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Geometric Complexity Theory, Tensor Rank, and Littlewood-Richardson Coefficients

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Abstract

We provide a thorough introduction to Geometric Complexity Theory, an approach towards computational complexity lower bounds via methods from algebraic geometry and representation theory. Then we focus on the relevant representation theoretic multiplicities, namely plethysm coefficients, Kronecker coefficients, and Littlewood-Richardson coefficients. These multiplicities can be described as dimensions of highest weight vector spaces for which explicit bases are known only in the Littlewood-Richardson case.

By explicit construction of highest weight vectors we can show that the border rank of $m \times m$ matrix multiplication is a least $\frac{3}{2}m^2 - 2$ and the border rank of 2×2 matrix multiplication is exactly seven. The latter gives a new proof of a result by Landsberg (J. Amer. Math. Soc., 19:447–459, 2005).

Moreover, we obtain new nonvanishing results for rectangular Kronecker coefficients and we prove a conjecture by Weintraub (J. Algebra, 129 (1): 103–114, 1990) about the nonvanishing of plethysm coefficients of even partitions.

Our in-depth study of Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$ yields a polynomial time algorithm for deciding $c_{\lambda\mu}^{\nu} \geq t$ in time polynomial in n and quadratic in t, where n denotes the number of parts of ν. For $t = 1$, i.e., for checking positivity of $c_{\lambda\mu}^{\nu}$, we even obtain a running time of $n^3 \log \nu_1$.

Moreover, our insights lead to a proof of a conjecture by King, Tollu, and Toumazet (CRM Proc. Lecture Notes, 34, Symmetry in Physics: 99–112), stating that $c_{\lambda\mu}^{\nu} = 2$ implies $c_{M\lambda M\mu}^{M\nu} = M + 1$ for all $M \in \mathbb{N}$.

Zusammenfassung

Diese Arbeit führt gründlich in die Geometrische Komplexitätstheorie ein, ein Ansatz, um untere Berechnungskomplexitätsschranken mittels Methoden aus der algebraischen Geometrie und Darstellungstheorie zu finden. Danach konzentrieren wir uns auf die relevanten darstellungstheoretischen Multiplizitäten, und zwar auf Plethysmenkoeffizienten, Kronecker-Koeffizienten und Littlewood-Richardson-Koeffizienten. Diese Multiplizitäten haben eine Beschreibung als Dimensionen von Höchstgewichtsvektorräumen, für welche konkrete Basen nur im Littlewood-Richardson-Fall bekannt sind.

Durch explizite Konstruktion von Höchstgewichtsvektoren können wir zeigen, dass der Grenzrang der $m \times m$ Matrixmultiplikation mindestens $\frac{3}{2}m^2 - 2$ ist, und der Grenzrang der 2×2 Matrixmultiplikation genau sieben ist. Dies liefert einen neuen Beweis für ein Ergebnis von Landsberg (J. Amer. Math. Soc., 19:447–459, 2005).

Desweiteren erhalten wir Nichtverschwindungsresultate für rechteckige Kronecker-Koeffizienten und wir beweisen eine Vermutung von Weintraub (J. Algebra, 129 (1): 103–114, 1990) über das Nicht-Verschwinden von Plethysmenkoeffizienten von geraden Partitionen.

Unsere eingehenden Untersuchungen zu Littlewood-Richardson-Koeffizienten $c_{\lambda\mu}^{\nu}$ ergeben einen Polynomialzeitalgorithmus zum Entscheiden von $c_{\lambda\mu}^{\nu} \geq t$ mit Laufzeit polynomiell in n und quadratisch in t, wobei n die Anzahl der Teile von ν ist. Für $t = 1$, also zum Testen der Positivität von $c_{\lambda\mu}^{\nu}$, bekommen wir sogar eine Laufzeit von $n^3 \log \nu_1$.

Darüber hinaus führen unsere Einsichten zu einem Beweis einer Vermutung von King, Tollu und Toumazet (CRM Proc. Lecture Notes, 34, Symmetry in Physics: 99–112), welche besagt, dass aus $c_{\lambda\mu}^{\nu} = 2$ immer $c_{M\lambda M\mu}^{M\nu} = M + 1$ für alle $M \in \mathbb{N}$ folgt.

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Key words and phrases. Geometric Complexity Theory, permanent versus determinant, tensor rank, matrix multiplication, Kronecker coefficients, Littlewood-Richardson coefficients, plethysm coefficients

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Last, but certainly not least, I want to say thank you to my parents, for their loving support in all areas of life.

Eidesstattliche Erklärung Hiermit versichere ich, dass ich die folgende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen als Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

Paderborn, den 7.1.2013

Christian Ikenmeyer

Wenn nicht der Herr das Haus baut,
müht sich jeder umsonst, der daran baut.

Ps. 127, 1

Contents

Chapter 1

Introduction

Computational complexity is concerned with the study of the inherent difficulties of computational problems. One main goal is the search for computational lower bounds that ultimately lead to the separation of complexity classes. The study of these problems began in the 50s and turned out to be extraordinarily difficult. One infamous example is the $\mathsf{P} \neq \mathsf{NP}$ problem, where the goal is to find superpolynomial complexity lower bounds for any arbitrary NP-complete problem, see e.g. [Coo03, Coo06]. Despite many attempts, progress on these questions has been slow. In fact, one breakthrough result is of negative nature, explaining why certain proof techniques are insufficient for proving $\mathsf{P} \neq \mathsf{NP}$, see [RR97]. Even "simpler" problems are still wide open, where the following is of high interest to us: It is unknown whether the permanent of a given integer matrix can be computed in polynomial time. Valiant [Val79b] showed that computing the permanent is complete for the complexity class $\#\mathsf{P}$. Therefore, if the permanent could be computed in polynomial time, then $\mathsf{FP} = \#\mathsf{P}$ and in particular $\mathsf{P} = \mathsf{NP}$. Notably, for nonnegative matrices, a randomized polynomial time approximation algorithm for the permanent is known, see [JSV04].

Another example of paramount importance is the computational complexity of the matrix multiplication map. Since we can multiply two $n \times n$ matrices in time polynomial in n, we sharpen our analysis and are interested in the *exponent* of the running time of the fastest algorithm multiplying two $n \times n$ matrices. It turns out that this exponent also occurs as the exponent of the running time of the fastest algorithms for many other problems in linear algebra, for example matrix inversion or the computation of the determinant, see [BCS97, Ch. 16] for a survey.

Both problems — the computational complexity of computing the permanent and the computational complexity of matrix multiplication — can be interpreted in an algebraic framework, which is called *algebraic complexity theory*. Algebraic complexity theory and the classical boolean complexity theory have several tight relations, see for example [Bür00a, Ch. 4], [Bür00b]. One big advantage of algebraic complexity theory is that in the algebraic framework, our two problems can be rephrased as a so-called *orbit closure problem*, i.e., the problem of deciding whether a point $\mathscr{h}$ is contained in an affine variety $\overline{G\mathscr{c}}$, where G is a general linear group or a threefold product of general linear groups (see Chapter 2), G acts regularly on a finite dimensional complex vector space, and $\overline{G\mathscr{c}}$ denotes the topological closure of the orbit $G\mathscr{c}$. For the matrix multiplication complexity, the orbit closure description was introduced by Strassen [Str87]. For the permanent complexity, this was proposed by Mulmuley and Sohoni [MS01, MS08] (see [Mul11] for recent pointers to the literature). Since both problems share a lot of structure, we will strive to present a unified viewpoint as often as possible.

Orbit closure problems can be studied with tools from algebraic geometry and representation theory as follows. First, we note that $\mathfrak{h} \in \overline{G\mathfrak{c}}$ iff $\overline{G\mathfrak{h}} \subseteq \overline{G\mathfrak{c}}$. But $\overline{G\mathfrak{h}} \subseteq \overline{G\mathfrak{c}}$ implies a surjection of coordinate rings

$$\mathbb{C}[\overline{G\mathfrak{c}}] \twoheadrightarrow \mathbb{C}[\overline{G\mathfrak{h}}]. \qquad (*)$$

Therefore, by Schur's Lemma, for each irreducible G-representation $\{\lambda\}$, the multiplicity of $\{\lambda\}$ in the left hand side of $(*)$ cannot exceed the multiplicity of $\{\lambda\}$ in the right hand side. An irreducible G-representation $\{\lambda\}$ that does not satisfy this inequality is called a *representation theoretic obstruction against* $\mathfrak{h} \in \overline{G\mathfrak{c}}$. These are the objects we are interested in.

Mulmuley and Sohoni coined the term *Geometric Complexity Theory* for the study of those G-irreducible representations $\{\lambda\}$ that occur in $\mathbb{C}[\overline{G\mathfrak{h}}]$ with positive multiplicity, but which do not occur in $\mathbb{C}[\overline{G\mathfrak{c}}]$. One central question is how to decide for a given $\{\lambda\}$ whether $\{\lambda\}$ occurs in $\mathbb{C}[\overline{G\mathfrak{c}}]$ (or $\mathbb{C}[\overline{G\mathfrak{h}}]$) with positive multiplicity.

A prototype of such representation theoretic multiplicities, which has many desirable properties, is the *Littlewood-Richardson coefficient* $c_{\lambda\mu}^{\nu}$ and for this coefficient, deciding positivity can be done in polynomial time. To the best of our knowledge, this fact was first pointed out in [DLM06, MS05]. Indeed, by the saturation property (see [KT99, Buc00]), we have that $c_{\lambda\mu}^{\nu} > 0$ is equivalent to $\exists N\, c_{N\lambda N\mu}^{N\nu} > 0$, which can be rephrased as the feasibility problem of a rational polyhedron, well-known to be solvable in polynomial time by the ellipsoid method [Kha80] or interior point methods [Kar84].

The starting point for our analyses of Littlewood-Richardson coefficients was a question in [MS05] asking for a combinatorial algorithm for deciding $c_{\lambda\mu}^{\nu} > 0$ in polynomial time, using ideas similar to the max-flow or weighted matching problems in combinatorial optimization. We quote [MS05] here, see also the journal version [MNS12].

> It is of interest to know if there is a purely combinatorial algorithm for this problem that does not use linear programming; i.e., one similar to the max-flow or weighted matching problems in combinatorial optimization. [...] It is reasonable to conjecture that there is a polynomial time algorithm that provides an integral proof of positivity of $c_{\lambda\mu}^{\nu}$ in the form of an integral point in P.

Among other insights, such an algorithm is introduced in Part II of this thesis.

Besides Geometric Complexity Theory, Littlewood-Richardson coefficients are of interest in several areas of mathematics. They appear not only in representation theory and algebraic combinatorics, but also in topology and enumerative geometry (Schubert calculus): for instance, they determine the multiplication in the cohomology ring of the Grassmann varieties. Littlewood-Richardson coefficients gained further prominence due to their role in the proof of Horn's conjecture [HR95, Kly98, KT99, KTW04] on the relation of the eigenvalues of a triple A, B, C of Hermitian matrices satisfying $C = A + B$. The latter problem is of relevance in perturbation and quantum information theory. We refer to Fulton [Ful00] for an excellent account of these more recent developments.

1.1 Main Results

We prove the following two nonvanishing results (Theorem 6.2.1 and Theorem 6.3.2), one for plethysm coefficients and one for Kronecker coefficients. Theorem 6.2.1 was conjectured by Weintraub in [Wei90].

Theorem 6.2.1 ([BCI11a]). *For all $k, n, d \in \mathbb{N}$ with $d \le k$ and for all partitions λ of dn with at most d parts, the irreducible GL_k-representation $\{2\lambda\}$ occurs in the plethysm $\mathsf{Sym}^d\mathsf{Sym}^{2n}\mathbb{C}^k$, where $2\lambda = (2\lambda_1, \dots, 2\lambda_d)$.*

The next result focuses on the nonvanishing of rectangular Kronecker coefficients and rectangular symmetric Kronecker coefficients.

Theorem 6.3.2 ([BCI11b]). *Fix n. There exists a stretching factor $\gamma \in \mathbb{N}$ such that for all $d \in \mathbb{N}$ and all partitions λ of dn with at most n^2 parts both the Kronecker coefficient $k(\gamma(n\times d); \gamma(n\times d); \gamma\lambda)$ and the symmetric Kronecker coefficient $\mathsf{sk}\big(2\gamma\lambda; \big(2\gamma(n\times d)\big)^2\big)$ do not vanish. Here $n\times d$ denotes the rectangular partition of nd with n rows and d columns and stretching is defined componentwise.*

In Chapter 7 we introduce a combinatorial interpretation of highest weight vectors in the vector space $\mathsf{Sym}^d \bigotimes^3(\mathbb{C}^n)^*$ of homogeneous polynomials of degree d, called *obstruction designs*. This viewpoint enables us to explicitly describe a family of highest weight vectors proving the following lower bound on the border rank $\underline{R}$ of $m \times m$ matrix multiplication $\mathcal{M}_m$, see Theorem 8.2.1:

$$\underline{R}(\mathcal{M}_m) \ge \tfrac{3}{2}m^2 - 2. \tag{1.1.1}$$

In a joint effort with Peter Bürgisser, Jon Hauenstein, and Joseph Landsberg, for $m = 2$ we even can prove $\underline{R}(\mathcal{M}_2) = 7$ by an explicit contruction of a homogeneous highest weight vector of degree 20 in the coordinate ring $\mathbb{C}[\bigotimes^3(\mathbb{C}^4)^*]$. So in this case our complexity lower bound is sharp. This is remarkable, since "$\underline{R}(\mathcal{M}_2) = 6$ or 7" was a long-standing open problem since the 70s, first proved in 2005 by Landsberg [Lan05] using very different methods than ours.

But we also contribute some negative results. We prove that "SL-obstructions" are not a sufficiently sharp tool to prove significant lower bounds on the border rank of matrix multiplication, see Theorem 9.1.1. For this we rely on original work presented in Chapter 5. This SL-no go result appeared in [BI11].

Regarding Littlewood-Richardson coefficients, we prove the following algorithmic results (Theorem 11.2.4, Theorem 11.3.2, and Theorem 11.3.3), and achieve a proof of a conjecture of King, Tollu, and Toumazet posed in [KTT04], see Theorem 12.4.1.

Theorem 11.2.4 ([BI12]). *Given partitions λ, μ, ν with $|\lambda|+|\mu| = |\nu|$, Algorithm 2 decides the positivity of the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ with $\mathcal{O}(n^3 \log \nu_1)$ arithmetic operations and comparisons, where n denotes the number of parts of ν.*

Algorithm 2 is basically a capacity scaling Ford-Fulkerson algorithm [FF62] on well-chosen residual networks.

Theorems 11.3.2 and 11.3.3 ([Ike12]). *Given partitions λ, μ, ν with $|\lambda|+|\mu| = |\nu|$ and a natural number $t \ge 1$, Algorithm 3 decides $c^\nu_{\lambda\mu} \ge t$ in time $\mathcal{O}\big(t^2 \cdot \mathsf{poly}(n)\big)$, where n denotes the number of parts of ν. A variant of Algorithm 3 computes $c^\nu_{\lambda\mu}$ in time $\mathcal{O}\big((c^\nu_{\lambda\mu})^2 \cdot \mathsf{poly}(n)\big)$.*

Algorithm 3 enumerates with polynomial delay the points in the hive flow polytope, whose integral points count the Littlewood-Richardson coefficient. With only minor modifications our algorithms can be used to enumerate efficiently all hive flows corresponding to a given tensor product $\{\lambda\} \otimes \{\mu\}$ of irreducible GL_n-representations, as asked in [KT01, p. 186].

Our algorithms are easy to state and implement and they are available online. We encourage the reader to try out our Java applet at
`http://www-math.upb.de/agpb/flowapplet/flowapplet.html`.

Moreover, our insights into the structure of the hive flow polytope lead to the proof of the following conjecture of King, Tollu, and Toumazet.

Theorem 12.4.1 ([Ike12]). *Given partitions λ, μ and ν such that $|\nu| = |\lambda| + |\mu|$. Then $c_{\lambda\mu}^{\nu} = 2$ implies $c_{M\lambda M\mu}^{M\nu} = M + 1$ for all $M \in \mathbb{N}$.*

We remark that [KTT04, Conj. 3.3] also contains the conjecture that $c_{\lambda\mu}^{\nu} = 3$ implies either $c_{M\lambda M\mu}^{M\nu} = 2M + 1$ or $c_{M\lambda M\mu}^{M\nu} = (M+1)(M+2)/2$. We think that a careful refinement of our methods can be used to prove this and similar conjectures as well.

1.2 Outline and Further Results

The thesis is divided into two parts, which can be studied independently of each other. The first part deals with the Geometric Complexity Theory of the permanent polynomial and the matrix multiplication map. The second part focuses on the in-depth study of Littlewood-Richardson coefficients.

1.2 (A) Part I: Geometric Complexity Theory

Part I starts with three preliminary chapters, each one containing material well-known to the experts of the particular fields. However, since Geometric Complexity Theory is an interdisciplinary approach, these chapters pursue the important goal to make this thesis accessible to a wide readership including theoretical computer scientists, algebraic geometers, representation theorists, and algebraic combinatorialists. We start in Chapter 2 by giving a brief summary of the connection between boolean complexity theory and algebraic complexity theory and show how the computational complexity of the permanent polynomial and the matrix multiplication map can be reformulated as orbit closure problems. In Chapter 3 we give an extremely brief introduction to classical algebraic geometry over $\mathbb{C}$ and the theory of obstructions. After that, in Chapter 4, we give details about the representation theory of the general linear and symmetric groups over $\mathbb{C}$ and the coefficients arising in decompositions of tensor products of irreducible representations, which are of interest in Geometric Complexity Theory.

Chapter 5 studies the coordinate rings of orbits and contains slightly extended material from [BI11], [BI10], and [BLMW11]. As a first simple application of our study, Claim 5.3.6 provides a simple new proof that the typical border rank of tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ equals 5. In Section 5.4 we generalize the study of exponents of regularity from the tensor scenario in [BI11] to the polynomial scenario in a uniform treatment. This is related to studies of Kumar, cp. [Kum10].

In Chapter 6 we present our contributions to the representation theory of the general linear and symmetric groups over $\mathbb{C}$. This includes, but is not limited to, the publications with Bürgisser and Christandl [BCI11b] and [BCI11a].

Additionally, using algorithmic insights, the kernel of the Foulkes-Howe map $\Psi_{5,5}$ is decomposed as an S_5-representation, see (6.1.4). This is joint work with Sevak Mkrtchyan. We thank Joseph Landsberg and Shrawan Kumar for helping us and introducing us to the problem.

Chapter 7 focuses on the theory of obstruction designs, which is a combinatorial description of highest weight vectors in the tensor scenario. This chapter consists of original content only.

We remark that the explicit construction of highest weight vectors in the determinant versus permanent scenario is pursued by Kumar in [Kum12].

Besides several calculations regarding plethysm coefficients and symmetric Kronecker coefficients, in Chapter 8 we prove the lower bound (1.1.1) on the border rank of matrix multiplication and its refinement $\underline{R}(\mathcal{M}_2) = 7$. Both proofs use the theory introduced in Chapter 7 to explicitly construct and evaluate highest weight vectors.

Chapter 9 presents results from [BI11] which show that some very optimistic approaches to Geometric Complexity Theory are doomed to fail.

1.2 (B) Part II: Littlewood-Richardson Coefficients

The second part of this thesis focuses on Littlewood-Richardson coefficients. The results are submitted, cf. [BI12] and [Ike12].

