion* $\jmath\colon S_{\underline{I}}^{[w]} \to S$ *is affine.*

Proof. By Prop. 2.2. *loc.cit.* the Γ-orbit of w is a locally closed subset of underlying topological space $\overline{\Xi}$ of $G\text{-Zip}_k^\chi \otimes \bar{k}$ with no two elements are comparable with respect to $\preceq$. Therefore, it is locally closed subset of $\overline{\Xi}$, and it can be described as a disjoint union of one-point reduced stacks (cf. [PWZ12, Subsection 2.2]).

Thus, just by the base change we obtain the following scheme-theoretically disjoint decomposition: $S_{\underline{I}}^{[w]} \otimes \bar{k} = \bigsqcup_{w'\in\Gamma w} S_{\underline{I}}^{w'}$.

Clearly, $\jmath$ is affine if and only if each $\jmath \otimes id_{\bar{k}}|_{S_{\underline{I}}^{w'}} \colon S_{\underline{I}}^{w'} \to S$ is affine for $w' \in \Gamma w$.

Thus, without loss of generality we can assume k is algebraically closed. In this case $\Gamma = 1$ and $[w]$ corresponds to a single orbit $\mathfrak{O}$ of G.

The quotient map $G \to G\text{-Zip}_k^\chi$ is a representable faithfully flat stack morphism. Then the affineness of the morphism $\mathfrak{O} \hookrightarrow G$ (see [GW10, ch. 12, Prop.12.3.(3)]) implies by faithfully flat descent that the schematic stack morphism $[w] \to G\text{-Zip}_k^\chi$ is affine.

So, we have the following diagram:

$$\begin{array}{ccc} S_{\underline{I}}^{[w]} & \overset{\jmath}{\hookrightarrow} & S \\ \downarrow & & \downarrow \zeta \\ [w] & \hookrightarrow & G\text{-Zip}_k^{\chi} \end{array}$$

Hence the morphism $\jmath \colon S_{\underline{I}}^{[w]} \to S$ is affine just by the definition. □

The affineness of the inclusion $\jmath$ implies the following result:

Corollary 3.2. *Suppose that S is a locally noetherian k-scheme, Z a closed subscheme of S of codimension ≥ 2, which contains no embedded components of S that the restriction of $\underline{I}$ to $S \setminus Z$ is fppf locally constant, then $\underline{I}$ is fppf locally constant.*

Proof. That Z contains no embedded components implies that the scheme theoretic closure of $S \setminus Z$ coincides with S. The claim of Corollary is then an immediate consequence from the following lemma and the fact that faithfully flat morphism preserves the codimensions. □

Lemma 3.3. *Let X be a scheme, Y be an locally-noetherian scheme, $X \hookrightarrow Y$ be an affine immersion. Denote by $\overline{X}$ the schematic closure of X in Y and let Z be an irreducible component of $\overline{X} \setminus X \neq \varnothing$. Then* $\operatorname{codim}(Z, \overline{X}) = 1$.

Proof. Since an affine morphism is quasi-compact, $X \hookrightarrow Y$ factorizes through the inclusion $\overline{X} \hookrightarrow Y$, and one has an affine open immersion $X \hookrightarrow \overline{X}$ (cf. [GW10, Remark 10.31.]).

First assume that $\operatorname{codim}(Z, \overline{X}) = 0$. As Z is closed in $\overline{X}$ of codimension 0 it must be an irreducible component of $\overline{X}$. But $Z \cap X = \varnothing$, hence X is not dense in $\overline{X}$. This gives a contradiction to the assumption.

Suppose there is a component Z of $\overline{X} \setminus X$ such that $\operatorname{codim}(Z, \overline{X}) \geq 2$. Replacing $\overline{X}$ by Spec $\mathcal{O}_{\overline{X},Z}$ and X respectively by $X \cap$ Spec $\mathcal{O}_{\overline{X},Z}$ we can assume that $\overline{X} =$ Spec A for a local noetherian ring A of dimension at least two and $X =$ Spec $A \setminus \{z\}$, where z is a closed point of the codimension at least two. Again replacing A with the quotient $A/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime ideal of A, we can furthermore assume that A is integral.

Next, we replace A with the completion $\hat{A}$ which is possible since the morphism Spec $\hat{A} \to$ Spec A is faithfully flat (cf.[GW10, Prop. B.40]), i.e. it preserves the codimensions, and affine morphisms are stable under base change, we can assume A is a complete local noetherian ring.

Then Spec A is excellent due to [GW10, Theorem 12.51]. This implies that the normalization Spec $A' \to$ Spec A is a finite morphism (cf.[GW10, Theorem 12.51]).

Replacing A with its normalization we can assume that A is normal.

Eventually, we get an affine inclusion $X :=$ Spec $A \setminus \{z\} \hookrightarrow$ Spec A of normal noetherian schemes. By an algebraic analogue of Hartogs' theorem [GW10, Theorem 6.45.] we conclude that $A \cong \Gamma(X, \mathcal{O}_X)$ which is clearly a contradiction to X is affine.

□

4. Applications

4.1. **Purity of level-1-stratification.** Let S be an $\mathbb{F}_p$ -scheme and X over S be a Barsotti-Tate group. Further let $X[1]$ be the corresponding truncated Barsotti-Tate group of the level 1, i.e. p-torsion of X . The strata of level-1-stratification of S corresponds to the loci of S where $X[1]$ has a constant isomorphism class.

In this subsection we illustrate an easy way to show the purity of such stratification just utilizing the fact that its covariant Diedonné crystal carries an F-zip structure. However, this approach turn out to be unsatisfactory for the study of the higher level stratifications as the corresponding Diedonné crystals do not carry F-zip structure any longer.

In the next subsection we will reprove and generalize this result for stratifications of the higher levels using explicit construction of certain moduli spaces of Barsotti-Tate groups. Despite of some redundancy we intend to show both approaches, one presented in this subsection is preferable for the level-1-stratification due to its simplicity.

We denote by $X[1]^\vee$ the Cartier dual of $X[1]$.

Let $\mathbb{D}(X)$ be its covariant Diedonné crystal and $\mathcal{M}(X) := \mathbb{D}(X)(S, S, 0)$ be its evaluation at the trivial object $(S, S, 0)$ of crystalline site.

$\mathcal{M}(X)$ is a local free $\mathcal{O}_S$- module of the rank equal to the height h of X.

Moreover, $\mathcal{M}(X)$ is endowed with an F-zip structure in the following way [PWZ12, Subsection 9.3]:

There is an exact sequence (cf. [BBM82, Corollaire 3.3.5., Proposition 5.3.6]) which is functorial in X and compatible with base change

$S' \to S$:

$$0 \longrightarrow \omega_{X[1]^\vee} \longrightarrow \mathcal{M}(X) \longrightarrow \mathrm{Lie}(X[1]) \longrightarrow 0,$$

where $\omega_{X[1]^\vee} = e^*\Omega_{X[1]^\vee/S}$ is the $\mathcal{O}_S$-module of invariant differentials of $X[1]^\vee$.

The relative Frobenius $F_{X/S}\colon X \to X^{(p)}$ and the Verschiebung $V_{X/S}\colon X^{(p)} \to X$ give rise to $\mathcal{O}_S$-linear homomorphisms $\mathcal{F} := \mathcal{M}(V) : \mathcal{M}(X)^{(p)} \to \mathcal{M}(X)$ resp. $\mathcal{V} := \mathcal{M}(F) : \mathcal{M}(X) \to \mathcal{M}(X)^{(p)}$.

Note that the roles of the Frobenius and the Verschiebung are switched in the covariant Diedonné theory.

Moreover, Im $\mathcal{V} = \ker \mathcal{F} = \omega^{(p)}_{X[1]^\vee}$ and Im $\mathcal{F} = \ker \mathcal{V}$ are local direct summands of $\mathcal{M}(X)^{(p)}$, respectively $\mathcal{M}(X)$.

