Moduli spaces of p-divisible groups of dimension 2 and height h with h odd

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1 Introduction

Let p be a prime, and denote by $\mathrm{Nilp}_{\mathbb{Z}_p}$ the category of schemes over $\mathrm{Spec}(\mathbb{Z}_p)$, on which p is locally nilpotent. Let $\mathbb{X}$ be a p-divisible group over $\mathbb{F}_p$. We want to study the functor

$$\begin{aligned} \mathcal{M}\colon \mathrm{Nilp}_{\mathbb{Z}_p} &\longrightarrow (\mathrm{Sets}) \\ S &\longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of } (X, \rho) \text{ where} \\ X/S \text{ is a } p\text{-divisible group, and} \\ \rho\colon \mathbb{X} \times_{\mathrm{Spec}\,\mathbb{F}_p} \bar{S} \longrightarrow X \times_S \bar{S} \text{ a quasi-isogeny} \end{array} \right\}, \end{aligned}$$

where $\bar{S}$ is the closed subscheme of S defined by the ideal sheaf $p \cdot \mathscr{O}_S$. This functor is an example of a Rapoport-Zink space. In general, Rapoport-Zink spaces are moduli spaces of p-divisible groups + quasi-isogenies to a given $\mathbb{X}$ with additional structures. They are representable by formal schemes locally formally of finite type over $\mathrm{Spf}(\mathbb{Z}_p)$.

In general, a Rapoport-Zink space is attached to a datum (G, K, b, μ), where

- $G/\mathbb{Q}_p$ is a classical group,
- $K \subset G(\mathbb{Q}_p)$ is a parahoric subgroup,
- b is a σ-conjugacy class in $G(L)$, where $L/\mathbb{Q}_p$ is an unramified finite extension and $\sigma\colon L \longrightarrow L$ is an automorphism of L over $\mathbb{Q}_p$ which induces the Frobenius automorphism $x \longmapsto x^p$ on $\mathcal{O}_L/p \cdot \mathcal{O}_L$,
- μ is a minuscule cocharacter of G defined over a finite extension $E/\mathbb{Q}_p$

satisfying certain conditions (see sections 1.38 and 3.17 in [RZ]).
If the σ-conjugacy class b is basic (for definition see e.g. Remark 1.15 in [RZ]), the Rapoport-Zink spaces uniformise the supersingular locus of the

parahoric reduction of some Shimura varieties of PEL-type.

The functor $\mathcal{M}$ defined above corresponds the the following datum:

- $G = \mathrm{GL}_h$, where h is the height of the p-divisible group $\mathbb{X}$
- $K = \mathrm{GL}_h(\mathbb{Z}_p)$,
- b can be obtained from the isocrystal $(N(\mathbb{X}), F)$ of $\mathbb{X}$ by fixing a basis of $N(\mathbb{X})$. It is then isomorphic to $(\mathbb{Q}_p^h, \tilde{b})$, where $\tilde{b}$ an element of $\mathrm{GL}_h(\mathbb{Q}_p)$. It is unique up to a base change of $N(\mathbb{X})$, that is, up to (σ-)conjugation with elements of $\mathrm{GL}_h(\mathbb{Q}_p)$, thus its ($\sigma$-)conjugacy class b is unique.
  Here the property of b to be basic is equivalent to the isocrystal $(N(\mathbb{X}), F)$ to be isoclinic.
- μ depends on the dimension d of $\mathbb{X}$: It sends a $t \in \mathbb{G}_m(R)$ to the diagonal matrix $\mathrm{diag(t, \ldots, t, 1, \ldots, 1)} \in \mathrm{GL_h(R)}$ for any $\mathbb{Q}_p$-algebra R. The number of t's is d and the number of 1 is $h - d$. The pair $(d, h - d)$ is called the signature of μ.

There are various results on the geometric structure of the underlying reduced subscheme $\mathcal{M}_{\mathrm{red}}$ of the Rapoport-Zink space $\mathcal{M}$ for some choices of (G, K, b, μ), that is, the reduced subscheme of $\mathcal{M}$ defined by its maximal ideal of definition.

- If $G = \mathrm{GL}_{h,\mathbb{Q}_p}$ and $b \in B(G, \mu)$ for a cocharacter μ of G (see, e.g section 4 in [Rap] for definition), then Viehmann has determined in [Vie] the irreducible components and the dimension of $\mathcal{M}_{\mathrm{red}}$ and computed its étale cohomology.
- For $G = \mathrm{Sp}_{2\mathrm{g}}$ and b basic Hoeve ([Hoe]) has considered the Ekedahl-Oort stratification of the Rapoport-Zink space. He has described each Ekedahl-Oort stratum in the moduli space of principally polarized abelian varieties in terms of fine Deligne-Lusztig varieties.
- Let $K/\mathbb{Q}_p$ be a quadratic unramified extension, $G/\mathbb{Q}_p$ the group of similitudes of a hermitian $(K, \mathbb{Q}_p)$-vector space, and $E/\mathbb{Q}_p$ a quadratic field extension with $G \times_{\mathbb{Q}_p} E \cong \mathrm{GL}_h \times \mathbb{G}_m$, in fact $E \cong K$. Let μ be a cocharacter of G over E which sends a $t \in \mathbb{G}_m(R)$ to the element $(\mathrm{diag}(t, 1, \ldots, 1), t) \in \mathrm{GL}_h(R) \times \mathbb{G}_m(R)$ for any E-algebra R.
  In this case, Vollaard and Wedhorn have defined a stratification of the underlying reduced subscheme of any connected component of the associated Rapoport-Zink space by locally closed subschemes $\mathcal{N}_\Lambda$, where Λ runs though the set of vertices of the Bruhat-Tits building of an inner

form of G. Furthermore, they showed that each stratum has the structure of a Deligne-Lusztig variety [VW].

In this thesis, we consider the same case of $G = \mathrm{GL}_{h,\mathbb{Q}_p}$ as Viehmann, but want to have a closer look at the scheme-theoretic structure of $\mathcal{M}_{\mathrm{red}}$. We restrict ourselves to the case of signature $(2, h)$, h being odd.
The simpler case would be that of signature $(1, h)$. The corresponding Shimura variety is the one used by Harris and Taylor in their proof of the Langlands correspondence for GL_n for p-adic fields in [HT]. In this case, the underlying topological space of any connected component of $\mathcal{M}_{\mathrm{red}}$ consists only of one point (see, for example, [Vie]). Thus, the next case to look at would be that of signature $(2, h)$.
We take here h to be odd because in this case, the isocrystal $N(\mathbb{X})$ of $\mathbb{X}$ is simple, and Oort and de Jong have shown that that the connected components of $\mathcal{M}_{\mathrm{red}}$ are irreducible.
Following the idea of Vollaard and Wedhorn, we define a stratification of the $\overline{\mathbb{F}_p}$-valued points of any connected component of $\mathcal{M}$ by closed subsets corresponding to certain pairs of lattices. We define projective schemes over $\mathbb{F}_{p^h}$ whose $\overline{\mathbb{F}_p}$-valued points are precisely these subsets.

We state the main results:

Proposition 1.1. *Let $\kappa\colon \mathcal{M} \longrightarrow \mathbb{Z}$ be the morphism which assigns to a point (X, ρ) in $\mathcal{M}(S)$ the height of the quasi-isogeny ρ. Then the fibers $\mathcal{M}(i) := \kappa^{-1}(i)$ are non-empty. They form the connected components of $\mathcal{M}$ and are isomorphic to each other (see Corollary 3.9).*

We set $\mathcal{N} := \mathcal{M}(0)$ to be the connected component of the identity morphism $\mathrm{id}\colon \mathbb{X} \longrightarrow \mathbb{X}$.

Theorem 1.2. (a) *There exists a finite stratification of $\mathcal{N}(\overline{\mathbb{F}_p})$ by locally closed irreducible subsets $\mathcal{N}^{\circ}_{j,l,\underline{\alpha_l}}(\overline{\mathbb{F}_p})$ with j and l integers satisfying the inequalities $0 \leq j \leq \frac{h-3}{2}$ and $0 \leq l \leq j$, and $\underline{\alpha_l} \in \mathbb{F}^l_{p^h}$ if $l > 0$ and $\underline{\alpha_l} = \emptyset$ if $l = 0$. The tuple $(j, l, \underline{\alpha_l})$ is a combinatorial datum which can be attached to each point of $\mathcal{N}(\overline{\mathbb{F}_p})$ (Lemma 3.13).*
Let $\mathcal{N}_{j,l,\underline{\alpha_l}}(\overline{\mathbb{F}_p})$ be the closure of $\mathcal{N}^{\circ}_{j,l,\underline{\alpha_l}}(\overline{\mathbb{F}_p})$ in $\mathcal{N}(\overline{\mathbb{F}_p})$. It is a union of other strata and we determine precisely the strata contributing to it.

(b) *There exist closed subfunctors $\mathcal{N}_{j,l,\underline{\alpha_l}} \longrightarrow \mathcal{N} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\mathbb{Z}_{p^h})$, whose $\overline{\mathbb{F}_p}$-valued points are precisely the closed subsets $\mathcal{N}_{j,l,\underline{\alpha_l}}(\overline{\mathbb{F}_p})$.*
The subfunctors $\mathcal{N}_{j,l,\underline{\alpha_l}}$ are represented by projective $\overline{\mathbb{F}}_{p^h}$-schemes (Lemmas 3.14 and 3.15).

(c) *There exists a projective $\mathbb{F}_{p^h}$-scheme Y and a closed immersion of formal $\mathbb{Z}_{p^h}$-schemes $\iota\colon Y \longrightarrow \mathcal{N} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\mathbb{Z}_{p^h})$, which induces a bijection*

$\iota(k)\colon Y(k) \longrightarrow \mathcal{N}(k)$ for any perfect extension $k \supset \mathbb{F}_{p^h}$. In particular, we have $Y_{red} \cong \mathcal{N}_{red}$ (Lemma 3.16).

We compute the tangent space at any point of $Y(\overline{\mathbb{F}_p})$ and show in Proposition 4.1 and Theorem 4.2:

Theorem 1.3. (a) *If $h > 5$, then Y is not smooth.*

(b) *We determine precisely the singularity locus of Y and show, that, in particular, Y is regular in codimension 1.*

(c) *Y is generically reduced, but not reduced in general.*

(d) *There exists an algebraic group H over $\mathrm{Spec}(\mathbb{F}_{p^h})$ which acts on Y such that the stratification of Y by H-orbits is the singularity stratification of Y (see Proposition 4.4).*

This thesis is organized as follows: In the second section we recall some facts on the moduli space $\mathcal{M}$ of quasi-isogenies of a given p-divisible group and describe the $\overline{\mathbb{F}_p}$-valued points via Dieudonné theory in terms of semi-linear algebra.
In the third section we study one connected component $\mathcal{N}$ of the moduli space $\mathcal{M}$. We attach to every point of $\mathcal{N}(\overline{\mathbb{F}_p})$ a combinatorial datum $(j, l, \underline{\alpha_l})$ and define a stratification of $\mathcal{N}(\overline{\mathbb{F}_p})$ in terms of this combinatorial datum by subsets $\mathcal{N}_{j,l,\underline{\alpha_l}}(\overline{\mathbb{F}_p})$. Furthermore, we describe the inclusion relations of these closed subsets.
We show that these closed subsets are in fact the $\overline{\mathbb{F}_p}$-valued points of subfunctors $\mathcal{N}_{j,l,\underline{\alpha_l}}$ of $\mathcal{N}$, and that these subfunctors are representable by projective $\mathbb{F}_{p^h}$-schemes. We determine one of these subfunctors $\mathcal{N}'$ and its representing projective $\mathbb{F}_{p^h}$-scheme Y, whose associated reduced subscheme is isomorphic to $\mathcal{N}_{\mathrm{red}}$.
Finally, in the fourth section we consider the scheme Y. We determine its singular locus and the singularity stratification of Y and show that the singularity stratification is in fact given by the action of an algebraic group.

2 Moduli space of p-divisible groups

Let k be a perfect field of characteristic p and $W = W(k)$ its ring of Witt vectors. Denote by $\mathrm{Nilp}_{W(k)}$ the category of schemes S over $W(k)$ such that p is locally nilpotent on S. For $S \in \mathrm{Nilp}_{W(k)}$, let $\bar{S}$ be the closed subscheme of S defined by the ideal sheaf $p \cdot \mathscr{O}_S$.
Let $\mathbb{X}$ be a p-divisible group over k. We consider the functor

$$\mathcal{M}\colon \mathrm{Nilp}_{W(k)} \longrightarrow (\mathrm{Sets}),$$

which assigns to S the set of isomorphism classes of pairs (X, ρ), where X is a p-divisible group over S and $\rho\colon \mathbb{X}_{\bar{S}} = \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S} \longrightarrow X \times_S \bar{S}$ a quasi-isogeny. Two such pairs (X_1, ρ_1) and (X_2, ρ_2) are isomorphic if $\rho_2 \circ \rho_1^{-1}$ can be lifted to an isomorphism $X_1 \longrightarrow X_2$.

If we replace the p-divisible group $\mathbb{X}$ by an isogeneous (even quasi-isogeneous) $\mathbb{X}'$, we get an isomorphism of the corresponding functors $\mathcal{M} \longrightarrow \mathcal{M}'$ by composition with the isogeny $\mathbb{X}' \longrightarrow \mathbb{X}$.
Let $\kappa\colon \mathcal{M} \longrightarrow \mathbb{Z}$ be a morphism, which maps a quasi-isogeny to its height. Then we can decompose $\mathcal{M}$ into open and closed formal subschemes $\mathcal{M}(i) = \kappa^{-1}(i)$, i. e.

$$\mathcal{M}(i)(S) = \{\text{isomorphism classes of } (X, \rho) \in \mathcal{M}(S) \mid \text{ height}(\rho) = i\}.$$

Set $\mathcal{N} := \mathcal{M}(0)$.

Dieudonné modules

Let k be a perfect field of characteristic $p > 0$ and $W(k)$ its Witt ring. Denote by σ the Frobenius automorphism on k as well as on $W(k)$ and $\mathrm{Frac}(W(k))$.

Definition 2.1. *A Dieudonné module over k is a finitely generated free $W(k)$-module M together with two mappings F and $V\colon M \longrightarrow M$ which satisfy*

$$F(am + n) = \sigma(a)F(m) + F(n) \text{ and}$$
$$V(\sigma(a)m + n) = aV(m) + V(n) \text{ for all } a \in W(k), m, n \in M,$$

and $FV = VF = p$.

By Dieudonné theory, one can assign to each p-divisible group X over k its Dieudonné module $\mathbb{D}(X)$. The modules used here will always be contravariant Dieudonné modules. With the following theorem, we can describe p-divisible groups with semi-linear algebra.

Theorem 2.2 (Dieudonné, [Gro]). *Let k be a perfect field. Then there is an equivalence of categories:*

$$(p\text{-divisible groups}/k) \longrightarrow (\text{Dieudonné-modules over } k)$$
$$X \longmapsto \mathbb{D}(X).$$

An isocrystal over k is a finite-dimensional $\mathrm{Frac}(W(k))$-vector space equipped with a bijective mapping F satisfying the same conditions as above. V is then given by $V = p \cdot F^{-1}$

Proposition 2.3 (Dieudonné, [Man]). *Let k be an algebraically closed field. Then the category of isocrystals over k is semisimple with simple objects parametrized by $\mathbb{Q}$ in the following manner:*
To $\lambda = \frac{r}{s} \in \mathbb{Q}$ with $(r, s) = 1$ corresponds the isocrystal N_λ with

$$N_\lambda = \mathrm{Frac}(W(k))^s, \quad F_\lambda = \begin{pmatrix} 0 & 0 & \dots & 0 & p^r \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \cdot \sigma.$$

The rational number λ is also called the slope *of the isocrystal N_λ.*

Let $\mathbb{X}$ be a p-divisible group over k of height h and dimension d. Then its isocrystal $N(\mathbb{X}) := \mathbb{D}(\mathbb{X}) \otimes_{W(k)} \mathrm{Frac}(W(k))$ has dimension h, and for any p-divisible group X over k which is quasi-isogeneous to $\mathbb{X}$ we have $N(X) \cong N(\mathbb{X})$, an isomorphism of isocrystals.