In contrast to the Kronecker coefficients, different combinatorial characterizations of the Littlewood-Richardson coefficients are known. The classic Littlewood-Richardson rule (cf. [Ful97, Ch. I.5]) counts certain skew tableaux, while in Berenstein and Zelevinsky [BZ92], the number of integer points of certain polytopes are counted. A beautiful characterization was given by Knutson and Tao [KT99], who characterized Littlewood-Richardson coefficients either as the number of honeycombs or hives with prescribed boundary conditions. We add to this the following new description, which is based on flows in networks and which serves several algorithmic purposes: We characterize $c_{\lambda\mu}^{\nu}$ as the number of *capacity achieving hive flows* on the honeycomb graph G, cf. Figures 10.1.i–10.1.ii. Besides capacity constraints given by λ, μ, ν, these flows have to satisfy rhombus inequalities corresponding to the ones considered in [KT99, Buc00].

Section 10.1 describes the setting and introduces the main terminology. We define the notion of hive flows on honeycomb graphs and associate with a triple λ, μ, ν of partitions the polytope $B(\lambda, \mu, \nu)$ of bounded hive flows, along with a linear function δ measuring the overall throughput of a flow. A flow $f \in B(\lambda, \mu, \nu)$ is called capacity achieving if $\delta(f) = |\nu|$. We denote by $P(\lambda, \mu, \nu)$ the polytope consisting of these flows and by $P(\lambda, \mu, \nu)_{\mathbb{Z}}$ the set of its integral points. It turns out that the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ counts the elements of $P(\lambda, \mu, \nu)_{\mathbb{Z}}$, see Proposition 10.1.11.

In Section 10.2 we obtain some structural insights into the set of hive flows. We show that $P(\lambda, \mu, \nu)_{\mathbb{Z}}$ is the vertex set of a natural connected graph (*Connectedness Theorem* 10.2.14). The connectedness immediately implies the property $c_{\lambda\mu}^{\nu} = 1 \Rightarrow \forall N\, c_{N\lambda N\mu}^{N\nu} = 1$, which was conjectured by Fulton and proved in [KTW04], see Corollary 10.2.16. The connectedness of $P(\lambda, \mu, \nu)_{\mathbb{Z}}$ is also relevant for efficiently enumerating the points of $P(\lambda, \mu, \nu)_{\mathbb{Z}}$ and for proving the conjecture by King, Tollu, and Toumazet (Theorem 12.4.1).

The above discussion suggests to decide $c_{\lambda\mu}^{\nu} > 0$ by optimizing the overall throughput function δ on the polytope $B(\lambda, \mu, \nu)$ of bounded hive flows. We imitate the basic Ford-Fulkerson idea and construct, for a given integral hive flow f, a "residual digraph" R_f, such that f optimizes δ on $B(\lambda, \mu, \nu)$ iff R_f does not contain an s-t-path. In Section 10.3 we define the residual digraph R_f and study the partition of the triangular graph into f-flatspaces. We then present and analyze Algorithm 1, a first max-flow algorithm for deciding $c_{\lambda\mu}^{\nu} > 0$ (see Theorem 11.1.2). The proof of correctness of this algorithm requires an in-depth understanding of the properties of hives and it has two main ingredients. The *Shortest Path Theorem* 10.3.9 states that the rhombus inequalities are not violated after augmenting the current flow f by a shortest path in the residual network R_f. This is remarkable since, unlike in the usual max-flow situation, the polytopes of hive flows are not integral, cf. [Buc00]. The other ingredient, needed for the correctness, is the *Rerouting Theorem* 10.3.22, which tells us how to replace an augmenting flow direction d by a flow in the residual network without changing the overall throughput. This amounts to a rerouting of d along the borders of the flatspaces of the current flow f (cf. Figure 10.3.ii). Here the main difficulty is the analysis of the degenerate situation of large flatspaces, a topic not pursued in detail in the previous papers [KT99, Buc00].

Chapter 11 is of algorithmic nature, where we also consider running time issues. In Section 11.2 we modify Algorithm 1 and obtain a polynomial time algorithm for deciding the positivity of Littlewood-Richardson coefficients, see Theorem 11.2.4. This algorithm is a considerable improvement over the one presented at the FPSAC conference 2009 [Ike08, BI09], both with regard to simplicity and running time. The reason is that there, before each augmentation step, the flow had to be substituted by a nondegenerate flow using a costly routine. (Nondegenerate meaning that small triangles and small rhombi are the only flatspaces, cf. [Buc00] and Remark 10.3.25.) The present algorithm does not suffer from this deficiency anymore.

Recall that the integral capacity achieving hive flows count the Littlewood-Richardson coefficient and form the vertices of the connected Littlewood-Richardson graph $P(\lambda, \mu, \nu)_{\mathbb{Z}}$. To compute $c^{\nu}_{\lambda\mu}$ we design a variant of breadth-first-search in Section 11.3 that lists all points in $P(\lambda, \mu, \nu)_{\mathbb{Z}}$ with only polynomial delay between the single outputs. This enables us to decide $c^{\nu}_{\lambda\mu} \geq t$ in time $\mathcal{O}\big(t^2 \cdot \mathsf{poly}(n)\big)$, see Theorem 11.3.2. Also we get an algorithm for computing $c^{\nu}_{\lambda\mu}$ which runs in time $\mathcal{O}\big((c^{\nu}_{\lambda\mu})^2 \cdot \mathsf{poly}(n)\big)$, see Theorem 11.3.3. This implies that "small" Littlewood-Richardson coefficients can be efficiently computed.

These algorithms are based on a subalgorithm for the efficient construction of all neighboring hive flows from a given integral hive flow. Such a subalgorithm is provided in Section 11.4.

Chapter 12 is devoted to the combinatorially quite intricate proofs of the Rerouting Theorem, the Shortest Path Theorem, the Connectedness Theorem, and the King-Tollu-Toumazet conjecture (Theorem 12.4.1). For proving Theorem 12.4.1, we heavily rely on machinery developed in the previous chapters of Part II.

Part I

Geometric Complexity Theory

"Sei $\Psi(F, n)$ die Anzahl der Schritte, die die Maschine dazu benötigt und sei $\varphi(n) = \max_F \Psi(F, n)$. Die Frage ist, wie rasch $\varphi(n)$ für eine optimale Maschine wächst. [...] Wenn es wirklich eine Maschine mit $\varphi(n) \sim Kn$ (oder auch nur $\sim Kn^2$) gäbe, hätte das Folgerungen von der grössten Tragweite."

— A letter sent from Gödel to von Neumann,
Princeton, 20 March 1956

Chapter 2

Preliminaries: Geometric Complexity Measures

In this chapter we show how fundamental lower bounds problems in computational complexity can be restated naturally as so-called *orbit closure problems*. These orbit closure problems are pure mathematical statements which will be analyzed in later chapters with methods from algebraic geometry and representation theory.

2.1 Circuits and Algebraic Complexity Theory

In this section we recall basic facts from circuit complexity and highlight the relevant links to algebraic complexity theory. We also describe several fundamental algebraic complexity classes.

2.1.1 Definition (Circuit)**.** Fix a set S and a set of functions of arbitrary arity $F = \{f_i \colon S^{a_i} \to S\}$, where $a_i \in \mathbb{N}_{\geq 1}$. (For example, for *boolean circuits*, choose $S = \{0, 1\}$ and $F = \{\texttt{and}, \texttt{or}, \texttt{not}\}$). A *circuit* C is a directed graph that contains no directed cycle such that the following properties hold (see Figure 2.1.i):

- A subset of the vertices with indegree 0 is labeled by indeterminates. These vertices are called the *input gates*. The other vertices with indegree 0 are labeled with elements of S and are called *constant gates*. All other vertices are called *computation gates*. The computation gates with outdegree 0 are also called *output gates*.
- Each computation gate v is labeled with a function f_i from F with arity a_i coinciding with the indegree of v. ■

A circuit C defines a function $C_v \colon S^n \to S$ for each output gate v in the natural way by induction over the structure of the digraph, where n is the number of input gates. We call a circuit a *single-output circuit*, if it has only one output gate and in this case $C \colon S^n \to S$ denotes the function of the output gate. The *size* $|C|$ of a circuit C is defined to be the number of its gates. The *circuit complexity* $c_{S,F}(f)$ of a function $f \colon S^n \to S$ is the minimal size of a circuit C defining f.

A *boolean circuit* is a circuit with $S = \{0, 1\}$ and $F = \{\texttt{and}, \texttt{or}, \texttt{not}\}$.

2.1.2 Definition. A function $f \colon \mathbb{N} \to \mathbb{N}$ is called *polynomially bounded* if there exists a univariate polynomial p such that for all $n \in \mathbb{N}$ we have $f(n) \leq p(n)$. The set $\mathsf{poly}(n)$ is defined as the set of all polynomially bounded functions, i.e., the union $\mathsf{poly}(n) = \bigcup_{k \in \mathbb{N}} \mathcal{O}(n^k)$. ■

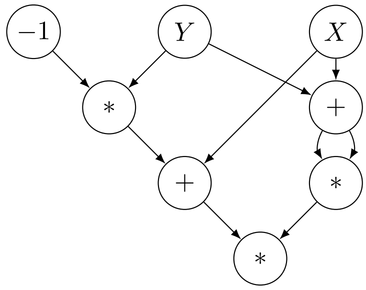


Figure 2.1.i: A circuit computing the polynomial $X^3+X^2Y-XY^2-Y^3$. Here $S=\mathbb{C}$ and $F=\{*,+\}$. The circuit has 2 input gates, one constant gate, 5 computation gates, and 1 output gate.

The following complexity class is the nonuniform analogue of the class **P** of languages decidable in polynomial time and is of fundamental interest in computational complexity theory.

2.1.3 Definition. Let $S := \{0,1\}$ and $F = \{\texttt{and}, \texttt{or}, \texttt{not}\}$. The class **P/poly** consists of all function families (χ_n) with $\chi_n \colon S^n \to S$ whose complexity sequence $c_{S,F}(\chi_n)$ is polynomially bounded. ■

We want to rephrase Definition 2.1.3 in a more algebraic manner. Let $S := \mathbb{F}$ denote a fixed field and let $F := \{+,*\}$, where "+" and "∗" have arity 2. Circuits corresponding to this S and F are called *arithmetic circuits over* $\mathbb{F}$.

The following rephrased version of Definition 2.1.3 has a more algebraic flavor.

2.1.4 Lemma (Arithmetic characterization of **P/poly**). *Let $S := \mathbb{F}_2$ and let $F := \{+,*\}$. The class* **P/poly** *consists of all function families (χ_n) with $\chi_n \colon S^n \to S$ whose sequence $c_{S,F}(\chi_n)$ is polynomially bounded.*

Proof. Addition computes the `xor` function, multiplication the `and` function. The set $\{\texttt{xor}, \texttt{and}, 1\}$ is a complete set in the sense of [End72, Sec. 1.5], because $A \;\texttt{xor}\; 1 = \texttt{not}\; A$ and the set $\{\texttt{and}, \texttt{not}\}$ is complete. Hence circuits using $\{\texttt{and}, \texttt{or}, \texttt{not}\}$ can be converted into circuits using $\{\texttt{xor}, \texttt{and}, 1\}$ by replacing gates with subcircuits of constant size, and vice versa. □

Given this algebraic characterization, it is straightforward to work over different fields $\mathbb{F}$ other than $\mathbb{F}_2$. In fact finite fields have the following disadvantage. Single-output arithmetic circuits with n input gates not only naturally define a function $\mathbb{F}^n \to \mathbb{F}$, but they also *compute a polynomial* in $\mathbb{F}[X_1,\dots,X_n]$ by induction on the circuit structure, see Figure 2.1.i. But single-output arithmetic circuits over finite fields, which compute different polynomials, can define the same function: Let $f(X,Y) = X^2Y$ and $g(X,Y) = XY^2$, then

$$\forall x \in \mathbb{F}_2^2 : f(x) = g(x) \text{ , but } f \neq g \text{ as polynomials.}$$

Infinite fields do not suffer from this deficiency, as the following easy lemma shows.

2.1.5 Lemma. *Let $\mathbb{F}$ be an infinite field. Then for two polynomials $f,g \in \mathbb{F}[X_1,X_2,\dots,X_n]$ we have*

$$f = g \quad \Leftrightarrow \quad f(x) = g(x) \textit{ for all } x \in \mathbb{F}^n.$$

Proof. We show by induction that a polynomial that vanishes on the whole $\mathbb{F}^n$ is the zero polynomial. For $n = 1$ the result follows easily from successive polynomial division by linear factors. For $n > 1$ we can decompose every f that vanishes on $\mathbb{F}^n$ as $f = \sum_{i=0}^{\deg f} g_i X_n^i \in \mathbb{F}[X_1, \ldots, X_{n-1}][X_n]$. Fix a point $(x_1, \ldots, x_{n-1}) \in \mathbb{F}^{n-1}$. Define $p(y) := f(x_1, \ldots, x_{n-1}, y) \in \mathbb{F}[y]$. Note that $p(y)$ vanishes on $\mathbb{F}$ and hence p is the zero polynomial. Equating coefficients of p yields that for all i we have $g_i(x_1, \ldots, x_{n-1}) = 0$. Since the point $(x_1, \ldots, x_{n-1})$ was chosen arbitrarily, all g_i vanish on the whole $\mathbb{F}^{n-1}$. By induction hypothesis each g_i is the zero polynomial. Therefore f is the zero polynomial. □

In the light of Lemma 2.1.5 we see that when working over an infinite field it is reasonable to focus on the polynomials computed by arithmetic circuits instead of the functions defined by them.

> **Ground Field:** For the sake of considerable simplification regarding questions from algebraic geometry and representation theory, we choose the complex numbers as our ground field in this whole thesis.

The polynomial family X^{2^n} has a circuit family (C_n) where each (C_n) consists of n consecutive multiplication gates, see Figure 2.1.ii. But computing X^{2^n} on some

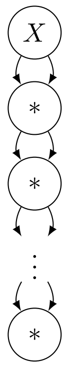


Figure 2.1.ii: A circuit family of polynomially bounded size computing X^{2^n}.

input $X \in \mathbb{Z}$ requires a Turing machine to make exponentially many steps due to the growth of the integers X^{2^n}. To ensure that small circuits can be evaluated efficiently we impose an upper bound on the degree of computed polynomials in the Definition 2.1.7 of the following algebraic complexity class.

A family (f_n) of multivariate polynomials is called a p-*family*, if the number of variables and the degree are both polynomially bounded.

2.1.6 Main Examples. Both the determinant family $(\det_n)$ with

$$\det_n := \sum_{\pi \in \mathsf{S}_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} X_{i,\pi(i)}$$

and the permanent family (per_n) with

$$\mathrm{per}_n := \sum_{\pi \in \mathsf{S}_n} \prod_{i=1}^{n} X_{i,\pi(i)}$$

are p-families. ■

2.1.7 Definition. The *arithmetic complexity* $L(f)$ of a polynomial f is the size of the smallest single-output circuit computing f.

The class **VP** consists of those p-families (f_n) for which $L(f_n)$ is polynomially bounded. ■

The class **VP** has one severe drawback: The only polynomials that are known to be complete for this class (as defined in the next section) are quite artificial. To get rid of this deficiency, in the following we will define subclasses that have more natural complete polynomials.

Computation gates labeled with "$*$" are called *multiplication gates*. In Section 2.4 we will need to distinguish the following subclass of multiplication gates: If one of the two parents of a multiplication gate m is a constant gate, then m is called a *scalar multiplication gate*, otherwise m is called a *nonscalar multiplication gate*. For example the leftmost multiplication gate in Figure 2.1.i is scalar, whereas the other two are nonscalar.

2.1.8 Definition. An arithmetic circuit is called *skew*, if for each multiplication gate at least one of the two parents is an input gate or a constant gate.

The *skew complexity* $L_{\mathsf{s}}(f)$ of a polynomial f is the size of the smallest single-output skew circuit computing f.

The class $\mathbf{VP_s}$ consists of the p-families with polynomially bounded skew complexity. ■

2.1.9 Definition. An arithmetic circuit is called *weakly skew*, if for each multiplication gate v one parent gate w of the two parent gates satisfies: The gate w and its ancestors form a subcircuit, i.e., the edge from w to v is the only edge connecting the subcircuit of w with the the rest of the circuit, see Figure 2.1.iii.

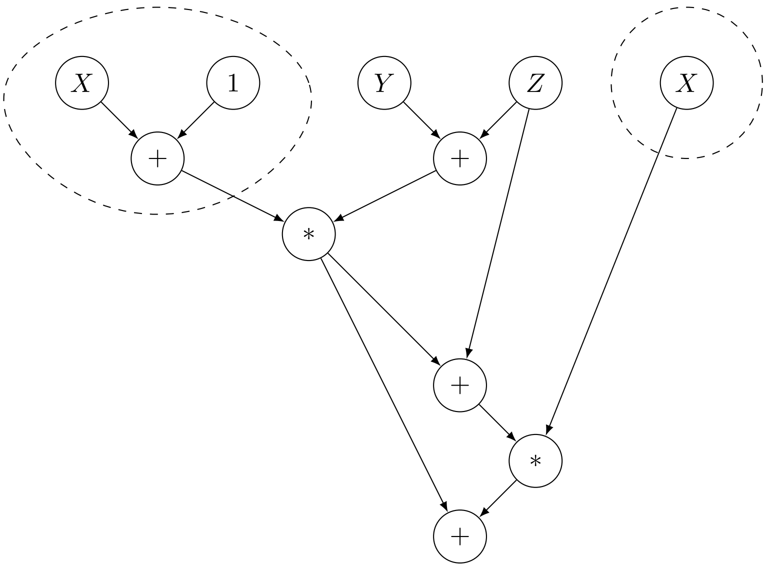


Figure 2.1.iii: A weakly skew arithmetic circuit. Each multiplication gate has an independent subcircuit, which is framed by a dashed curve.

The *weakly skew complexity* $L_{\mathsf{ws}}(f)$ of a polynomial f is the size of the smallest weakly skew single-output circuit computing f.

The class $\mathbf{VP_{ws}}$ consists of the p-families with polynomially bounded weakly skew complexity. ■

2.1.10 Definition. An arithmetic circuit is called a *formula*, if, when we interpret the circuit as an undirected graph, it has no cycles.

The *formula complexity* $L_{\mathsf{e}}(f)$ of a polynomial f is the size of the smallest single-output formula computing f.

The class $\mathbf{VP_e}$ consists of the p-families with polynomially bounded formula complexity. ■

The next definition will be used to compare different complexity measures.

2.1.11 Definition. Let X be any set on which we want to measure complexity of some sort, i.e., the set of homogeneous polynomials. Two functions $c, c' \colon X \to \mathbb{N}$ are called *polynomially equivalent* if there exist two univariate polynomials p and q such that for all $x \in X$ we have $c(x) \le p(c'(x))$ and $c'(x) \le q(c(x))$. ■

2.1.12 Theorem ([MP08])**.** *L_{s} and L_{ws} are polynomially equivalent. In particular* $\mathbf{VP_s} = \mathbf{VP_{ws}}$.

We will see in Lemma 2.1.13 that we can restrict our considerations to circuits that compute only homogeneous polynomials. The *homogeneous skew complexity* $H_{\mathsf{s}}(f)$ of a polynomial f is the size of the smallest multiple-output skew circuit computing all the homogeneous parts of f.

2.1.13 Lemma. *L_{s} and H_{s} are polynomially equivalent.*

Proof. One direction is clear by appending addition gates to a skew circuit which computes the homogeneous parts of f at its output gates. For the other direction, given a skew circuit C computing a polynomial f and let d denote the degree of f. Since f is skew, we have $d \le |C|$. We construct a skew circuit C' computing each homogeneous part $f^{(m)}$ of f, $1 \le m \le d$. Each computation gate v is replaced by a subcircuit consisting of d gates $v_1, \ldots, v_d$, such that v_i computes the ith homogeneous part of the intermediate computation result of v, $1 \le i \le d$. The subcircuit for an addition gate v with parents v^1 and v^2 consists of d addition gates v_i with parents v_i^1 and v_i^2. The subcircuit for a multiplication gate v with parents v^1 and α, where α is a constant gate, consists of d multiplication gates $v_1, \ldots, v_d$, where v_i has v_i^1 and α as a parent. The subcircuit for a multiplication gate v with parents v^1 and X, where X is an input gate, consists of d multiplication gates $v_1, \ldots, v_d$, where v_i has v_{i-1}^1 and X as a parent. The correctness of this construction follows by induction on the structure. The new circuit C' has size $|C'| \le d \cdot |C| \le |C|^2$. Moreover, C' is skew. □

In the next section we see that the determinant family is complete for $\mathbf{VP_{ws}}$.