One obtains the corresponding F -zip $\underline{\mathcal{M}(X)} = (\mathcal{M}(X), C^\bullet, D_\bullet, \varphi_\bullet)$ with a descending filtration $C^0 = \mathcal{M}(X)$, $C^1 = \omega_{X[1]^\vee}$ and $C^2 = 0$ and an ascending filtration $D_{-1} = 0$, $D_0 = \ker \mathcal{V}$ and $D_1 = \mathcal{M}(X)$ with the isomorphisms $\varphi_0\colon \left(C^0/C^1\right)^{(p)} = \mathcal{M}(X)^{(p)}/\ker \mathcal{F} \underset{\sim}{\overset{\mathcal{F}}{\to}} \mathrm{Im}\ \mathcal{F} = \ker \mathcal{V} \cong D_0/D_{-1}$. and $\varphi_0\colon \left(C^1/C^2\right)^{(p)} = \mathrm{Im}\ \mathcal{V} \underset{\sim}{\overset{V^{-1}}{\to}} D_1/D_0$.

As explained in the previous section, an F-zip structure gives a GL_h-zip structure that also gives a decomposition 3.1 of S.

Recall that there is an equivalence of categories between truncated Barsotti-Tate groups over a perfect field and a the modulo p reductions of the covariant Diedonné modules.

Therefore the decomposition pieces $S^{[w]}$ where $\mathcal{M}(X)$ has fppf-locally a constant isomorphism class are exactly the loci where $X[1]$ has a constant one.

Now let S be a locally noetherian $\mathbb{F}_p$ -scheme. The purity of the inclusion $S^{[w]} \hookrightarrow S$ implies in particular that whenever $X[1]$ has a constant isomorphism class over $S \setminus S'$ where S' is closed of codimension at least two in S, it has a constant isomorphism class over S overall.

4.2. **Purity of level-m-stratifications.** Let k be a perfect field of characteristic $p > 0$, and $\mathcal{BT}^{n,d}$ resp. $\mathcal{BT}^{n,d}_m$ be the moduli spaces of Barsotti-Tate groups of dimension d and codimension $n-d$ over k resp. m-truncated Barsotti-Tate groups of same dimension and codimension over k. That is, for each k-scheme S the groupoids $\mathcal{BT}^{n,d}(S)$ resp. $\mathcal{BT}^{n,d}_m(S)$ are the categories of the Barsotti-Tate groups resp. m-truncated Barsotti-Tate groups over S as above with morphisms in

$\mathcal{BT}^{n,d}(S)$ resp. $\mathcal{BT}^{n,d}_m(S)$ being isomorphisms of (truncated) Barsotti-Tate groups. Moreover, by [Wed01, Prop 1.8.] and by base change, $\mathcal{BT}^{n,d}_m$ is a smooth algebraic stack of finite type over k.

Our goal here is to sketch briefly the construction of a quotient stack closely related to the stack $\mathcal{BT}^{n,d}_m$, to relate latter to some algebraic zip datum, and to deduce certain purity results.

For a commutative $\mathbb{F}_p$-algebra R and $m \in \mathbb{N}$ denote by $W(R)$ and $W_m(R)$ the ring of Witt-vectors resp. the ring of Witt-vectors of length m with coefficients in R. Furthermore, let $\sigma_R \colon W(R) \to W(R)$ by the Frobenius endomorphism induced by the Frobenius endomorphism (i.e. p-power map) on R and set $\sigma := \sigma_k$. Let by $p = (0, 1, 0, 0, \ldots)$ be the standard uniformizer of the discrete valuation ring $W(k)$.

Let $\mathcal{K}$ be a smooth affine group scheme of finite type over Spec $(W(k))$. We denote by $W_m(\mathcal{K})$ the smooth affine algebraic group over k which represents the functor $R \mapsto \mathcal{K}(W_m(R))$, and by $W(\mathcal{K})$ the corresponding pro-algebraic group. Let $\mathcal{T}_m$ be the kernel of the projection : $W(\mathcal{K}) \to W_m(\mathcal{K})$ given on the k-valued points by the reduction modulo p^m.

Let now D a Barsotti-Tate group over k of dimension d and codimension $n-d$. The height of D is n. Let $(\mathbf{M}, \phi)$ be a contravariant Diedonné module of D, i.e. a free $W(k)$ module of rank n together with σ-linear endomorphism $\phi \colon \mathbf{M} \to \mathbf{M}$ such that it holds $\phi(\mathbf{M}) \supseteq p\mathbf{M}$.

Moreover, one has a direct sum decomposition $\mathbf{M} = \mathbf{M}_0 \oplus \mathbf{M}_1$ such that $\mathbf{M}_1/p\mathbf{M}_1$ is isomorphic to the kernel of the $\bar{\phi} \colon \mathbf{M}/p\mathbf{M} \to \mathbf{M}/p\mathbf{M}$ which is the reduction of ϕ modulo p. Note that $\mathbf{M}_0$ and $\mathbf{M}_1$ are free $W(k)$-modules of rank d resp. $n-d$. It follows that $\phi(\mathbf{M}_0 \oplus \mathbf{M}_1) = \mathbf{M}_0 \oplus p\mathbf{M}_1$.

Let $\mathcal{K} = \mathrm{GL}_{\mathbf{M}}$ and take $K := W(\mathcal{K})(k) = \mathrm{GL}_{\mathbf{M}}(W(k))$. Thus, by fixing a basis of $\mathbf{M}$ we can write $\phi = b \circ \sigma$ with $b \in K\mu(p)K$, where $\mu(p)$ is given by the matrix $\begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & p\mathbb{1}_{n-d} \end{pmatrix}$ and σ is applied coordinatewise. A change of basis amounts to σ-conjugating by element $a \in K$, $b \mapsto ab\sigma(a)^{-1}$.

Thus the objects of $\mathcal{BT}^{n,d}(k)$ are given by the set K-σ conjugation classes of $K\mu(p)K$.

From the surjectivity of the Frobenius in this case follows that each of the orbits can by parametrized by the elements in $K\mu(p)$. The K-σ conjugation on $K\mu(p)K$ descends to K_μ-σ conjugation on $K\mu(p)$, where

K_μ is the normalizer of the set $K\mu(p) \subset K\mu(p)K$ with respect to the K-σ conjugation.

We see that:

$$K_\mu = \{a \in K : aK\mu(p)\sigma(a)^{-1} \in K\mu(p)\} = \{a \in K : {}^{\mu(p)}a \in K\}$$
$$= \left\{ \left(\begin{array}{c|c} A_{d\times d} & pB_{d\times n-d} \\ \hline C_{n-d\times d} & D_{n-d\times n-d} \end{array} \right) \in K \right\}, \text{where } A_{d\times d}, B_{d\times n-d},$$

$C_{n-d\times d}, D_{n-d\times n-d}$ are sub-matrices over $W(k)$ of size as specified by the lower indices.

Note that K_μ is σ-invariant.

As explained in [Vie11, section 1.4.] the isomorphism classes of m-truncated Barsotti-Tate groups can be described by classifying the reductions of the Diedonné module (M, ϕ) as above modulo p^m. Translated into the group actions it means the following:

Given two K-σ conjugation classes, say $[g]$ and $[g']$ for some $g, g' \in K\mu(p)K$, have the same p^m-reduction of their Diedonné modules if and only if there are $t, t' \in T_m := \mathcal{T}_m(k)$ such that $[tgt'] = [g']$. As expected, since T_m is a normal subgroup of K, this relation remains stable under K-σ conjugation that corresponds to the basis change of $\mathbf{M}$.

Thus, one has a bijection:

$$\{\text{objects of } \mathcal{BT}^{n,d}_m(k)\} \simeq \{K\text{-}\sigma \text{ conjugacy classes of } T_m\backslash K\mu(p)K/T_m\}.$$

Since T_m is a normal subgroup of K we can as before parametrize such orbits by the elements of $K\mu(p)$. Note that that yields a bijection between a K-σ orbit of $T_m\backslash\ K\mu(p)K/T_m$ and the corresponding K_μ-σ-orbit of $T_m\backslash\ K\mu(p)$.