An isocrystal N over k is called *isoclinic of slope* λ if $N \otimes \bar{k}$ is the direct sum of copies of N_λ. If $l \supset k$ is an algebraic extension, denote by N_l the isocrystal $(N(\mathbb{X}) \otimes_{\mathrm{Frac}(W(k))} \mathrm{Frac}(W(l)), F_l = F \otimes \sigma)$ over l. Thus, if $\mathcal{M}$ is the functor of p-divisible groups quasi-isogeneous to $\mathbb{X}$ as before, then we get via this equivalence of categories:

$$\mathcal{M}(\bar{k}) = \{F\text{- and } V\text{-invariant } W(\bar{k})\text{-lattices } M \subset N_{\bar{k}}\}.$$

3 Structure of $\mathcal{N}(k)$

Valuations on Dieudonné modules

From now on let $\mathcal{N}$ be the open and closed formal subscheme $\mathcal{M}(0)$ of $\mathcal{M}$. We want to consider $\mathcal{N}$ first, and then show that every other $\mathcal{M}(i)$ is isomorphic to $\mathcal{N}$.

To define a stratification on the moduli space $\mathcal{N}$, we consider valuations on Dieudonné modules as defined by Lau, Nicole and Vasiu in [LNV].
Let k again be a perfect field of characteristic $p > 0$, $W(k)$ its ring of Witt vectors and denote by $v_p : W(k) \longrightarrow \mathbb{R} \cup \{\infty\}$ the p-adic valuation on $W(k)$.

Definition 3.1. *A valuation on a $W(k)$-module M is a map $w : M \longrightarrow \mathbb{R} \cup \{\infty\}$ that has the following properties:*

(i) $w(ax) = v_p(a) + w(x)$ for all $a \in W(k)$ and $x \in M$,

(ii) $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in M$.

The valuation w is called non-trivial if $w(x) \neq \infty$ for some $x \in M$. It is called *non-degenerate* if $w(x) = \infty$ implies $x = 0$.

Definition 3.2. *Let F be a σ-linear endomorphism of M. A valuation w on M is called an F-valuation of slope $\lambda \in \mathbb{R}$ if for all $x \in M$ one has*

$$w(Fx) = w(x) + \lambda.$$

Let now N be an isocrystal over k. The following lemma describes the existence and uniqueness of F-valuations on isocrystals.

Lemma 3.3 (Lemma 5.3 in [LNV])**.** *There exists a non-degenerate F-valuation of slope λ on N if and only if N is isoclinic of slope λ. When N is simple of slope λ, any two non-trivial F-valuations of slope λ on N differ by the addition of a constant.*

Setup

Now we want to fix the setup of this thesis.

Let h be odd, and fix the standard basis $\{e_0, \ldots, e_{h-1}\}$ of $\mathbb{Q}_p^h$. We define an isocrystal over $\mathbb{F}_p$ by

$$F(e_i) = \begin{cases} e_{i+2}, & 0 \leq i \leq h-3, \\ \\ p \cdot e_{i-h+2}, & i = h-2, h-1. \end{cases}$$

If we change the basis of $\mathbb{Q}_p^h$ to $\{e_0, e_2, \ldots, e_{h-1}, p \cdot e_1, p \cdot e_3, \ldots, p \cdot e_{h-2}\}$, then according to this basis F is given by

$$F = \begin{pmatrix} 0 & 0 & \ldots & 0 & p^2 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \cdot \sigma.$$

Thus, $(\mathbb{Q}_p^h, F)$ is a simple isocrystal over $\mathbb{F}_p$ of slope $\lambda = \frac{2}{h}$

Define a valuation $w\colon \mathbb{Q}_p^h \longrightarrow \mathbb{R} \cup \{\infty\}$ of slope λ on $(\mathbb{Q}_p^h, F)$ by setting

$$w(e_i) := \frac{i}{h}, \quad i = 0, \ldots, h-1$$

and for any $v = \sum_{i=0}^{h-1} a_i e_i$ set $w(v) = \min_i\{v_p(a_i) + \frac{i}{h}\}$.

Fix a p-divisible group $\mathbb{X}$ over $\mathbb{F}_p$ with this isocrystal, such that the Dieudonné module of $\mathbb{X}$ is $\mathbb{D}(\mathbb{X}) = \bigoplus_{i=0}^{h-1} \mathbb{Z}_p \cdot e_i = \{x \in \mathbb{Q}_p^h \mid w(x) \geq 0\}$.

Let $\mathcal{M}/\operatorname{Spf}(\mathbb{Z}_p)$ be the associated moduli space of p-divisible groups quasi-isogeneous to $\mathbb{X}$ as defined in the introduction. It is representable by a formal scheme locally formally of finite type over $\operatorname{Spf}(\mathbb{Z}_p)$ (see Theorem 2.16 in [RZ] and note, that $\mathbb{X}$ is decent, i.e. its isocrystal is generated by elements x with $F^h(x) = p^2 x$).

From now on let $k \supset \mathbb{F}_p$ be an algebraic closure, $W(k)$ its Witt ring.

Set $N := \mathbb{Q}_p^h \otimes_{\mathbb{Q}_p} \operatorname{Frac}(W(k))$ and let F on N be given on the base vectors $e_0, \ldots, e_{h-1}$by the same assignment as before.

Set $\mathbb{M} := \mathbb{D}(\mathbb{X}) \otimes_{\mathbb{Z}_p} W(k) = \bigoplus_{i=0}^{h-1} W(k) \cdot e_i$.

We use the same basis $\{e_0, \ldots, e_{h-1}\}$ for N as before and set $e_{i+nh} := p^n \cdot e_i$ for any $n \in \mathbb{Z}$. This gives a system of elements $\{e_j, j \in \mathbb{Z}\}$ in N such that every element $v \in N$ can be uniquely written as $v = \sum_{j\in\mathbb{Z}}[a_j]e_j$, with $[a_j]$ the Teichmüller representative in $W(k)$ of an $a_j \in k$ and $a_j = 0$ for j small enough.

The semimodule of M

Oort and de Jong defined in [dJO] a combinatorial invariant for k-valued points of $\mathcal{M}_{\text{red}}$.
Recall that N is a simple isocrystal over k of slope $\lambda = \frac{2}{h}$. For any element $v = \sum_{k\in\mathbb{Z}}[a_k]e_k$ in N there is a $j \in \mathbb{Z}$ with $a_j \neq 0$ and $a_k = 0$ for all $k < j$. It is called the *first index* of v.

Definition 3.4. *A subset $A \subset \mathbb{Z}$ is called a* semimodule *if it is bounded below and satisfies $2 + A \subset A$ and $(h-2) + A \subset A$.*
If $M \subset N$ is the Dieudonné module of some $X \in \mathcal{M}(k)$, then

$$A(M) := \{j \in \mathbb{Z} \mid j \textit{ is the first index of some } v \in M\}$$

is a semimodule in $\mathbb{Z}$, called the semimodule of M.

Denote again by w the extension of the fixed valuation w on $\mathbb{Q}_p^h$ to the isocrystal $N = \mathbb{Q}_p^h \otimes_{\mathbb{Q}_p} \mathrm{Frac}(W(k))$ over k.
For any F- and V-invariant lattice M in N we have

$$A(M) = h \cdot w(M) \subset \mathbb{Z}.$$

This follows directly from the observation that for $v \in M$ we have: j is the first index of v, if and only if $v = \sum_{k \geq j} [a_k] e_k$ with $a_j \neq 0$, if and only if $w(v) = \min_k \{v_p([a_k]) + \frac{k}{h}\} = \frac{j}{h}$ since all $[a_k]$ are either 0 or units in $W(k)$.

$A(M)$ is bounded from below and there exists a $N \in \mathbb{N}$ with $N + \mathbb{N} \subset A(M)$, since for $j \in A(M)$ we have $j+2 \in A(M)$ and $j + (h-2) \in A(M)$. So, since h is odd, we have $j + (h-3) + \mathbb{N} \subset A(M)$.
A lattice M in N which is also a Dieudonné module of some p-divisible group $X \in \mathcal{M}(k)$ is called a Dieudonné lattice (equivalently: M is F- and V-invariant).

Definition 3.5. *For $M, M' \subset N$ lattices set $|M/M'| :=$ index of M' in M, i.e.*

$$|M/M'| = lg_{W(k)}\left({}^{M}/_{(M \cap M')}\right) - lg_{W(k)}\left({}^{M'}/_{(M \cap M')}\right).$$

The index $|M/M'|$ is finite, since both M and M' are of full rank in N.

The following lemma allows us to compute some invariants of the Dieudonné lattices using their semimodules.

Lemma 3.6. *Let $(0) \neq M$ be an F- and V-invariant lattice in N with semimodule $A(M)$.*
(1) *For any F- and V-invariant sublattice $(0) \neq M' \subsetneq M$ we have:*

$$|M/M'| = \#\{A(M) \setminus A(M')\}.$$

(2) *For any F- and V-invariant lattice $(0) \neq M'$ we have:*

$$|M/M'| = \#\{A(M) \setminus A(M')\} - \#\{A(M') \setminus A(M)\}.$$

Proof. (1) Let $S = A(M) \setminus A(M')$. Suppose first that $S \neq \emptyset$.
Since both are bounded below and there exists an integer $n \in \mathbb{N}$ with $n + \mathbb{N} \subset A(M') \subset A(M)$, this difference set is finite, so $S = \{s_1, \ldots, s_n\}$ with $s_1 < s_2 < \ldots < s_n$. For $s \in S$ let $m_s = e_s + \sum_{j>s} [\alpha_j] e_j \in M$ be an element of valuation $w(m_s) = \frac{s}{h}$ in M.

Then

$$M' \subsetneq M' + \langle m_{s_n} \rangle_{W(k)} \subsetneq M' + \langle m_{s_n} \rangle_{W(k)} + \langle m_{s_{n-1}} \rangle_{W(k)} \subsetneq \cdots$$
$$\ldots \subsetneq M' + \sum_{i=2}^{n} \langle m_{s_i} \rangle_{W(k)} \subsetneq M' + \sum_{i=1}^{n} \langle m_{s_i} \rangle_{W(k)} \subseteq M$$

is a chain of submodules of length $\#\{S\} = \#\{A(M) \setminus A(M')\}$.

To show the other inequality, consider first the case of M and M' both F- and V-invariant lattices in N with $M' \subseteq M$ and $A(M) = A(M')$, that is, $S := A(M) \setminus A(M') = \emptyset$. We want to show that in this situation only the case $M = M'$ is possible.
Let $0 \neq m \in M$ and let $m' \in M'$ be an element of the same valuation. By multiplying both with suitable units in W, we can achieve for both to look like

$$m = e_j + \sum_{i>j} [\alpha_i] e_i \quad \text{and} \quad m' = e_j + \sum_{i>j} [\beta_i] e_i.$$

Set $m_1 = m - m'$. Then $m = m' + m_1$ and the valuation of m_1 is strictly bigger than that of m and m'. Since $A(M') = A(M)$, we can find an m_1' in M' of the same valuation as m_1, such that their difference $m_2 = m_1 - m_1'$ has again a strictly bigger valuation. By proceeding, we get two sequences of elements (m_i) and (m_i') with $m_i' \in M'$ such that

$$m = m' + m_1 = m' + m_1' + m_2 = \ldots = m' + m_1' + \ldots + m_n' + m_{n+1} = \ldots$$

and $w(m_n) \to \infty$ and $w(m_n') \to \infty$ for $n \to \infty$. Since $W(k)$ is complete, so are M and M', thus $m \in M'$ and, consequently, $M' = M$.

If we have a chain of submodules $M' \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M$, as in the case of the lemma, the consideration above gives us a chain of semimodules $A(M') \subsetneq A(M_1) \subsetneq A(M_2) \subsetneq \ldots \subsetneq A(M)$ and the length of this chain is always less or equal to $\#\{A(M) \setminus A(M')\}$.

(2) Follows directly from (1), since $A(M \cap M')$ is a subset of $\big(A(M) \cap A(M')\big)$ with finite complement and the subtraction of this complement in both terms cancels out.

□

A different way to compute the index of two Dieudonné lattices is given by the following remark.

Remark 3.7. *Let M, M' lattices in N and $g \in \mathrm{GL}_h(\mathrm{Frac}(W(k)))$ an automorphism of the underlying vector space N with $g(M) = M'$. Then*

$$|M/M'| = v_p(\det(g)).$$

Proof. g is uniquely determined up to multiplication from left and right with elements of $\mathrm{GL}_h(W(k))$, thus $v_p(\det(g))$ is well-defined. If $M' \subset M$, one can find a basis $\{m_1, \ldots, m_h\}$ of M and integers $0 \leq i_1 \leq \ldots \leq i_h$ such that $\{p^{i_j} \cdot m_j, 1 \leq j \leq h\}$ is a basis of M'. Then $v_p(\det(g)) = i_1 + \ldots + i_h = |M/M'|$. □

Theorem 3.8 (Section 3 in [Vie])**.** *The open and closed formal subschemes $\mathcal{M}(i)$ are connected.*

Proof. According to Theorem 3.1 of [Vie], the set of connected components of $\mathcal{M}$ can be identified with the set $\Delta = \mathcal{M}(\mathbb{X}_{\mathrm{m}}) \times \mathcal{M}(\mathbb{X}_{\text{ét}}) \times \mathbb{Z}$, where $\mathbb{X} = \mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\text{ét}}$ is the decomposition of $\mathbb{X}$ into its multiplicative, bi-infinitesimal and étale parts, and $\mathcal{M}(\mathbb{X}_{\mathrm{m}})$ and $\mathcal{M}(\mathbb{X}_{\text{ét}})$ are the moduli spaces corresponding to $\mathbb{X}_{\mathrm{m}}$ and $\mathbb{X}_{\text{ét}}$. If $S \in \mathrm{Nilp}_{W(k)}$ and $\rho\colon \mathbb{X}_{\bar{S}} \longrightarrow X_{\bar{S}}$ a quasi-isogeny to a p-divisible group X over S, we get a morphism $X \longrightarrow X_{\text{ét}}$ with $X_{\text{ét}}$ étale over S, and a quasi-isogeny $\rho_{\text{ét}}\colon \mathbb{X}_{\text{ét}} \longrightarrow X_{\text{ét}}$ which is functorially in ρ. The assignment $\rho \longmapsto \rho_{\text{ét}}$ gives a morphism $\kappa_{\text{ét}}\colon \mathcal{M} \longrightarrow \mathcal{M}(\mathbb{X}_{\text{ét}})$.
By duality one obtains also a morphism $\kappa_{\mathrm{m}}\colon \mathcal{M} \longrightarrow \mathcal{M}(\mathbb{X}_{\mathrm{m}})$, and by combining both these morphisms with the locally constant height morphism, one gets

$$\kappa\colon \mathcal{M} \longrightarrow \Delta,$$

which sends a pair $(X,\rho) \in \mathcal{M}(S)$ to $(\rho_{\mathrm{m}}, \rho_{\text{ét}}, \mathrm{ht}(\rho))$ and identifies the set of connected components of $\mathcal{M}$ with Δ.

However, in our case the p-divisible group $\mathbb{X}$ has only trivial multiplicative and étale parts, since its isocrystal $N(\mathbb{X})$ is simple, and, if $\mathbb{X}$ were multiplicative or étale there would exist a basis of $N(\mathbb{X})$ such that $F = p^{\alpha}\sigma$ with $\alpha \in \{0,1\}$ according to this basis. Since this is not the case here, we have $\Delta = \mathbb{Z}$, $\kappa = \mathrm{ht}$ and $\mathcal{M}(i) = \kappa^{-1}(i)$ is connected.