2.2 Completeness and Reduction

In this section we recall the notions of reduction and completeness of polynomials in algebraic complexity theory. See [Bür00a] for an in-depth treatment.

2.2.1 Definition. Let $f \in \mathbb{C}[X_1, \ldots, X_n]$, $g \in \mathbb{C}[Y_1, \ldots, Y_m]$. Then f is called a *projection of* g, written $f \le g$, if there exist $a_1, \ldots, a_m \in \mathbb{C} \cup \{X_1, \ldots, X_n\}$ such that $g(a_1, \ldots, a_m) = f$.

A family (f_n) of polynomials is called a p-projection of a family (g_n) of polynomials, if there exists a polynomially bounded function t such that $f_n \le g_{t(n)}$ for all n. In this situation we write $f \le_{\mathsf{p}} g$.

Let $\mathbf{C}$ be a class of p-families. A p-family g is called $\mathbf{C}$-*complete*, if $g \in \mathbf{C}$ and

$$\forall f \in \mathbf{C} : \; f \le_{\mathsf{p}} g.$$

■

2.2.2 Theorem ([MP08]). *The determinant family* $(\det_n)$ *is* $\mathsf{VP_{ws}}$*-complete. More precisely,*

(1) $L_{\mathsf{ws}}(\det_n)$ *is polynomially bounded* [MP08, Prop. 5].

(2) If $L_{\mathsf{ws}}(f) = n$*, then* $f \leq \det_{n+1}$.[MP08, Lemma 6].

Theorem 2.2.2 continues Valiant's work: He showed in [Val79a] that

$$\text{if } L_{\mathsf{e}}(f) = n, \text{ then } f \leq \det_{n+1}. \tag{2.2.3}$$

A natural complete problem for $\mathsf{VP_e}$ is the iterated 3×3 matrix multiplication $\mathrm{IMM}_{3,n}$, defined as follows. $\mathrm{IMM}_{3,n} :=$

$$\mathrm{tr}\Bigg(\begin{pmatrix} X^{(1)}_{1,1} & X^{(1)}_{1,2} & X^{(1)}_{1,2} \\ X^{(1)}_{2,1} & X^{(1)}_{2,2} & X^{(1)}_{2,2} \\ X^{(1)}_{3,1} & X^{(1)}_{3,2} & X^{(1)}_{3,2} \end{pmatrix} \begin{pmatrix} X^{(2)}_{1,1} & X^{(2)}_{1,2} & X^{(2)}_{1,2} \\ X^{(2)}_{2,1} & X^{(2)}_{2,2} & X^{(2)}_{2,2} \\ X^{(2)}_{3,1} & X^{(2)}_{3,2} & X^{(2)}_{3,2} \end{pmatrix} \cdots \begin{pmatrix} X^{(n)}_{1,1} & X^{(n)}_{1,2} & X^{(n)}_{1,2} \\ X^{(n)}_{2,1} & X^{(n)}_{2,2} & X^{(n)}_{2,2} \\ X^{(n)}_{3,1} & X^{(n)}_{3,2} & X^{(n)}_{3,2} \end{pmatrix} \Bigg)$$

$\in \mathbb{C}[\{X^{(k)}_{i,j} \mid 1 \leq i,j \leq 3,\ 1 \leq k \leq n\}]$.

2.2.4 Theorem ([BOC92]). *The family* $(\mathrm{IMM}_{3,n})$ *is* $\mathsf{VP_e}$*-complete.*

Every polynomial f has a trivial formula computing it: Just write f as a sum of monomials and compute each monomial separately by iteratively multiplying with one variable. Eq. (2.2.3) implies that every polynomial is a projection of $\det_n$ for n big enough. This leads to the following definition.

2.2.5 Definition. The *determinantal complexity* $\mathsf{dc}(f)$ of a polynomial f is defined to be the minimal n such that $f \leq \det_n$. ■

Definition 2.2.5 and Theorem 2.2.2 give the following equivalence.

2.2.6 Corollary. L_{s}*,* L_{ws}*,* H_{s}*, and* dc *are pairwise polynomially equivalent.*

Hence the class $\mathsf{VP_{ws}}$ is the class of p-families with polynomially bounded determinantal complexity.

2.2.7 Remark. (1) As seen in Theorem 2.2.4, we can also define the IMM-complexity of a polynomial. This notion is polynomially equivalent to the formula complexity. In fact, the whole Geometric Complexity Theory program also works if we replace the determinant by iterated 3×3 matrix multiplication. Instead of proving lower bounds for the weakly skew complexity we then try to show lower bounds for the formula complexity. In this thesis we focus on the determinant family though.

(2) There are polynomials that are not a projection of the iterated 2×2 matrix multiplication $\mathrm{IMM}_{2,n}$, no matter how large n is, see [AW11]. ■

2.2.8 Conjecture (Valiant's conjecture). $\mathsf{dc}(\mathrm{per}_n)$ *is not polynomially bounded.*

Conjecture 2.2.8 is equivalent to saying that $(\mathrm{per}_n) \notin \mathsf{VP_{ws}}$. In fact, the best known upper bound is due to Ryser [Rys63] and yields $L(\mathrm{per}_n) \leq \mathcal{O}(n2^n)$ and $\mathsf{dc}(\mathrm{per}_n) \leq \mathcal{O}(n^2 2^n)$. On the other hand, we only have marginal lower bounds: Mignon and Ressayre [MR04] showed that $\mathsf{dc}(\mathrm{per}_n) \geq \frac{1}{2}n^2$.

The natural complexity class for the permanent family is the following.

2.2.9 Definition. A p-family (f_n) is called p-*definable*, if there exists a p-family $(g_n) \in \mathsf{VP}$ and polynomially bounded functions $k, m \colon \mathbb{N} \to \mathbb{N}$ such that for all n:

$$f_n(X_1, \ldots, X_{m(n)}) = \sum_{e \in \{0,1\}^{k(n)}} g_n(X_1, \ldots, X_{m(n)}, e_1, \ldots, e_{k(n)}).$$

The class of all p-definable families is called VNP. ■

Clearly, $\mathsf{VP_e} \subseteq \mathsf{VP_{ws}} = \mathsf{VP_s} \subseteq \mathsf{VP} \subseteq \mathsf{VNP}$.

2.2.10 Theorem ([Val79a]). *The permanent family* (per_n) *is* VNP*-complete.*

From Theorem 2.2.10 it follows that Conjecture 2.2.8 is equivalent to $\mathsf{VP_{ws}} \neq \mathsf{VNP}$.

2.2.11 Remark. If $\mathsf{NP} \not\subseteq \mathsf{P/poly}$, then Valiant's conjecture is true [Bür00a, Ch. 4], assuming the generalized Riemann hypothesis. In a sense this implication gives some hint that it should be easier to prove Valiant's conjecture than proving $\mathsf{NP} \not\subseteq \mathsf{P/poly}$. Moreover, Valiant's conjecture can be defined over any field, not just over the complex numbers, and it is easy to see that $\mathsf{NP} \not\subseteq \mathsf{P/poly}$ implies Valiant's conjecture over any finite field. ∎

2.3 Approximating Polynomials

Instead of attacking Conjecture 2.2.8 directly, Geometric Complexity Theory aims to resolve a slightly different, but more geometric conjecture, which we will present in this section.

The degree d part $\mathbb{C}[X_1,\dots,X_N]_d$ of the polynomial ring $\mathbb{C}[X_1,\dots,X_N]$ is isomorphic to $\mathbb{C}^D$ as a $\mathbb{C}$-vector space, where $D = \binom{N+d-1}{d}$. But $\mathbb{C}^D$ is endowed with a topology induced by the euclidean norm. Via this vector space isomorphism, $\mathbb{C}[X_1,\dots,X_N]_d$ obtains a topology, which we will call the $\mathbb{C}$*-topology*. The $\mathbb{C}$-topology on $\mathbb{C}[X_1,\dots,X_N]$ is defined to be the product topology of its homogeneous components.

2.3.1 Definition. The *approximate determinantal complexity* $\overline{\mathsf{dc}}(f)$ of a polynomial f is defined to be the minimal n such that there exists a sequence of polynomials (g_k) with $\mathsf{dc}(g_k) \leq n$ such that $\lim_{k\to\infty} g_k = f$.

Alternatively, $\overline{\mathsf{dc}}(f)$ is the minimal n such that

$$f \in \overline{\{g \mid \mathsf{dc}(g) \leq n\}},$$

where the bar denotes the topological closure. ∎

Clearly, $\overline{\mathsf{dc}}(f) \leq \mathsf{dc}(f)$ for all polynomials.

2.3.2 Conjecture ([MS01, Conj. 4.3]). $\overline{\mathsf{dc}}(\mathrm{per}_n)$ *is not polynomially bounded.*

Clearly, an answer in the affirmative to Conjecture 2.3.2 implies an answer in the affirmative to Valiant's Conjecture 2.2.8.

2.3.3 Definition. The class $\overline{\mathsf{VP_{ws}}}$ consists of those p-families (f_n) that have polynomially bounded approximate determinantal complexity. ∎

The VNP-completeness of the permanent family (Thm. 2.2.10) implies that Conjecture 2.3.2 is equivalent to $\mathsf{VNP} \not\subseteq \overline{\mathsf{VP_{ws}}}$. See [Bür01, hypothesis (7)] for a slightly weaker conjecture and some implications. We now show that Conjecture 2.3.2 can be rephrased as an orbit closure problem.

2.3.4 Definition. An *action* (X, ρ_X) of a monoid G is a set X endowed with a monoid homomorphism $\rho_X \colon G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ denotes the group of permutations of elements of X. We also say that G *acts on* X *via* ρ_X. For $g \in G$ and $x \in X$ we use the short notation

$$gx := \big(\rho_X(g)\big)(x).$$

∎

The matrix monoid $\mathbb{C}^{n^2 \times n^2}$ acts on $X = \mathbb{C}[X_{1,1}, X_{1,2}, \ldots, X_{n,n}]_n$ by

$$(gf)(x) := f({}^t g x),$$

where ${}^t g$ denotes the transposed matrix of g.

2.3.5 Definition. For an action (X, ρ_X) of a monoid G we define the *orbit* Gx of a point $x \in X$ as

$$Gx := \{gx \mid g \in G\}. \qquad \blacksquare$$

2.3.6 Definition. For $M < n^2$, interpret $\{X_0, X_1, \ldots, X_M\}$ as a subset of $\{X_{1,1}, X_{1,2}, \ldots, X_{n,n}\}$ with $X_i \neq X_j$ for all $i \neq j$. For $m < n$ let $f \in \mathbb{C}[X_1, \ldots, X_M]_m$. The *determinantal orbit closure complexity* $\mathsf{docc}(f)$ of f is defined to be the minimal n such that $z^{n-m} f \in \overline{\mathsf{GL}_{n^2} \det_n}$ for $z := X_0$. $\blacksquare$

We will only need docc for homogeneous polynomials, albeit it can be easily defined for arbitrary polynomials by homogenizing with z.

The starting point of Geometric Complexity Theory is the following proposition.

2.3.7 Proposition. $\overline{\mathsf{dc}}$ *and* docc *are polynomially equivalent.*

Proof. First of all, we look at how to transform a skew circuit C of minimal size which computes $\det_n$ into a skew circuit computing $g\det_n$ for $g \in \mathsf{GL}_{n^2}$. This is done by preceding C by the skew circuit computing the matrix-vector product $X \mapsto {}^t g X$, see Figure 2.3.i. Let $p(n)$ denote the size of this combined circuit. Clearly, $p(n)$ is

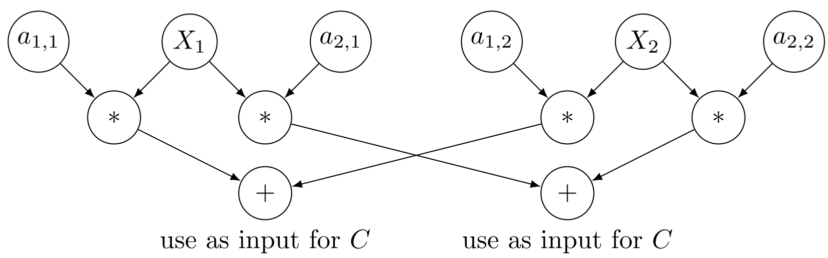


Figure 2.3.i: Preceding a circuit C that has two input gates with the matrix-vector product circuit for a matrix $A = (a_{i,j}) \in \mathbb{C}^{2 \times 2}$.

polynomially bounded, because of Theorem 2.2.2(1).

Fix a homogeneous polynomial f of degree m in M variables.

Let $n := \mathsf{docc}(f)$, so $z^{n-m} f \in \overline{\mathsf{GL}_{n^2} \det_n}$. Hence there exists a sequence $(g_k)_k$ in GL_{n^2} such that $\lim_{k\to\infty} g_k \det_n = z^{n-m} f$. As noted above, $\mathsf{dc}(g_k \det_n) \leq p(n)$. Let $g_k \det_n(z{\leftarrow}1)$ denote the evaluation of $g_k \det_n$ at $z = 1$. Now $\lim_{k\to\infty} \big(g_k \det_n(z{\leftarrow}1)\big) = f$ and also $\mathsf{dc}\big(g_k \det_n(z{\leftarrow}1)\big) \leq p(n)$. Thus $\overline{\mathsf{dc}}(f) \leq p(n)$.

For the other direction let $n := \overline{\mathsf{dc}}(f)$. Hence there exists a sequence f_k of polynomials with $\mathsf{dc}(f_k) \leq n$ such that $\lim_{k\to\infty} f_k = f$. The homogeneous degree m part of f_k is denoted by $f_k^{(m)}$. Note that $\lim_{k\to\infty} f_k^{(m)} = f$. According to Corollary 2.2.6 there exists a polynomial p such that for all k we have $\mathsf{dc}(f_k^{(m)}) \leq p(n)$. Hence each $f_k^{(m)}$ is a projection of the determinant: $f_k^{(m)} = \det(A_k)$ with $A_k \in \big(\{X_1, \ldots, X_M\} \cup \mathbb{C}\big)^{p(n)\times p(n)}$. When replacing all constants c in A_k by cz, $z = X_0$ we obtain new matrices B_k with $z^{p(n)-m} f_k^{(m)} = \det(B_k)$. The entries of B_k are linear forms in the variables X_i and thus we can write $\det(B_k) = g_k \det_{p(n)}$ with $g_k \in \mathbb{C}^{p(n)^2 \times p(n)^2}$. Since $\mathsf{GL}_{p(n)^2 \times p(n)^2}$ lies dense in $\mathbb{C}^{p(n)^2 \times p(n)^2}$ we can choose $\tilde{g}_k \in \mathsf{GL}_{p(n)^2 \times p(n)^2}$ such that both matrices differ only marginally: For the entry α at position i, j in g_k and the entry β at position i, j in $\tilde{g}_k$ we can assume $|\alpha - \beta| \leq \frac{1}{k}$. Recall that f is homogeneous of degree m. Hence $\lim_{k\to\infty} \tilde{g}_k \det_{p(n)} = z^{p(n)-m} f$, which shows $z^{p(n)-m} f \in \overline{\mathsf{GL}_{p(n)^2} \det_{p(n)}}$ and therefore $\mathsf{docc}(f) \leq p(n)$. $\square$

Geometric Complexity Theory studies the orbit closure complexity docc, because it turns out that this is a geometric question that can be analyzed with algebraic geometry and representation theory, see Chapter 3.

2.4 Complexity of Bilinear Maps

In this section we analyze the complexity of bilinear maps, notably matrix multiplication, whose study is our main motivation. The results in this and the following section are mainly due to Strassen, see [Str73] and [Str87]. We additionally recommend Chapter 14 of [BCS97], which serves as an excellent survey.

Let $f\colon \mathbb{C}^{m_1} \to \mathbb{C}^{m_2}$ be a function given by polynomials $f_1, \ldots, f_{m_2}$ via $f(x) = (f_1(x), \ldots, f_{m_2}(x))$. We say that an arithmetic circuit C *computes* f, if

- C has input gates labeled with m_1 indeterminates.
- C has m_2 output gates computing the polynomials f_i, see Figure 2.4.i for an example.

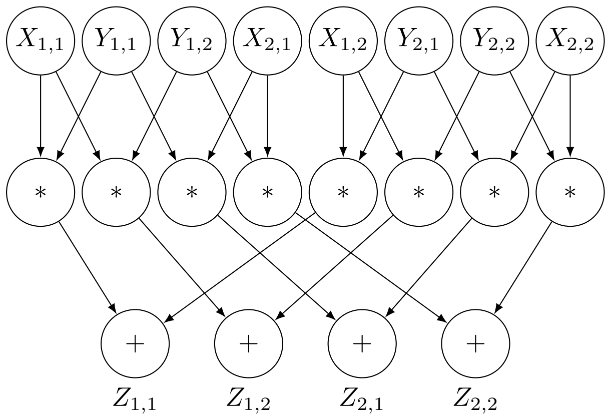


Figure 2.4.i: An arithmetic circuit computing the 2×2 matrix multiplication map $f\colon \mathbb{C}^8 \to \mathbb{C}^4$. This easily generalizes to a family of arithmetic circuits which compute the $m \times m$ matrix multiplication map with m^3 nonscalar multiplications.

Strassen [Str69] realized that there exists an arithmetic circuit which computes the 2×2 matrix multiplication with only 7 instead of 8 nonscalar multiplications. The implications are severe.

2.4.1 Theorem ([Str69])**.** *There exists a family of arithmetic circuits of size $\mathcal{O}(n^{2.81})$ computing the $n \times n$ matrix multiplication.*

Proof. The proof is based on the observation that block matrices can be multiplied blockwise:

$$\begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \cdot \begin{pmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2} \end{pmatrix} = \begin{pmatrix} X_{1,1}Y_{1,1} + X_{1,2}Y_{2,1} & X_{1,1}Y_{1,2} + X_{1,2}Y_{2,2} \\ X_{2,1}Y_{1,1} + X_{2,2}Y_{2,1} & X_{2,1}Y_{1,2} + X_{2,2}Y_{2,2} \end{pmatrix}$$

for matrices $X_{i,j}$ and $Y_{i,j}$, where for all i, j the matrix $X_{i,j}$ has the same format as $Y_{i,j}$. In our setting, X and Y are $2^n \times 2^n$ matrices which have variables as entries. We divide both X and Y into four blocks, each of size $2^{n-1} \times 2^{n-1}$. To calculate the block matrix product we use Strassen's 2×2 matrix multiplication algorithm. The occuring smaller matrix multiplications are handled recursively also with Strassen's algorithm. This gives a circuit of some size $T(2^n)$. By construction we have $T(2^n) \le 7 \cdot T(2^{n-1}) + a \cdot (2^{n-1})^2$, where a is the number of additions in Strassen's circuit. Solving this recursion yields a circuit size of $T(n) = \mathcal{O}(n^{\log_2 7})$. □

In the proof of Theorem 2.4.1 the exponent of the running time was only determined by the number of multiplications, independent of the number of additions. This justifies the following definition.

2.4.2 Definition. The *multiplicative complexity* $L^*(f)$ of a map $f\colon \mathbb{C}^{k_1} \to \mathbb{C}^{k_2}$ is the minimum number of nonscalar multiplications used by an algebraic circuit computing f. ■

Coppersmith and Winograd [CW90] showed that $n \times n$ matrix multiplication can be done with circuits of size $\mathcal{O}(n^{2.376})$, which was recently improved by Stothers [Sto10] to $\mathcal{O}(n^{2.374})$ and Vassilevska Williams [Vas12] to $\mathcal{O}(n^{2.373})$.