Moreover, one has the bijection between the sets of orbits. Thus, we get:

$$\{\text{objects of } \mathcal{BT}^{n,d}_m(k)\} \simeq \{\ K_\mu\text{-}\sigma \text{ conjugacy classes of } T_m\backslash K\mu(p)\}.$$

This allows to construct the quotient stack $\mathfrak{C}_m(\mu)$ in the following manner: Let $\mathcal{K}_\mu$ be a smooth affine subgroup scheme of $\mathcal{K}$ over Spec $(W(k))$ such that $\mathcal{K}_\mu(W(k)) = K_\mu$.

Note that

$${}^{\mu(p)}K_\mu = \left\{ \left(\begin{array}{c|c} A_{d\times d} & B_{d\times n-d} \\ \hline pC_{n-d\times d} & D_{n-d\times n-d} \end{array} \right) \in K \right\} \subset K.$$

Similarly denote by ${}^{\mu(p)}\mathcal{K}_\mu$ a smooth affine subgroup scheme of $\mathcal{K}$ over Spec $(W(k))$ such that ${}^{\mu(p)}\mathcal{K}_\mu(W(k)) = {}^{\mu(p)}K_\mu$.

In an analogous fashion to an zip group 1.1 we define an affine smooth group scheme $\mathcal{E}$ of finite type over Spec $W(k)$ as follows. Let T be a finite generated $W(k)$-algebra. Denote by $\lambda_{\mu(p)}$, $\rho_{\mu(p)}$ $\tilde{\sigma}\colon \mathcal{K} \to \mathcal{K}$ the scheme endomorphisms given on T-valued points by the right translation by $\mu(p)$ resp. left translation by $\mu(p)$ resp. by the Frobenius σ. By some abuse of notations denote by the same letters restrictions of these maps to $\mathcal{K}_\mu$, ${}^{\mu(p)}\mathcal{K}_\mu$.

Let $\mathcal{E}$ be the following fiber product:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & {}^{\mu(p)}\mathcal{K}_\mu \\ \downarrow & & \downarrow \rho_{\mu(p)} \\ \mathcal{K}_\mu & \xrightarrow{\lambda_{\mu(p)}\circ\tilde{\sigma}} & \mathcal{K} \end{array}$$

Note that $\mathcal{E}$ is a group scheme, and its $W(k)$-valued points are given by:

$$\mathcal{E}(W(k)) := \{(x', x'') \in K_\mu \times {}^{\mu(p)}K_\mu : {}^{\mu(p)}\sigma(x') = x''\}.$$

Let $\mathfrak{C}_m(\mu)$ be the quotient stack of $W(\mathcal{K}_\mu)$-σ-action on $\mathcal{T}_m\backslash\mathcal{K} \cong W_m(\mathcal{K})$. This action is given on k-valued points by:

$$(x', [x]) \mapsto [x' \cdot x \cdot {}^{\mu(p)}\left(\sigma(x')^{-1}\right)], x \in K, x' \in K_\mu. \tag{4.1}$$

Note that it is equivalent to the $W(\mathcal{E})$-action given by:

$$((x', x''), [x]) \mapsto [x' \cdot x \cdot x''^{-1}] \tag{4.2}$$

Unless stated otherwise, for the rest this subsection we will assume that $k = \bar{k}$.

Example 4.1. Let $m = 1$. Consider the $W(\mathcal{K}_\mu)$-σ-action on $\mathcal{T}_1\backslash\mathcal{K} \cong W_1(\mathcal{K}) \cong \mathrm{GL}_{\mathbf{M}/p\mathbf{M}}$. As explained before this action is equivalent to the action of $W(\mathcal{E})$ on $\mathrm{GL}_{\mathbf{M}/p\mathbf{M}}$. Note that the $W(\mathcal{E})$-action on $\mathrm{GL}_{\mathbf{M}/p\mathbf{M}}$ factors through $W_1(\mathcal{E})$.

Let P' and P be the images of $W(\mathcal{K}_\mu) \to W_1(\mathcal{K}_\mu)$ resp. $W({}^{\mu(p)}\mathcal{K}_\mu) \to W_1({}^{\mu(p)}\mathcal{K}_\mu)$ under the reduction modulo p. Obviously, they are opposite parabolic group with the common Levi factor $L = P' \cap P$.

Now consider an zip datum $\mathcal{Z} = (\mathrm{GL}_{\mathbf{M}/p\mathbf{M}}, P, P', \bar{\sigma})$, where $\bar{\sigma}\colon L \to L$ is the reduction modulo p of σ.

Note that $W_1(\mathcal{E})$ is exactly the zip group associated to $\mathcal{Z}$. Thus, the orbits of $W(\mathcal{K}_\mu)$-σ-action are affine due to Theorem 2.9.

This example reproves in particular purity of the level-1-stratification.

In order to show the purity of level-m-stratification for $m \geq 2$ consider the smooth morphism $\mathcal{T}_m\backslash\mathcal{K} \cong W_m(\mathcal{K}) \to \mathcal{T}_1\backslash\mathcal{K} \cong W_1(\mathcal{K})$, of algebraic groups over k which is just induced on k-valued points by the reduction modulo p^{m-1}. Moreover, this morphism is equivariant with respect to $W(\mathcal{E})$-action, and therefore it maps each orbit in $W_m(\mathcal{K})$ onto an orbit in $W_1(\mathcal{K})$. Note that for all $s \in \mathbb{N}$, $W(\mathcal{E})$-action on $W_s(\mathcal{K})$ factors through $W_s(\mathcal{E})$- action.

Our aim is to establish the affineness of these orbits.

The following general fact [SR05, Ch. 11, Theorem 8.4.] will be further useful:

Lemma 4.2. *Let $\mathcal{G}$ be a smooth affine algebraic group over an algebraically closed field k and $\mathcal{U} \subset \mathcal{G}$ be a closed smooth connected unipotent subgroup.*

Then the homogeneous space $\mathcal{G}/\mathcal{U}$ is an affine variety if and only if there exist a morphism of varieties $\Phi\colon \mathcal{G} \to \mathcal{U}$ such that $\Phi(xu) = \Phi(x)u$ for all $x \in \mathcal{G}$ and $u \in \mathcal{U}$.

The proof of the next proposition is largely influenced by [NVW10, Subsection 5.1.].

Proposition 4.3. *$W_m(\mathcal{E})$-orbits in $W_m(\mathcal{K})$ are affine for all $m \in \mathbb{N}$.*

Proof. Let $\mathfrak{O}$ be an $W_m(\mathcal{E})$-orbit in $W_m(\mathcal{K})$ that maps to some $E_{\mathcal{Z}}$-orbit $\mathfrak{o}$ in $\mathrm{GL}_{\mathbf{M}/p\mathbf{M}}$. Let $\mathcal{I}_m$ be the stabilizer of some closed point x of $\mathfrak{O}$ and $\mathcal{I}$ be the stabilizer of the image of x in $\mathfrak{o}$. As explained before (cf. remark 2.8, proof of Theorem 2.9), the orbit $\mathfrak{O}$ is affine if and only if ${\mathcal{I}_m}^0_{\mathrm{red}}$ is exact in $W_m(\mathcal{E})$. We also know that $\mathcal{I}^0_{\mathrm{red}}$ is exact in $W_1(\mathcal{E})$.

Consider the exact sequence of smooth affine algebraic groups:

$$1 \longrightarrow \mathcal{N} \longrightarrow W_m(\mathcal{E}) \xrightarrow{\text{mod } p^{m-1}} W_1(\mathcal{E}) \longrightarrow 1$$

where $\mathcal{N}$ is the kernel of $\ker(W_m(\mathcal{E}) \to W_1(\mathcal{E}))$, which is smooth connected unipotent algebraic group.

Note that $W_m(\mathcal{E})$ is $\mathcal{N}$-torsor over $W_1(\mathcal{E})$ and it is trivial as a torsor of algebraic unipotent group over affine k-scheme. Hence we have $W_m(\mathcal{E}) \cong \mathcal{N} \rtimes W_1(\mathcal{E})$.

Pulling back this exact sequence by the inclusion $\mathcal{I}^0_{\mathrm{red}} \hookrightarrow W_1(\mathcal{E})$ we get the following exact sequence of smooth affine algebraic groups:

$$1 \longrightarrow \mathcal{N} \longrightarrow \mathcal{U} \longrightarrow \mathcal{I}^0_{\mathrm{red}} \longrightarrow 1$$

As before we have $\mathcal{U} \cong \mathcal{N} \rtimes \mathcal{I}^0_{\text{red}}$, and $\mathcal{U}$ is a smooth unipotent connected algebraic group.