□

A semimodule $A \subset \mathbb{Z}$ is called *normalized* if $|A \setminus \mathbb{N}| = |\mathbb{N} \setminus A|$. One sees immediately that there are only finitely many normalized semimodules in $\mathbb{Z}$: if $j \in A$ is the minimal element of A (it exists since A is bounded below), we have $j + (h-3) + \mathbb{N} \subset A$ because h is odd, so the number of elements $k > j$ but $k \notin A$ is bounded by $\frac{h-3}{2}$.

Thus, for (X,ρ) in $\mathcal{M}(k)$, we have

$$(X,\rho) \in \mathcal{M}(i)(k) \iff \mathrm{height}(\rho) = i = v_p(\det(\mathbb{D}(\rho))) = |\mathbb{D}(X)/\mathbb{M}|.$$

Corollary 3.9. *The open and closed subschemes $\mathcal{M}(i)$ are non-empty and isomorphic to each other.*

Proof. For any $i \in \mathbb{Z}$ let $M_i = \langle e_i, \ldots, e_{i+h-1}\rangle_{\mathbb{Z}_p}$. This is an F- and V-invariant lattice in $N(\mathbb{X})$ with semimodule $A(M_i) = i + \mathbb{N}$, so, $M_i \otimes_{\mathbb{Z}_p} W(k)$ is of index i in $\mathbb{M}$, thus the corresponding p-divisible group $(X_i, \rho) \in \mathcal{M}(\mathbb{F}_p)$ lies in $\mathcal{M}(i)(\mathbb{F}_p)$.

To show the isomorphism between $\mathcal{M}(0)$ and $\mathcal{M}(i)$, we have to find a quasi-isogeny $\rho_i\colon \mathbb{X} \longrightarrow \mathbb{X}$ of height i. Any quasi-isogeny of $\mathbb{X}$ to itself gives us an automorphism of the isocrystal (N, F), and this group is described by

$$\mathrm{Aut}_{isoc}(N,F) \cong D_{\frac{2}{h}} = \mathbb{Q}_{p^h}[\Pi]/\big(\Pi^h = p^2, \Pi \cdot a = a^\sigma \cdot \Pi, a \in \mathbb{Q}_{p^h}\big).$$

Here $D_{\frac{2}{h}}$ denotes the division algebra over $\mathbb{Q}_p$ with invariant $\frac{2}{h}$ and σ is the automorphism of $\mathbb{Q}_{p^h}$ which induces $x \longmapsto x^p$ on $\mathbb{F}_{p^h}$ as before. Thus, we have to find an $f \in D_{\frac{2}{h}}$ with $v_p(\det(f)) = v_p(f \cdot \sigma(f) \cdot \ldots \cdot \sigma^{h-1}(f)) = i$. Set $f = p^a \cdot \Pi^b$ with $a, b \in \mathbb{Z}$ such that $h \cdot a + 2 \cdot b = i$. This is possible, since h is odd. Then $v_p(\det(f)) = h \cdot v_p(f) = h \cdot (a + \frac{2}{h} \cdot b) = i$. And the composition with ρ_i gives an isomorphism

$$\mathcal{M}(0)(S) \longrightarrow \mathcal{M}(i)(S), \quad (X, \rho) \longmapsto (X, \rho_{i,\bar{S}} \circ \rho \colon \mathbb{X}_{\bar{S}} \longrightarrow X_{\bar{S}}).$$

□

Structure of lattices $M \in \mathcal{N}(k)$

Let M be a Dieudonné lattice in N which is the Dieudonné module of some p-divisible group in $\mathcal{N}(k)$, that is, M has index 0 in $\mathbb{M}$. Then its semimodule $A(M)$ is normalized, which means $|A(M) \setminus \mathbb{N}| = |\mathbb{N} \setminus A(M)|$. As we have seen in the previous section, the number of integers which are bigger than the minimal element of $A(M)$, but are not contained in $A(M)$ is limited by $\frac{h-3}{2}$. Thus, there exists a $0 \leq j = j(M) \leq \frac{h-3}{2}$, such that

$$A(M) = \{-j, -j+2, -j+4, \ldots, j-4, j-2, j, j+1, j+2, j+3, \ldots\}$$

$$= (-j + 2\mathbb{N}) \cup (j + \mathbb{N}).$$

In fact, there is an $m = e_{-j} + \sum_{i>-j}[a_i]e_i$ in M, such that

$$M = \langle m, F(m), F^2(m), \ldots, F^{j-1}(m), e_j, e_{j+1}, e_{j+2}, \ldots \rangle_{W(k)}.$$

The following picture shows the semimodule of a F- and V-invariant lattice M with index 0 in $\mathbb{M} = \mathbb{D}(\mathbb{X}) \otimes W(k)$, where the dots stand for points in $A(M)$ and the boxes for points which are not in $A(M)$, also called "gaps".

…	□	•	□	•	alternating dots and gaps	□	•		•	…
…	$-j-1$	$-j$	…			$j-1$	j		$j+1$	…

By multiplying the elements $F^i(m), 0 < i < j$, and the $e_k, k \geq j$, with suitable scalars and subtracting them from m, we can achieve for m to have the form:

$$m = e_{-j} + [\alpha_{-j+1}]e_{-j+1} + [\alpha_{-j+3}]e_{-j+3} + \ldots + [\alpha_{j-1}]e_{j-1}$$

with $\alpha_i \in k$. Thus, we have seen the following

Proposition 3.10. *Let $M \subset N$ be a Dieudonné lattice. Then there exists an index $j \in \mathbb{Z}$ and an element $m \in M$ with*

$$m = e_{-j} + [\alpha_{-j+1}]e_{-j+1} + [\alpha_{-j+3}]e_{-j+3} + \ldots + [\alpha_{j-1}]e_{j-1}$$

with $a_i \in k$, such that

$$M = \langle m, F(m), \ldots, F^{j-1}(m), e_j, e_{j+1}, \ldots \rangle_{W(k)}$$

and

$$A(M) = (-j + 2\mathbb{N}) \cup (j + \mathbb{N}).$$

τ-invariant lattices

Following the idea of [VW], we first want to define a stratification of $\mathcal{N}(k)$ by suitable pairs of lattices in N. Here, we choose τ-invariant lattices, where τ is the operator on N given by

$$\tau := p^{-2}F^h.$$

If $\{e_i, i \in \mathbb{Z}\}$ is the system of elements in N defined above then τ acts on an element $m = \sum_{k\in\mathbb{Z}}[\alpha_k]e_k$ as $\tau(m) = \sum_{k\in\mathbb{Z}}[\sigma^h(\alpha_k)]e_k$.

To any Dieudonné lattice $M \subset N$ we attach the following two τ-invariant lattices:

$$\Lambda^+M := \sum_{i\geq 0} \tau^i(M),$$
$$\Lambda^-M := \bigcap_{i\geq 0} \tau^i(M).$$

These are again F- and V-invariant lattices in N for τ commutes with F and V, but they are, in general, not attached to some p-divisible groups in $\mathcal{N}$, since their index in $\mathbb{M}$ can differ from 0.

We now determine the pairs of lattices (M^+, M^-) that can occur as maximal resp. minimal τ-invariant lattices of a Dieudonné lattice $M \subset N$.

Let $M \in \mathcal{N}(k)$ be a Dieudonné lattice in N and

$$m = e_{-j} + [\alpha_{-j+1}]e_{-j+1} + [\alpha_{-j+3}]e_{-j+3} + \ldots + [\alpha_{j-1}]e_{j-1}$$

be the element with $M = \langle m, F(m), \ldots, F^{j-1}(m), e_j, e_{j+1}, \ldots \rangle_{W(k)}$ as in Proposition 3.10.
Now

$$\tau(m) = \tau(e_{-j}) + \sum_{i=0}^{j-1} \tau([a_{-j+2i+1}]e_{-j+2i+1}) = e_{-j} + \sum_{i=0}^{j-1}[a^{p^h}_{-j+2i+1}]e_{-j+2i+1},$$

thus it depends on the coefficients $a_i \in k$ whether M is τ-invariant.

Let now $l \in [0, j-1] \cap \mathbb{N}$ be the smallest integer with $\alpha_{-j+2l+1} \in k \setminus \mathbb{F}_{p^h}$ if such an l exists, and set $l := j$ if all α_i are in $\mathbb{F}_{p^h}$. Then

$$m - \tau(m) = ([\alpha_{-j+2l+1}] - [\alpha_{-j+2l+1}^{p^h}])e_{-j+2l+1} + \sum_{i > -j+2l+1} [\beta_i] e_i$$

is an element of $\Lambda^+ M$ with valuation $\frac{-j+2l+1}{h}$, and thus gives the element $-j+2l+1 \in A(M^+)$. By F-invariance of M^+ we now have

$$\begin{aligned} A(\Lambda^+ M) = \{ & -j, -j+2, -j+4, \ldots, -j+2l-2, -j+2l, \\ & -j+2l+1, -j+2l+2, -j+2l+3, \ldots \} \end{aligned}$$

$$= \ (-j + 2\mathbb{N}) \ \cup \ \big((-j+2l) + \mathbb{N}\big).$$

If $l < j$, none of the elements $m, F(m), \ldots, F^{j-l-1}(m)$ are τ-invariant, but a suitable linear combination of $F^{j-l}(m)$ and the $e_i, i \geq j$ is, so

$$A(\Lambda^- M) = \{j-2l, j-2l+2, \ldots, j-2, j, j+1, j+2, \ldots\}$$

$$= \big((j-2l) + 2\mathbb{N}\big) \ \cup \ (j + \mathbb{N}).$$

So, $\Lambda^- M \subset M \subset \Lambda^+ M$ is a chain of F- and V-invariant lattices in N with indexes $|\Lambda^+ M / M| = |M / \Lambda^- M| = j - l$.
We can now give the precise description of the occurring pairs of lattices $(\Lambda^+ M, \Lambda^- M)$:

Lemma 3.11. *Let $0 \leq j \leq \frac{h-3}{2}$ and $0 \leq l \leq j$ be integers, and fix a tuple $\underline{\alpha_l} = (\alpha_1, \ldots, \alpha_l) \in \mathbb{F}_{p^h}^l$. If $l = 0$, then $\underline{\alpha_l} := \emptyset$.*

For each tuple $(j, l, \underline{\alpha_l})$ define two $\mathbb{Z}_{p^h}$-lattices $\Lambda^+_{j,l,\underline{\alpha_l}}$ and $\Lambda^-_{j,l,\underline{\alpha_l}}$ in the isocrystal $(\mathbb{Q}_p^h, F) \otimes \mathbb{Q}_{p^h}$ over $\mathbb{F}_{p^h}$:

$$\begin{aligned} \Lambda^+_{j,l,\underline{\alpha_l}} := \ & \langle v = v(j,l,\underline{\alpha_l}) = e_{-j} + [\alpha_1]e_{-j+1} + [\alpha_2]e_{-j+3} + \ldots + [\alpha_l]e_{-j+2l-1}, \\ & F(v), F^2(v), \ldots, F^l(v), e_{-j+2l+1}, e_{-j+2l+2}, \ldots \rangle_{\mathbb{Z}_{p^h}} \end{aligned}$$

$$\begin{aligned} \Lambda^-_{j,l,\underline{\alpha_l}} := \ & \langle w = w(j,l,\underline{\alpha_l}) = e_{j-2l} + [\alpha_1^{p^{j-l}}]e_{j-2l+1} + [\alpha_2^{p^{j-l}}]e_{j-2l+3} + \ldots \\ & \ldots + [\alpha_l^{p^{j-l}}]e_{j-1}, \\ & F(w), F^2(w), \ \ldots, \ F^l(w), e_{j+1}, e_{j+2}, \ldots \rangle_{\mathbb{Z}_{p^h}} \end{aligned}$$

Then for every $M \in \mathcal{N}(k)$, there exists a tuple $(j, l, \underline{\alpha_l})$ with $0 \leq j \leq \frac{h-3}{2}$, $0 \leq l \leq j$ and $\underline{\alpha_l} \in \mathbb{F}_{p^h}^l$, such that

$$\Lambda^+ M = \Lambda^+_{j,l,\underline{\alpha_l}} \otimes_{\mathbb{Z}_{p^h}} W(k) \text{ and } \Lambda^- M = \Lambda^-_{j,l,\underline{\alpha_l}} \otimes_{\mathbb{Z}_{p^h}} W(k).$$

Proof. Take the element $m = e_{-j} + [\alpha_{-j+1}]e_{-j+1} + [\alpha_{-j+3}]e_{-j+3} + \ldots + [\alpha_{j-1}]e_{j-1}$ as in Prop 3.10 with $M = \langle m, F(m), \ldots, F^{j-1}(m), e_j, \ldots \rangle_{W(k)}$, and let $l \in [0, j-1] \cap \mathbb{N}$ be as before the smallest integer with $\alpha_{-j+2l+1} \in k \setminus \mathbb{F}_{p^h}$ if such an l exists, and set $l := j$ if all α_i are in $\mathbb{F}_{p^h}$. Then for $\underline{\alpha_l} = (\alpha_{-j+1}, \alpha_{-j+3}, \ldots, \alpha_{-j+2l-1}) \in \mathbb{F}_{p^h}^l$ we have the desired equations of lattices. □

Now define a stratification of the set $\mathcal{N}(k)$ by subsets of the form

$$\mathcal{N}_{j,l,\underline{\alpha_l}}(k) := \left\{ M \in \mathcal{N}(k) \mid \begin{array}{c} \Lambda^-_{j,l,\underline{\alpha_l}} \otimes_{\mathbb{Z}_{p^h}} W(k) \subset \Lambda^- M \\ \text{and} \\ \Lambda^+ M \subset \Lambda^+_{j,l,\underline{\alpha_l}} \otimes_{\mathbb{Z}_{p^h}} W(k) \end{array} \right\}.$$

Before showing some properties of these subsets, we recall some facts on finite locally free groups over an $\mathbb{F}_p$-scheme S from [dJ].

Let S be a scheme over $\mathrm{Spec}(\mathbb{F}_p)$, and denote by f_S the absolute Frobenius endomorphism on S. For G a finite locally free group over S denote by

$$G^D = \mathcal{H}om(G, \mathbb{G}_{m,S})$$

its Cartier dual. The assignment $G \longrightarrow G^D$ is a contravariant auto-equivalence of the category of finite locally free group schemes over S.
Let $F_G \colon f_S^* G =: G^{(p)} \longrightarrow G$ be the Frobenius morphism of the scheme G over S and $V_G := \left(F_{G^D}\right)^D \colon G \longrightarrow G^{(p)}$ the Verschiebung morphism of G. These two morphisms satisfy $F_G \circ V_G = p \cdot \mathrm{id}_G$ and $V_G \circ F_G = p \cdot \mathrm{id}_{G^{(p)}}$.
There is the following result by de Jong:

Proposition 3.12 (Section 2 in [dJ]). *Let S be a scheme over* $\mathrm{Spec}\,\mathbb{F}_p$. *Then there is a contravariant equivalence of categories*

$$\left(\begin{array}{l} \textit{finite locally free group schemes} \\ G \textit{ over } S \textit{ with } V_G = 0 \end{array} \right) \to \left(\begin{array}{l} \textit{locally free } \mathcal{O}_S\textit{-modules } M + \\ F \colon M^{(p)} \longrightarrow M \ \mathcal{O}_S - \textit{linear} \end{array} \right)$$

$$G \mapsto (\alpha_G,\ F \colon \alpha_G^{(p)} \cong \alpha_{G^{(p)}} \longrightarrow \alpha_G)$$

where $M^{(p)} = f_S^*(M)$ is the pullback of the $\mathcal{O}_S$-module M via the Frobenius morphism f_S on $\mathcal{O}_S$ and for G as before the $\mathcal{O}_S$-module α_G is defined as $\alpha_G = \mathcal{H}om_{\mathcal{O}_S}(G, \mathbb{G}_{a,S})$.