2.5 Tensor Rank and Border Rank

Remarks on Dirac's bra-ket notation. In the following sections and in later chapters, in particular in Chapters 4, 6, 7, and 8, we will use Dirac's bra-ket notation, see e.g. [Gri02, Ch. 3 and 6]. Elementary properties of tensors can be found in [Nor84, Ch. 1 and 2].

Let V be a finite dimensional complex vector space with Hermitian inner product $\langle , \rangle$. Let V^* denote the dual vector space of V consisting of linear forms $V \to \mathbb{C}$. The map $V \to V^*, v \mapsto \langle v, \cdot\rangle =: v^*$ is antilinear. The vector v^* is called the *adjoint* of v. We write elements of V as $|v\rangle$ and define $\langle v| := v^*$. Contraction of tensors has a short notation: $\langle f|v\rangle := \langle f|(|v\rangle)$. The composition $(\langle f|\circ g)(|v\rangle)$ with $g \in \mathrm{End}(\mathbb{C}^N)$ is written as $\langle f|g|v\rangle$. We write $\bigotimes^d V := \underbrace{V \otimes V \otimes \cdots \otimes V}_{d \text{ factors}}$.

If $V = \mathbb{C}^N$ with standard basis and standard Hermitian inner product, then elements $v \in V$ are given by column vectors and v^* is the row vector obtained by transposition and complex conjugation. The composition $\langle f|g|v\rangle$ and can now be interpreted as a vector-matrix-vector product.

The standard basis vectors of $\mathbb{C}^N$ are written as $|1\rangle, |2\rangle$ etc. with dual basis vectors $\langle 1|, \langle 2|$. We abbreviate the tensor product notation $|ij\rangle := |i\rangle \otimes |j\rangle$, whose dual is $\langle ij|$. We can also have more than two tensor factors in this notation, so e.g. $|11212\rangle$ is an element of $\bigotimes^5 \mathbb{C}^2$.

Sometimes we write the basis vectors of $\mathbb{C}^{N^2}$ as $|(i,j)\rangle$, where $1 \le i,j \le N$ with dual vectors $\langle (i,j)|$. Their tensor product is analogously abbreviated as $|(i,j)(k,l)\rangle := |(i,j)\rangle \otimes |(k,l)\rangle$.

This completes our notational remarks. ■

Let $W := \mathbb{C}^M$. There is a natural linear isomorphism

$$\begin{aligned} \Phi\colon W^* \otimes W^* \otimes W &\to \{f \mid f\colon W \times W \to W \text{ bilinear}\} \\ \varphi \otimes \psi \otimes w &\mapsto ((u,v) \mapsto \varphi(u) \cdot \psi(v) \cdot w) \end{aligned}$$

Choosing bases, we can identify W with its dual W^* and can interpret the matrix multiplication map as a tensor in $\bigotimes^3 W := W \otimes W \otimes W$ as follows.

2.5.1 Main Example. The vector space $\mathbb{C}^{m^2}$ has a basis $|(i,j)\rangle$, $1 \le i,j \le m$. The tensor in $V := \bigotimes^3 \mathbb{C}^{m^2}$ corresponding to the $m \times m$ matrix multiplication map is the following:

$$\mathcal{M}_m := \sum_{i,j,k=1}^{m} |(i,j)\rangle \otimes |(j,k)\rangle \otimes |(k,i)\rangle.$$

■

A way to measure complexity of elements in $\bigotimes^3\mathbb{C}^M$ is the *tensor rank*, which is defined as follows.

2.5.2 Definition. The tensor rank $R(v)$ of $v \in \bigotimes^3\mathbb{C}^M$ is defined as the minimum $n \in \mathbb{N}$ such that there is a representation

$$v = \sum_{i=1}^{n} w_{1,i} \otimes w_{2,i} \otimes w_{3,i}, \quad \text{where all } w_{j,i} \in \mathbb{C}^M.$$

■

Implicitly using the isomorphism Φ, we can speak of the rank of bilinear maps. Tensor rank is a generalization of the well-known rank of matrices, but computing the tensor rank is **NP**-hard [Hås90]. The following lemma shows that there is an extremely tight relation, much stronger than mere polynomial equivalence, between rank and multiplicative complexity.

2.5.3 Lemma ([Str73, §4], see also [BCS97, 14.8])**.** *For arbitrary bilinear maps f we have $L^*(f) \leq R(f) \leq 2 \cdot L^*(f)$.*

In terms of tensor rank, Strassen's algorithm for 2×2 matrix multiplication can be reinterpreted as follows.

2.5.4 Proposition. $R(\mathcal{M}_2) \leq 7$.

Proof. Let $\{|(0,0)\rangle, |(0,1)\rangle, |(1,0)\rangle, |(1,1)\rangle\}$ denote the standard basis of $\mathbb{C}^4$.

$$\begin{aligned}
\mathcal{M}_2 &= \sum_{i,j,k=0}^{1} |(i,j)\rangle|(j,k)\rangle|(k,i)\rangle \\
&= |-(0,1)+(1,0)+(1,1)\rangle \otimes |(0,0)+(0,1)-(1,0)\rangle \otimes |(0,0)+(1,1)\rangle \\
&+ |(0,0)+(0,1)-(1,0)-(1,1)\rangle \otimes |(0,0)+(0,1)\rangle \otimes |(0,0)\rangle \\
&+ |-(1,0)\rangle \otimes |(0,0)-(1,0)\rangle \otimes |-(0,1)+(1,1)\rangle \\
&+ |(0,1)\rangle \otimes |(1,1)\rangle \otimes |(1,0)+(1,1)\rangle \\
&+ |-(0,1)+(1,1)\rangle \otimes |-(0,0)-(0,1)+(1,0)+(1,1)\rangle \otimes |(1,1)\rangle \\
&+ |(0,0)\rangle \otimes |-(0,1)\rangle \otimes |(0,0)-(1,0)\rangle \\
&+ |(1,0)+(1,1)\rangle \otimes |(1,0)\rangle \otimes |(0,0)+(0,1)\rangle,
\end{aligned}$$

as one checks easily via multilinearity of the tensor product. Here we used the short notation $|(i,j)+(k,l)\rangle := |(i,j)\rangle + |(k,l)\rangle$. □

It is known that $R(\mathcal{M}_2) = 7$, as discovered in [HK71] and [Win71].

In Section 2.2 we linked skew complexity to projections of determinants. Here we will do something similar: We will link (see Thm. 2.5.7) the rank of a tensor to *restrictions* of the so-called *unit tensor*, defined below.

Let $V_n := \bigotimes^3\mathbb{C}^n$ and let $\{|i\rangle\}_{1\leq i\leq n}$ denote a basis of $\mathbb{C}^n$.

2.5.5 Definition. The tensor

$$\mathcal{E}_n := \sum_{i=1}^{n} |i\rangle \otimes |i\rangle \otimes |i\rangle \in V_n$$

is called the *n-th unit tensor*. ■

2.5.6 Definition. A tensor $v \in V_M$ is called a *restriction* of $\mathcal{E}_n$, written $v \leq \mathcal{E}_n$, if there exist linear maps g_1, g_2 and $g_3 \colon \mathbb{C}^n \to \mathbb{C}^M$ such that $v = (g_1 \otimes g_2 \otimes g_3)\mathcal{E}_n$. ■

The following characterization by Strassen brings together rank and restrictions of the unit tensor.

2.5.7 Theorem ([Str87]). *For all $v \in V_M$ we have $R(v) \leq n \Leftrightarrow v \leq \mathcal{E}_n$.*

Proof. The following three statements are equivalent: $R(v) \leq n \iff$ there exist $w_{j,i} \in \mathbb{C}^M$, $j = 1, \dots, 3$, $i = 1, \dots, n$ with $v = \sum_{i=1}^{n} w_{1,i} \otimes w_{2,i} \otimes w_{3,i} \iff$ there exist linear maps $g_1, g_2, g_3 : \mathbb{C}^n \to \mathbb{C}^M$ with $g_j(|i\rangle) = w_{j,i}$ such that $v = \sum_{i=1}^{r} g_1(|i\rangle) \otimes g_2(|i\rangle) \otimes g_3(|i\rangle)$. But the latter means $(g_1 \otimes g_2 \otimes g_3)\mathcal{E}_n = v$. □

Although $\mathcal{E}_n$ depends on the choice of basis for $\mathbb{C}^n$, from Theorem 2.5.7 it follows that the set of projections of $\mathcal{E}_n$ is independent of the choice of basis.

From the viewpoint of Geometric Complexity Theory the situation is similar to the one in Section 2.3: It turns out that the *approximate* version of rank, the so-called *border rank*, can be studied with methods from algebraic geometry and representation theory.

2.5.8 Definition. For a tensor $v \in V_M$, the *border rank* $\underline{R}(v)$ is defined as the smallest n such that there exists a sequence (v_k) of tensors in V_M with $\lim_{k\to\infty} v_k = v$ and $R(v_k) \leq n$.

Equivalently, the border rank is the smallest n such that $v \in \overline{\{w \in V_M \mid R(w) \leq n\}}$. ■

Clearly, we always have $\underline{R}(v) \leq R(v)$.

2.5.9 Example. We consider the tensor corresponding to 2×2-matrix multiplication where the lower right entry of the left matrix is 0, i.e., the following tensor (using the same syntax as in Proposition 2.5.4):

$$\begin{aligned} ◰ \times \square := &|(0,0)(0,0)(0,0)\rangle + |(0,1)(1,0)(0,0)\rangle + |(0,1)(1,1)(1,0)\rangle \\ &+ |(0,0)(0,1)(1,0)\rangle + |(1,0)(0,0)(0,1)\rangle + |(1,0)(0,1)(1,1)\rangle. \end{aligned}$$

It is known that $R(◰ \times \square) = 6$, cf. [BCS97, Ex. 17.10]. But $\underline{R}(◰ \times \square) \leq 5$, which can be seen as follows (cf. [BCS97, p. 378]):

$$\begin{aligned} |v_\varepsilon\rangle := &|(0,1) + \varepsilon(0,0)\rangle && \otimes|(0,1) + \varepsilon(1,1)\rangle && \otimes|(1,0)\rangle \\ &+|(1,0) + \varepsilon(0,0)\rangle && \otimes|(0,0)\rangle && \otimes|(0,0) + \varepsilon(0,1)\rangle \\ &-|(0,1)\rangle && \otimes|(0,1)\rangle && \otimes|(0,0) + (1,0) + \varepsilon(1,1)\rangle \\ &-|(1,0)\rangle && \otimes|(0,0) + (0,1) + \varepsilon(1,0)\rangle && \otimes|(0,0)\rangle \\ &+|(0,1) + (1,0)\rangle && \otimes|(0,1) + \varepsilon(1,0)\rangle && \otimes|(0,0) + \varepsilon(1,1)\rangle, \end{aligned}$$

where we used the same notation as in the proof of Proposition 2.5.4. Note that $R(|v_\varepsilon\rangle) \leq 5$ and that $\lim_{\varepsilon\to 0} \varepsilon^{-1}|v_\varepsilon\rangle = ◰ \times \square$. Hence $\underline{R}(◰ \times \square) \leq 5$. ■

2.5.10 Remark. Until 2011 the best known lower bounds for rank and border rank of matrix multiplication were as follows: Bläser [Blä99] showed $R(\mathcal{M}_n) \geq \frac{5}{2}n^2 - 3n$ and Lickteig [Lic84] showed $\underline{R}(\mathcal{M}_n) \geq \frac{3}{2}n^2 + \frac{1}{2}n - 1$. Very recently, Landsberg & Ottaviani and Landsberg improved on both results: [LO11] show $\underline{R}(\mathcal{M}_n) \geq 2n^2 - n$, and [Lan12] shows $R(\mathcal{M}_n) \geq 3n^2 - 4n^{\frac{3}{2}} + n$. ■

We now will pose the question of determining the border rank of a specific tensor as an orbit closure problem. Recall Definitions 2.3.4 and 2.3.5.

The group $G_n := \mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$ acts on the space $V_n := \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ via

$$(g_1, g_2, g_3) \cdot (v_1 \otimes v_2 \otimes v_3) := (g_1 v_1) \otimes (g_2 v_2) \otimes (g_3 v_3).$$

We can embed $\mathbb{C}^M \subseteq \mathbb{C}^n$ if $M \leq n$ and get an embedding $\iota_M^n : V_M \hookrightarrow V_n$.

2.5.11 Definition. The *unit tensor orbit closure complexity* $\mathsf{utocc}(v)$ of a tensor $v \in V_M$ is the minimal n such that $\iota_M^n(v) \in \overline{G_n \mathcal{E}_n}$. ■

We will not make much use of the term "utocc" because of the following Theorem 2.5.12 that shows that $\underline{R}$ and utocc are not just polynomially equivalent, but equal. Note that $|222\rangle \in V_2$ has rank 1, but is not contained in $\overline{G_1 \mathcal{E}_1}$. Therefore we need a technical restriction.

2.5.12 Theorem ([Str87]). *If for $v \in V_M$ we have $\underline{R}(v) \geq M$, then $\underline{R}(v) = \mathsf{utocc}(v)$.*

Proof. Fix $v \in V_M$. For all $n \geq M$ we prove: $\underline{R}(v) \leq n \Leftrightarrow \iota_M^n(v) \in \overline{G_n \mathcal{E}_n}$.

First, assume $\iota_M^n(v) \in \overline{G_n \mathcal{E}_n}$. Hence $\iota_M^n(v) = \lim_{k\to\infty} g_k \mathcal{E}_n$ with all $g_k \in G_n$. To get rid of the embedding, let $p : V_n \to V_M$ denote a linear projection onto V_M such that $p \circ \iota_M^n = \mathrm{id}$. Then $v = \lim_{k\to\infty}(p \circ g_k)\mathcal{E}_n$. But $R\big((p \circ g_k)\mathcal{E}_n\big) \leq n$ and hence $\underline{R}(v) \leq n$.

Now let $\underline{R}(v) \leq n$. It follows that $v = \lim_{k\to\infty} v_k$ with $R(v_k) \leq n$. By Theorem 2.5.7 this means that $v_k \leq \mathcal{E}_n$, i.e., $v_k = g_k \mathcal{E}_n$ with $g_k = g_k^{(1)} \otimes g_k^{(2)} \otimes g_k^{(3)}$, where $g_k^{(i)} \in \mathbb{C}^{M\times n}$. Since $M \leq n$, we can embed $\mathbb{C}^{M\times n} \subseteq \mathbb{C}^{n\times n}$ by embedding $M \times n$ submatrices into the first M rows, filling the remaining $n-M$ rows with zeros. The image of $g_k^{(i)}$ under this embedding shall be called ${g'}_k^{(i)}$. Note that, as linear maps, we have ${g'}_k^{(i)} = \iota_M^n \circ g_k^{(i)}$. We can approximate ${g'}_k^{(i)} \in \mathbb{C}^{n\times n}$ componentwise by $\tilde{g}_k^{(i)} \in \mathsf{GL}_n$ such that each matrix entry in ${g'}_k^{(i)}$ differs from the corresponding entry in $\tilde{g}_k^{(i)}$ by at most $\frac{1}{k}$. Let $\tilde{g}_k := \tilde{g}_k^{(1)} \otimes \tilde{g}_k^{(2)} \otimes \tilde{g}_k^{(3)}$. Hence $\iota_M^n(v) = \lim_{k\to\infty} \tilde{g}_k \mathcal{E}_n$. Therefore $\iota_M^n(v) \in \overline{G_n \mathcal{E}_n}$. □

The orbit closures $\overline{G_n \mathcal{E}_n}$ are also called *(higher) Secant varieties of Segre varieties* and have been the focus of many research attempts, see for example [BCS97, Ch. 20] or [Lan11] for a survey.

2.6 Summary and Unifying Notation

Our goal is to analyze the approximate determinantal complexity of polynomials and the border rank of (matrix multiplication) tensors. Both situations have much in common and we strive to treat both cases simultaneously as often as possible. For fixed n and m we want to use the following notation.

notation	determinantal complexity ($n \geq m+1$)	border rank ($n \geq m^2$)
G	GL_{n^2}	$\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$
V	$\mathsf{Sym}^n \mathbb{C}^{n^2}$	$\bigotimes^3 \mathbb{C}^n$
$\eta = \dim V$	$\binom{n^2+n-1}{n}$	n^3
$W \subseteq V$	$\mathsf{Sym}^n \mathbb{C}^{m^2+1}$	$\bigotimes^3 \mathbb{C}^{m^2}$
$\mathfrak{h} := \mathfrak{h}_{m,n} \in W$	$z^{n-m}\mathrm{per}_m$	$\mathcal{M}_m$
$\mathfrak{c} := \mathfrak{c}_n \in V$	$\det_n$	$\mathcal{E}_n$

Here, $\mathsf{Sym}^n \mathbb{C}^{n^2}$ denotes the homogeneous part of degree n of the polynomial ring $\mathbb{C}[X_1, \ldots, X_{n^2}]$ and $\mathsf{Sym}^n \mathbb{C}^{m^2+1}$ is defined analogously in m^2+1 variables. The point $\mathfrak{h}$ resembles the hard problem for which we want to prove lower bounds and the orbit closure $\overline{G\mathfrak{c}_n}$ is exactly the set of all points with complexity at most n. In later chapters we will refer to these notations as the *tensor scenario* or the *polynomial scenario*, respectively. In both scenarios, for a given m, we try to find n as large as possible such that

$$\mathfrak{h}_{m,n} \notin \overline{G\mathfrak{c}_n}.$$

Since the orbit closure is the smallest closed set that contains the orbit, this is equivalent to proving

$$\overline{G\mathfrak{h}_{m,n}} \nsubseteq \overline{G\mathfrak{c}_n}.$$

If we want to treat $G\mathfrak{c}$ and $G\mathfrak{h}$ simultaneously, we just write Gv, as in the following chapter.

Chapter 3

Preliminaries: The Flip via Obstructions

In this chapter we see that our objects of study — the orbit closures $\overline{Gv}$ — have the additional crucial property of being *Zariski closed*, which enables their study with tools from algebraic geometry. This is the prime reason the for studying $\overline{\mathsf{dc}}$ instead of dc and border rank instead of rank. Moreover, we see that the coordinate ring $\mathbb{C}[\overline{Gv}]$ of $\overline{Gv}$ is a graded $\mathbb{C}$-algebra and that each homogeneous component $\mathbb{C}[\overline{Gv}]_d$ is a G*-representation*, which enables us to define *representation theoretic obstructions*.

The content of this chapter is classical and well-known to the experts. Many statements hold in far greater generality, as the expert will easily recognize.

3.1 Classical Algebraic Geometry

For very good introductions to quasi-affine varieties, see e.g. [Sha94, Sec. I.2], [TY05, Ch. 11 and 12], or [Har77, Ch. 1].

We call the standard topology on V, which is induced by the euclidean norm, the $\mathbb{C}$*-topology on* V. We call the closure of a subset $X \subseteq V$ w.r.t. the $\mathbb{C}$-topology the $\mathbb{C}$*-closure of* X. We now present a different topology, the *Zariski topology*, and highlight their relationship with the $\mathbb{C}$-topology.

3.1.1 Definition. The set $\mathbb{A}^\eta := \mathbb{C}^\eta$ endowed with the following *Zariski topology* is called the η*-dimensional affine space*: A subset $X \subseteq \mathbb{A}^\eta$ is defined to be *closed*, iff there exists $r \in \mathbb{N}$ and polynomials $f_1, \ldots, f_r \in \mathbb{C}[T_1, \ldots, T_\eta]$ such that

$$x \in X \;\Leftrightarrow\; \forall 1 \le i \le r : f_i(x) = 0.$$

We say that $f_1, \ldots, f_r$ *cut out* X. A subset $U \subseteq \mathbb{A}^\eta$ is called *open*, iff $\mathbb{A}^\eta \setminus U$ is closed. ■

From now on we use the short notation $V := \mathbb{A}^\eta$ and $\mathbb{C}[V] := \mathbb{C}[T_1, \ldots, T_\eta]$. All topological notions shall refer to the Zariski topology unless otherwise stated.