Since $\mathcal{I}^0_{\text{red}}$ is exact in $W_1(\mathcal{E})$, there exists by 4.2 a morphism of varieties $\Phi\colon \mathcal{I}^0_{\text{red}} \to W_1(\mathcal{E})$ such that $\Phi(xu) = \Phi(x)u$ for all $x \in W_1(\mathcal{E})$, $u \in \mathcal{I}^0_{\text{red}}$. It yields the morphism of varieties $\Phi'\colon \mathcal{N} \rtimes \mathcal{I}^0_{\text{red}} \to \mathcal{N} \rtimes W_1(\mathcal{E})$ given by (Id, Φ). Φ' obviously satisfies the condition of Lemma 4.2. This implies affineness of the quotient $W_m(\mathcal{E})/\mathcal{U}$. Therefore $\mathcal{U}$ is exact in $W_m(\mathcal{E})$.

Now observe that ${\mathcal{I}_m}^0_{\text{red}} \subset \mathcal{U}$ and hence unipotent. So ${\mathcal{I}_m}^0_{\text{red}}$ is exact in $\mathcal{U}$ (cf. remark 2.8). Thus we conclude ${\mathcal{I}_m}^0_{\text{red}}$ is exact in $W_m(\mathcal{E})$ by transitivity of induction. Hence $\mathfrak{O}$ is affine. □

E. Lau shows in [Lau10, Lemma 3.5] that the category $\text{Disp}_m(k)$ of truncated displays of level m over k is equivalent to truncated Diedonné modules of level m over k. Moreover, fixing the dimension d one gets $\text{Disp}_{m,d} \cong \mathfrak{C}_m(\mu)$ (see the construction in the proof of *loc. cit.* Prop. 3.15). The main result of the same paper (see *loc. cit.* Theorem 4.5.) establishes a connection between the stack of truncated displays and the stack of truncated Barsotti-Tate groups: It implies there exists a smooth morphism $\Lambda\colon \mathcal{BT}^{n,d}_m \to \text{Disp}_{m,d} \cong \mathfrak{C}_m(\mu)$ which is an equivalence on geometric points.

Note that the underlying topological spaces of stacks $\mathfrak{C}_m(\mu)$ and $\mathcal{BT}^{n,d}_m$ for $m > 1$ contain infinitely many points, hence we cannot just tacitly repeat the arguments of Section 3 passing from the affineness of the orbits to the purity of the corresponding strata.

Nevertheless, a slight modification of the arguments makes it possible. In this part we follow closely to [NVW10, Subsections 2.2-2.3].

Let X' be an object in $\mathcal{BT}^{n,d}_m(k)$, and $X := \Lambda X'$. Note that (X', k) and (X, k) define the points of $\mathcal{BT}^{n,d}_m$ resp. $\mathfrak{C}_m(\mu)$ in sense of [LMB91, Definition 5.2]. The corresponding fppf 1-morphisms of stacks over k $X'\colon \text{Spec } k \to \mathcal{BT}^{n,d}_m$ and $X\colon \text{Spec } k \to \mathfrak{C}_m(\mu)$ admit canonical factorizations $\text{Spec } k \overset{\bar{X}'}{\twoheadrightarrow} \mathring{X}' \overset{\imath'}{\hookrightarrow} \mathcal{BT}^{n,d}_m$ and $\text{Spec } k \overset{\bar{X}}{\twoheadrightarrow} \mathring{X} \overset{\imath}{\hookrightarrow} \mathfrak{C}_m(\mu)$, where $\mathring{X}'$ and $\mathring{X}$ are residue gerbes of points X' and X, see [LMB91, Section 11]. Note that $\bar{X}'$ and $\bar{X}$ are fppf epimorphisms, and $\imath$, $\imath'$ are monomorphisms.

As $\mathcal{BT}^{n,d}_m$ and $\mathfrak{C}_m(\mu)$ are locally noetherian stacks over k, the points X', X are algebraic by [LMB91, Théorème 11.3]. Thus $\mathring{X}'$ and $\mathring{X}$ are fppf gerbes of finite type over Spec k.

Further let $\mathfrak{X}_m$ be a m-truncated Barsotti-Tate group over S. Thus $\mathfrak{X}_m$ defines the stack morphism $\zeta_{\mathfrak{X}_m} \colon S \to \mathcal{BT}^{n,d}_m$.

Essentially, our situation is summarized by the following diagram with 2-Cartesian squares:

$$\begin{array}{ccccccc}
S^m_{X'} & \longrightarrow & \mathring{X}' & \longrightarrow & \mathring{X} & \longleftarrow & \mathfrak{O} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
S & \xrightarrow{\zeta_{\mathfrak{X}_m}} & \mathcal{BT}^{n,d}_m & \xrightarrow{\varLambda} & \mathfrak{C}_m(\mu) & \longleftarrow & W_m(\mathcal{K})
\end{array}$$

Let us explain it in more detail:

1) By $\mathfrak{O}$ is denoted the $W_m(\mathcal{E})$-orbit of an arbitrary lift of X in $W_m(\mathcal{K})$.
2) $\mathfrak{O}$ is the fiber product in the right square of the above diagram by [LMB91, Exemple 11.2.2].
3) The $\mathfrak{O}$ is affine by Prop. 4.3 and smooth over k, and $\mathfrak{O} \hookrightarrow W_m(\mathcal{K})$ is an affine immersion of noetherian schemes since $\mathfrak{O}$ is affine and $W_m(\mathcal{K})$ separated (cf. [GW10, ch. 12, Prop.12.3.(3)]).
4) The quotient map $W_m(\mathcal{K}) \to \mathfrak{C}_m(\mu)$ is smooth and surjective, in particular a faithfully flat stack morphism.
5) By faithfully flat descent $\mathring{X} \hookrightarrow \mathfrak{C}_m(\mu)$ is representable by an affine immersion of finite presentation.
6) $\mathfrak{O} \to \mathring{X}$ is a smooth and surjective stack morphism by base change.
7) $\mathring{X}$ is smooth over k since $\mathfrak{O}$ is smooth over k and by 6).
8) Similarly by smoothness $\varLambda$, $\mathring{X}'$ is smooth over k since $\mathring{X}' \hookrightarrow \mathring{X}$ is so, and by 6).
9) Consider the fiber product $\mathring{X} \underset{\mathfrak{C}_m(\mu)}{\times} \mathcal{BT}^{n,d}_m$. Since $\varLambda$ is smooth, and by base change of 5) follows that it is a reduced substack of $\mathcal{BT}^{n,d}_m$. Moreover, since $\varLambda$ induces an equivalence on the geometrical points, the middle square commutes, so there is a 1-morphism $\mathring{X} \to \mathring{X} \underset{\mathfrak{C}_m(\mu)}{\times} \mathcal{BT}^{n,d}_m$ which is an isomorphism of the reduced one-point stacks.
10) The *level m stratum $S^m_{X'}$ of $\mathfrak{X}_m/S$ with respect to X'* is defined by the fiber product in the left square of the above diagram. By the definition of a residue gerbe, the morphism of k-schemes $f \colon T \to S$ factors through $S^m_{X'}$ if and only if $f^*\mathfrak{X}_m$ is locally for

fppf topology isomorphic to $X' \underset{k}{\times} T$ as m-truncated Barsotti-Tate groups over T.

11) $S^m_{X'} \hookrightarrow S$ is an affine immersion of finite presentation by the base change of the morphism in 4). In particular one can view $S^m_{X'}$ as a subscheme of S.

Let k be again an arbitrary perfect field of characteristic $p > 0$. In this case we can also define the level m stratum $S^m_{X'}$ as in 9).