Lemma 3.13. *For every* $0 \leq j \leq \frac{h-3}{2}, 0 \leq l \leq j$ *and* $\underline{\alpha_l} \in \mathbb{F}^l_{p^h}$ *the subset* $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ *is a closed irreducible subset of* $\mathcal{N}(k)$ *of dimension* $j - l$.

Proof. In the last section we have seen that the Dieudonné module M of every $X \in \mathcal{N}(k)$ is contained in the lattice $\mathbb{M}^+ \otimes W(k)$, where

$$\mathbb{M}^+ := \langle e_{-\frac{h-3}{2}}, e_{-\frac{h-3}{2}+1}, \ldots, e_{\frac{h-3}{2}-1} \rangle_{\mathbb{Z}_{p^h}}$$

since $j \leq \frac{h-3}{2}$. It also must contain the lattice $\mathbb{M}^- \otimes W(k)$, where

$$\mathbb{M}^- := \langle e_{\frac{h-3}{2}}, e_{\frac{h-3}{2}+1}, \ldots, e_{\frac{h-3}{2}+(h-1)} \rangle_{\mathbb{Z}_{p^h}}.$$

Now $p \cdot \mathbb{M}^+ = \langle e_{-\frac{h-3}{2}+h}, e_{-\frac{h-3}{2}+h+1}, \ldots \rangle_{\mathbb{Z}_{p^h}} \subset \mathbb{M}^-$, so that their quotient $\big(\mathbb{M}^+/\mathbb{M}^-\big)$ is an $\mathbb{F}_{p^h}$-vector space.

Let $\mathbb{X}^+$ and $\mathbb{X}^-$ be p-divisible groups over $\mathbb{F}_{p^h}$ with Dieudonné modules

$$\mathbb{D}(\mathbb{X}^+) = \mathbb{M}^+ \text{ and } \mathbb{D}(\mathbb{X}^-) = \mathbb{M}^-.$$

From the inclusions of Dieudonné modules $\mathbb{M}^- \hookrightarrow \mathbb{D}(\mathbb{X}) \otimes \mathbb{Z}_{p^h} \hookrightarrow \mathbb{M}^+$ we get an isogeny $\tilde{\rho}\colon \mathbb{X}^+ \longrightarrow \mathbb{X}^-$ of height $h - 3$ and also two isogenies

$$\rho^+\colon \mathbb{X}^+ \longrightarrow \mathbb{X} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^h}, \qquad \rho^-\colon \mathbb{X} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^h} \longrightarrow \mathbb{X}^-$$

of height $\frac{h-3}{2}$.

Let $\mathcal{N}'$ be the functor, which assigns to a scheme S over $\mathrm{Spec}(\mathbb{F}_{p^h})$ the set

$$\mathcal{N}'(S) = \left\{ (X,\rho) \in \mathcal{N}(S) \;\middle|\; \begin{array}{l} \rho \circ (\rho_S^+)^{-1}\colon \mathbb{X}^+ \times S \longrightarrow \mathbb{X} \times S \longrightarrow X \text{ and} \\ \rho_S^- \circ (\rho)^{-1}\colon X \longrightarrow \mathbb{X} \times S \longrightarrow \mathbb{X}^- \times S \\ \text{are isogenies} \end{array} \right\}$$

Then, according to Proposition 2.9 in [RZ], $\mathcal{N}'$ is a closed subfunctor of $\mathcal{N} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\mathbb{Z}_{p^h})$ with $\mathcal{N}'(k) \cong \mathcal{N}(k)$, since for $X \in \mathcal{N}(k)$ with Dieudonné module $\mathbb{D}(X) = M$ we have the inclusions $\mathbb{M}^- \otimes W(k) \subset M \subset \mathbb{M}^+ \otimes W(k)$, thus the isogenies $\mathbb{X}^+ \times_{\mathrm{Spec}(\mathbb{F}_{p^h})} \mathrm{Spec}(k) \longrightarrow X \longrightarrow \mathbb{X}^- \times_{\mathrm{Spec}(\mathbb{F}_{p^h})} \mathrm{Spec}(k)$.
Let $X \in \mathcal{N}'(R)$ for an $\mathbb{F}_{p^h}$-algebra R. Then from the definition of $\mathcal{N}'$ we get two exact sequences

$$0 \longrightarrow \mathrm{Ker}(\rho_X^+) \longrightarrow \mathbb{X}_R^+ \xrightarrow{\rho_X^+} X \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ker}(\rho_X^-) \longrightarrow X \xrightarrow{\rho_X^-} \mathbb{X}_R^- \longrightarrow 0$$

with $\mathrm{Ker}(\rho_X^+) \subset \mathrm{Ker}(\tilde{\rho}_R \colon \mathbb{X}_R^+ \longrightarrow \mathbb{X}_R^-)$, because the composition $\mathbb{X}_R^+ \longrightarrow X \longrightarrow \mathbb{X}_R^-$ is an isogeny. Since the isogeny ρ_X^+ is determined up to isomorphism by its kernel, we have to describe the subgroups of $\mathrm{Ker}(\tilde{\rho}_R)$ of height $\frac{h-3}{2} = \mathrm{height}(\rho_X^+)$.
The Dieudonné module $\mathbb{D}(\mathrm{Ker}(\tilde{\rho})) = \mathbb{M}^+/\mathbb{M}^-$ of $\mathrm{Ker}(\tilde{\rho})$ is annihilated by V, since

$$V(\mathbb{M}^+) = \langle e_{\frac{h-1}{2}}, \ldots, e_{3.\frac{h-1}{2}} \rangle_{\mathbb{Z}_{p^h}} \subset \mathbb{M}^-,$$

so by Lemma 3.12 the subgroup $\mathrm{Ker}(\rho_X^+)$ of $\mathrm{Ker}(\tilde{\rho}_R)$ is uniquely determined by the associated surjective morphism of Dieudonné modules

$$\mathbb{D}(\mathrm{Ker}(\tilde{\rho}_R)) \longrightarrow \mathbb{D}(\mathrm{Ker}(\rho_X^+)).$$

This one, being surjective, is again uniquely described by its kernel

$$\mathrm{Ker}\left(\mathbb{D}(\mathrm{Ker}(\rho_X^+) \hookrightarrow \mathrm{Ker}(\tilde{\rho}_R))\right),$$

which is a locally free direct summand of $\mathbb{D}(\mathrm{Ker}(\tilde{\rho}_R))$ of rank $\frac{h-3}{2}$ because $\mathbb{D}(\mathrm{Ker}(\rho_X^+))$ is a locally free R-module of rank $\frac{h-3}{2}$.
Thus, we can define a morphism of functors $\mathcal{N}' \longrightarrow \mathrm{Grass}_{\frac{h-3}{2}}(\mathbb{M}^+/\mathbb{M}^-)$ by the prescription on R-valued points for any $\mathbb{F}_{p^h}$-algebra R:

$$\begin{aligned} \mathcal{N}'(R) &\longrightarrow \mathrm{Grass}_{\frac{h-3}{2}}\left(\mathbb{M}^+/\mathbb{M}^-\right)(R), \\ (X, \rho) &\longmapsto \mathrm{Ker}\left(\mathbb{D}(\mathrm{Ker}(\rho_X^+) \hookrightarrow \mathrm{Ker}(\tilde{\rho}_R))\right) \end{aligned}$$

where $\mathrm{Grass}_{\frac{h-3}{2}}\left(\mathbb{M}^+/\mathbb{M}^-\right)(R)$ denotes the R-valued points of Grassmannian variety over $\mathbb{F}_{p^h}$, that is, the set of locally free direct summands of the R-module $\left((\mathbb{M}^+/\mathbb{M}^-) \otimes_{\mathbb{F}_{p^h}} R\right)$ of rank $\frac{h-3}{2}$.

So, since $\mathcal{N}(k) = \mathcal{N}'(k)$, we get a morphism

$$\begin{aligned} \mathcal{N}(k) &\longrightarrow \mathrm{Grass}_{\frac{h-3}{2}}\left(\mathbb{M}^+/\mathbb{M}^-\right)(k), \\ X &\longmapsto \mathbb{D}(X)/(\mathbb{M}^- \otimes W(k)), \end{aligned}$$

which comes from the morphism of functors $\mathcal{N}' \longrightarrow \mathrm{Grass}_{\frac{h-3}{2}}(\mathbb{M}^+/\mathbb{M}^-)$.

We can simplify the conditions for $M \in \mathcal{N}(k)$ being in $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ to

$$M \in \mathcal{N}_{j,l,\underline{\alpha_l}}(k) \Longleftrightarrow \Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k) \subset M \subset \Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k),$$

since both lattices $\Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k)$ and $\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)$ are τ-invariant. $\Lambda^+ M$ being the minimal τ-invariant lattice containing M must therefore be contained in $\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)$ and $\Lambda^- M$, which is the maximal τ-invariant lattice contained in M, must also contain $\Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k)$.

Both $\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)$ and $\Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k)$ contain $\mathbb{M}^- \otimes W(k)$ and are contained in $\mathbb{M}^+ \otimes W(k)$, thus correspond to certain subspaces in $(\mathbb{M}^+/\mathbb{M}^-) \otimes k$. So, the conditions for M being in $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ above transform into inclusion conditions for subspaces, and both of them define closed subsets in the Grassmannian variety of $\frac{h-3}{2}$-dimensional subspaces in $\big(\mathbb{M}^+/\mathbb{M}^-\big) \otimes k$.

Let $\mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k)$ be the subset of $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ defined by the condition

$$\begin{aligned}\mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k) = \{M \in \mathcal{N}(k) | \Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k) = \Lambda^- M, \\ \Lambda^+ M = \Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)\}.\end{aligned}$$

Then every $M \in \mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k)$ is of the form

$$\begin{aligned}M = \langle m = v(j,l,\underline{\alpha_l}) + [\beta_{l+1}]e_{-j+2l+1} + \ldots + [\beta_j]e_{j-1}, \\ F(m), \ldots, F^{j-1}(m), e_j, e_{j+1}, \ldots\rangle_W\end{aligned}$$

with $\beta_{l+1} \in k \setminus \mathbb{F}_{p^h}$. One sees that $m - \tau(m) = \big[\beta_{l+1} - \beta_{l+1}^{p^h}\big]e_{-j+2l+1} + \ldots$ is an element of valuation $\frac{-j+2l+1}{h}$ in $\Lambda^+ M$, so $\Lambda^+(M)$ is a submodule of $\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)$ with the same semimodule as $\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)$ and by the proof of Lemma 3.6 we have the equality of lattices $\Lambda^+ M = \Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)$. So $\mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k) \cong \mathbb{A}^{j-l-1}(k) \times (\mathbb{A}^1(k) \setminus \mathbb{A}^1(\mathbb{F}_{p^h}))$ is irreducible of dimension $j-l$.

The quotient $\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)/\Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k)$ is a k-vector space of dimension $2(j-l)$ and for any $M \in \mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ the quotient $\overline{M} := M/\Lambda^-_{j,l,\underline{\alpha_l}}$ is a subspace of $\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}$ of dimension $j-l$.
But, of course, not every subspace of dimension $j-l$ in $\big(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\big) \otimes k$ corresponds to a lattice $M \in \mathcal{N}(k)$, for it has to be also F-invariant. The V-invariance does not matter here, because V is zero on the subquotient $(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}) \otimes k$ of $(\mathbb{M}^+/\mathbb{M}^-) \otimes k$.
This invariance condition is again a closed condition, so we can view $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ as a closed subvariety of $\mathrm{Grass}_{j-l}(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}})(k)$.

If we take the residue classes in $\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}$ of the basis elements of $\Lambda^+_{j,l,\underline{\alpha_l}}$ as above, they form a generating system of the quotient $\big(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\big) \otimes k$ consisting of τ- stable elements. Choosing a basis $\{v_i\}$ in this generating system, the operator τ acts on $\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}} \otimes k$ as $\tau(\sum_{i=1}^{2(j-l)} a_i v_i) = \sum_{i=1}^{2(j-l)} a_i^{p^h} v_i$.

If $M \in \mathcal{N}(k)$ is an element of $\mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k)$, we see that

$$\Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k) = \Lambda^+(M) = M + \tau(M),$$

since $m - \tau(m)$ is an element of valuation $\frac{-j+2l+1}{h}$ in $M + \tau(M)$, $F(m) - \tau(F(m))$ has valuation $\frac{-j+2l+3}{h}$, and so on. Thus $M + \tau(M)$ is an F- and V-invariant sublattice of $\Lambda^+ M$ with the same semimodule as $\Lambda^+ M$. The equality follows again from the proof of Lemma 3.6 .

This means that the two subspaces $\overline{M} := M/\big(\Lambda^-_{j,l,\underline{\alpha_l}} \otimes k\big)$ and $\overline{\tau(M)} := \tau(M)/\big(\Lambda^-_{j,l,\underline{\alpha_l}} \otimes k\big)$ of $\big(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\big) \otimes k$ have the property:

$$\overline{M} \cap \overline{\tau(M)} = \{0\}, \quad \text{or, equivalently,} \quad \overline{M} + \overline{\tau(M)} = \big(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\big) \otimes k.$$

Thus, $\mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k)$ is the intersection of the closed subvariety $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ of $\mathrm{Grass}_{j-l}\left(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\right)$ with the one Deligne-Lusztig variety of maximal dimension in $\mathrm{Grass}_{j-l}\left(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\right)$ given by the Weil group element of maximal length, that is, the set of all subspaces U of dimension $j-l$ in $(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}) \otimes k$ fulfilling $U \cap \sigma^h(U) = \{0\}$. The latter is an open subvariety of $\mathrm{Grass}_{j-l}\left(\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}}\right)$, so is $\mathcal{N}^\circ_{j,l,\underline{\alpha_l}}(k)$ in $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$.
The irreducibility of the $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ will be shown later in the Corollary 3.17. □

Inclusion relations between $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$

Let $(j, l, \underline{\alpha_l})$ and $(j', l', \underline{\beta_{l'}})$ be tuples as before and $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ and $\mathcal{N}_{j',l',\underline{\beta_{l'}}}(k)$ the associated closed subsets of $\mathcal{N}(k)$.
Then for $\mathcal{N}_{j,l,\underline{\alpha_l}}(k) \subset \mathcal{N}_{j',l',\underline{\beta_{l'}}}(k)$ we have to require $j \leq j'$ and distinguish the two cases:

(a) $j' - j$ odd, i.e. there exists a $k \in \mathbb{N}$ such that $j' = j + 2k + 1$.
To get $\Lambda^+_{j,l,\underline{\alpha_l}} \subset \Lambda^+_{j',l',\underline{\beta_{l'}}}$ and $\Lambda^-_{j',l',\underline{\beta_{l'}}} \subset \Lambda^-_{j,l,\underline{\alpha_l}}$, we then have to require $l' \leq k$ with no further conditions on the coefficients α_i and β_i.

This is clear if we consider the semimodules of $\Lambda^+_{j,l,\underline{\alpha_l}}$ and $\Lambda^+_{j',l',\underline{\beta_{l'}}}$:
The inclusion $\Lambda^+_{j,l,\underline{\alpha_l}} \subset \Lambda^+_{j',l',\underline{\beta_{l'}}}$ of lattices gives an inclusion of semimodules $A(\Lambda^+_{j,l,\underline{\alpha_l}}) \subset A(\Lambda^+_{j',l',\underline{\beta_{l'}}})$, so the elements of $A(\Lambda^+_{j,l,\underline{\alpha_l}})$ must already be contained in $A(\Lambda_{j',l',\underline{\beta_{l'}}})$. This means that the gaps of $A(\Lambda^+_{j',l',\underline{\beta_{l'}}})$ can only occur to the left side of the first dot of $A(\Lambda^+_{j,l,\underline{\alpha_l}})$, since the gaps of both semimodules do not overlap. In particular, the number l' of gaps in $A(\Lambda^+_{j',l',\underline{\beta_{l'}}})$ is less than half the distance of the first dots j' and j, which is k.
The other inclusion $\Lambda^-_{j',l',\underline{\beta_{l'}}} \subset \Lambda^-_{j,l,\underline{\alpha_l}}$ does not impose further conditions.