It is easy to see that finite unions of closed sets are closed: If we take two closed subsets $X, Y \subseteq V$ cut out by $\{f_1, \ldots, f_r\}$ and $\{g_1, \ldots, g_s\}$, respectively, then the union $X \cup Y$ is cut out by the set of products $\{f_i \cdot g_j \mid 1 \le i \le r, 1 \le j \le s\}$.

For a subset $X \subseteq \mathbb{A}^\eta$ we define the *vanishing ideal*

$$I(X) := \{f \in \mathbb{C}[V] \mid \forall x \in X : f(x) = 0\}.$$

The celebrated Hilbert's basis theorem (e.g. [AM69, Thm. 7.5]) states that all ideals of $\mathbb{C}[V]$ are finitely generated and hence we can equivalently define a subset $X \subseteq V$ to be closed iff there exists an ideal $I \subseteq \mathbb{C}[V]$ such that $x \in X \Leftrightarrow \forall f \in I : f(x) = 0$. This implies that infinite intersections of closed sets are again closed: If $\{X_i\}_i$ is a set of closed subsets of V, then the vanishing ideal $I(\cap_i X_i)$ is the sum of the vanishing ideals $I(\cap_i X_i) = \sum_i I(X_i)$.

The above properties imply that Definition 3.1.1 satisfies the axioms of a topological space. The following easy lemma states that the $\mathbb{C}$-topology is a refinement of the Zariski topology.

3.1.2 Lemma. *If $X \subseteq V$ is Zariski closed, then X is closed in the $\mathbb{C}$-topology.*

Proof. X is Zariski closed $\Leftrightarrow$ $X = \{x \mid f_1(x) = \cdots = f_r(x) = 0\} = f_1^{-1}(0) \cap \ldots \cap f_r^{-1}(0)$, which is closed in the $\mathbb{C}$-topology, because the f_i are polynomials and hence continuous in the $\mathbb{C}$-topology and $\{0\}$ is closed in the $\mathbb{C}$-topology. □

We remark that all $\mathbb{C}$-closed sets that we analyze will also be Zariski-closed, see the upcoming Theorem 3.2.6.

The *closure* $\overline{U}$ of a subset $U \subseteq V$ is defined to be the intersection of all closed sets in V containing U, or equivalently as the smallest closed set in V which contains U. We remark that $\overline{U}$ is cut out by the vanishing ideal $I(U)$.

3.1.3 Definition. A subset $X \subseteq V$ is called *locally closed*, if X is the intersection of an open and a closed set. We also call locally closed sets *quasi-affine varieties*. ■

3.1.4 Example. The subset $\mathsf{GL}_N \subseteq \mathbb{A}^{N^2}$ is defined as the complement of the set cut out by the determinant polynomial $\det\colon \mathbb{A}^{N^2} \to \mathbb{C}$ and thus carries the structure of a quasi-affine variety. ■

3.1.5 Definition. A *regular function* f on a quasi-affine variety X is a function $f\colon X \to \mathbb{C}$ defined on the whole of X as follows: There exist finitely many fractions of polynomials $\frac{f_i}{g_i}$ with $f_i, g_i \in \mathbb{C}[V]$ such that

$$\forall x \in X \ \exists i : g_i(x) \neq 0$$

and

$$\forall x \in X \ \forall i : \text{either } g_i(x) = 0 \text{ or } \frac{f_i(x)}{g_i(x)} = f(x).$$ ■

3.1.6 Example. Due to the far-reaching consequences of Definition 3.1.5, we provide a nontrivial and hopefully illuminating example, a *glued double cusp*, shown to me by Prof. Dr. Eike Lau.

Let the closed set $X \subseteq \mathbb{A}^5$ be cut out by the polynomials $T_1^3 - T_2^2$, $T_3^3 - T_4^2$, $T_1T_3 - T_5^2$, and $T_2T_4 - T_5^3$. Since $\{0\} \subseteq X$ is closed, the set $X \setminus \{0\}$ is a quasi-affine variety. One can check that $X \setminus \{0\}$ is parametrized by two variables as follows:

$$\begin{aligned} X \setminus \{0\} = \{(t_1, t_2, t_3, t_4, t_5) \mid{} & \alpha, \beta \in \mathbb{C}, \alpha \neq 0 \text{ or } \beta \neq 0, \\ & t_1 = \alpha^2, t_2 = \alpha^3, t_3 = \beta^2, t_4 = \beta^3, t_5 = \alpha\beta\}. \end{aligned}$$

Now consider the following regular function defined by two fractions of polynomials:

$$f = \frac{t_5 t_1}{t_2} = \frac{t_4}{t_3},$$

whose value is just β in the above syntax. Although f is defined on the whole $X \setminus \{0\}$, the two fractions of polynomials are not, because their denominators both have zeros in $X \setminus \{0\}$. In fact, one can show that f cannot be written as a single fraction of polynomials. ■

3.1.7 Definition. A *regular map* from a quasi-affine variety $X \subseteq V$ to $\mathbb{A}^\tau$ is a map $f: X \to \mathbb{A}^\tau$ given by a regular function in each of the τ coordinates. ■

3.1.8 Example. The inversion map $\mathsf{GL}_N \to \mathsf{GL}_N$, $g \mapsto g^{-1}$ is a regular map, because $g^{-1} = \frac{1}{\det(g)}\mathrm{Adj}(g)$, where $\mathrm{Adj}(g)$ is the adjoint matrix (sometimes also called the adjugate matrix), see e.g. [HJ85, 0.8.2]. ■

We remark that regular maps are continuous in the Zariski topology and in the $\mathbb{C}$-topology and they form the morphisms in the category of quasi-affine varieties. It follows that the composition of regular maps is again regular.

The next proposition shows that regular maps from closed sets can be defined by polynomials instead of fractions of polynomials and that additionally only a single polynomial is needed for each coordinate.

3.1.9 Proposition (e.g. [Bro89, 6.14])**.** *Regular maps from closed sets $X \subseteq V$ to $\mathbb{A}^\tau$ are given by τ restrictions of polynomials from $\mathbb{C}[V]$.*

Since regular functions are just regular maps to $\mathbb{A}^1$, each regular function on a closed set $X \subseteq V$ is given by a single restriction of a polynomial.

3.1.10 Definition (Coordinate ring)**.** The set $\mathbb{C}[X]$ of regular functions on a quasi-affine variety X forms a $\mathbb{C}$-algebra, called the *coordinate ring of X*. Addition and multiplication are defined pointwise. ■

From Proposition 3.1.9 we get the following corollary.

3.1.11 Corollary. *For a closed set $X \subseteq V$ we have $\mathbb{C}[X] \simeq \mathbb{C}[V]/I(X)$.*

We will study coordinate rings with the help of representation theory in Section 3.3.

We remark that we always study quasi-affine varieties together with their specific embedding. Indeed, often the *closure* of a quasi-affine variety in the surrounding space is the more interesting object in Geometric Complexity Theory. This follows from the considerations in Chapter 2 and the fact that in all cases that we study, Zariski closure and $\mathbb{C}$-closure coincide, see the forthcoming Theorem 3.2.6.

3.2 Linear Algebraic Groups and Polynomial Obstructions

In order to define the term *linear algebraic group* we need the following easy topological statement, Claim 3.2.1.

We have a bijection map

$$\iota : \mathbb{A}^{\tau_1} \times \mathbb{A}^{\tau_2} \to \mathbb{A}^{\tau_1+\tau_2},\ (x,y) \mapsto (x_1,\dots,x_{\tau_1},y_1,\dots,y_{\tau_2}).$$

The following claim, whose straightforward proof we omit, endows the product of two quasi-affine varieties with the structure of a quasi-affine variety.

3.2.1 Claim. *Given two quasi-affine varieties $X \subseteq \mathbb{A}^{\tau_1}$ and $Y \subseteq \mathbb{A}^{\tau_2}$, the image $\iota(X,Y) \subseteq \mathbb{A}^{\tau_1+\tau_2}$ is again a quasi-affine variety.*

Abusing notation, for two quasi-affine varieties $X \subseteq \mathbb{A}^{\tau_1}$ and $Y \subseteq \mathbb{A}^{\tau_2}$, we denote by $X \times Y$ the quasi-affine variety $\iota(X,Y) \subseteq \mathbb{A}^{\tau_1+\tau_2}$.

3.2.2 Example. The set $\mathsf{GL}_N \times \mathsf{GL}_N$ is a quasi-affine variety. The multiplication map $\mathsf{GL}_N \times \mathsf{GL}_N \to \mathsf{GL}_N$, $(g_1,g_2) \mapsto g_1g_2$ is a regular map, given by polynomials in each coordinate. ■

We can now turn to two fundamental definitions.

3.2.3 Definition. A subgroup of GL_N that is a closed subset of $\mathbb{A}^{N^2}$ is called a *linear algebraic group*. ■

Since the set $\mathsf{GL}_N \subseteq \mathbb{A}^{N^2}$ is open, it follows that every linear algebraic group is a quasi-affine variety. It also follows that for every linear algebraic group G the multiplication and inversions maps are regular maps.

For example $\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$ is a linear algebraic group if we interpret it as a subgroup of the group GL_{3n} via the block diagonal embedding.

3.2.4 Definition. A quasi-affine variety X with an action of a linear algebraic group G is called a *G-variety*, if the action map $G \times X \to X$ is a regular map. ■

We note that if we fix $g \in G$ for a G-variety X, then the map $X \to X$, $x \mapsto gx$ is regular. We further note that in both scenarios from Section 2.6, the affine space V is a G-variety.

The following crucial theorem endows orbits of regular group actions with the structure of a quasi-affine variety.

3.2.5 Theorem (see e.g [Kra85, II.2.2 c])**.** *Given an algebraic group G and an action of G on some affine space V such that V is a G-variety. For $v \in V$ the orbit Gv is locally closed.*

The action of G on V restricts to the orbit Gv and hence makes Gv a G-variety. To prove that $\overline{Gv}$ is also a G-variety it remains to show that $gx \in \overline{Gv}$ for all $g \in G$, $x \in \overline{Gv}$. If $x \in \overline{Gv}$, then there exists a sequence (x_k) with $x_k \in Gv$ such that $\lim_{k\to\infty} x_k = x$. But $\lim_{k\to\infty} gx_k = g \lim_{k\to\infty} x_k = gx$ by continuity of the G-action.

The following Theorem 3.2.6 lets us draw the connection between algebraic geometry and complexity theory.

A subset $X \subseteq V$ is called *constructible*, if X is a finite union of locally closed sets.

3.2.6 Theorem ([Kra85, AI.7.2 Folgerung])**.** *For constructible sets, Zariski closure and $\mathbb{C}$-closure coincide.*

Theorem 3.2.6 implies that the $\mathbb{C}$-closure $\overline{Gv}$ is also Zariski closed. In particular the orbit closures in Section 2.6 are Zariski closed, according to Theorem 3.2.5.

Polynomial Obstructions. We can draw the following crucial conclusion that states that polynomials can be used to separate points $\mathfrak{h} \in V$ from $\overline{G\mathfrak{c}} \subseteq V$.

3.2.7 Corollary. *Let $\mathfrak{h} \in V$. If $\mathfrak{h} \notin \overline{G\mathfrak{c}} \subseteq V$, then there exists a polynomial $f \in \mathbb{C}[V]$ that vanishes on $\overline{G\mathfrak{c}}$ but not on $\mathfrak{h}$.*

Proof. The following holds for any closed set $X \subseteq V$. Let X be cut out by $f_1, \dots, f_r$, i.e., $x \in X \Leftrightarrow f_1(x) = \cdots = f_r(x) = 0$. Then $x \notin X \Leftrightarrow \exists 1 \le i \le r : f_i(x) \neq 0$. □

3.2.8 Definition (Polynomial Obstruction)**.** We call the polynomials from Corollary 3.2.7 that separate $\mathfrak{h}$ from $\overline{G\mathfrak{c}}$ by satisfying $f(\overline{G\mathfrak{c}}) = 0$ and $f(\mathfrak{h}) \neq 0$ *polynomial obstructions.* ■

Corollary 3.2.7 states that polynomial obstructions are *guaranteed to exist* if $\mathfrak{h} \notin \overline{G\mathfrak{c}}$.

One major questions is, whether there are short encodings of such polynomials f and if there are short proofs that f is an obstruction.

The Flip. Although it is clear from the viewpoint of algebraic geometry, we want to emphasize that Definition 3.2.8 enables us to perform the so-called *flip*: We can prove the *absence* of efficient algorithms for computational problems by proving the *existence* of obstructions. And these obstructions are guaranteed to exist if and only if there are no efficient algorithms. ■

We want to study polynomial obstructions in a structured way. If we look at the projection map from $\mathbb{C}[V]$ to $\mathbb{C}[V]/I(G\mathfrak{c}) \simeq \mathbb{C}[\overline{G\mathfrak{c}}]$, then a polynomial vanishing on $\overline{G\mathfrak{c}}$ is the zero polynomial in $\mathbb{C}[\overline{G\mathfrak{c}}]$. Our search for a polynomial obstruction is therefore equivalent to searching for a polynomial in the vanishing ideal $I(G\mathfrak{c})$ which does not vanish on $\mathfrak{h}$. The coordinate ring $\mathbb{C}[\overline{G\mathfrak{c}}]$ is a graded ring (see Prop. 3.3.8) and each homogeneous component is a G-representation as defined Section 3.3. We use representation theory for its study.

3.3 Representation Theoretic Obstructions

A *representation* $(\mathscr{V}, \rho_{\mathscr{V}})$ *of a group* G is a finite dimensional complex vector space $\mathscr{V}$ endowed with a group homomorphism $\rho_{\mathscr{V}} : G \to \mathsf{GL}(\mathscr{V})$. We omit $\rho_{\mathscr{V}}$ if it is clear which morphism is meant. For $v \in \mathscr{V}$ we use the short notation

$$gv := \big(\rho_{\mathscr{V}}(g)\big)(v).$$

A subspace $\mathscr{W}$ of $\mathscr{V}$ that satisfies

$$\forall w \in \mathscr{W} : Gw \subseteq \mathscr{W}$$

is called a *subrepresentation of* G. Each subrepresentation is again a representation.

If $G \subseteq \mathbb{A}^{N^2}$ is a linear algebraic group, then we call $(\mathscr{V}, \rho_{\mathscr{V}})$ a *rational representation*, if $\rho_{\mathscr{V}}$ is a regular map when regarding $\mathscr{V}$ as an affine space. A special case are the *polynomial representations*, where we require that $\rho_{\mathscr{V}}$ is the restriction of a regular function from the affine space $\mathbb{A}^{N^2}$, which means that all entries in the representation matrices $\rho_{\mathscr{V}}(g)$ are given by polynomials in the entries of g.

We want to introduce our prime example here, which are homogeneous components of coordinate rings of G-varieties.

3.3.1 Claim. *Let X be a G-variety. Then G acts on $\mathbb{C}[X]$ via $(gf)(x) := f(g^{-1}x)$. Let $\mathbb{C}[X]$ have a grading that is respected by the action of G. Then every homogeneous component $\mathbb{C}[X]_d$ is a rational G-representation.*

Proof. We first note that $gf \in \mathbb{C}[X]$, because $x \mapsto f(g^{-1}x)$ is the composition of the regular map $x \mapsto g^{-1}x$ and the regular function f, and is hence also regular.

For $g, \tilde{g} \in G$ and $f \in \mathbb{C}[X]_d$ we have

$$((g\tilde{g})f)(x) = f(\tilde{g}^{-1}g^{-1}x) = (\tilde{g}f)(g^{-1}x) = (g(\tilde{g}f))(x),$$

hence $g \mapsto (f \mapsto gf)$ is a group homomorphism from G to $\mathsf{GL}(\mathbb{C}[X]_d)$. □

For every representation $\mathscr{V}$, the zero vector space and $\mathscr{V}$ itself are two subrepresentations. If a representation $\mathscr{V}$ has only these two subrepresentations, then $\mathscr{V}$ is called *irreducible*. Given two representations $(\mathscr{V}, \rho_{\mathscr{V}})$ and $(\mathscr{W}, \rho_{\mathscr{W}})$ of G, then a linear map $\varphi : \mathscr{V} \to \mathscr{W}$ is called a *G-morphism* or G-equivariant, if

$$\forall g \in G, \forall v \in \mathscr{V} : g\varphi(v) = \varphi(gv),$$

which means $\forall g \in G, \forall v \in \mathscr{V} : \big(\rho_{\mathscr{W}}(g)\big)\big(\varphi(v)\big) = \varphi\big(\big(\rho_{\mathscr{V}}(g)\big)(v)\big)$. If φ is a G-morphism and additionally an isomorphism of vector spaces, then φ is called a G-isomorphism and the representations $\mathscr{V}$ and $\mathscr{W}$ are called *isomorphic*. The inverse map φ^{-1} is again a G-isomorphism. The kernel and the image of every G-morphism are always subrepresentations. Subrepresentations of rational representations are rational. If $\mathscr{V}$ is a representation of a group G, then the dual space $\mathscr{V}^*$ is a G-representation again via

$$(gf)(v) := f(g^{-1}v) \quad \text{for all } f \in \mathscr{V}^*,\ g \in G,\ v \in \mathscr{V}. \tag{3.3.2}$$

This representation is called the *dual representation* or *contragredient representation*. The following theorem, known as Schur's lemma, is easy to prove, but plays a prominent role in Geometric Complexity Theory.

3.3.3 Lemma (Schur's Lemma)**.** *Let $\mathscr{V}$ and $\mathscr{W}$ be irreducible representations of a group G and let $\varphi\colon \mathscr{V} \to \mathscr{W}$ be a G-morphism. Then either φ is a G-isomorphism or $\varphi = 0$ is the zero map.*

Proof. Since the kernel of φ is a subrepresentation of the irreducible representation $\mathscr{V}$, we have $\ker\varphi = \{0\}$ or $\ker\varphi = \mathscr{V}$. Hence φ is an isomorphism or $\varphi = 0$, respectively. □

The linear algebraic groups we are interested in behave very nicely from the viewpoint of representation theory, as can be seen in the upcoming Theorem 3.3.5.

3.3.4 Definition. A representation $\mathscr{V}$ is called *completely reducible*, if $\mathscr{V}$ can be written as a direct sum of irreducible G-subrepresentations.

A linear algebraic group G is called *linearly reductive*, if every rational G-representation is completely reducible. ■

The literature also knows the notion of a linear algebraic group G being *reductive*, which coincides with G being linearly reductive in characteristic 0, see e.g. [Spr77, Lemma 2.1.4].

The following theorem combines several fundamental results from representation theory.

3.3.5 Theorem. *(1) Every representation of a finite group is completely reducible. (Maschke's Theorem)*

(2) The groups GL_n and SL_n are linearly reductive.

(3) Normal subgroups and homomorphic images of linearly reductive groups are linearly reductive.

(4) If $H \subseteq G$ is a normal subgroup and if both H and G/H are linearly reductive, then G is linearly reductive.