Then we have:

Theorem 4.4. *The immersion $S^m_{X'} \hookrightarrow S$ is affine.*

Proof. By base change and by 11), we conclude that $(S^m_{X'})_{\bar{k}} \hookrightarrow (S)_{\bar{k}}$ is an affine immersion of finite presentation, and that it by faithfully flat descent implies that $S^m_{X'} \hookrightarrow S$ is so as well and, in particular, pure. □

5. F-zip structures on de Rham cohomology

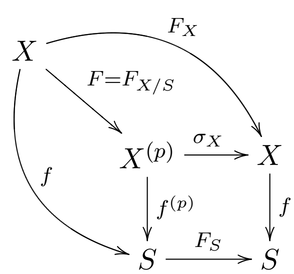
Let S be an $\mathbb{F}_p$-scheme throughout this subsection.

A vast amount of geometric examples of F-zips comes from the structures which naturally arise on the de Rham cohomology.

For an arbitrary $\mathbb{F}_p$ scheme Y denote by $F_Y : Y \to Y$ the absolute Frobenius.

Furthermore let X be a smooth proper scheme over S, and denote by $f : X \to S$ a structure morphism, and by $F = F_{X/S} : X \to X^{(p)}$ the relative Frobenius.

Thus, we have the following commutative diagram.



We start with a recollection the basic facts (cf. [MW04, Sect. 6], [Wed08, Subsect. 1.1.]).

The de Rham cohomology $H^\bullet_{\mathrm{dR}}(X/S) := \mathbf{R}f_*\Omega^\bullet_{X/S}$ is the hypercohomology of the complex $\Omega^\bullet_{X/S}$ with respect to the left exact functor f_* going from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_S$-modules.

Note that the coboundary maps of $\Omega^\bullet_{X/S}$ are $f^{-1}(\mathcal{O}_S)$-linear but not $\mathcal{O}_X$-linear.

There are two exact sequences converging to $H^\bullet_{\mathrm{dR}}(X/S)$, namely the Hodge-de-Rham sequence

$$ {}_HE_1^{ab} = R^bf_*(\Omega^a_{X/S}) \Rightarrow H^{a+b}_{\mathrm{dR}}(X/S) $$

and the conjugate spectral sequence

$$ {}_{conj}E_2^{ab} = R^af_*(\mathcal{H}^b(\Omega^\bullet_{X/S})) \Rightarrow H^{a+b}_{\mathrm{dR}}(X/S). $$

Recall there is a morphism of the graded $\mathcal{O}_{X^{(p)}}$-algebras:

$$ \gamma\colon \bigoplus_{i\in\mathbb{N}_0} \Omega^i_{X^{(p)}/S} \to \bigoplus_{i\in\mathbb{N}_0} \mathcal{H}^iF_*\Omega^\bullet_{X/S}. $$

Moreover, if f is smooth, γ is an isomorphism denoted by C^{-1} and called the *Cartier isomorphism* (cf. [Ill96, Section 3]).

In addition, it has the following properties:

(i) C^{-1} restricts on the zero-graded piece to the algebra isomorphism $F^*\colon \mathcal{O}_{X^{(p)}} \to F_*\mathcal{O}_X$.

(ii) C^{-1} maps $d(\sigma^{-1}(x)) \in \Omega^1_{X^{(p)}/S}$ to the class of $x^{p-1}dx$ in $\mathcal{H}^1F_*\Omega^\bullet_{X/S}$.

Note, that C^{-1} induces an isomorphism of $\mathcal{O}_S$-modules

$$ R^af^{(p)}_*\Omega^b_{X^{(p)}/S} \xrightarrow{\sim} R^af^{(p)}_*(\mathcal{H}^bF_*\Omega^\bullet_{X/S}) = R^af^{(p)}_*F_*(\mathcal{H}^b\Omega^\bullet_{X/S}) = $$
$$ R^af_*(\mathcal{H}^b\Omega^\bullet_{X/S}) = {}_{conj}E_2^{ab} $$

Moreover, if we assume that $\mathcal{O}_S$-modules $R^af_*\Omega^b_{X/S}$ are flat (this holds in particular if they are locally free), then we have:

$$ R^af^{(p)}_*\Omega^b_{X^{(p)}/S} \cong R^af^{(p)}_*\sigma^*_X\Omega^b_{X/S} \cong F^*_Sf_*R^a\Omega^b_{X/S} \cong \left({}_HE_1^{ba}\right)^{(p)} $$

Thus under this assumption we get an isomorphism:

$$ \varphi^{ab}\colon \left({}_HE_1^{ba}\right)^{(p)} \xrightarrow{\sim} {}_{conj}E_2^{ab} \tag{5.1} $$

Remark 5.1. Fix an integer $n \in \mathbb{N}_0$. The definition of spectral sequence, applied to the case of Hodge-de Rham spectral sequence implies that the limit term $M := H^n_{\mathrm{dR}}(X/S)$ is endowed with a descending filtration $\mathrm{Fil}^\bullet$ such that $\mathrm{Fil}^\bullet$ such that $\mathrm{Fil}^kM/\mathrm{Fil}^{k+1}M \cong {}_HE_\infty^{k,n-k}$. We call $\mathrm{Fil}^\bullet$ the *Hodge filtration*.

On the other hand, the conjugate spectral sequence furnishes us with the second descending filtration $\mathrm{Fil}'^{\bullet}$ such that $\mathrm{Fil}'^{k}M/\mathrm{Fil}'^{k+1}M \cong {}_{conj}E_{\infty}^{k,n-k}$.

Note that ${}_{H}E_{\infty}^{k,n-k}$ and ${}_{conj}E_{\infty}^{k,n-k}$ are in general the subquotients of ${}_{H}E_{1}^{k,n-k}$ resp. ${}_{conj}E_{2}^{k,n-k}$.

In the classical situation, there are some discrete conditions for degeneration at E_1, which arise directly from the construction of the spectral sequence:

Remark 5.2. Let K be an arbitrary field. Suppose $\bar{X}$ is a proper scheme over K, and let $b_n := \dim_K H^n_{\mathrm{dR}}(\bar{X})$. We define Hodge numbers $h^{a,b}$ for $a, b \in \mathbb{N}_0$ by and $h^{a,b} := \dim_K H^b(\bar{X}, \Omega^a_{\bar{X}/K})$, $a, b \in \mathbb{N}_0$. They satisfy the following inequalities: $b_n \leq \sum_{a,b\in\mathbb{N}_0, a+b=n} h^{a,b}$ for all $n \in \mathbb{N}_0$, and the Hodge-de Rham spectral sequence degenerates in E_1 if and only if the latter inequalities are equalities for all $n \in \mathbb{N}_0$.

The following remark generalizes the above one:

Remark 5.3. Let R be a commutative ring, and $f\colon \tilde{X} \to \tilde{S}$ be a proper smooth scheme over $\tilde{S} := \mathrm{Spec}\ R$. Denote by R-$\mathrm{MOD}_{\mathrm{fl}}$ the category of R-modules of finite length. We recall that R-$\mathrm{MOD}_{\mathrm{fl}}$ is an abelian category, and its objects are both Noetherian and Artinian R-modules, or equivalently, finitely generated and Artinian ones.

Length $\lg : R\text{-}\mathrm{MOD}_{\mathrm{fl}} \to \mathbb{N}_0$ is additive on exact sequences of objects in R-$\mathrm{MOD}_{\mathrm{fl}}$: Hence for two objects M and N in R-$\mathrm{MOD}_{\mathrm{fl}}$ such that N is proper subquotient of M holds: $\lg N < \lg M$.

Suppose that ${}_{H}E_1^{ab} = R^b f_*(\Omega^a_{\tilde{X}/\tilde{S}})$ and $H^n_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ are objects in $\mathcal{O}_{\tilde{S}}$-$\mathrm{MOD}_{\mathrm{fl}}$ for all $a, b \in \mathbb{N}_0$, e.g. it is a case if $\mathcal{O}_{\tilde{S}}$ is an Artinian ring.

In view of the above and Remark 5.1 we arrive at the following criterion for degeneration the Hodge-de Rham spectral sequence at E_1:

We have:

$$\lg H^n_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \leq \sum_{a,b\in\mathbb{N}_0, a+b=n} \lg R^b f_*(\Omega^a_{\tilde{X}/\tilde{S}}),$$

and the Hodge-de Rham spectral sequence degenerates in E_1 if and only if the latter inequalities are equalities for all $n \in \mathbb{N}_0$.