(b) $j' - j$ even, i.e. there exists a $k \in \mathbb{N}$ such that $j' = j + 2k$.
Then we have to require $l' \le l + k$ and $\alpha_1 = \beta_1^{p^k}, \alpha_2 = \beta_2^{p^k}, \dots, \alpha_{l'-k} = \beta_{l'-k}^{p^k}$ if $l' - k \ge 1$ for the lattices to be contained in each other.

In this case, the gaps of the semimodules might overlap, but since $\Lambda^+_{j',l',\underline{\beta_{l'}}}$ is the bigger lattice, it must contain all the dots of the lesser lattice $\Lambda^+_{j,l,\underline{\alpha_l}}$, which limits the number l' of gaps in $A(\Lambda^+_{j',l',\underline{\beta_{l'}}})$ to $l + k =$ (half the distance of j' and j) + (number of gaps in $A(\Lambda^+_{j,l,\underline{\alpha_l}})$).
Also, the coefficients in the gaps of both semimodules are not independent. From Lemma 3.11 we have the generators of both lattices: $\Lambda^+_{j',l',\underline{\beta_{l'}}} = \langle v' = v(j', l', \underline{\beta_{l'}}), F(v'), \dots, e_{-j'+2l'+1}, \dots \rangle_{\mathbb{Z}_{p^h}}$ and $\Lambda^+_{j,l,\underline{\alpha_l}} = \langle v = v(j, l, \underline{\alpha_l}), F(v), \dots, e_{-j+2l+1}, \dots \rangle_{\mathbb{Z}_{p^h}}$. So, the vector $v = v(j, l, \underline{\alpha_l})$ must either be in the span of the $\{e_{-j'+2l'+i}, i \ge 1\}$, then the coefficients $\alpha_1, \dots, \alpha_l$ of $v(j, l, \underline{\alpha_l})$ do not depend on the β_i, or $v(j, l, \underline{\alpha_l}) = F^k(v') + \sum_{i \ge 1} [\gamma_i] e_{-j'+2l'+i}$. In this case the first $l' - k$ coefficients $\alpha_1, \dots, \alpha_{l'-k}$ of v must coincide with the first $l' - k$ coefficients of $F^k(v')$, which are $\beta_1^{p^k}, \dots, \beta_{l'-k}^{p^k}$.

Therefore, $\mathcal{N}_{\frac{h-3}{2},0,\emptyset}(k)$ is the biggest of these subsets, containing all other $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ and $\mathcal{N}(k) = \mathcal{N}_{\frac{h-3}{2},0,\emptyset}(k)$.

The subfunctors $\mathcal{N}_{j,l,\underline{\alpha_l}}$

Let $(j, l, \underline{\alpha_l})$ be tuples and $\Lambda^+_{j,l,\underline{\alpha_l}}$ and $\Lambda^-_{j,l,\underline{\alpha_l}}$ be the F- and V-invariant lattices in $(\mathbb{Q}_p, F) \otimes \mathbb{Q}_{p^h}$ associated to these tuples defined in the chapters before.
Let $X^+_{j,l,\underline{\alpha_l}}$ and $X^-_{j,l,\underline{\alpha_l}}$ be the p-divisible groups of dimension 2 and height h over $\mathbb{F}_{p^h}$ with Dieudonné modules $\Lambda^+_{j,l,\underline{\alpha_l}}$, resp. $\Lambda^-_{j,l,\underline{\alpha_l}}$, and denote by $\rho^+ \colon X^+_{j,l,\underline{\alpha_l}k} \longrightarrow \mathbb{X} \times \operatorname{Spec}(\mathbb{F}_{p^h})$ and $\rho^- \colon \mathbb{X} \times \operatorname{Spec}(\mathbb{F}_{p^h}) \longrightarrow X^-_{j,l,\underline{\alpha_l}k}$ the associated quasi-isogenies of height $j - l$.

By the same idea as in the proof of Lemma 3.13 we define a subfunctor $\mathcal{N}_{j,l,\underline{\alpha_l}}$ of $\mathcal{N} \times_{\operatorname{Spf} \mathbb{Z}_p} \operatorname{Spec} \mathbb{F}_{p^h}$ by

$$\begin{aligned}\mathcal{N}_{j,l,\underline{\alpha_l}}(R) := \{(X, \rho) \in \mathcal{N}(R) \mid & \rho \circ \left(\rho^+_R\right)^{-1} : X^+_{j,l,\underline{\alpha_l}} \times R \longrightarrow \mathbb{X}_R \longrightarrow X \text{ and} \\ & \rho^-_R \circ (\rho)^{-1} : X \longrightarrow \mathbb{X}_R \longrightarrow X^-_{j,l,\underline{\alpha_l}} \times R \\ & \text{are isogenies}\}\end{aligned}$$

for every $\mathbb{F}_{p^h}$-algebra R.

Lemma 3.14. *The functor $\mathcal{N}_{j,l,\underline{\alpha_l}}$ is a closed subfunctor of $\mathcal{N}$.*

Proof. This follows from Proposition 2.9 in [RZ]. □

Proposition 3.15. *Let $0 \leq j \leq \frac{h-3}{2}, 0 \leq l \leq j$ and $\underline{\alpha_l} \in \mathbb{F}_{p^h}^l$ as before. The functor $\mathcal{N}_{j,l,\underline{\alpha_l}}$ is representable by a projective $\mathbb{F}_{p^h}$-scheme.*

Proof. We show the representability similarly as the proof of Lemma 3.13 and give now the description of the projective $\mathbb{F}_{p^h}$-scheme representing $\mathcal{N}_{j,l,\underline{\alpha_l}}$.
Denote by $K = K_{j,l,\underline{\alpha_l}}$ the kernel of the isogeny $X^+_{j,l,\underline{\alpha_l}} \longrightarrow X^-_{j,l,\underline{\alpha_l}}$ of p-divisible groups over $\mathbb{F}_{p^h}$ given by the inclusion of Dieudonné modules $\Lambda^-_{j,l,\underline{\alpha_l}} \subset \Lambda^+_{j,l,\underline{\alpha_l}}$.
Let R be an $\mathbb{F}_{p^h}$-algebra and $(X, \rho) \in \mathcal{N}_{j,l,\underline{\alpha_l}}(R)$. From the description of the R-valued points of $\mathcal{N}_{j,l,\underline{\alpha_l}}$ we have the two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_R & \longrightarrow & X^+_{j,l,\underline{\alpha_l}R} & \longrightarrow & X^-_{j,l,\underline{\alpha_l}R} & \longrightarrow & 0 \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & K_X & \longrightarrow & X^+_{j,l,\underline{\alpha_l}R} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

and $K_X \hookrightarrow K_R$, since $X \longrightarrow X^-_R$ is an isogeny.

Since an isogeny is determined by its kernel up to isomorphism, a pair (X, ρ) lying in $\mathcal{N}_{j,l,\underline{\alpha_l}}(R)$ is determined by a subgroup K_X in K_R. Now K is annihilated by V, since $V(\Lambda^+_{j,l,\underline{\alpha_l}}) \subset \Lambda^-_{j,l,\underline{\alpha_l}}$, so according to Proposition 3.12 the inclusion $K_X \subset K_R$ is determined by the induced morphism $\mathbb{D}(K_X) \longrightarrow \mathbb{D}(K_R)$ of Dieudonné modules. Since $K_X \longrightarrow K_R$ is injective, the corresponding morphism of Dieudonné modules is surjective, and therefore uniquely determined by its kernel $\operatorname{Ker}(\mathbb{D}(K_R) \longrightarrow \mathbb{D}(K_X))$.

Now $p\Lambda^+_{j,l,\underline{\alpha_l}} \subset \Lambda^-_{j,l,\underline{\alpha_l}}$, so

$$\mathbb{D}(K) = \Lambda^+_{j,l,\underline{\alpha_l}} / \Lambda^-_{j,l,\underline{\alpha_l}} =: W_{j,l,\underline{\alpha_l}}$$

is a $2(j-l)$-dimensional $\mathbb{F}_{p^h}$-vector space and $\mathbb{D}(K_R) = W_{j,l,\underline{\alpha_l}} \otimes_{F_{p^h}} R$. Since the height of the isogeny $X^+_{j,l,\underline{\alpha_l}} \times_{\operatorname{Spec}(\mathbb{F}_{p^h})} \operatorname{Spec}$ ® $\longrightarrow X$ is $j-l$ and $\mathbb{D}(K_X)$ is a locally free R-module of rank $j-l$, the kernel of $\mathbb{D}(K_X \longrightarrow K_R)$ is a direct summand of rank $j-l$ in $W_{j,l,\underline{\alpha_l}} \otimes R$.
Since $F(\Lambda^-_{j,l,\underline{\alpha_l}}) \subset \Lambda^-_{j,l,\underline{\alpha_l}}$, we get an action of the Frobenius endomorphism F on the quotient $\Lambda^+_{j,l,\underline{\alpha_l}} / \Lambda^-_{j,l,\underline{\alpha_l}}$, and will denote this σ-linear operator again by F. Since the quotient $\mathbb{D}(K_X)$ of $\mathbb{D}(K_R)$ also carries an F-action, the kernel $\operatorname{Ker}\big(\mathbb{D}(K_X \longrightarrow K_R)\big)$ of this projection must be F-invariant.

Let $Y_{j,l,\underline{\alpha_l}} \in \mathrm{Grass}_{j-l}(W_{j,l,\underline{\alpha_l}})$ be the closed subscheme defined on R-valued points, R an $\mathbb{F}_{p^h}$-algebra, by:

$$Y_{j,l,\underline{\alpha_l}}(R) := \left\{ \begin{array}{ll} U \subset W_{j,l,\underline{\alpha_l}} \otimes R \text{ locally free } \mid & \mathrm{rk}(U) = j-l, \\ \text{direct summand} & F(U^\sigma) \subset U \end{array} \right\}$$

where $\mathrm{Grass}_{j-l}(W_{j,l,\underline{\alpha_l}})$ denotes the projective scheme over F_{p^h} whose R-valued points are the locally direct summands of rank $j-l$ in $W_{j,l,\underline{\alpha_l}} \otimes R$.

Thus, by sending a pair (X, ρ) to the kernel of $\mathbb{D}(K_R) \longrightarrow \mathbb{D}(K_X)$ we get a morphism $\mathcal{N}_{j,l,\underline{\alpha_l}} \longrightarrow \mathrm{Grass}_{j-l}(W_{j,l,\underline{\alpha_l}})$ which is an isomorphism onto $Y_{j,l,\underline{\alpha_l}}$. □

Let $d := \frac{h-3}{2}$, $W := W_{d,0,\emptyset}$ and $Y := Y_{d,0,\emptyset}$.

The subfunctor $\mathcal{N}'$ defined in the proof of Lemma 3.13 is now the subfunctor $\mathcal{N}_{d,0,\emptyset}$ of $\mathcal{N} \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\mathbb{Z}_{p^h}$, since $\mathbb{M}^- = \Lambda_{d,0,\emptyset}$ and $\mathbb{M}^+ = \Lambda_{d,0,\emptyset}$, and we denote by $\iota\colon \mathcal{N}' \longrightarrow \mathcal{N} \times \mathrm{Spf}(\mathbb{Z}_{p^h})$ the closed immersion of $\mathrm{Spf}(\mathbb{Z}_{p^h})$-schemes. We have again a bijection $\iota(k)\colon \mathcal{N}'(k) \longrightarrow \mathcal{N}(k)$ for all perfect fields $k/\mathbb{F}_{p^h}$, since for every lattice $M \in \mathcal{N}(k)$ we have

$$\left(\mathbb{M}^- = \Lambda^-_{d,0,\emptyset}\right) \otimes W(k) \subset M \subset \left(\Lambda^+_{d,0,\emptyset} = \mathbb{M}^+\right) \otimes W(k).$$

With the definition of the scheme Y, we obtain the following

Corollary 3.16. *There is a closed immersion $Y \cong \mathcal{N}' \longrightarrow \mathcal{N} \times \mathrm{Spf}(\mathbb{Z}_{p^h})$ of the $\mathbb{F}_{p^h}$-scheme Y into the formal scheme $\mathcal{N}$, which is a bijection on k-valued points for every perfect field $k \supset \mathbb{F}_{p^h}$.*
In particular, we get an isomorphism of the associated reduced subschemes $Y_{red} \longrightarrow \mathcal{N}_{red} = \mathcal{M}(0)_{red}$.

Furthermore, with the precis description of the functors $\mathcal{N}_{j,l,\underline{\alpha_l}}$ we get the following

Corollary 3.17. *The closed subsets $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ of $\mathcal{N}(k)$ are irreducible.*

Proof. For $d = \frac{h-3}{2}$ and $l = 0$ we have $\mathcal{N}_{d,0,\emptyset} = \mathcal{N}' \cong Y$ with $Y_{\mathrm{red}} \cong \mathcal{M}(0)_{\mathrm{red}}$ irreducible. So, Y is irreducible itself.
For general choices of $(j, l, \underline{\alpha_l})$ we have seen in Proposition 3.15 that the subfunctors $\mathcal{N}_{j,l,\underline{\alpha_l}}$ of $\mathcal{N} \times \mathrm{Spf}(\mathbb{Z}_{p^h})$ are isomorphic to the $\mathbb{F}_{p^h}$-schemes $Y_{j,l,\underline{\alpha_l}}$, where

$$Y_{j,l,\underline{\alpha_l}}(R) := \left\{ \begin{array}{ll} U \subset W_{j,l,\underline{\alpha_l}} \otimes R \text{ locally free } \mid & \mathrm{rk}(U) = j-l, \\ \text{direct summand} & F(U^\sigma) \subset U \end{array} \right\}.$$

The σ-linear endomorphism F is given on $W_{j,l,\underline{\alpha_l}}$ by the matrix

$$F = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right) \in \mathrm{M}_{2(j-l)}(\mathbb{F}_{p^h})$$

according to the basis given by the residue classes of $\{v = v(j, l, \underline{\alpha_l}), F(v), \ldots, F^{j-l-1}(v), e_{-j+2l+1}, e_{-j+2l+3}, \ldots, e_{j-1}\}$ in $\Lambda^+_{j,l,\underline{\alpha_l}}/\Lambda^-_{j,l,\underline{\alpha_l}} = W_{j,l,\underline{\alpha_l}}$. So, for different j and l only the size of F changes, but not the general shape of the matrix. So, for every choice of $(j, l, \underline{\alpha_l})$ we have the same modular description for the subscheme $Y_{j,l,\underline{\alpha_l}} \subset \mathrm{Grass}_{j-l}(W_{j,l,\underline{\alpha_l}})$ only with different vector spaces $W_{j,l,\underline{\alpha_l}}$. Thus, all $Y_{j,l,\underline{\alpha_l}}$ are irreducible, and so are the subsets $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ of $\mathcal{N}(k)$. □

4 The scheme Y

Let Y as before be the projective scheme over $\mathbb{F}_{p^h}$ given by

$$Y(R) = \left\{ \begin{array}{ll} U \subset \left(\mathbb{M}^+/\mathbb{M}^-\right) \otimes_{\mathbb{F}_{p^h}} R \text{ locally} \mid & \mathrm{rk}(U) = d, \\ \text{direct summand} & F(U^\sigma) \subset U \end{array} \right\}$$

for an $\mathbb{F}_{p^h}$-algebra R, and let $W := \mathbb{M}^+/\mathbb{M}^-$ be an $\mathbb{F}_{p^h}$-vector space of dimension $2d = h - 3$.

We have defined a closed immersion $\iota\colon Y \longrightarrow \mathcal{N} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(\mathbb{Z}_{p^h})$ with $\iota(k)\colon Y(k) \longrightarrow \mathcal{N}(k)$ a bijection for all perfect fields $k \supset \mathbb{F}_{p^h}$.