Proof. For (1) see e.g. [Kra85, AII.4 Beispiel b]. For (2) see e.g. [Kra85, AII.5 Satz 4]. For (3) and (4) see e.g. [Kra85, II.3.5 Satz 2]. □

3.3.6 Example. Since $\mathbb{C}^\times = \mathsf{GL}_1$ is linearly reductive according to Theorem 3.3.5, each $\mathbb{C}^\times$-representation decomposes into irreducibles. The irreducible rational representations $\mathscr{V}_d$ of $\mathbb{C}^\times$ are known to be indexed by integers $d \in \mathbb{Z}$:

$$\alpha \odot f := \alpha^d.f \quad \text{for all } f \in \mathscr{V}_d,\ \alpha \in \mathbb{C}^\times,$$

where the group action is denoted by "$\odot$" and the period "." stands for multiplication with scalars. ■

For a linearly reductive group G, every representation $\mathscr{V}$ can be written as

$$\mathscr{V} = \bigoplus_i \mathscr{V}_i,$$

where $\mathscr{V}_i$ is irreducible. We remark here that in general this decomposition is not unique and in fact in our cases of interest it will most often not be unique. However, the following *isotypic decomposition* is unique.

3.3.7 Definition. For a given isomorphy type λ of a linearly reductive group G and a G-representation $\mathscr{V}$ the linear span of all irreducible subrepresentations of type λ is called the *isotypic component of $\mathscr{V}$ of type λ.* ■

If $\mathscr{W}_\lambda$ denotes the isotypic component of $\mathscr{V}$ of type λ, then we call

$$\mathscr{V} = \bigoplus_\lambda \mathscr{W}_\lambda$$

the *isotypic decomposition* of $\mathscr{V}$; here the sum is over all distinct isomorphy types λ of irreducible G-representations. By definition, $\mathscr{W}_\lambda$ decomposes (in a non-unique way) into $\operatorname{mult}_\lambda(\mathscr{V}) \in \mathbb{N}$ many copies of the irreducible G-representations $\mathscr{V}_\lambda$ of type λ. We call $\operatorname{mult}_\lambda(\mathscr{V})$ the *multiplicity of $\mathscr{V}_\lambda$ in $\mathscr{V}$*, and we use the following short notation:

$$\mathscr{V} \simeq \bigoplus_\lambda \operatorname{mult}_\lambda(\mathscr{V}) \cdot \mathscr{V}_\lambda.$$

The following short excursion is an application of the isotypic decomposition.

Excursion: Graded Coordinate Rings. In this short paragraph we assume that $X \subseteq V$ is a *cone*, which is a quasi-affine variety that is closed under scaling as provided by the $\mathbb{C}$-vector space V. We write $\alpha.v$ for the multiplication of a scalar $\alpha \in \mathbb{C}$ with some $v \in V$. All quasi-affine varieties of interest in both scenarios from Section 2.6 are cones. The product of field elements $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ shall also be written as $\alpha.\beta$.

3.3.8 Proposition. *Given a cone $X \subseteq V$. A regular function $f \in \mathbb{C}[X]$ is called* homogeneous of degree $d \in \mathbb{Z}$, *if*

$$\forall \alpha \in \mathbb{C}^\times, x \in X : f(\alpha.x) = \alpha^d.f(x).$$

This definition makes $\mathbb{C}[X]$ into a graded $\mathbb{C}$-algebra.

Proof. The space

$$W_d := \{f \in \mathbb{C}[X] \mid \forall \alpha \in \mathbb{C}^\times, x \in X : f(\alpha.x) = \alpha^d.f(x)\}$$

is a linear subspace of $\mathbb{C}[X]$. Assume there are finitely many different $d_i \in \mathbb{Z}$ and $f_i \in W_{d_i}$ such that $\sum_i f_i = 0$. Fix $x \in X$. For all $\alpha \in \mathbb{C}^\times$ we have

$$0 = \sum_i f_i(\alpha x) = \sum_i \alpha^{d_i} f_i(x).$$

This can be interpreted as an element in the ring $\mathbb{C}[\alpha, \alpha^{-1}]$. Equating coefficients yields $f_i(x) = 0$ for all i. Since $x \in X$ was arbitrary, it follows $f_i = 0$ for all i.

We now show that every $f \in \mathbb{C}[X]$ can be decomposed into a sum $f = \sum_{d \in \mathbb{Z}} f_d$ with $f_d \in W_d$. Let W denote the linear span of the orbit $\mathbb{C}^\times f$. Then W is a finite dimensional $\mathbb{C}$-vector space and a $\mathbb{C}^\times$-representation via

$$(\alpha \odot f)(x) := f(\alpha.x).$$

Now we decompose f according to the isotypic decomposition w.r.t. the linearly reductive group $\mathbb{C}^\times$. Example 3.3.6 shows that we get the desired sum $f = \sum_{d \in \mathbb{Z}} f_d$. □

This completes our excursion on graded coordinate rings. ■

Back to our scenarios in Section 2.6, we see that Gv and $\overline{Gv}$ are both cones and hence $\mathbb{C}[Gv]$ and $\mathbb{C}[\overline{Gv}]$ are graded coordinate rings. The homogeneous components $\mathbb{C}[Gv]_d$ and $\mathbb{C}[\overline{Gv}]_d$ are finite dimensional vector spaces. For example, for $\mathfrak{c} \in V$, the fact that G is linearly reductive (Theorem 3.3.5) allows us to decompose as follows:

$$\mathbb{C}[\overline{G\mathfrak{c}}]_d = \bigoplus_{\lambda} \mathrm{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{c}}]_d) \cdot \mathscr{V}_\lambda.$$

Recall from Section 2.6 that we want to show that $\overline{G\mathfrak{h}_{m,n}} \not\subseteq \overline{G\mathfrak{c}_n}$. Assume the contrary, namely

$$\overline{G\mathfrak{h}_{m,n}} \subseteq \overline{G\mathfrak{c}_n}.$$

From Corollary 3.1.11 we see that $\mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]$ is a factor ring of $\mathbb{C}[\overline{G\mathfrak{c}_n}]$. Hence we get a surjection of representations in each degree d by restriction of functions:

$$\mathbb{C}[\overline{G\mathfrak{c}_n}]_d \twoheadrightarrow \mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]_d.$$

Since this surjection is just the restriction of functions, it is clearly a G-morphism. For all d and λ, Schur's Lemma 3.3.3 implies the inequality

$$\mathrm{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{c}_n}]_d) \geq \mathrm{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]_d).$$

3.3.9 Definition. A *representation theoretic obstruction for* $\mathfrak{h}_{m,n} \in \overline{G\mathfrak{c}_n}$ is a tuple consisting of a type λ of irreducible G-representations and a degree d that satisfies

$$\mathrm{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{c}_n}]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]_d).$$ ■

The above discussion shows that the existence of a representation theoretic obstruction proves $\overline{G\mathfrak{h}_{m,n}} \not\subseteq \overline{G\mathfrak{c}_n}$.

3.3.10 Definition. If for a representation theoretic obstruction (λ, d) we have $\mathrm{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{c}_n}]_d) = 0$, then we call (λ, d) an *occurence obstruction*, otherwise a *multiplicity obstruction*. ■

By Corollary 3.2.7 we know that if computational lower bounds exist, then polynomial obstructions must exist for them, "certifying the lower bound". Sadly, nothing similar is known for representation theoretic obstructions and we state the following problem.

3.3.11 Problem. *For the scenarios in Section 2.6, if $\mathfrak{h}_{m,n} \notin \overline{G\mathfrak{c}_n}$, is there an occurence obstruction proving this? Is there at least a multiplicity obstruction proving this?*

Mulmuley and Sohoni conjecture that Conjecture 2.3.2 can be proved with occurence obstructions, see [MS08, Def. 1.2].

3.4 Coordinate Rings of Orbits

In Geometric Complexity Theory we are mostly interested in the coordinate ring $\mathbb{C}[\overline{Gv}]$ of orbit closures. However, the coordinate ring $\mathbb{C}[Gv]$ of orbits is much easier to study than $\mathbb{C}[\overline{Gv}]$. This is what we want to analyze in this section, postponing more insights concerning the scenarios of Section 2.6 to Chapter 5.

3.4.1 Claim. *In the scenarios of Section 2.6 we have an inclusion of homogeneous parts of coordinate rings*

$$\mathbb{C}[\overline{Gv}]_d \overset{\iota}{\hookrightarrow} \mathbb{C}[Gv]_d,$$

given by restriction of functions. This inclusion is a G-morphism.

Claim 3.4.1 and Schur's Lemma 3.3.3 immediately imply that for all λ we have

$$\operatorname{mult}_\lambda(\mathbb{C}[\overline{Gv}]_d) \le \operatorname{mult}_\lambda(\mathbb{C}[Gv]_d). \tag{3.4.2}$$

Proof of Claim 3.4.1. We show that the kernel of the linear restriction map ι is zero. To achieve this we need to show that each homogeneous polynomial $f \in \mathbb{C}[V]_d$ that vanishes on the whole Gv also vanishes on $\overline{Gv}$. This is true, because f is continuous.

The map ι is a G-morphism, because it is just the restriction of functions. □

In the two scenarios, for representation theoretic obstructions (λ, d) we need to prove upper bounds on $\operatorname{mult}_\lambda(\mathbb{C}[\overline{G\mathfrak{c}_n}]_d)$. If (3.4.2) is used to prove such an upper bound, then we say that (λ, d) has an *orbit-wise upper bound proof.*

3.4 (A) Geometric Invariant Theory

In this subsection we give an explicit realization of a crucial result that can be used to calculate the decomposition of $\mathbb{C}[G\mathfrak{c}_n]_d$.

An important notion is the following: For a group H acting on a set X we denote by X^H the set of H-*invariants*, i.e., the set of elements $x \in X$ satisfying $hx = x$ for all $h \in H$.

For a linear algebraic group G, the linear algebraic product group $G \times G$ acts on $\mathbb{C}[G]$ via

$$\big((g_1, g_2)f\big)(g) \coloneqq f(g_1^{-1} g g_2),$$

for all $f \in \mathbb{C}[G]$ and $g, g_1, g_2 \in G$. Fix $v \in V$ and let $H \coloneqq \mathsf{stab}_G(v)$ be its stabilizer. Let $\epsilon \in G$ denote the neutral element and define $\vec{H} \coloneqq \{\epsilon\} \times H \subseteq G \times G$. We call

$$\mathbb{C}[G]^{\vec{H}} = \{f \in \mathbb{C}[G] \mid \forall h \in H : (\epsilon, h)f = f\}$$

the *ring of right H-invariants.*

3.4.3 Theorem. *Let G be a linear algebraic group with identity element ϵ and let $v \in V$ for a G-variety V. Then the map*

$$\varphi \colon \mathbb{C}[Gv] \to \mathbb{C}[G]^{\vec{H}}, \ f \mapsto \big(g \mapsto f(gv)\big)$$

is an isomorphism of $\mathbb{C}$-algebras. Moreover, φ is G-equivariant, where G acts on $\mathbb{C}[G]^{\vec{H}}$ via $(gf)(g') \coloneqq f(g^{-1}g')$. Additionally, if both Gv and G are cones such that for all $\alpha \in \mathbb{C}^\times$ we have $\operatorname{diag}(\alpha)\epsilon = \alpha^\ell.\epsilon$, then φ maps $\mathbb{C}[Gv]_d$ to $\mathbb{C}[G]^{\vec{H}}_{\ell d}$.

Proof. According to [TY05, 25.4.6 Prop.], the map

$$\varphi \colon \mathbb{C}[Gv] \to \mathbb{C}[G]^{\vec{H}}$$

is a ring isomorphism. The rest of the proof is straightforward. It is easy to see that φ is $\mathbb{C}$-linear. It remains to show that φ is G-equivariant and that the gradings are respected up to the scalar ℓ. For the G-equivariance we have to show that $g(\varphi(f)) = \varphi(gf)$.

We have $\varphi(f) = \big(g' \mapsto f(g'v)\big)$ and hence $g(\varphi(f)) = \big(g' \mapsto f((g^{-1}g')v)\big)$.

On the other hand $\varphi(gf) = \big(g' \mapsto (gf)(g'v)\big) = \big(g' \mapsto f(g^{-1}g'v)\big) = g(\varphi(f))$.

We now consider the gradings, which exist according to Proposition 3.3.8. Let $f \in \mathbb{C}[Gv]_d$ and let $g \in G$, $\alpha \in \mathbb{C}^\times$. Then

$$(\varphi(f))(\alpha.g) = f((\alpha.g)v) = f(\alpha^\ell.gv) = \alpha^{\ell d}.f(gv) = \alpha^{\ell d}.(\varphi(f))(g).$$ □

3.4 (B) Peter-Weyl Theorem

The representation theory of $\mathbb{C}[G]$ is quite well-understood and beautiful as seen in the following algebraic version of the Peter-Weyl Theorem. We state it without proof.

3.4.4 Theorem (Algebraic Peter-Weyl Theorem, cf. e.g. [Kra85, II.3.1 Satz 3], [GW09, Thm. 4.2.7], or [Pro07, Ch. 7, 3.1 Thm.]). *Let G be linearly reductive. The isotypic decomposition of $\mathbb{C}[G]$ as a $G \times G$-representation is given by*

$$\mathbb{C}[G] = \bigoplus_{\lambda} \mathscr{V}_\lambda^* \otimes \mathscr{V}_\lambda,$$

where λ runs over all G-isomorphy types, $\mathscr{V}_\lambda$ denotes the irreducible representation of type λ, and $\mathscr{V}_\lambda^*$ denotes its dual. The group $G \times G$ acts canonically on the right hand side via $(g_1, g_2)(f \otimes v) := (g_1 f) \otimes (g_2 v)$.

From Theorem 3.4.3 and Theorem 3.4.4 we can immediately conclude that in both scenarios we have

$$\mathbb{C}[Gv] \simeq \mathbb{C}[G]^{\vec{H}} \simeq \bigoplus_{\lambda} \{\lambda^*\} \otimes \{\lambda\}^H \simeq \bigoplus_{\lambda} \dim(\{\lambda\}^H)\{\lambda^*\} \tag{3.4.5}$$

as G-representations, where $\{\lambda\}$ denotes an irreducible G-representation of type λ. We will make use of this fact in Chapter 5. To analyze the dimension of $\{\lambda\}^H$, we study representation theory in Chapters 4 and 6.

Chapter 4

Preliminaries: Classical Representation Theory

This chapter gives details about the representation theory of the general linear and symmetric groups over $\mathbb{C}$. We study those coefficients arising in decompositions of tensor products of irreducible representations which are of interest in Geometric Complexity Theory, namely plethysm coefficients, Kronecker coefficients, and Littlewood-Richardson coefficients.

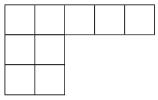
4.1 Young Tableaux

In this section we very briefly summarize the representation theory of the symmetric group S_d and the general linear group GL_n. See [Sag01], [Ful97, parts I and II], and [FH91, §4 and §15] for proofs, details and further reading.

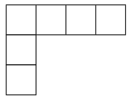
Combinatorics. An *ℓ-generalized partition* λ is defined to be a finite sequence of nonincreasing integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. We write (d), $d \in \mathbb{Z}$ for the 1-generalized partition consisting of the single element d. The set of ℓ-generalized partitions is closed under elementwise addition and elementwise scaling with natural numbers. We write $\lambda + \mu$ and $a\lambda$, $a \in \mathbb{N}$, for these operations, respectively. We write $\ell{\times}k$ for the ℓ-generalized partition having all ℓ elements equal to k. We remark that $a(\ell{\times}k) = \ell{\times}(ak)$. The *dual ℓ-generalized partition* λ^* of λ is defined via $\lambda^* := (-\lambda_{\ell(\lambda)}, \ldots, -\lambda_1)$. We say that λ is *regular*, if all parts of λ are pairwise distinct. (The naming has nothing to do with the term "regular function" from Section 3.1.5.)

If $\lambda_\ell \geq 0$, then we call λ a *partition*. The number of nonzero elements in λ is called its *length* $\ell(\lambda)$. Like the set of ℓ-generalized partitions, the set of partitions is closed under elementwise addition and elementwise scaling with natural numbers. We call $|\lambda| := \sum_i \lambda_i$ the *weight of* λ. If λ satisfies $|\lambda| = d$ and $\ell(d) \leq n$, then we write $\lambda \vdash_{\overline{n}} d$. If we do not specify the weight, we just write $\lambda \vdash_{\overline{n}}$, and if we do not specify the length, we write $\lambda \vdash d$.

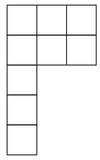
A pictorial description of partitions is given by *Young diagrams*, which are upper-left-justified arrays having λ_i boxes in the ith row, see Figure 4.1.i(a). Formally, the Young diagram to the partition λ is the set $\{(r, c) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0} \mid c \leq \lambda_r\}$, where the first coordinate specifies the row from top to bottom and the second one specifies the column from left to right. We identify partitions with their Young diagrams. The partitions $\ell{\times}k$ correspond to rectangular Young diagrams with ℓ rows and k columns. In particular $(r) = 1{\times}r$ stands for the partition with a single row and r boxes. We want to introduce a notation for Young diagrams with a single row and a single column, called *hook partitions*: $c \ulcorner r$ shall denote the partition $(c{\times}1){+}(r{-}1)$,



(a) The Young diagram corresponding to the partition $\lambda = (5,2,2) \vdash_3 9$.



(b) The Young diagram corresponding to the hook partition $3 \ulcorner 4$.



(c) The Young diagram corresponding the transposed partition ${}^t\lambda$ of λ in Figure (a).

2	2	1	1	1
1	2			
2	2			

(d) A filling of λ with entries from $\{1,2\}$ and content $(4,5)$.

1	8	6	4	3
5	9			
2	7			

(e) A numbering of λ.

1	1	1	2	4
2	2			
3	4			

(f) A semistandard tableau of shape λ with entries from $\{1,2,3,4\}$.

1	4	7	8	9
2	5			
3	6			

(g) A standard tableau of shape λ.

Figure 4.1.i: Young diagrams, fillings, numberings, and Young tableaux.

see Figure 4.1.i(b). We have $(c \ulcorner r) \vdash_c r+c-1$. When reflecting a Young diagram λ at the diagonal from the upper left to the lower right we get a Young diagram again, which we call the *transposed Young diagram* ${}^t\lambda$, see Figure 4.1.i(c). Note that the the number of boxes of λ in column i equals ${}^t\lambda_i := ({}^t\lambda)_i$. For two partitions λ and μ we write $\mu \subseteq \lambda$ if the Young diagram of μ is contained in the Young diagram of λ, or equivalently, $\forall i : \mu_i \le \lambda_i$.

For partition triples $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ we use the short notation $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} \vdash_{n_1,n_2,n_2} d_1, d_2, d_3$, which stands for $\lambda^{(k)} \vdash_{n_k} d_k$ for all $k \in \{1,2,3\}$. If $n_1 = n_2 = n_3 =: n$ and $d_1 = d_2 = d_3 =: d$ we use the even shorter syntax $\lambda \vdash^*_n d$.

Given a partition λ, a *filling of λ with entries from a set S* of natural numbers is a map $\varphi\colon \lambda \to S$. This is visualized by writing elements of S into the boxes of the Young diagram, see Figure 4.1.i(d). The *content* z of a filling is defined as the finite sequence $(z_1, z_2, \ldots)$, where $z_i := |\varphi^{-1}(i)|$ denotes the number of occurences of i in the filling. Note that the content is not necessarily a partition. The *length* $\ell(z)$ is defined as the largest i such that $z_i \neq 0$. A filling is called a *numbering of* λ, if $S = \{1, \ldots, |\lambda|\}$ and φ is bijective, see Figure 4.1.i(e). The content of a numbering is always $(1,1,\ldots,1)$. A filling of λ is called a *semistandard Young tableau of shape* λ, or simply a *semistandard tableau of shape* λ, if the entries are nondecreasing in each row from left to right and increasing in each column from top to bottom, see Figure 4.1.i(f). If this filling is also a numbering, then it is called a *standard tableau*, see Figure 4.1.i(g).