Now we recall the following definition.

Definition 5.4. Let $f\colon X \to S$ be a smooth proper morphism of arbitrary schemes. We say f *satisfies condition* (D) if the following two conditions hold:

(a) The $\mathcal{O}_S$-modules ${}_HE_1^{ab} = R^b f_*(\Omega^a_{X/S})$ are locally free of finite rank for all $a, b \in \mathbb{N}_0$.

(b) The Hodge-de Rham spectral sequence degenerates at E_1.

Remark 5.5. The part (a) of condition (D) implies that the formation of ${}_HE_1^{ab}$ commutes with an arbitrary base change $S' \to S$.

The condition (D) remains true after an *arbitrary* base change $S' \to S$ (see [Kat72, 2.2.1.11]).

Remark 5.6. Condition (D) implies the isomorphism 5.1, and the conjugate spectral sequence also degenerates at E_2 (see [Kat72, Proposition 2.3.2]).

Since, by a general principle, the formation of ${}_HE_r^{ab}$ commutes with any *flat* base change, and a condition of the degeneration at E_1 expressed as ${}_HE_1^{ab} = {}_HE_2^{ab}$ for all $a, b \in \mathbb{N}_0$ is stable under faithfully flat descent, holds the following:

Remark 5.7. Let $f\colon X \to S$ as in the above definition, and $S' \to S$ be an fpqc morphism. Then $f_{S'}\colon X \underset{S}{\times} S' \to S'$ satisfies (D) iff f satisfies (D).

These nice properties of the spectral sequence provided f satisfies condition (D) give birth the following F-zip structure on $H^\bullet_{\mathrm{dR}}(X/S)$.

Construction 5.8. Fix an integer $0 \leq n \leq 2\dim(X/S)$.

Suppose $f\colon X \to S$ satisfies condition (D). We associate to f an F-zip $(\mathcal{M}, C^\bullet, D_\bullet, \varphi_\bullet)$ as follows: Set $M = H^n_{\mathrm{dR}}(X/S)$. As the Hodge-de Rham spectral sequence degenerates at E_1 we have ${}_HE_\infty^{k,n-k} = {}_HE_1^{k,n-k}$, by Remark 5.6 we also have ${}_{conj}E_\infty^{k,n-k} = {}_{conj}E_2^{k,n-k}$.

Let a descending filtration $C^\bullet$ be the Hodge filtration, and we define an ascending filtration $D_\bullet$ by $D^iM = \mathrm{Fil}'^{n-i}M$, $i \in \mathbb{Z}$, where Fil' is defined as in Remark 5.1.

Note that φ is given by the isomorphisms 5.1 just by setting $\varphi_i = \varphi^{n-i,i}$.

Definition 5.9. Let $f\colon X \to S$ be a smooth proper morphism.

We say $\tilde{X}$ is a *lift of X in zero characteristic* if there exist a scheme $\tilde{S}$ *flat* over $\operatorname{Spec} \mathbb{Z}_p$, and a scheme morphism $S \to \tilde{S}$ such that one has the following diagram with a *Cartesian* square:

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} \\ S & \longrightarrow & \tilde{S} \\ & & \downarrow{\scriptstyle flat} \\ & & \operatorname{Spec} \mathbb{Z}_p \end{array}$$

Presuming the existence of a lift in zero characteristic we are looking for easily testable sufficient conditions under which the condition (D) is met. First we will need a few technical facts which essentially only rephrase the content of [Mum70, Ch.2, §5].

Lemma 5.10. *Let $f\colon X \to S$ be a proper smooth morphism of locally noetherian schemes with $S = \operatorname{Spec} A$ affine. Furthermore let B be an arbitrary A-algebra, and $Y = \operatorname{Spec} B$. Then there is a finite complex $0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \ldots \to \mathcal{F}^m \to 0$ of finitely generated projective A-modules such that one has for all $n \in \mathbb{N}_0$ the natural isomorphism of B-modules: $H^n_{\mathrm{dR}}(X \underset{S}{\times} Y/Y) \cong H^n(\mathcal{F}^\bullet \underset{A}{\otimes} B)$.*

Proof. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a finite affine cover of X, and consider the finite Čech bicomplex $\check{C}^{\bullet\bullet} = \check{C}^\bullet(\mathfrak{U}, \Omega^\bullet_{X/S})$. Further let $\mathcal{F}^\bullet$ be a total complex associated to the bicomplex $\check{C}^{\bullet\bullet}$. As $\Omega^a_{X/S}$ are locally free $\mathcal{O}_X$-modules, and since X is flat over S, they are flat over S. Moreover, as $f\colon X \to S$ is separated, hence $\mathcal{F}^\bullet$ is a complex of flat A-modules, which represents the complex $\mathbf{R}f_*\Omega^\bullet_{X/S}$ in the derived category.

Moreover, for all A-algebras B, $\{U_i \underset{S}{\times} Y\}_{i\in I}$ is the cover of $X \underset{S}{\times} Y$, and $\check{C}^\bullet(\mathfrak{U}, \Omega^\bullet_{X/S}) \underset{A}{\otimes} B$ is the corresponding Čech bicomplex. The associated total complex is just $\mathcal{F}^\bullet \underset{A}{\otimes} B$, and so we have: $H^n_{\mathrm{dR}}(X \underset{S}{\times} Y/Y) = \mathbf{R}^n f_*\Omega^\bullet_{X \underset{S}{\times} Y/Y} \cong H^n(\mathcal{F}^\bullet \underset{A}{\otimes} B)$ as required. Moreover, this isomorphism is obviously functorial in B. □

The previous lemma leads us to the following semi-continuity result for the dimension of the cohomology groups of the fibers whose proof follows verbatim along the same lines as [Mum70, Ch. II, §5, Corollary]. For the reader's convenience, we will sketch it.

Proposition 5.11. *Let $f\colon X \to S$ be a proper smooth morphism of locally noetherian schemes. Then we have:*

i) For each $n \in \mathbb{N}_0$, the function $S \to \mathbb{Z}$ defined by $s \mapsto \dim_{\kappa(s)} H^n_{\mathrm{dR}}(X_s)$ is upper semi-continuous.

ii) The Euler characteristic $\chi\colon S \to \mathbb{Z}$, $s \mapsto \sum_{j\in\mathbb{N}_0} (-1)^j \dim_{\kappa(s)} H^j_{\mathrm{dR}}(X_s)$ is locally constant on S.

Proof. Since the question is local on S, we may assume $S = \mathrm{Spec}\ A$, where A is a local ring. Since all projective modules over a local ring are free, we can pick a complex $\mathcal{F}^\bullet$ of finitely generated free A-modules which furnishes us with the isomorphism in Lemma 5.10. Let $d^i\colon \mathcal{F}^i \to \mathcal{F}^{i+1}$ be the coboundary maps of $\mathcal{F}^\bullet$. Then by the previous lemma holds:

$$\dim_{\kappa(s)} H^n_{\mathrm{dR}}(X_s) = \dim_{\kappa(s)}\ \ker d^n \otimes_A \kappa(s) - \dim_{\kappa(s)} \mathrm{Im}\ d^{n-1} \otimes_A \kappa(s) =$$
$$\dim_{\kappa(s)} \mathcal{F}^n \otimes_A \kappa(s) - \dim_{\kappa(s)} \mathrm{Im}\ d^n \otimes_A \kappa(s) - \dim_{\kappa(s)} \mathrm{Im}\ d^{n-1} \otimes_A \kappa(s). \quad (\dagger)$$

$\dim_{\kappa(s)} \mathcal{F}^n \otimes_A \kappa(s)$ is constant in s; therefore it amounts to show that the function $\rho\colon S \to \mathbb{Z}$, $s \mapsto \dim_{\kappa(s)} \mathrm{Im}\ d^i \otimes_A \kappa(s)$ is lower semi-continuous for each $i \in \mathbb{N}_0$, i.e. the set $M_\rho = \{s \in S : \rho(s) < r\}$ is closed in S for each $r \in \mathbb{N}_0$.