A basis of W is given by the images of the elements $e_{-d}, e_{-d+1}, \ldots, e_{d-2}, e_{d-1}$ in $\mathbb{M}^+/\mathbb{M}^-$. A matrix of $F\colon W \longrightarrow W$ according to this basis is given by

$$F = \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & 0 \end{pmatrix} \circ \sigma.$$

Examples $h = 5$ and $h = 7$

If $h = 5$, then $d = \frac{h-3}{2} = 1$, $W \cong \mathbb{F}^2_{p^h}$ and $F = 0$. So Y is the whole $\mathrm{Grass}_1(W) = \mathbb{P}^1_{\mathbb{F}_{p^h}}$.

If $h = 7$, then $d = 2$ and we fix a basis of $W = W_{d,0,\emptyset}$ given by the images of the elements $\{e_{-2}, e_{-1}, e_0, e_1\}$ in the quotient $W = \mathbb{M}^+/\mathbb{M}^-$ and also an isomorphism $W \cong (\mathbb{F}_{p^h})^4$ given by this basis.

To determine the subvariety Y, we consider the open affine covering $\{U_{ij}, 1 \leq i < j \leq 4\}$ of $\mathrm{Grass}_2(W)$, where a direct summand $U \subset R^4$ lies in $U_{ij}(R)$, if U is given as the image of a matrix $A \in \mathrm{M}_{2\times 4}(R)$ whose (i,j)-minor is invertible, for any $\mathbb{F}_{p^h}$-algebra R.

There are six of these open affine subsets and we want to determine the conditions on the closed subscheme Y inside every one of them.

- $(i,j) = (1,2)$:
  Let $U \subset R^4$ be a locally free direct summand of rank 2 with $U \in U_{12}(R)$. Then there exists a unique matrix of the form

  $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2\times 4}(R)$$

  such that the columns of A form a basis of $U \subset R^4$.
  In order for U to be F-invariant, the images under F of these base vectors have to be linear combinations of the base vectors, in other words, there has to be a matrix $C \in \mathrm{M}_2(R)$ such that

  $$F \cdot A^\sigma = A \cdot C.$$

  Thus we get:

  $$\begin{pmatrix} 0 & & & \\ 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  and see, that the columns of the latter matrix can not be a linear combination of the column vectors of A. So, Y does not intersect U_{12} at all.

- $(i,j) = (1,3)$:
  Let U be the image of $A = \begin{pmatrix} 1 & 0 \\ a & b \\ 0 & 1 \\ c & d \end{pmatrix}$. By the same computations as

before we get

$$F \cdot A^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ a^\sigma & b^\sigma \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 0 \\ a & b \\ 0 & 1 \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

if $b = 0$ and $d = a^\sigma$.
Thus, $(Y \cap U_{13})(R) \cong \left\{ B = \begin{pmatrix} a & 0 \\ c & a^\sigma \end{pmatrix} \in \mathrm{M}_2(R) \right\} \cong \mathbb{A}^2(R)$.

- $(i, j) = (1, 4)$:

  Let U be the image of $A = \begin{pmatrix} 1 & 0 \\ a & b \\ c & d \\ 0 & 1 \end{pmatrix}$. We get again

$$F \cdot A^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ a^\sigma & b^\sigma \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 0 \\ a & b \\ c & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ d^{-1} & 0 \end{pmatrix}$$

  if $b = 0$ and $d^{-1} = a^\sigma$.
  Thus, $(Y \cap U_{14})(R) \cong \left\{ B = \begin{pmatrix} a & 0 \\ c & (a^\sigma)^{-1} \end{pmatrix} \in \mathrm{M}_2(R) \right\} \cong \mathbb{G}_m(R) \times \mathbb{A}^1(R)$.

- : $(i, j) = (2, 3)$:

  Let U be the image of $A = \begin{pmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{pmatrix}$. We have

$$F \cdot A^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ a^\sigma & b^\sigma \\ 1 & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ d^{-1} & 0 \end{pmatrix}$$

  if $b = 0$ and $d^{-1} = a^\sigma$.
  Thus, $(Y \cap U_{23})(R) \cong \left\{ B = \begin{pmatrix} a & 0 \\ c & (a^\sigma)^{-1} \end{pmatrix} \in \mathrm{M}_2(R) \right\} \cong \mathbb{G}_m(R) \times \mathbb{A}^1(R)$.

- : $(i, j) = (2, 4)$:

  Let U be the image of $A = \begin{pmatrix} a & b \\ 1 & 0 \\ c & d \\ 0 & 1 \end{pmatrix}$. By the same computations as before we get

$$F \cdot A^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ a^\sigma & b^\sigma \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \\ c & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

if $b = 0$ and $d = a^\sigma$.
Thus, $(Y \cap U_{24})(R) \cong \left\{ B = \begin{pmatrix} a & 0 \\ c & a^\sigma \end{pmatrix} \in \mathrm{M}_2(R) \right\} \cong \mathbb{A}^2(R)$.

- : $(i, j) = (3, 4)$:

  Let U be the image of $A = \begin{pmatrix} a & b \\ c & d \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. By the same computations as before we get

$$F \cdot A^\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix}$$

  if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} = 0$.

  Thus, $(Y \cap U_{34})(R) \cong \{B \in \mathrm{M}_2(R) \mid B \cdot B^\sigma = 0\}$

From the computations above we see that the singular locus of Y is contained in $Y \cap U_{34} := Y_{34}$ whose R valued points are $Y_{34}(R) = \{B \in M_2(R) \mid B \cdot B^\sigma = 0\}$, so $Y_{34} = \mathrm{Spec}\left(k[a, b, c, d]/I\right)$, where the defining ideal I of Y_{34} is given by $I = (a^{p+1} + bc^p, ab^p + bd^p, a^p c + c^p d, b^p c + d^{p+1})$.

Let us compute the points in which Y is not smooth.

Let $B \in Y_{34}(k)$ with $B \neq 0$. Since $B \cdot B^\sigma = 0$, we have that its determinant fulfills $\det(B)^{p+1} = 0$, thus $\det(B) = 0$, since $\det(B) \in k$. So, the image of B is a one-dimensional subspace of k^2.

Denote by $k[\varepsilon]$ the ring $k[T]/(T^2)$, ε being the residue class of T. Then

$$
\begin{aligned}
T_Y(B) &= \{C \in \mathrm{M}_2(k) \mid B + \varepsilon C \in Y(k[\varepsilon])\} \\
&= \{C \in \mathrm{M}_2(k) \mid (B + \varepsilon C) \cdot (B + \varepsilon C)^\sigma = (B + \varepsilon C) \cdot (B^\sigma + \varepsilon^p C^\sigma) = 0\} \\
&= \{C \in \mathrm{M}_2(k) \mid (B + \varepsilon C) \cdot B^\sigma = B \cdot B^\sigma + \varepsilon C \cdot B^\sigma = 0\} \\
&= \{C \in \mathrm{M}_2(k) \mid C \cdot B^\sigma = 0\}.
\end{aligned}
$$

The last condition $C \cdot B^\sigma = 0$ means that the image of B^σ is contained in the kernel of C. Let $\langle v \rangle \subset k^2$ be the image of B^σ, then

$$
\begin{aligned}
T_Y(B) &= \{C \in \mathrm{M}_2(k) \mid C \cdot B^\sigma = 0\} \\
&= \{C \in \mathrm{M}_2(k) \mid \langle v \rangle \subset \mathrm{Ker}(C)\} \\
&\cong \mathrm{Hom}_k\left(k^2/\langle v \rangle, k^2\right) \cong k^2.
\end{aligned}
$$

So, Y is smooth in any point $B \in Y(k) \setminus \{0\}$.

If we take $B = 0$, then, following the computations of $T_Y(B)$ above, we see that the condition $C \cdot B^\sigma = 0$ is satisfied for any $C \in \mathrm{M}_2(k)$, so $T_Y(0) = \mathrm{M}_2(k) \cong k^4$.
Thus, the only singular point of Y_{34} is $B = 0 \in \mathrm{M}_2(k)$ corresponding to the maximal ideal $\mathfrak{m} = (a, b, c, d) \subset k[a, b, c, d]/I$. This maximal ideal $\mathfrak{m}$ is generated by zero divisors in $k[a, b, c, d]/I$, since

$$
\begin{aligned}
a \cdot (a^p d - a^{p-1} bc) &= a(a^p d + bc^p d - bc^p d - a^{p-1} bc) \\
&= (a^{p+1} + bc^p) \cdot d - (a^p c + c^p d) \cdot b \in I,
\end{aligned}
$$

and $b \cdot (ab^{p-1} + d^p) \in I$ and $c \cdot (a^p + c^{p-1} d) \in I$, and also

$$
\begin{aligned}
d \cdot (ad^p - bcd^{p-1}) &= d \cdot (ad^p + ab^p c - ab^p c - bcd^{p-1}) \\
&= a \cdot (b^p c + d^{p+1}) - c \cdot (ab^p + bd^p) \in I,
\end{aligned}
$$

but none of the polynomials $a^p d - a^{p-1} bc$, $ab^{p-1} + d^p$, $a^p + c^{p-1} d$, $ad^p - bcd^{p-1}$ are contained in I.

Proposition 4.1. *Y is generically reduced, but not reduced. In particular, it is not Cohen-Macaulay.*

Proof. Y is irreducible, so the generic point in Y corresponds to the unique minimal prime ideal in $k[a,b,c,d]/I = \Gamma(Y \cap U_{34}, \mathscr{O}_Y)$. But the maximal ideal $\mathfrak{m} = (a,b,c,d)$ is generated by zero divisors, so, it is an associated prime ideal, but not the minimal one (since $\dim(Y) = 2$). Thus, the open subset $Y_{34} \backslash V(\mathfrak{m}) \subset Y_{34}$ does not contain all associated prime ideals of $\Gamma(Y_{34}, \mathscr{O}_Y) = k[a,b,c,d]/I$, and therefore, it is not schematically dense (Lemma 9.23 in [GW]). Since $Y_{34} \setminus V(\mathfrak{m})$ is reduced (it is even regular), its closure in Y_{34} is also reduced, but is not the whole Y_{34}. Thus Y cannot be reduced. □

Smooth locus of Y

Recall that for any perfect field $k \supset \mathbb{F}_{p^h}$ we have a stratification of $\mathcal{N}(k)$ by subsets of the form $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$, where

$$\mathcal{N}_{j,l,\underline{\alpha_l}}(k) = \{X \in \mathcal{N}(k) \ \mid \Lambda^-_{j,l,\underline{\alpha_l}} \otimes W(k) \subset \mathbb{D}(X) \subset \Lambda^+_{j,l,\underline{\alpha_l}} \otimes W(k)\}.$$

We also have the equality $\mathcal{N}(k) = \mathcal{N}_{d,0,\emptyset}(k)$ and an isomorphism $\mathcal{N}_{d,0,\emptyset} \cong Y$ of $\mathbb{F}_{p^h}$-schemes. We will denote the stratification of Y given via this isomorphism again by $\mathcal{N}_{j,l,\underline{\alpha_l}}$ The $\mathcal{N}_{j,l,\underline{\alpha_l}}$ are closed subschemes of Y of dimension $j-l$. The following theorem describes the smooth locus of Y and also computes the tangent space $T_U Y$ in every point $U \in Y(k)$, k being an algebraically closed field.

Theorem 4.2. *Let $k \supset \mathbb{F}_{p^h}$ an algebraically closed field and $U \in Y(k)$. Then Y is smooth in all points $U \in Y(k) \setminus \mathcal{N}_{d-2,0,\emptyset}(k)$. In particular, Y is regular in codimension* 1.

Proof. Denote by W the F_{p^h}-vector space $\mathbb{M}^+/\mathbb{M}^-$ and by F again the projection of the σ-linear morphism $F \colon \mathbb{M}^+ \longrightarrow \mathbb{M}^+$ to W. Fix an isomorphism $W \cong (\mathbb{F}_{p^h})^{2d}$ given by the basis $\{e_d, e_{-d+1}, \ldots, e_{d-1}\}$ of W, where the e_i are residue classes of the base vectors e_i in $\mathbb{M}^+$ and denote by $\mathrm{Grass}(d,2d)$ the Grassmannian over $\mathbb{F}_{p^h}$ whose R-valued points are the locally free direct summands of rank d in R^{2d} for any $\mathbb{F}_{p^h}$-algebra R.
Let $U \in \mathrm{Grass}(d,2d)(k)$. The tangent space of $\mathrm{Grass}(d,2d)$ in U can be computed as

$$T_U \,\mathrm{Grass}(d,2d) = \mathrm{Hom}_k(U, (W \otimes k)/U).$$

by sending a linear map $w \colon U \longrightarrow W \otimes k = W_k$ to the image of $\tilde{w}$ in $W_{k[\varepsilon]} = W_k \otimes_k k[\varepsilon] \cong W_k \oplus \varepsilon W_k$, where

$$\tilde{w} = \begin{pmatrix} \iota & 0 \\ w & \iota \end{pmatrix} : U_{k[\varepsilon]} = U \oplus \varepsilon U \longrightarrow W_{k[\varepsilon]} = W_k \oplus \varepsilon W_k$$

with $\iota \colon U \longrightarrow W_k$ the inclusion map.
The map $w \longmapsto \mathrm{Im}(\tilde{w})$ is linear and defines a surjective morphism of vector

spaces $\mathrm{Hom}_k(U, W_k) \longrightarrow T_U \,\mathrm{Grass}(d, 2d)$, whose kernel consists of those homomorphisms w with $w(U) \subset U$.

Now let $U \in Y(k)$. Since $F(U^\sigma) \subset U$, we get a homomorphism

$$\bar{F} \colon (W_k/U)^\sigma \longrightarrow W_k/U$$

and claim that

$$T_U Y = \{w \in \mathrm{Hom}_k(U, W_k/U) \mid \bar{F} \circ w^\sigma = w \circ F\}.$$

Let $Z \subset W_k$ be a complement of U, and let $\hat{w} \in \mathrm{Hom}_k(U, Z)$ be the unique representative of a morphism $w \in \mathrm{Hom}_k(U, W_k/U)$. Then $\tilde{w} \colon U_{k[\varepsilon]} = U \oplus \varepsilon U \longrightarrow W_k \oplus \varepsilon W_k$ is given by

$$\tilde{w}(u_1, u_2) = (u_1, \hat{w}(u_1) + u_2).$$

So, the submodule $\mathrm{Im}(\tilde{w})$ of $W_{k[\varepsilon]}$ lies in $Y(k[\varepsilon])$ if $F_{k[\varepsilon]}(\mathrm{Im}(\tilde{w}))^\sigma \subset \mathrm{Im}(\tilde{w})$, which means

$$F_{k[\varepsilon]}(u_1, \hat{w}(u_1) + u_2) = (F(u_1), F(\hat{w}(u_1)) + F(u_2)) \in \mathrm{Im}(\tilde{w}),$$

that is, if there exist $x, y \in U$ with $(F(u_1), F(\hat{w}(u_1)) + F(u_2)) = (x, \hat{w}(x) + y)$. So, putting $x := F(u_1)$, we have to find a $y = F(\hat{w}(u_1)) + F(u_2) - \hat{w}(F(u_1))$ in U.

Write $F \colon W_k^\sigma \longrightarrow W_k$ as $F = (F|_U, F|_Z) \colon (U \oplus Z)^\sigma \longrightarrow U \oplus Z$. Then $F|_Z$ decomposes into $F|_Z = (f_U, f_Z) \colon Z^\sigma \longrightarrow U \oplus Z$, and $f_Z \colon Z^\sigma \longrightarrow Z$ is the unique representative of $\bar{F} \colon (W_k/U)^\sigma \longrightarrow W_k/U$ in $\mathrm{Hom}_k(Z^\sigma, Z)$.
So, for $y = F(\hat{w}(u_1)) + F(u_2) - \hat{w}(F(u_1))$ to be in U, or, equivalently, $y - F(u_2) = F(\hat{w}(u_1)) - \hat{w}(F(u_1)$ to be in U, we have to require that the Z-part of it is zero, meaning: $f_Z(\hat{w}(u_1)) - \hat{w}(F(u_1)) = 0$. This shows the condition for $w \in \mathrm{Hom}_k(U, W_k/U)$ to lie in $T_U Y$.