Irreducible S_d-Representations. The irreducible representations of S_d are indexed by the partitions $\lambda \vdash d$. We write $[\lambda]$ for the irreducible representation of type λ, called the *Specht module*. The trivial representation corresponds to the partition (d). The representation $[d \times 1]$ is 1-dimensional and defined as follows:

$\forall v \in [d{\times}1]\ \forall \pi \in \mathsf{S}_d : \pi v := \mathrm{sgn}(\pi).v$. Hence $[d{\times}1]$ is called the *sign representation* or the *alternating representation.* Given two irreducible S_d-representations $[\lambda]$ and $[\mu]$, then their tensor product $[\lambda] \otimes [\mu]$ is an $\mathsf{S}_d \times \mathsf{S}_d$-representation via $(\pi_1, \pi_2)(v_1 \otimes v_2) := (\pi_1 v_1) \otimes (\pi_2 v_2)$, where $v_1 \in [\lambda]$, $v_2 \in [\mu]$, and $\pi_i \in \mathsf{S}_d$. The $\mathsf{S}_d \times \mathsf{S}_d$-representation $[\lambda] \otimes [\mu]$ becomes an S_d-representation via the diagonal embedding $\mathsf{S}_d \hookrightarrow \mathsf{S}_d \times \mathsf{S}_d$, $\pi \mapsto (\pi, \pi)$. In general $[\lambda] \otimes [\mu]$ is not irreducible as an S_d-representation. [BK99] show that the cases where λ or μ are (d) or $1{\times}d$ are the only ones in which $[\lambda] \otimes [\mu]$ is irreducible. Clearly, $[n{\times}1] \otimes [n{\times}1] = [(n)]$ is trivial. Moreover, one can show via character theory that $[d{\times}1] \otimes [\mu] \simeq [{}^t\mu]$, which is irreducible.

Irreducible GL_n-Representations. The irreducible rational representations $\{\lambda\}$ of GL_n, sometimes called the *Weyl-modules* or *Schur-modules*, are indexed by n-generalized partitions λ. The polynomial representations $\{\lambda\}$ are indexed by partitions $\lambda \vdash_{\overline{n}}$.

Given a GL_n-representation $\mathscr{V}$, a vector $v \in \mathscr{V}$ on which the diagonal matrices act via

$$\mathrm{diag}(\alpha_1, \dots, \alpha_n)v = \alpha_1^{z_1} \alpha_2^{z_2} \cdots \alpha_n^{z_n} v$$

for some $z \in \mathbb{Z}^n$ is called a *weight vector of weight z.* Every rational GL_n-representation $\mathscr{V}$ is a rational representation of the *torus* $(\mathbb{C}^\times)^n$ via the embedding as diagonal matrices. Since $(\mathbb{C}^\times)^n$ is reductive (Thm. 3.3.5), $\mathscr{V}$ decomposes as a $(\mathbb{C}^\times)^n$-representation:

$$\mathscr{V} = \bigoplus_{z \in \mathbb{Z}^n} \mathscr{V}^z, \tag{4.1.1}$$

where $\mathscr{V}^z$ denotes the isotypic component of type z. We call $\mathscr{V}^z$ the *weight space of weight z*, its elements are called *weight vectors of weight z*, and the decomposition (4.1.1) is called the *weight decomposition of $\mathscr{V}$.*

For GL_n^3 we also have a weight decomposition by choosing the torus $((\mathbb{C}^\times)^n)^3$ and proceeding analogously.

Tableau Calculus. Let $F(n)$ denote the infinite set of fillings of Young diagrams of arbitrary shape filled with elements from $\{1, \dots, n\}$. The group S_n acts on $F(n)$ by changing each entry i in a filling to $\pi(i)$ for $\pi \in \mathsf{S}_n$. Let $\mathbb{C}^{F(n)}$ be the infinite dimensional complex vector space with basis $F(n)$. The action of S_n on $F(n)$ extends to a linear action on $\mathbb{C}^{F(n)}$. Let $F(\lambda, z) \subseteq F(n)$ denote the finite set of fillings of shape $\lambda \vdash_{\overline{n}}$ and content $z \in \mathbb{N}^n_{\geq 0}$, where $|z| = |\lambda|$. The stabilizer $\mathsf{stab}_{\mathsf{S}_n}(z)$ of the natural action of S_n on $\mathbb{N}^n_{\geq 0}$ leaves the set $F(\lambda, z)$ invariant. Hence $\mathsf{stab}_{\mathsf{S}_n}(z)$ acts on the complex vector space $\mathbb{C}^{F(\lambda,z)}$ with basis $F(\lambda, z)$.

The following three S_n-subrepresentations of $\mathbb{C}^{F(n)}$ are of importance, where the most complicated one — $K_3(n)$ — will play for us only a minor role.

(1) The subspace $K_1(n)$ generated by the sums $T_1 + T_2$, where T_1 and T_2 are two fillings that differ only in one column by a single transposition of entries.

(2) The subspace $K_2(n)$ generated by the differences $T_1 - T_2$, where T_1 and T_2 are two fillings that differ by switching two columns of the same length.

(3) The subspace $K_3(n)$ generated by the sums $T - \sum_S S$, where the sum is over all fillings S that arise from T by exchanging for some j and k the top k elements from the $(j+1)$th column with any selection of k elements in the jth column, preserving their vertical order, see [Ful97, p. 110].

We note $K_2(n) \subseteq K_3(n)$. We call the sum $K(n) := K_1(n) + K_2(n) + K_3(n)$ the *tableau kernel.* Note that $K(\lambda, z) := K(n) \cap \mathbb{C}^{F(\lambda,z)}$ is a $\mathsf{stab}_{\mathsf{S}_n}(z)$-subrepresentation, called the tableau kernel with respect to the content z. The kernel $K(\lambda, z)$ is defined in a way such that the cosets of semistandard tableaux with entries from $\{1,\dots,n\}$ form a basis of the factor space $\mathbb{C}^{F(\lambda,z)}/K(\lambda,z)$ for every partition $\lambda \vdash_{\overline{n}}$ and content $z \in \mathbb{N}^n_{\geq 0}$, $|z| = |\lambda|$, see e.g. [Ful97, §8.1, proof of Thm. 1, p. 110]. Expressing a filling T over the basis of semistandard tableaux is called *straightening the filling* T.

4.1.2 Definition. If a filling T can be converted to a semistandard tableau by using only permutations in columns and switching whole columns, then we say that T is *easy to straighten*. ∎

The irreducible S_n-representation $[\lambda]$, where $\lambda \vdash n$, can be defined as $\mathbb{C}^{F(\lambda,n\times 1)}/K(\lambda,n\times 1)$, so $[\lambda]$ has as a formal basis the standard tableaux of shape λ. For example, the basis of the trivial representation $[(n)]$ is the standard tableau $\boxed{1}\boxed{2}\boxed{3}\boxed{\cdot\cdot\cdot}\boxed{n}$. It is easy to see that this tableau generates the trivial representation, because every filling of the shape (n) is easy to straighten. Analogously, we see that $[n\times 1]$ is the alternating representation.

Fix $n \in \mathbb{N}$ and a partition $\lambda \vdash_{\overline{n}}$. Let $F(\lambda) \subseteq F(n)$ denote the set of fillings of shape λ filled with elements from $\{1,\dots,n\}$. Let $K(\lambda) := K(n) \cap \mathbb{C}^{F(\lambda)}$. As a vector space, the irreducible GL_n-representation $\{\lambda\}$ of type $\lambda \vdash_{\overline{n}}$ can be defined as $\mathbb{C}^{F(\lambda)}/K(\lambda)$. So $\{\lambda\}$ has as a formal basis the semistandard tableaux of shape λ and content $z \in \mathbb{N}^n_{\geq 0}$ with $|z| = |\lambda|$, see [Ful97, §8.1, Thm. 1, p. 110]. We now describe the action of GL_n on $\{\lambda\}$.

We interpret fillings as mappings from the Young diagram to the natural numbers. For $\lambda \vdash d$ choose a fixed numbering $T_0 \colon \lambda \to \mathbb{N}$. To each filling T of shape λ there corresponds a vector $v_T \in \bigotimes^d \mathbb{C}^n$, defined as

$$v_T := |T(T_0^{-1}(1))\rangle \otimes |T(T_0^{-1}(2))\rangle \otimes \cdots \otimes |T(T_0^{-1}(d))\rangle.$$

For example, if $T_0 = \begin{smallmatrix}1&3\\2\end{smallmatrix}$ and $T = \begin{smallmatrix}3&3\\1\end{smallmatrix}$, then $v_T = |313\rangle$. This gives a linear isomorphism $\mathbb{C}^{F(\lambda)} \xrightarrow{\sim} \bigotimes^d \mathbb{C}^n, T \mapsto v_T$. Using this isomorphism, the natural action of GL_n on $\bigotimes^d \mathbb{C}^n$ induces a GL_n-action on $\mathbb{C}^{F(\lambda)}$. It is readily checked that the tableau kernel $K(\lambda) \subseteq \mathbb{C}^{F(\lambda)}$ is a GL_n-subrepresentation. Hence $\{\lambda\} = \mathbb{C}^{F(\lambda)}/K(\lambda)$ is a well-defined GL_n-representation.

Note that the GL_n-action is consistent with the S_n-action that we defined earlier on $\{\lambda\}$ via the action of S_n on fillings, where $\mathsf{S}_n \subseteq \mathsf{GL}_n$ is embedded via permutation matrices.

For a diagonal matrix $A := \mathrm{diag}(\alpha_1, \dots \alpha_n)$ and a semistandard tableau T with content z we have $AT = \alpha_1^{z_1} \cdots \alpha_n^{z_n} T$. It follows that a basis of the z-weight space $\{\lambda\}^z$ is given by the semistandard tableaux of shape λ and content z. Its dimension, i.e., the number of semistandard tableaux of a given shape λ and content z, is called the *Kostka number* $K_{\lambda,z}$. We will now state the simple condition for checking nonzeroness of a Kostka number. The set $\mathbb{Z}^n$ carries the partial *dominance order* $\preceq$, defined via $z \preceq \lambda$ iff $\sum_{i=1}^j z_i \leq \sum_{i=1}^j \lambda_i$ for all j.

4.1.3 Proposition (see e.g. [FH91, p. 457, Ex. A.11], [Ful97, p. 26, Ex. 2], or [Mac95, I.7 Exa. 9])**.** *The Kostka number $K_{\lambda,z}$ is nonzero iff $z \preceq \lambda$.*

Proposition 4.1.3 is sometimes referred to as the Gale-Ryser theorem.

Highest Weight Vectors. An irreducible representation of GL_n has the following crucial characterizing property (Lemma 4.1.5).

4.1.4 Definition. Let $U_n \subseteq \mathsf{GL}_n$ denote the group of upper triangular matrices with 1s on the main diagonal, the so-called *maximal unipotent group*. Given a rational GL_n-representation $\mathscr{V}$. A weight vector $v \in \mathscr{V}$ that is fixed under the action of U_n, i.e., $\forall u \in U_n : uv = v$, is called a *highest weight vector of* $\mathscr{V}$. The vector space of highest weight vectors of weight λ is denoted by $\mathsf{HWV}_\lambda(\mathscr{V})$. ∎

From Definition 4.1.4 it follows that the weights of highest weight vectors are n-generalized partitions.

4.1.5 Lemma. *Each irreducible GL_n-representation $\mathscr{V}$ contains, up to scalar multiples, exactly one highest weight vector v. If $\mathscr{V}$ has isomorphy type λ, then the highest weight vector of $\mathscr{V}$ has weight λ. The representation $\mathscr{V}$ is the linear span of the GL_n-orbit of v.*

Proof. See e.g. [Kra85, III.1.4 Satz 1]. □

It is easy to check that the highest weight vector in $\{\lambda\}$ is the unique semistandard tableau of shape λ and content λ that contains only entries i in each row i.

Lemma 4.1.5 gives a convenient way to describe the multiplicities of irreducibles occuring in a GL_n-representation $\mathscr{V}$:

$$\mathrm{mult}_\lambda(\mathscr{V}) = \dim(\mathsf{HWV}_\lambda(\mathscr{V})). \tag{4.1.6}$$

The following easy proposition shows that when searching for polynomial obstructions (see Def. 3.2.8), we can restrict our search to highest weight vectors.

4.1.7 Proposition. *In both scenarios from Section 2.6, if there exists a polynomial $f \in \mathbb{C}[V]$ satisfying $f(\overline{G\mathfrak{c}}) = 0$ and $f(g\mathfrak{h}) \neq 0$ for some $g \in G$, then there exists a highest weight vector $f_\lambda \in \mathbb{C}[V]$ satisfying $f_\lambda(\overline{G\mathfrak{c}}) = 0$ and $f_\lambda(g'\mathfrak{h}) \neq 0$ for some $g' \in G$ and some G-isomorphy type λ.*

Proof. The fact $f(\overline{G\mathfrak{c}}) = 0$ means that f is contained in the vanishing ideal $I(\overline{G\mathfrak{c}})$. But $I(\overline{G\mathfrak{c}})$ is a graded G-representation. Hence we can write $f = \sum_{d,\lambda} f_{d,\lambda}$, where $f_{d,\lambda} \in I(\overline{G\mathfrak{c}})_d$ are elements from the isotypic component of type λ in the homogeneous part $I(\overline{G\mathfrak{c}})_d$. By Lemma 4.1.5, it follows that we can write $f_{d,\lambda} = \sum_i g_{d,\lambda,i} f_{d,\lambda,i}$, where $g_{d,\lambda,i} \in G$ and $f_{d,\lambda,i}$ is a highest weight vector in $I(\overline{G\mathfrak{c}})_d$ of weight λ.

Let $g \in G$ with $f(g\mathfrak{h}) \neq 0$. Then $g_{d,\lambda,i} f_{d,\lambda,i}(g\mathfrak{h}) \neq 0$ for some d, λ, i. This means $f_{d,\lambda,i}(g_{d,\lambda,i}^{-1} g\mathfrak{h}) \neq 0$, which proves the proposition. □

The following lemma states that highest weight vectors are preserved under GL_n-morphisms.

4.1.8 Lemma. *If $v \in \mathscr{V}$ is a highest weight vector of weight λ and $\varphi\colon \mathscr{V} \to \mathscr{W}$ is a GL_n-morphism, then $\varphi(v) \in \mathscr{W}$ is zero or a highest weight vector of weight λ.*

Proof. Let $u \in U_n$. Then $\varphi(v) = \varphi(uv) = u\varphi(v)$. Hence $\varphi(v)$ is U_n-invariant. Let $t := \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ and $\alpha^\lambda := \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n} \in \mathbb{C}$. Then $\alpha^\lambda.\varphi(v) = \varphi(\alpha^\lambda.v) = \varphi(tv) = t\varphi(v)$. Hence $\varphi(v)$ is a weight vector of weight λ. □

If $\mathscr{V}$ is a GL_n-representation, then $\mathscr{V}$ is a $\mathsf{GL}_\ell \times \mathsf{GL}_{n-\ell}$-representation via embedding as block diagonal matrices. Restricting $\mathsf{GL}_{n-\ell}$ to unit matrices makes $\mathscr{V}$ a GL_ℓ-representation.

4.1.9 Theorem (Pieri rule, see e.g. [Ful97, p. 24, p. 121] or [FH91, Ex. 4.44, p. 59])**.** *For $\lambda \vdash_{\overline{n}}$ we have*

$$\{\lambda\}\downarrow^{\mathsf{GL}_n}_{\mathsf{GL}_{n-1}} \simeq \bigoplus_\mu \{\mu\},$$

where the sum is over all partitions $\mu \vdash_{\overline{n-1}}$ that are obtained from λ by removing boxes, but at most one in each column.

Definition 4.1.4 implies the following claim.

4.1.10 Claim. *Fix an ℓ-generalized partition λ and append $n-\ell$ zeros to obtain $\bar{\lambda}$. A highest weight vector $v \in \mathscr{V}$ of weight $\bar{\lambda}$ with respect to GL_n is also a highest weight vector of weight λ with respect to GL_ℓ.*

The GL_n-representations $\{n\times d\}$, $d \in \mathbb{Z}$, are 1-dimensional and are defined via

$$gv := \det^d(g).v \quad \text{for all } g \in G,\ v \in \{n\times d\}.$$

Note that this is compatible with Example 3.3.6. Tensoring the irreducible GL_n-representation $\{\lambda\}$ with $\{n\times d\}$ gives an irreducible GL_n-representation of type $\lambda+n\times d$. In terms of Young diagrams, this corresponds to adding/removing columns of length n, see Figure 4.1.ii for an example.

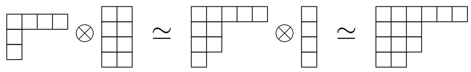


Figure 4.1.ii: Tensoring with rectangular partitions in GL_4.

Every GL_n-representation is naturally an SL_n-representation. We have the following nice property.

4.1.11 Lemma. *Irreducible GL_n-representations are also irreducible SL_n-representations. Two irreducible GL_n-representations $\{\lambda\}$ and $\{\mu\}$ are isomorphic as SL_n-representations if their Young diagrams differ by adding/removing columns of n boxes.*

The irreducible representations of $\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$ are the tensor products $\{\lambda^{(1)}\} \otimes \{\lambda^{(2)}\} \otimes \{\lambda^{(3)}\}$, where each $\lambda^{(k)}$, $1 \le k \le 3$, has at most n rows.

An analogous statement to Lemma 4.1.11 also holds for GL_n^3 and SL_n^3, but columns can be added and removed in each of the three components.

Dual Representations. Recall the bra-ket notation from Section 2.5. If $|v\rangle \in \mathscr{V}$ is a highest weight vector of $\{\lambda\}$, then the adjoint vector $\langle v| \in \mathscr{V}^*$ is a highest weight vector of $\{\lambda\}^*$: Indeed, for all $u \in U_n$ (see Def. 4.1.4), $|v\rangle \in \mathscr{V}$ a highest weight vector, and for all $|w\rangle \in \mathscr{V}$ we have

$$\big(u^{-1}\langle v|\big)|w\rangle = \langle v|u|w\rangle = \overline{\langle w|u|v\rangle} = \overline{\langle w|v\rangle} = \langle v|w\rangle.$$

The weight can be checked in the same manner.

For partition triples λ the dual $\{\lambda\}^*$ is given by $\{\lambda\}^* = \{\lambda^{(1)}\}^* \otimes \{\lambda^{(2)}\}^* \otimes \{\lambda^{(3)}\}^*$. We write $\{\lambda^*\} := \{\lambda\}^*$.

The following claim classifies the dual representations of S_d and GL_n, respectively.

4.1.12 Claim. $[\lambda]^* \simeq [\lambda]$ *and* $\{\lambda\}^* \simeq \{\lambda^*\}$.

Proof. The first statement can be easily seen with character theory. We omit the proof. The second statement follows from the statement about highest weight vectors right before this claim. □

4.2 Explicit Highest Weight Vectors

For a finite group G let $\mathbb{C}[G]$ denote its *group algebra*, i.e., the set of formal $\mathbb{C}$-linear combinations $\sum_{g\in G} \alpha_g g$, where $\alpha_g \in \mathbb{C}$. If G acts on a vector space $\mathscr{V}$, then for $p = \sum_{g\in G} \alpha_g g \in \mathbb{C}[G]$ and $v \in \mathscr{V}$ we write $pv := \sum_{g\in G} \alpha_g gv$. This defines an *action of the group algebra on $\mathscr{V}$*. The symbol for the group algebra resembles the one of the coordinate ring of G, because both algebras are isomorphic as G-representations.

In this section we will make heavy use of Dirac's bra-ket notation, explained in Section 2.5. If we have a left-action of a group G on $\mathscr{V}$, then we get a right-action of G on $\mathscr{V}^*$ defined via: $(\langle f|g)|v\rangle := \langle f|(g|v\rangle)$ for $\langle f| \in \mathscr{V}^*$, $g \in G$, and $|v\rangle \in \mathscr{V}$. Omitting unneccesary brackets we use the following short notation, consistent with the notation we already introduced: $\langle f|g|v\rangle$. Clearly, a right action of G on $\mathscr{V}^*$ gives rise to the left action of G on $\mathscr{V}^*$ via $g\langle f| := \langle f|g^{-1}$.