Consider now the A-linear map $\wedge^r d^i\colon \bigwedge^r K^i \to \bigwedge^r K^{i+1}$ between free A-modules of finite rank induced by d^i. Clearly, then we have: $M_\rho = \{s \in S : \wedge^r d^i \otimes_A \kappa(s) = 0\}$. Moreover, the map $\wedge^r d^i$ is given by a matrix with entries in A, which correspond to the global sections of the structure sheaf on S. Their common zero locus defines a closed set in S.

The second assertion follows on taking alternating sum of † over j. □

The following proposition *loc. cit.* will be further useful.

Proposition 5.12. *Let $f\colon X \to S$ be a proper morphism of locally noetherian schemes, and F a coherent sheaf on X, flat over S. Assume S is reduced. For each $b \in \mathbb{N}_0$ the following conditions are equivalent:*

(i) $s \mapsto \dim_{\kappa(s)} H^b(X_s, F_s)$ *is a locally constant function on S.*

(ii) $R^b f_*(F)$ *is a locally free sheaf on* S*, and for all* $s \in S$*, the natural map* $R^b f_*(F) \underset{\mathcal{O}_S}{\otimes} \kappa(s) \longrightarrow H^b(X_s, F_s)$ *is an isomorphism.*

Proof. The claim is a part of [Mum70, Ch.2, Corollary 2]. □

The Mumford's proof of Proposition 5.12 relies solely on the fact that $H^b(X \underset{\text{Spec } A}{\times} \text{Spec } B/\text{Spec } B, F \underset{A}{\otimes} B)$ with A and B as in Proposition 5.10 can be computed as $H^b(\mathcal{F}^\bullet \underset{A}{\otimes} B)$, where $F^\bullet$ is a perfect complex of A-modules. By Proposition 5.10 the same holds true for $H^n_{\text{dR}}(X \underset{S}{\times} Y/Y)$. Thus we have:

Corollary 5.13. *Let* $f\colon X \to S$ *be a smooth proper morphism of locally noetherian schemes, and* S *be reduced. For each* $n \in \mathbb{N}_0$ *the following conditions are equivalent:*

(i) $s \mapsto \dim_{\kappa(s)} H^n_{\text{dR}}(X_s)$ *is a locally constant function on* S*.*

(ii) $H^n_{\text{dR}}(X/S)$ *is a locally free* $\mathcal{O}_S$*-module, and for all* $s \in S$*, the natural map* $H^n_{\text{dR}}(X/S) \underset{\mathcal{O}_S}{\otimes} \kappa(s) \longrightarrow H^n_{\text{dR}}(X_s)$ *is an isomorphism.*

The constancy of Hodge numbers in fibers of the lift in zero characteristic turns out to be a sufficient for the condition (D) to be met:

Proposition 5.14. *Let* $f\colon X \to S$ *be a smooth proper morphism. Suppose that there is a lift of* X *in zero characteristic,* $\tilde{f}\colon \tilde{X} \to \tilde{S}$ *such that* $\tilde{X}$ *and* $\tilde{S}$ *are locally noetherian schemes,* $\tilde{f}$ *is proper and smooth, and* $\tilde{S}$ *reduced.*

Further assume the functions $\tilde{s} \mapsto \dim_{\kappa(\tilde{s})} H^b(\tilde{X}_{\tilde{s}}, \Omega^a_{\tilde{X}_{\tilde{s}}/\kappa(\tilde{s})})$ *are locally constant on* $\tilde{S}$ *for all* $a, b \in \mathbb{N}_0$*.*

Then f *satisfies condition (D).*

Proof. It suffices to prove the assertion of the proposition for $\tilde{S}$ is connected. We complete the right column of diagram 5.9 with Cartesian squares by taking generic fibers of $\tilde{X}$, $\tilde{S}$, Spec $\mathbb{Z}_p$:

$$
\begin{array}{ccccc}
X & \longrightarrow & \tilde{X} & \hookleftarrow & X' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f'} \\
S & \longrightarrow & \tilde{S} & \overset{\jmath}{\hookleftarrow} & S' \\
& & \downarrow{\scriptstyle flat} & & \downarrow \\
& & \text{Spec } \mathbb{Z}_p & \hookleftarrow & \text{Spec } \mathbb{Q}_p
\end{array}
$$

Note that $(\Omega^a_{\tilde{X}/\tilde{S}})_{\tilde{s}} \cong \Omega^a_{\tilde{X}_{\tilde{s}}/\kappa(\tilde{s})}$ (e.g. [Ill96, 1.3]) for all $a \in \mathbb{N}_0$.

Since the open immersions are preserved by a base change, $\jmath\colon S' \hookrightarrow \tilde{S}$ is an open immersion again; moreover $\jmath$ is dominant by [Gro, IV/2, Proposition 2.3.7(i)] because Spec $\mathbb{Q}_p \hookrightarrow$ Spec $\mathbb{Z}_p$ is dominant and quasi-compact.

By Prop. 5.12 $R^b\tilde{f}_*(\Omega^a_{\tilde{X}/\tilde{S}})$ are locally free modules of finite rank for all $a, b \in \mathbb{N}_0$. In particular, their formation commutes with an arbitrary base change.

Thus $R^b f'_*(\Omega^a_{X'/S'}) \cong \jmath^* R^b\tilde{f}_*(\Omega^a_{\tilde{X}/\tilde{S}})$ are locally free $\mathcal{O}_{S'}$-modules of the same rank. Note that the last isomorphism is true in general since $\jmath$ is a flat morphism.

As S' is of zero characteristic the corresponding Hodge-de Rham sequence degenerates at E_1 by [DI87]. On the other hand, by Proposition 5.11 the function $\tilde{s} \mapsto \dim_{\kappa(\tilde{s})} H^n_{\mathrm{dR}}(X_{\tilde{s}})$ is upper semi-continuous on $\tilde{S}$.

Let $s' \in \mathrm{Im}\ \jmath$. As Hodge-de Rham sequence degenerates at E_1 in zero characteristic, for $K = \kappa(s')$ holds the equality of Remark 5.2.

Since by Prop. 5.12 the Hodge numbers are constant on $\tilde{S}$, we have $\dim_{\kappa(\tilde{s})} H^n_{\mathrm{dR}}(\tilde{X}_{\tilde{s}}) \le \dim_{\kappa(s')} H^n_{\mathrm{dR}}(\tilde{X}_{s'})$. On the other hand Im $\jmath$ is open and dense in $\tilde{S}$, so Proposition 5.11 forces the equality for all $n \in \mathbb{N}_0$. Thus $\tilde{s} \mapsto \dim_{\kappa(\tilde{s})} H^n_{\mathrm{dR}}(\tilde{X}_{\tilde{s}})$ is a constant function on $\tilde{S}$.

Thus Prop. 5.13 implies that $H^n_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ is locally free $\mathcal{O}_{\tilde{S}}$-module.

In fact, one can test the degeneration of Hodge-de Rham sequence locally on $\tilde{S}$, moreover it is sufficient to prove it for $\tilde{S} = \mathrm{Spec}\ R$, where R is a local Artinian ring, see the proof of [Kat72, Proposition 2.3.2].

Note that in this case for the only point $\tilde{s} \in \tilde{S}$ holds $\lg H^n_{\mathrm{dR}}(\tilde{X}/\tilde{S}) = \dim_{\kappa(\tilde{s})} H^n_{\mathrm{dR}}(\tilde{X}_{\tilde{s}}) \cdot \lg R$ and $\lg R^b\tilde{f}_*(\Omega^a_{\tilde{X}/\tilde{S}}) = \dim_{\kappa(\tilde{s})} H^b(\tilde{X}_{\tilde{s}}, \Omega^a_{\tilde{X}_{\tilde{s}}/\kappa(\tilde{s})}) \cdot \lg R$

Thus, by of Remark 5.3, the Hodge-de Rham sequence is degenerate for $\tilde{f}\colon \tilde{X} \to \tilde{S}$.

We see that $\tilde{f}\colon \tilde{X} \to \tilde{S}$ satisfies condition (D) except for $\tilde{S}$ is not in characteristic p. But Remark 5.5 remains true also in this case, hence $f\colon X \to S$ also satisfies condition (D).