We now compute the tangent space at a point $U \in Y(k)$.

First case: $j \leq d-2$: Let $U \in Y(k)$ correspond to an $M \in \mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ with $j \leq d-2$. Recall that M is generated as a $W(k)$-module by the elements

$$M = \langle m = e_{-j} + \sum_{i=1}^{j} [\gamma_i] e_{-j+2i-1}, F(m), \ldots, F^{j-1}(m), e_j, e_{j+1}, \ldots, e_{h-1} \rangle_{W(k)}.$$

Take as a basis of U the residue classes in W_k of the basis elements of M. Then $U = \langle u = e_{-j} + \sum_{i=1}^{j} \gamma_i e_{-j+2i-1}, F(u), \ldots, F^{j-1}(u), e_j, \ldots, e_{d-1} \rangle_k$ and we can take as complement the subspace Z generated by

$$Z = \langle e_{-d}, \ldots, e_{-j-1}, e_{-j+1}, e_{-j+3}, \ldots, e_{j-1} \rangle_k.$$

If we rearrange the basis in another order by taking first e_{-d} and its images under F: e_{-d+2}, e_{-d+4} and so on, as long as the images are in Z, and then e_{-d+1} and its images under F as long as they are in Z, then, according to this basis, $f_Z\colon Z^\sigma \longrightarrow Z$ has the shape:

$$f_Z = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right) : Z^\sigma \longrightarrow Z.$$

where the upper left corner is a $(\frac{d-j}{2}) \times (\frac{d-j}{2})$-matrix and the lower right corner a $\left(\frac{d+j}{2}\right) \times \left(\frac{d+j}{2}\right)$-matrix if $d-j$ is even, and, if $d-j$ is odd, the upper left corner is a $\left(\frac{d+j+1}{2}\right) \times \left(\frac{d+j+1}{2}\right)$-matrix and the lower right a $\left(\frac{d-j-1}{2}\right) \times \left(\frac{d-j-1}{2}\right)$-matrix. In any of these cases, f_Z has rank $d-2$.

The basis of U can also be changed in such a manner that

$$U = \left\langle \begin{array}{l} u, F(u), \ldots, F^{j-1}(u), F^j(u), \ldots, \begin{cases} F^{\frac{d+j+1}{2}}(u), & d-j \text{ odd}, \\ F^{\frac{d+j}{2}}(u), & d-j \text{ even}, \end{cases} \\ e_{j+1}, e_{j+3}, \ldots, \begin{cases} e_{d-2}, & d-j \text{ odd}, \\ e_{d-1}, & d-j \text{ even}. \end{cases} \end{array} , \right\rangle$$

Then according to this basis, $F|_U$ is given by

$$F|_U = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right),$$

with the upper left corner a $\left(\frac{d+j+1}{2}\right) \times \left(\frac{d+j+1}{2}\right)$-matrix and the lower right corner a $\left(\frac{d-j-1}{2}\right) \times \left(\frac{d-j-1}{2}\right)$-matrix if $d-j$ is odd, and the upper left corner of size $\left(\frac{d+j}{2}\right) \times \left(\frac{d+j}{2}\right)$ and the lower right corner of size $(\frac{d-j}{2}) \times (\frac{d-j}{2})$ if $d-j$ is even. Just remark that we get a proper decomposition into two blocks since $j < d-1$, so neither of the blocks has size d.

We can now compute the tangent space $T_U Y = \{w \in \mathrm{Hom}_k(U, W_k/U) \mid$

$\bar{F} \circ w^{\sigma} = w \circ F|_U\}$.

First case of "$j \le d-2$" : $d-j$ odd:
Let $w \in \mathrm{Hom}_k(U, W_k/U)$, and let its unique representative $\hat{w} \colon U \longrightarrow Z$ be given by a $d \times d$-matrix $A = (a_{ij}) \in \mathrm{M}_d(k)$ according to the last chosen bases of U and Z.
Then $f_Z \circ \hat{w}^{\sigma} =$

$$\left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right) \cdot (a_{ij}^p) = \left(\begin{array}{cccc} 0 & & & 0 \\ a_{11}^p & \cdots & \cdots & a_{1d}^p \\ \vdots & & & \vdots \\ a^p_{\frac{d+j+1}{2}-1,1} & \cdots & \cdots & a^p_{\frac{d+j+1}{2}-1,d} \\ \hline 0 & & & 0 \\ a^p_{\frac{d+j+1}{2}+1,1} & \cdots & \cdots & a^p_{\frac{d+j+1}{2}+1,d} \\ \vdots & & & \vdots \\ a^p_{d,1} & \cdots & \cdots & a^p_{dd} \end{array}\right)$$

and $\hat{w} \circ F|_U =$

$$(a_{ij}) \cdot \left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right) =$$

$$\left(\begin{array}{cccc|cccc} a_{12} & \cdots & a_{1,\frac{d+j+1}{2}} & 0 & a_{1,\frac{d+j+1}{2}+2} & \cdots & a_{1d} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{d2} & \cdots & a_{d,\frac{d+j+1}{2}} & 0 & a_{d,\frac{d+j+1}{2}+2} & \cdots & a_{dd} & 0 \end{array}\right).$$

Comparing the coefficients we get conditions on all entries of $\hat{w} = (a_{ij})$ but $a_{11}, \ldots, a_{\frac{d+j+1}{2},1}$, $a_{j+2,\frac{d+j+1}{2}+1}, \ldots, a_{\frac{d+j+1}{2},\frac{d+j+1}{2}+1}$, $a_{\frac{d+j+1}{2}+1,1}, \ldots, a_{d1}$ and $a_{\frac{d+j+1}{2}+1,\frac{d+j+1}{2}+1}, \ldots, a_{d,\frac{d+j+1}{2}+1}$, which gives us $\left(\frac{d+j+1}{2}\right) + \left(\frac{d-j-1}{2}\right) + \left(\frac{d-j-1}{2}\right) + \left(\frac{d-j-1}{2}\right) = \frac{1}{2}(4d-2j-2) = 2d-j-1$ free entries of $\hat{w}$. Thus, $\dim(T_U Y) = 2d-j-1 > d = \dim Y$, since $j < d-1$.

Second case of "$j \leq d-2$": $d-j$ even:
Let $w \in \mathrm{Hom}_k(U, W_k/U)$, and let its unique representative $\hat{w}\colon U \longrightarrow Z$ be given by a $d \times d$-matrix $A = (a_{ij}) \in \mathrm{M}_d(k)$ according to the last chosen bases of U and Z.
Then $f_Z \circ \hat{w}^\sigma =$

$$\left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right) \cdot (a_{ij}^p) = \left(\begin{array}{cccc} 0 & & & 0 \\ a_{11}^p & \cdots & \cdots & a_{1d}^p \\ \vdots & & & \vdots \\ a_{\frac{d-j}{2}-1,1}^p & \cdots & \cdots & a_{\frac{d-j}{2}-1,d}^p \\ \hline 0 & & & 0 \\ a_{\frac{d-j}{2}+1,1}^p & \cdots & \cdots & a_{\frac{d-j}{2}+1,d}^p \\ \vdots & & & \vdots \\ a_{d,1}^p & \cdots & \cdots & a_{dd}^p \end{array}\right)$$

and $\hat{w} \circ F|_U =$

$$(a_{ij}) \cdot \left(\begin{array}{ccc|ccc} 0 & & & & & \\ 1 & \ddots & & & 0 & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ & 0 & & 1 & \ddots & \\ & & & & 1 & 0 \end{array}\right) =$$

$$\left(\begin{array}{cccc|cccc} a_{12} & \cdots & a_{1,\frac{d+j}{2}} & 0 & a_{1,\frac{d+j}{2}+2} & \cdots & a_{1d} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{d2} & \cdots & a_{d,\frac{d+j}{2}} & 0 & a_{d,\frac{d+j}{2}+2} & \cdots & a_{dd} & 0 \end{array}\right).$$

Comparing the coefficients of these two matrices we get conditions on all entries of $\hat{w} = (a_{ij})$ but $a_{11}, \ldots, a_{\frac{d-j}{2},1}$, $a_{1,\frac{d+j}{2}+1}, \ldots, a_{\frac{d-j}{2},\frac{d+j}{2}+1}$, $a_{\frac{d-j}{2}+1,1}, \ldots, a_{d1}$ and $a_{\frac{d+j}{2}+1,\frac{d+j}{2}+1}, \ldots, a_{d,\frac{d+j}{2}+1}$, which gives us $\left(\frac{d-j}{2}\right) + \left(\frac{d-j}{2}\right) + \left(\frac{d+j}{2}\right) + \left(\frac{d-j}{2}\right) = \frac{1}{2}(4d-2j) = 2d-j$ free entries of $\hat{w}$. Thus, $\dim(T_U Y) = 2d-j > d = \dim Y$, since $j < d-1$.

Second case: $j = d$ or $d-1$:
Let $U \in Y(k)$ correspond to an $M \in \mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ with $j = d$ or $d-1$. Take for Z again the subspace given by either $Z = \langle e_{-d+1}, e_{-d+3}, \dots, e_{d-1}\rangle_k$ if $j = d$ or $Z = \langle e_{-d}, e_{-d+2}, \dots, e_{d-2}\rangle_k$ if $j = d-1$. Then Z is an F-invariant subspace itself and according to this basis, f_Z is given by

$$f_Z = \begin{pmatrix} 0 & & & \\ 1 & & & \\ 0 & & & \\ & \ddots & & \\ & & 0 \quad 1 & 0 \end{pmatrix} : Z^\sigma \longrightarrow Z.$$

If we take for U again the basis given by residue classes of the base vectors of the associated Dieudonné lattice M, that is

$$U = \langle u = e_{-j} + \sum_{i=1}^{j} \gamma_i e_{-j+2i-1}, F(u), \dots F^{j-1}(u), e_j, e_{j+1}, \dots, e_{d-1}\rangle_k,$$

then we see that, since $j = d$ or $d-1$, in both cases the last vector in this basis is $F^{d-1}(u)$. It is immediately clear if $j = d$, and if $j = d-1$ then $F^{d-1}(u) = F(F^{d-2}(u)) = F(e_{d-3} + \gamma_1^{p^{d-2}} e_{d-2}) = e_{d-1}$. Thus, in both cases we have

$$F|_U = \begin{pmatrix} 0 & & & \\ 1 & & & \\ 0 & & & \\ & \ddots & & \\ & & 0 \quad 1 & 0 \end{pmatrix} : U^\sigma \longrightarrow U.$$

Now, having computed both $F|_U$ and f_Z, which is the unique representative in $\mathrm{Hom}_k(Z^\sigma, Z)$ of $\bar{F}\colon (W_k/U)^\sigma \longrightarrow (W_k/U)$, we can compute the vector space $T_U Y = \{w \in \mathrm{Hom}_k(U, W_k/U) \mid \bar{F} \circ w^\sigma = w \circ F\}$.

Let $w \in \mathrm{Hom}_k(U, W_k/U)$ and let its representative $\hat{w}\colon U \longrightarrow Z$ be given by $A = (a_{ij}) \in \mathrm{M}_d(k)$ according to the chosen bases of U and Z. Then we have

$$f_Z \circ \hat{w}^\sigma = \begin{pmatrix} 0 & & & \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix} \cdot (a_{ij}^p) = \begin{pmatrix} 0 & \dots & 0 \\ a_{11}^p & \dots & a_{1d}^p \\ \vdots & & \vdots \\ a_{d-1,1}^p & \dots & a_{d-1,d}^p \end{pmatrix}.$$

and $\hat{w} \circ F|_U =$

$$(a_{ij}) \cdot \begin{pmatrix} 0 & & & \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{12} & \dots & a_{1d} & 0 \\ a_{22} & \dots & a_{2d} & 0 \\ \vdots & & \vdots & \vdots \\ a_{d2} & \dots & a_{dd} & 0 \end{pmatrix}.$$

So, by comparing the coefficients we get d free entries $a_{11}, \ldots, a_{d1}$ of the matrix $A = (a_{ij})$, and, consequently, $\dim(T_U Y) = d = \dim Y$.

Thus, the singular locus of Y_k consists of those $F-$invariant subspaces $U \subset W \otimes k$, which correspond to p-divisible groups $X \in \mathcal{N}(k)$ which lie in $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ with $j \leq d-2$. Since all these $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ with $j \leq d-2$ are contained in $\mathcal{N}_{d-2,0,\emptyset}(k)$, we get that the set of singular points in $Y(k)$ is precisely $\mathcal{N}_{d-2,0,\emptyset}(k)$. □

Now

$$\begin{aligned}\mathcal{N}_{d-2,0,\emptyset}(k) = \{U \in W \otimes k \text{ subspace of dimension } d \mid F(U^\sigma) \subset U \\ \text{and } \langle e_{d-2}, e_{d-1}\rangle_k \subset U \subset \langle e_{-d+2}, e_{-d+3}, \ldots, e_{d-2}, e_{d-1} \rangle_k\}\end{aligned}$$

and the restriction of the morphism F on W to the subquotient $W_{d-2,0,\emptyset} = \langle e_{-d+2}, \ldots, e_{d-1}\rangle / \langle e_{d-2}, e_{d-1}\rangle$ has the same shape as F, namely

$$F = \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & 0 \end{pmatrix} : W^\sigma_{d-2,0,\emptyset} \longrightarrow W_{d-2,0,\emptyset}.$$

So, by the same computation as above, we would get $\mathcal{N}_{d-4,0,\emptyset}(k)$ as the set of points in which $\mathcal{N}_{d-2,0,\emptyset}$ is not regular. Thus we get the following

Corollary 4.3 (Singularity stratification of Y)**.** *We get the stratification of Y by locally closed regular subschemes S_i of dimension i*

$$\begin{aligned} Y = \big(Y \setminus \mathcal{N}_{d-2,0,\emptyset}\big) \ &\cup \big(\mathcal{N}_{d-2,0,\emptyset} \setminus \mathcal{N}_{d-4,0,\emptyset}\big) \ \cup \ldots \cup \begin{cases} \mathcal{N}_{1,0,\emptyset}, & d \text{ odd}, \\ \mathcal{N}_{0,0,\emptyset}, & d \text{ even}, \end{cases} \\ = S_d \qquad\qquad &\cup S_{d-2} \qquad\qquad\qquad \cup \ldots \cup \begin{cases} S_1, & d \text{ odd}, \\ S_0, & d \text{ even}, \end{cases}\end{aligned}$$

such that the closure of any stratum S_i consists of the union of this stratum with all strata of smaller dimension: $\overline{S_i} = \bigcup_{j \leq i} S_j$, and such that the smooth locus of $\overline{S_i}$ is precisely S_i.

Singularity stratification of Y

Denote again by W the $(h-3)$-dimensional $\mathbb{F}_{p^h}$-vector space $\mathbb{M}^+/\mathbb{M}^-$. For any $\mathbb{F}_{p^h}$-algebra R let W_R and $F_R \colon W_R^\sigma \longrightarrow W_R$ denote the base changes. We want to describe the singularity stratification of Y as a stratification given by the action of an algebraic group $H / \operatorname{Spec} \mathbb{F}_{p^h}$.

Consider the algebraic group H over $\mathbb{F}_{p^h}$ given by:

$$H(R) = \left\{ g \in \mathrm{GL}(W_R) \mid g \cdot F_R \cdot (g^\sigma)^{-1} = F_R \right\}.$$

Then H acts on Y and we want to understand the stratification on Y given by this group action.

The group H

Choose again a basis for W given by the images of the elements $e_{-d}, e_{-d+1}, \ldots, e_{d-1}$ in $\mathbb{M}^+/\mathbb{M}^-$. We will denote this basis of W again by $\{e_{-d}, \ldots, e_{d-1}\}$. Then F is given by

$$F = \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 0 \end{pmatrix} \circ \sigma,$$

with respect to this basis.

Let $k \supset \mathbb{F}_{p^h}$ be an algebraically closed field and let $g = (a_{ij}) \in \mathrm{GL}_{2d}(k)$ an invertible matrix. Then g is in $H(k)$ if and only if $g \cdot F = F \cdot g^\sigma$, i.e.

$$(a_{ij}) \cdot \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 0 \end{pmatrix} \cdot (a_{ij}^p),$$

which means

$$\begin{pmatrix} a_{13} & \ldots & a_{1,2d} & 0 & 0 \\ a_{23} & \ldots & a_{2,2d} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ & & & & \\ a_{2d,3} & \ldots & a_{2d,2d} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ a_{11}^p & a_{12}^p & \ldots & a_{1d}^p \\ \vdots & \vdots & & \vdots \\ a_{2d-2,1}^p & a_{2d-2,2}^p & \ldots & a_{2d-2,2d}^p \end{pmatrix}.$$

So, comparing the coefficients on both sides, we get that g is an element of $H(k)$ if and only if g is of the form

$$g = \begin{pmatrix} A_1 & 0 & & \dots & 0 \\ A_2 & A_1^\sigma & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_{d-1} & A_{d-2}^\sigma & \dots & A_1^{\sigma^{d-2}} & 0 \\ A_d & A_{d-1}^\sigma & \dots & A_2^{\sigma^{d-2}} & A_1^{\sigma^{d-1}} \end{pmatrix},$$

with $A_1 \in \mathrm{GL}_2(k)$ and $A_2, \dots, A_d \in \mathrm{M}_2(k)$.

Proposition 4.4. *The stratification on Y by H-orbits is the singularity stratification of Y.*

Proof. Consider the subspace $U = \langle e_{-d+1}, e_{-d+3}, \dots, e_{d-3}, e_{d-1}\rangle_k$ of W_k. This subspace lies in the stratum $\mathcal{N}_{d-1,0,\emptyset}(k) \subset Y(k)$, and its $H(k)$-orbit consists of subspaces of the form

$$\begin{aligned} H(k).U &= \left\{ U' = \langle u' = a\, e_{-d} + b\, e_{-d+1} + \sum_{k=-d+2}^{d-1} a_k e_k, F(u'), \dots, F^{d-1}(u')\rangle_k \right\} \\ &= Y(k) \setminus \mathcal{N}_{d-2,0,\emptyset}(k) \\ &= \text{ smooth locus of } Y(k) \end{aligned}$$

since the coefficients a and b form the second column of the upper-left 2×2-corner A_1 of an element $g \in H(k)$, which is invertible, so a and b are not both equal to 0.

Denote by U_j the subspace of the form

$$U_j = \langle e_{-j}, e_{-j+2}, \dots, e_{j-2}, e_j, e_{j+1}, \dots, e_{d-1}\rangle_k \in \mathcal{N}_{j,0,\emptyset}(k),$$

then by the same computation as above its $H(k)$-orbit will consist either of $\mathcal{N}_{j+1,0,\emptyset}(k) \setminus \mathcal{N}_{j-1,0,\emptyset}(k) = $ smooth locus of $\mathcal{N}_{j+1,0,\emptyset}(k)$ if $d-j$ is odd, or $\mathcal{N}_{j,0,\emptyset}(k) \setminus \mathcal{N}_{j-2,0,\emptyset}(k) = $ smooth locus of $\mathcal{N}_{j,0,\emptyset}(k)$ if $d-j$ is even.

Thus, the H-action on Y gives us a stratification of Y of the following type:

$$
\begin{aligned}
Y = {} & H \cdot U_{d-1} \quad \cup \; H \cdot U_{d-3} \qquad\qquad \cup \ldots \cup \;\; H \cdot U_0 \\
= {} & \bigcup_{\substack{j=d,d-1\\ 0\le l\le j-1\\ \underline{\alpha_l}\in\mathbb{F}^l_{p^h}}} \mathcal{N}^{\circ}_{j,l,\underline{\alpha_l}} \cup \bigcup_{\substack{j=d-2,d-3\\ 0\le l\le j-1\\ \underline{\alpha_l}\in\mathbb{F}^l_{p^h}}} \mathcal{N}^{\circ}_{j,l,\underline{\alpha_l}} \quad \cup \ldots \cup \begin{cases} \mathcal{N}_{1,0,\emptyset}, & \text{if } d \text{ odd,} \\ \mathcal{N}_{0,0,\emptyset}, & \text{if } d \text{ even.} \end{cases} \\
= {} & \left(Y \setminus \mathcal{N}_{d-2,0,\emptyset}\right) \cup \left(\mathcal{N}_{d-2,0,\emptyset} \setminus \mathcal{N}_{d-4,0,\emptyset}\right) \cup \ldots \cup \begin{cases} \mathcal{N}_{1,0,\emptyset}, & \text{if } d \text{ odd,} \\ \mathcal{N}_{0,0,\emptyset}, & \text{if } d \text{ even.} \end{cases}
\end{aligned}
$$

□

The dimensions of the orbits decrease by 2 with every step, and the orbit of minimal dimension has dimension 1 if d is odd, or 0 if d is even. In any case, the smallest stratum $\mathcal{N}_{0,0,\emptyset}$ is contained in the orbit of minimal dimension.

Due to the following proposition, we can restrict ourselves to the smallest stratum $\mathcal{N}_{0,0,\emptyset}(k)$ which consists of only one point $U_0 = \langle e_0, \ldots, e_{d-1}\rangle_k$ in order to study the regularity properties of Y.

Proposition 4.5. *Let X be a scheme and G an algebraic group which acts on X.*
Let $U \subset X$ be an open subset, such that U is G-invariant (for example, $U = X_{reg}$ or the set of points in which X is Cohen-Macaulay). Then, if U contains all closed G-orbits in X, then $U = X$.

Proof. Let $x \in X$. Then the closure $\overline{Gx}$ of its orbit Gx contains at least one closed G-orbit in X, so $U \cap \overline{Gx} \neq \emptyset$. But since $U \cap \overline{Gx}$ is also open in $\overline{Gx}$, it has to intersect the orbit Gx. And since U is G-invariant, it contains the whole orbit Gx, thus it contains also x. □

Let $\mathcal{U}$ denote the open subset of the Grassmannian variety $\mathrm{Grass}(d, 2d)$ consisting of the subspaces U, which are images of matrices $A \in \mathrm{M}_{2d\times d}(k)$ whose lower half minor is invertible. Then $\mathcal{U} \cong \mathbb{A}^{d^2}$, the isomorphism being given on R-valued points by

$$\mathbb{A}^{d^2}(R) \cong \mathrm{M}_d(R) \ni B \longmapsto U = \mathrm{Im}\begin{pmatrix} B \\ I_d \end{pmatrix} \in \mathcal{U}(R),$$

and U_0 corresponds to the matrix $B = 0$.

To determine the definition ideal of $Y \cap \mathcal{U} \subset \mathcal{U}$, we compute the conditions on the matrix $B \in \mathrm{M}_d(k)$ which imply $F(U^\sigma) \subset U$.

The column vectors of $\begin{pmatrix} B \\ I_d \end{pmatrix}$ form a basis of U, so their images under F lie in U if and only if there exists a matrix $C \in \mathrm{M}_d(k)$ with

$$F \cdot \begin{pmatrix} B \\ I_d \end{pmatrix}^{\sigma} = \begin{pmatrix} B \\ I_d \end{pmatrix} \cdot C.$$

Let $B = (b_{ij}) \in \mathrm{M}_d(k)$. Then there exists a $C \in \mathrm{M}_d(k)$ with

$$F \cdot \begin{pmatrix} B \\ I_d \end{pmatrix}^{\sigma} = \begin{pmatrix} 0 & & & & & \\ 0 & \ddots & & & & \\ 1 & 0 & \ddots & & & \\ 0 & 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 & 0 \end{pmatrix} \cdot \left(\begin{array}{ccc} & b_{ij}^p & \\ \hline 1 & & \\ & \ddots & \\ & & 1 \end{array}\right)$$

$$= \begin{pmatrix} 0 & & \dots & & 0 \\ 0 & & \dots & & 0 \\ & & B^{\sigma} & & \\ 1 & 0 & \dots & & 0 \\ & \ddots & & & \\ & & 1 & 0 & 0 \end{pmatrix} = \left(\begin{array}{ccc} & b_{ij} & \\ \hline 1 & & \\ & \ddots & \\ & & 1 \end{array}\right) \cdot C$$

if and only if

$$C = \begin{pmatrix} b_{d-1,1}^p & & \dots & b_{d-1,d}^p \\ b_{d,1}^p & & \dots & b_{dd}^p \\ 1 & 0 & \dots & 0 \\ & \ddots & & \\ & & 1 \; 0 & 0 \end{pmatrix} \text{ and } B \cdot C = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ b_{11}^p & \dots & b_{1d}^p \\ \vdots & & \vdots \\ b_{d-2,1}^p & \dots & b_{d-2,d}^p \end{pmatrix},$$

the first equation given by the lower half of the matrix $F \cdot \begin{pmatrix} B^{\sigma} \\ I_d \end{pmatrix}$ and the second condition given by the upper half of the same matrix.

From these conditions on the matrix $B \in \mathrm{M}_d(k)$ we get the definition ideal

$I \subset k[T_{ij}]$ of $Y \cap \mathcal{U} \subset \mathcal{U}$. It is generated by the polynomials:

$$T^p_{d-1,j} \cdot T_{i1} + T^p_{dj} \cdot T_{i2} + T_{i,j+2}, \qquad i = 1, 2, \quad j = 1, \dots, d-2$$

$$T^p_{d-1,j} \cdot T_{i1} + T^p_{dj} \cdot T_{i2}, \qquad i = 1, 2, \quad j = d-1, d$$

$$T^p_{d-1,j} \cdot T_{i1} + T^p_{dj} \cdot T_{i2} + T_{i,j+2} - T^p_{i-2,j}, \quad i = 3, \dots, d, \quad j = 1, \dots, d-2$$

$$T^p_{d-1,j} \cdot T_{i1} + T^p_{dj} \cdot T_{i2} - T^p_{i-2,j}, \qquad i = 3, \dots d, \quad j = d-1, d$$

and $U_0 = \mathcal{N}_{0,0,\emptyset(k)}$ corresponds to the maximal ideal $\mathfrak{m} = (T_{ij}, 1 \leq 1, j \leq d)$ in $\Gamma(Y \cap \mathcal{U}, \mathcal{O}_Y) = k[T_{ij}]/I$.

In the example for $h = 7$ computed above, we have determined the special case of this ideal I, namely the case of $d = \frac{h-3}{2} = 2$. In that special case, the polynomials generating I were all homogeneous of degree $p + 1$ and the maximal ideal $\mathfrak{m} = (a, b, c, d)$ was generated by zero divisors in $k[a, b, c, d]/I$.
In the general case however, that is $h \geq 11$, some of the generating polynomials of I also have terms of degree 1 and in particular none of them decomposes as a product of polynomials of smaller degree. Unfortunately, I was not able to show neither that the ideal $\mathfrak{m} = (T_{ij}, 1 \leq i, j \leq d)$ contains at least one regular element in $k[T_{ij}, 1 \leq i, j \leq d]/I$, nor that all T_{ij} were all zero divisors in $k[T_{ij}]/I$.
If one were to find a regular element in $\mathfrak{m}$, it would mean the existence of a regular sequence of length at least 1 in the local ring $\mathcal{O}_{Y,U_0} = \big(k[T_{ij}]/I\big)_{\mathfrak{m}}$ and thus imply the Serre condition S_1 for this local ring. Together with the Serre condition R_0, which means that Y is generically regular and which holds, since Y is even regular in codimension 1, this would mean that Y is reduced in the point U_0. And by Proposition 4.5 we would have that Y is reduced and thus have described $\mathcal{M}(0)_{\mathrm{red}}$.

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Zusammenfassung

Sei $\mathrm{Nilp}_{\mathbb{Z}_p}$ die Kategorie der Schemata über $\mathrm{Spec}(\mathbb{Z}_p)$ auf denen p nilpotent operiert und sei $\mathbb{X}$ eine p-divisible Gruppe über $\mathrm{Spec}(\mathbb{F}_p)$ der Dimension 2 und Höhe h mit h ungerade. Wir betrachten den Funktor

$$\begin{aligned}\mathcal{M}\colon\ \mathrm{Nilp}_{\mathbb{Z}_p} &\longrightarrow (\mathrm{Sets})\\ S &\longmapsto \left\{\begin{array}{l}\text{Isomorphieklassen von Paaren } (X,\rho),\ \text{wobei}\\ X \text{ eine } p\text{-divisible Gruppe über } S \text{ und}\\ \rho\colon \mathbb{X}\times_{\mathrm{Spec}(\mathbb{F}_p)}\bar{S}\longrightarrow X\times_S \bar{S} \text{ eine Quasiisogenie}\end{array}\right\},\end{aligned}$$

wobei $\bar{S}$ das durch das Ideal $p\cdot\mathscr{O}_S$ definierte abgeschlossene Unterschema von S bezeichnet. Dieser Funktor ist darstellbar durch ein formales Schema lokal formal von endlichem Typ über $\mathrm{Spf}(\mathbb{Z}_p)$.
In dieser Arbeit wollen wir genauer das reduzierte Unterschema $\mathcal{M}_{\mathrm{red}}$ von $\mathcal{M}$ beschreiben. Oort und de Jong haben in ihrer Arbeit [dJO] gezeigt, dass jede Zusammenhangskomponente von $\mathcal{M}_{\mathrm{red}}$ irreduzibel ist und Viehmann hat in [Vie] die Zusammenhangskomponenten von $\mathcal{M}_{\mathrm{red}}$ bestimmt.
Sei $\mathcal{N}$ die Zusammenhangskomponente der Identität $\mathrm{id}\colon \mathbb{X}\longrightarrow\mathbb{X}$. Sei $k\supset\mathbb{F}_p$ ein algebraisch abgeschlossener Körper. Für die Beschreibung von k-wertigen Punkten von $\mathcal{M}_{\mathrm{red}}$ benutzen wir eine von Oort eingeführte Invariante: den Semimodul assoziiert zum Dieudonné-Modul einer p-divisiblen Gruppe.
Wir führen eine Stratifizierung von $\mathcal{N}(k)$ durch endlich viele irreduzible lokal abgeschlossen Teilmengen $\mathcal{N}^{\circ}_{j,l,\underline{\alpha_l}}(k)$ ein, die die Stratifizierung durch Semimoduln verfeinert. Wir bestimmen die Abschlüsse $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ dieser lokal abgeschlossenen Teilmengen, und auch die Strata, die zum Abschluss beitragen. Desweiteren definieren wir Unterfunktoren $\mathcal{N}_{j,l,\underline{\alpha_l}}$ von $\mathcal{N}\times\mathrm{Spf}(\mathbb{Z}_{p^h})$, deren k-wertige genau die abgeschlossenen Teilmengen $\mathcal{N}_{j,l,\underline{\alpha_l}}(k)$ von $\mathcal{N}(k)$ sind. Wir zeigen, dass die Unterfunktoren $\mathcal{N}_{j,l,\underline{\alpha_l}}$ darstellbar sind durch projektive $\mathbb{F}_{p^h}$-Schemata und untersuchen im folgenden diese Schemata.
Dabei bestimmen wir den glatten Ort dieser projektiven $\mathbb{F}_{p^h}$-Schemata und zeigen, dass sie, zumindest in Spezialfällen nicht reduziert und auch nicht Cohen-Macaulay sind.

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