□

5.1. **Applications of Proposition 5.14.**

K3 schemes over S. First we recall the definitions (see [Riz06]).

Definition 5.15. Let K be an arbitrary field, and Y be an arbitrary base scheme.

i) A smooth proper geometrically connected scheme X over K of dimension 2 is called a *K3 surface* if $\Omega^2_{X/K} \cong \mathcal{O}_X$, and $H^1(X, \mathcal{O}_X) = 0$.

ii) A *polarization on a K3 surface* X is a global section $\lambda \in \mathrm{Pic}_{X/K}(K)$ that over $\bar{K}$ is the class of an ample line bundle $\mathcal{L}_{\bar{K}}$. The degree of $\mathcal{L}_{\bar{K}}$, which is by definition the selfintersection index $(\mathcal{L}_{\bar{K}}, \mathcal{L}_{\bar{K}})_{X_{\bar{K}}}$, is called the *polarization degree* of λ. A polarization degree is always an even number.

iii) A *K3 scheme* over Y is a scheme X together with proper, smooth morphism $f\colon X \to Y$ whose geometric fibers are K3 surfaces.

iv) A *K3 space* over a scheme Y is an algebraic space X together with a proper and smooth morphism $f\colon X \to Y$ such that there is an étale cover $Y' \to Y$ of Y for which $f_{Y'}\colon X \underset{Y}{\times} Y' \to Y'$ is a K3 scheme over Y'.

v) A *polarization on a K3 space* $f\colon X \to Y$ is a global section $\lambda \in \mathrm{Pic}_{X/Y}(Y)$ such that for every geometric point $\bar{y}$ of Y the section $\lambda_{\bar{y}} \in \mathrm{Pic}_{X_{\bar{y}}/\kappa(\bar{y})}(\kappa(\bar{y}))$ is a polarization of $X_{\bar{y}}$, see [Riz06, section 1.3.1] for a definition of relative Picard functor $\mathrm{Pic}_{X/Y}$.

We recall also a well-known fact about the Hodge diamond of a K3 surface X, see [Del, Proposition 1.1]:

Remark 5.16. For the Hodge numbers of X/K holds:
$h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0$;
$h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$;
$h^{1,1} = 20$.

Note that in particular they do not depend on the field K.

J. Rizov constructs in *loc.cit.* a separated Deligne-Mumford stack $\mathcal{M}_{2d}$ fibered over (**Sch**) of polarized pairs $(f\colon X \to Y, \lambda)$, where X is a K3 space over Y, and λ is a polarization of the (constant) degree $2d$ on it. Moreover he shows in [Riz06, Proposition 1.4.15] that the moduli stack $\mathcal{M}_{2d}$ is smooth of relative dimension 19 over $\mathbb{Z}[\frac{1}{2d}]$.

This result establishes the following method of the construction of a lift in zero characteristic:

Corollary 5.17. *Let $(f\colon X \to S, \lambda)$ be a polarized K3 scheme with a polarization of degree $2d$. Assume $p \nmid 2d$.*

Then f satisfies condition (D).

Proof. Denote by $\mathcal{M}_{2d}^p = \mathcal{M}_{2d} \underset{\mathbb{Z}[\frac{1}{2d}]}{\otimes} \mathbb{Z}_p$ which is separated, smooth, in particular flat, Deligne-Mumford stack over $\mathbb{Z}_p$. In particular, one has a étale surjection $M'_{2d} \to \mathcal{M}_{2d}^p$ with M'_{2d} is a smooth scheme over $\mathbb{Z}_p$.

Let $\mathfrak{X} \to \mathcal{M}_{2d}^p$ be the universal K3 space with a polarization of degree $2d$. $(f\colon X \to S, \lambda)$ corresponds to the stack morphism $F\colon S \to \mathcal{M}_{2d}^p$ such that one has the following Cartesian diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \xrightarrow{F} & \mathcal{M}_{2d}^p \end{array}$$

Let $X' := X \underset{\mathcal{M}_{2d}^p}{\times} M'_{2d}$, $S' := S \underset{\mathcal{M}_{2d}^p}{\times} M'_{2d}$, $\mathfrak{X}' := \mathfrak{X} \underset{\mathcal{M}_{2d}^p}{\times} M'_{2d}$ be the étale covers of X resp. S resp. $\mathfrak{X}$.

By the base change by $M'_{2d} \to \mathcal{M}_{2d}^p$ we obtain the following Cartesian diagram:

$$\begin{array}{ccc} X' & \longrightarrow & \mathfrak{X}' \\ {\scriptstyle f_{M'_{2d}}}\downarrow & & \downarrow \\ S' & \xrightarrow{F_{M'_{2d}}} & M'_{2d} \end{array}$$

Note that of the diagonal $\mathcal{M}_{2d}^p \to \mathcal{M}_{2d}^p \times \mathcal{M}_{2d}^p$ of a Deligne-Mumford stack being a schematic morphism implies that $X' \to S'$ is a K3 scheme, and $\mathfrak{X}' \to M'_{2d}$ is a K3 space.

Note that by Remark 5.7 it suffices to test condition (D) for a K3 scheme $X' \to S'$.

Now by the definition of K3 space we can choose an étale cover $M''_{2d} \to M'_{2d}$ such that $\mathfrak{X}'' := \mathfrak{X}' \underset{M'_{2d}}{\times} M''_{2d} \to M''_{2d}$ is a K3 scheme.

By the same reasoning as above we can test condition (D) for a K3 scheme $X'' := X' \underset{M'_{2d}}{\times} M''_{2d} \to S'' := S' \underset{M'_{2d}}{\times} M''_{2d}$.

But $\mathfrak{X}'' \to M''_{2d}$ is a lift in zero characteristic of X''/S'' in sense of Definition 5.9 which clearly satisfies the assumptions of Proposition 5.14. This proves the claim. □

Remark 5.18. By utilizing the existence and the regularity properties of the moduli stack of polarized abelian schemes over $\mathbb{Z}$, see e.g [dJ93], one can show in a similar vein that an polarized abelian scheme X over

S under certain restrictions also satisfies condition (D). In fact, the polarization assumption can be abandoned, and condition (D) holds true for an arbitrary abelian scheme over S, see [BBM82, § 2.5. Prop 2.5.2].

Lift over Spec $W(k)$**.** Here we list some examples of proper smooth schemes X over k where condition (D) holds, i.e., $S =$ Spec k and $\tilde{S} =$ Spec $W(k)$:

(1) X/k is a *proper smooth curve*: In fact, there is a proper smooth lift $\tilde{X} \to$ Spec $W(k)$, since the obstructions lie in $H^2(X, \Omega^\vee_{X/k})$, and $H^2(X, \mathcal{O}_X)$, see e.g. [Gro, III/1, Théorème 5.1.4].

By Serre duality for Hodge numbers holds $h^{0,0} = h^{1,1} = t$, $h^{1,0} = h^{0,1} = g$, and they are the same on the generic and the special fiber since the Euler characteristic $\chi = 2t - 2g$ is constant on $\tilde{S}$ by Proposition 5.11(ii), and t, which is the number of geometric components, is also constant by [Gro, IV/3, Proposition 15.5.9(ii)].

(2) X/k is a *K3 surface*: By [Del] there exists a lift to a K3 scheme over $W(k)$, and the Hodge numbers do not depend on X and on the ground field.

(3) X/k is an *Enriques surface* if char$(k) \neq 2$. In this case $H^2(X, \mathcal{O}_X) = 0$ and it has an étale cover by a K3 surface Y. By Serre duality $H^2(X, \Omega^\vee_{X/k}) = 0$ if and only if $H^0(X, \Omega_{X/k} \otimes \omega_X) = 0$. The last equality is true since it holds for an étale cover Y, see [Lan83, Theorem 1.1.]. Thus a lift exists for the same reason as in (1).

Note that similar to K3 surfaces the Hodge numbers for Enriques surfaces over K in char$(K) \neq 2$ do not depend on any choice.

(4) X/k is a *smooth complete intersection* in $\mathbb{P}^n_k$, see [DK69, Exposé XI, Théorème 1.5].

(5) X/k is a *smooth proper toric variety*, see [Bli01].

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