For natural numbes $1 \le i_1, \dots, i_\ell, i'_1, \dots, i'_\ell \le N$ and for $p = \sum_{g\in G} \alpha_g g \in \mathbb{C}[\mathsf{S}_\ell]$ we use the short notation

$$\langle i_1 \cdots i_\ell | p | i'_1 \cdots i'_\ell\rangle := \sum_{g\in G} \alpha_g \langle i_1 \cdots i_\ell | g | i'_1 \cdots i'_\ell\rangle.$$

4.2 (A) Polarization, Restitution, and Projections

The symmetric group S_D acts on the space of tensors $\bigotimes^D \mathbb{C}^N$ via

$$\pi(v_1 \otimes v_2 \otimes \cdots \otimes v_D) := v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(D)}.$$

The subspace of tensors that are invariant under this action is denoted by

$$\mathsf{Sym}^D \mathbb{C}^N := (\textstyle\bigotimes^D \mathbb{C}^N)^{\mathsf{S}_D}.$$

We will see in Lemma 4.2.2 below how this is correlated to our use of Sym in Section 2.6. The projection map $\mathcal{P}_D \colon \bigotimes^D \mathbb{C}^N \twoheadrightarrow \mathsf{Sym}^D \mathbb{C}^N$ is given by

$$\mathcal{P}_D := \frac{1}{D!} \sum_{\pi\in\mathsf{S}_D} \pi \in \mathbb{C}[\mathsf{S}_D]. \tag{4.2.1}$$

We note that the image of the projection $\frac{1}{D!}\sum_{\pi\in\mathsf{S}_D} \mathrm{sgn}(\pi)\pi$, is denoted by $\bigwedge^D \mathbb{C}^N$, the space of *alternating tensors*.

The following easy and well-known lemma from multilinear algebra (e.g. [Dol03, Ch. 1.2]) builds a link between homogeneous polynomials and multilinear forms.

4.2.2 Lemma. *Let $W := \mathbb{C}^N$. Let $\langle f| \in \bigotimes^D W^*$. Then define the homogeneous polynomial $f \in \mathbb{C}[W]_D$ via*

$$\forall w \in W : f(w) = \langle f|w^{\otimes D}\rangle.$$

The map

$$\phi\colon \textstyle\bigotimes^D W^* \to \mathbb{C}[W]_D,\ \langle f| \mapsto f$$

is linear and induces a linear isomorphism

$$\phi_0 \colon \mathsf{Sym}^D W^* \xrightarrow{\sim} \mathbb{C}[W]_D.$$

The map ϕ is called the restitution map, *while the map $\psi := \phi_0^{-1}$ is called the* polarization map.

The map $\psi \circ \phi\colon \bigotimes^D W^* \to \mathsf{Sym}^D W^*$ is the projection $\mathcal{P}_D$ defined in (4.2.1).

We remark that $f(w) = \langle f|\mathcal{P}_D|w^{\otimes D}\rangle = \langle f|w^{\otimes D}\rangle$, because $\mathcal{P}_D|w^{\otimes D}\rangle = |w^{\otimes D}\rangle$.

Next we want to analyze the natural projection $\mathcal{P}_{d[n]} \colon \bigotimes^{dn} \mathbb{C}^N \twoheadrightarrow \mathsf{Sym}^d\mathsf{Sym}^n \mathbb{C}^N$, defined as the composition of projections

$$\mathcal{P}_{d[n]} \colon \textstyle\bigotimes^{dn}\mathbb{C}^N \simeq \bigotimes^d\bigotimes^n \mathbb{C}^N \xrightarrow{\mathcal{P}^{\mathrm{inner}}_{d[n]}} \bigotimes^d \mathsf{Sym}^n\mathbb{C}^N \xrightarrow{\mathcal{P}^{\mathrm{outer}}_{d[n]}} \mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^N. \tag{4.2.3}$$

The representation $\mathsf{Sym}^d\mathsf{Sym}^n(\mathbb{C}^N)^*$ is an example of a *plethysm*.

To describe $\mathcal{P}^{\text{inner}}_{d[n]}$ and $\mathcal{P}^{\text{outer}}_{d[n]}$ in more detail, we first fix the following bijection

$$\varphi\colon \{1,\ldots,d\} \times \{1,\ldots,n\} \xrightarrow{\sim} \{1,\ldots,dn\}, \quad (j,i) \mapsto (j-1)n+i.$$

Using φ the map $\mathcal{P}_{d[n]}$ can be explicitly described as follows. The projection $\mathcal{P}^{\text{inner}}_{d[n]}$ is given by $\mathcal{P}^{\text{inner}}_{d[n]} := \bigotimes^d \mathcal{P}_n := \frac{1}{(n!)^d} \sum_{\pi \in Y_{\text{inner}}} \pi \in \mathbb{C}[\mathsf{S}_{dn}]$, where

$$Y_{\text{inner}} := \mathsf{S}_{\{(1,1),\ldots,(1,n)\}} \times \mathsf{S}_{\{(2,1),\ldots,(2,n)\}} \times \cdots \times \mathsf{S}_{\{(d,1),\ldots,(d,n)\}}$$

is a Young subgroup whose elements leave the first component fixed (row permutations). The group $Y_{\text{outer}} \subseteq \mathsf{S}_{dn}$ is defined as the group of permutations π for which a permutation $\tilde{\pi} \in \mathsf{S}_d$ exists such that

$$\pi((j,i)) = (\tilde{\pi}(j), i).$$

Hence the elements of Y_{outer} leave the second component fixed and permute the first components simultaneously. Note that $Y_{\text{outer}} \simeq \mathsf{S}_d$. The projection $\mathcal{P}^{\text{outer}}_{d[n]}$ is given by $\mathcal{P}^{\text{outer}}_{d[n]} := \frac{1}{d!} \sum_{\pi \in Y_{\text{outer}}} \pi$. The group generated by Y_{inner} and Y_{outer} is called the *wreath product* $\mathsf{S}_n \wr \mathsf{S}_d$. See [Rot95, Ch. 7, p.172] for more information on wreath products. We obtain

$$\mathcal{P}_{d[n]} = \frac{1}{(n!)^d d!} \sum_{\pi \in \mathsf{S}_n \wr \mathsf{S}_d} \pi \in \mathbb{C}[\mathsf{S}_{dn}].$$

4.2.4 Remark. Let $|w\rangle = |\underbrace{w_1 w_1 \cdots w_1}_{n \text{ times}} \underbrace{w_2 w_2 \cdots w_2}_{n \text{ times}} \cdots \underbrace{w_d w_d \cdots w_d}_{n \text{ times}}\rangle$. Then $\mathcal{P}_{d[n]}|w\rangle = \frac{1}{d!} \sum_{\sigma \in \mathsf{S}_d} \sigma |w\rangle$, where σ interchanges the blocks of size n. We remark that the inner symmetry of $|w\rangle$ enables us to write $\mathcal{P}_{d[n]}|w\rangle$ with $d!$ summands instead of $(n!)^d d!$ ones. ■

Coordinate Rings. The group GL_N acts on the tensor space $\bigotimes^D \mathbb{C}^N$ via

$$g|w_1 w_2 \cdots w_D\rangle := g|w_1\rangle \otimes g|w_2\rangle \otimes \cdots \otimes g|w_D\rangle, \quad g \in \mathsf{GL}_N.$$

It is crucial to see that this action commutes with the action of S_D. A first consequence is that all projections from this subsection are GL_N-equivariant.

According to Lemma 4.2.2, in the scenarios from Section 2.6 the homogeneous components of the coordinate rings $\mathbb{C}[V]$ are isomorphic as G-representations to the following spaces:

$$\mathbb{C}[\mathsf{Sym}^n \mathbb{C}^{n^2}]_d \simeq \mathsf{Sym}^d \mathsf{Sym}^n (\mathbb{C}^{n^2})^* \quad \text{and} \quad \mathbb{C}[\textstyle\bigotimes^3 \mathbb{C}^n]_d \simeq \mathsf{Sym}^d \bigotimes^3 (\mathbb{C}^n)^*, \tag{4.2.5}$$

omitting unnecessary brackets.

The fact that the actions of GL_N and S_D commute has the following important consequence.

4.2.6 Corollary. *The group S_D stabilizes every highest weight vector space* $\mathsf{HWV}_\lambda(\bigotimes^d \mathbb{C}^N)$.

4.2 (B) Schur-Weyl Duality and Highest Weight Vectors

The vector space $\bigotimes^D \mathbb{C}^N$ is a $\mathsf{GL}_N \times \mathsf{S}_D$-representation. The following crucial theorem is known as the *Schur-Weyl duality*.

4.2.7 Theorem (Schur-Weyl duality, see e.g. [Pro07, Ch. 9, eq. (3.1.4)] or [GW09, eq. (9.1)]).

$$\bigotimes^D \mathbb{C}^N \simeq \bigoplus_{\lambda \vdash_N D} \{\lambda\} \otimes [\lambda]$$

as $\mathsf{GL}_N \times \mathsf{S}_D$*-representations.*

Our aim is now to construct the highest weight vectors in $\bigotimes^D(\mathbb{C}^N)^*$ explicitly, for the following two reasons (cf. Lemma 4.1.8 and (4.2.5)):

(1) Let $N = n^3$ and $D = d$. The images of the highest weight vectors $\bigotimes^d(\bigotimes^3\mathbb{C}^n)^*$ under the projection $\mathcal{P}_d$ span the highest weight vector space in $\mathbb{C}[\bigotimes^3\mathbb{C}^n]_d$.

(2) Let $N = n^2$ and $D = nd$. The images of the highest weight vectors $\bigotimes^{nd}(\mathbb{C}^{n^2})^*$ under the projection $\mathcal{P}_{d[n]}$ span the highest weight vector space in $\mathbb{C}[\mathsf{Sym}^n\mathbb{C}^{n^2}]_d$.

Let $W := \mathbb{C}^N$. Dualizing and using Claim 4.1.12 we get from Theorem 4.2.7

$$\bigotimes^D W^* \simeq \bigoplus_{\lambda \vdash_N D} \{\lambda^*\} \otimes [\lambda].$$

For a natural number $\ell \leq N$ we define the wedge product

$$\langle\widehat{\ell}| := \langle 1| \wedge \langle 2| \wedge \cdots \wedge \langle \ell| := \langle 123\cdots\ell|\Big(\sum_{\pi\in\mathsf{S}_\ell} \operatorname{sgn}(\pi)\pi\Big) = \sum_{\pi\in\mathsf{S}_\ell} \operatorname{sgn}(\pi)\langle 123\cdots\ell|\pi \in \bigwedge^\ell W^*,$$

sometimes called the *Slater-determinant*. We note the algorithmically interesting fact that the scalar product $\langle\widehat{\ell}|x_1 \otimes x_2 \otimes \cdots \otimes x_\ell\rangle$, where all $x_i \in \mathbb{C}^N$, is just the determinant of the matrix

$$\begin{pmatrix} \langle 1|x_1\rangle & \langle 1|x_2\rangle & \cdots & \langle 1|x_\ell\rangle \\ \langle 2|x_1\rangle & \langle 2|x_2\rangle & \cdots & \langle 2|x_\ell\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \ell|x_1\rangle & \langle \ell|x_2\rangle & \cdots & \langle \ell|x_\ell\rangle \end{pmatrix}. \tag{4.2.8}$$

4.2.9 Claim. *$\langle\widehat{\ell}|$ is a highest weight vector of weight $\ell\times(-1)$.*

Proof. Let $u = (u_{i,j})_{i,j} \in U_N$ be an upper triangular matrix with 1s on the main diagonal. We calculate

$$u^{-1}\langle\widehat{\ell}| = \sum_{k_1\leq k_2\leq\ldots\leq k_\ell} \left(\prod_{i=1}^n u_{i,k_i}\right) \langle k_1 \wedge k_2 \wedge \cdots \wedge k_\ell| \overset{(*)}{=} \langle 1 \wedge 2 \wedge \cdots \wedge \ell|,$$

where $(*)$ holds, because $\langle i| \wedge \langle i| = 0$, $i \in \mathbb{N}$. Hence $\langle\widehat{\ell}|$ is invariant under the maximal unipotent group U_N.

The diagonal matrices $\operatorname{diag}(t_1,\ldots,t_N)$ act as follows:

$$\begin{aligned}\operatorname{diag}(t_1,\ldots,t_N)\langle\widehat{\ell}| &= \langle\widehat{\ell}|\operatorname{diag}(t_1,\ldots,t_N)^{-1} \\ &= \langle 1|t_1^{-1} \wedge \langle 2|t_2^{-1} \wedge \cdots \wedge \langle \ell|t_\ell^{-1} = (t_1t_2\cdots t_\ell)^{-1}.\langle\widehat{\ell}|\end{aligned}$$ □

Let $\mu \vdash_N D$ be a partition with column lengths ${}^t\mu_1,\ldots,{}^t\mu_{\mu_1}$. Then define

$$\langle\widehat{\mu}| := \bigotimes_{i=1}^{\mu_1} \langle\widehat{{}^t\mu_i}| \in \bigotimes^D\mathbb{C}^N. \tag{4.2.10}$$

4.2.11 Claim. *$\langle\widehat{\mu}|$ is a highest weight vector of weight μ^*.*

Proof. Since each tensor factor is U_N-invariant by Claim 4.2.9, $\langle\widehat{\mu}|$ is U_N-invariant as well. Moreover, since each tensor factor is a weight vector, $\langle\widehat{\mu}|$ is a weight vector of weight μ^*, which is the sum of the weights of its tensor factors. □

4.2.12 Claim. *Let $W := \mathbb{C}^N$ and $\mu \vdash_N D$. The tensors $\langle\widehat{\mu}|\pi$ with $\pi \in \mathsf{S}_D$ generate the vector space $\mathsf{HWV}_{\mu^*}(\bigotimes^D W^*)$.*

Proof. Corollary 4.2.6 and Schur-Weyl duality imply that $\mathsf{HWV}_{\mu^*}\left(\bigotimes^D W^*\right) \simeq \mathsf{HWV}_{\mu^*}(\{\mu\}) \otimes [\mu]$. But $\mathsf{HWV}_{\mu^*}(\{\mu\})$ is 1-dimensional (Lemma 4.1.5) and so, as S_D-representations, we have $\mathsf{HWV}_{\mu^*}\left(\bigotimes^D W^*\right) \simeq [\mu]$, which is irreducible. The linear span of the S_D-orbit of $\langle\widehat{\mu}|$ is an S_D-representation. It lies in the irreducible S_D-representation $\mathsf{HWV}_{\mu^*}\left(\bigotimes^D W^*\right)$ and hence equals it. □

The following claim gives a recipe to find highest weight vectors in $\mathsf{Sym}^d\mathsf{Sym}^n(\mathbb{C}^{n^2})^*$.

4.2.13 Claim. *The tensors $\langle\widehat{\mu}|\pi\mathcal{P}_{d[n]}$ with $\pi \in \mathsf{S}_{dn}$ generate the vector space $\mathsf{HWV}_{\mu^*}(\mathsf{Sym}^d\mathsf{Sym}^n(\mathbb{C}^N)^*)$.*

Proof. Follows from Claim 4.2.11, Lemma 4.1.8, and the fact that $\mathcal{P}_d[n]$ is a GL_N-morphism. □

The Tensor Scenario. In analogy to the above discussion, we now treat the tensor scenario. For a partition triple $\lambda \vdash^*_n d$ we define

$$\langle\widehat{\lambda}| := \mathsf{reorder}_{3,n}\left(\langle\widehat{\lambda^{(1)}}| \otimes \langle\widehat{\lambda^{(2)}}| \otimes \langle\widehat{\lambda^{(3)}}|\right), \tag{4.2.14}$$

where for $a, b \in \mathbb{N}$ the linear isomorphism $\mathsf{reorder}_{a,b}\colon \bigotimes^a\bigotimes^b\mathbb{C}^n \to \bigotimes^b\bigotimes^a\mathbb{C}^n$ is defined on rank 1 tensors as follows:

$$\bigotimes_{i=1}^{a}\left(\bigotimes_{j=1}^{b} v_{ij}\right) \mapsto \bigotimes_{j=1}^{b}\left(\bigotimes_{i=1}^{a} v_{ij}\right), \quad v_{ij} \in \mathbb{C}^n. \tag{4.2.15}$$

Analogously to Claim 4.2.11, $\langle\widehat{\lambda}|$ is a highest weight vector of weight λ^*, where $(g^{(1)}, g^{(2)}, g^{(3)}) \in \mathsf{GL}_n^3$ acts via $g^{(1)} \otimes g^{(2)} \otimes g^{(3)} \otimes g^{(1)} \otimes g^{(2)} \otimes g^{(3)} \otimes \cdots$. In analogy to Claim 4.2.12 and Claim 4.2.13, respectively, we obtain the following two results.

4.2.16 Claim. *The highest weight vector space $\mathsf{HWV}_{\lambda^*}(\bigotimes^d\bigotimes^3(\mathbb{C}^n)^*)$ is generated by $\langle\widehat{\lambda}|\pi$ with $\pi \in \mathsf{S}_d^3$.*

Applying Lemma 4.1.8 to the projection $\mathcal{P}_d\colon \bigotimes^d\bigotimes^3(\mathbb{C}^n)^* \twoheadrightarrow \mathsf{Sym}^d\bigotimes^3(\mathbb{C}^n)^*$ we can draw the following important conclusion.

4.2.17 Claim. *The tensors $\langle\widehat{\lambda}|\pi\mathcal{P}_d$ with $\pi \in \mathsf{S}_d^3$ generate $\mathsf{HWV}_{\lambda^*}(\mathsf{Sym}^d\bigotimes^3(\mathbb{C}^n)^*)$.*

4.2.18 Remark. We can improve on Claim 4.2.16 by giving an explicit basis of the vector space $\mathsf{HWV}_{\lambda^*}(\bigotimes^d\bigotimes^3(\mathbb{C}^n)^*)$: Using Schur-Weyl duality (Theorem 4.2.7) we see $\mathsf{HWV}_{\lambda^*}(\bigotimes^d\bigotimes^3(\mathbb{C}^n)^*) \simeq [\lambda^{(1)}] \otimes [\lambda^{(2)}] \otimes [\lambda^{(3)}]$ has a basis given by tensor products of triples of standard tableaux of shape $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$. Hence a basis of $\mathsf{HWV}_{\lambda^*}(\mathsf{Sym}^d\bigotimes^3(\mathbb{C}^n)^*)$ is given by the S_d-invariants in $[\lambda^{(1)}] \otimes [\lambda^{(2)}] \otimes [\lambda^{(3)}]$. The fundamental problem is that no explicit description of this invariant space is known. We will see that its dimension is the Kronecker coefficient, cf. Proposition 4.4.4.

We have an analogous problem for the vector space $\mathsf{HWV}_{\lambda^*}(\mathsf{Sym}^d\mathsf{Sym}^n(\mathbb{C}^{n^2})^*)$, where the dimension is the plethysm coefficient, treated in the next section. ■

4.3 Plethysm Coefficients

In this section we analyze basic properties of plethysms.

The plethysm $\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^d$ is a GL_d-representation and decomposes as follows:

$$\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^d \simeq \bigoplus_\lambda p_\lambda(d[n])\{\lambda\},$$

where $p_\lambda(d[n]) \in \mathbb{N}$ is called a *plethysm coefficient.* Notation throughout the literature has been inconsistent. What is consistent is the notion of the *outer symmetry* and the *inner symmetry.* This is reflected in our syntax by an outer element d and an inner element n.

The following two questions are of interest in Geometric Complexity Theory and are wide open.

4.3.1 Questions.
(1) Is the function $(\lambda, n, d) \mapsto p_\lambda(d[n])$ in $\#\mathsf{P}$?
(2) Can positivity of $p_\lambda(d[n])$ be decided in polynomial time?

There is no special symbol for the coefficients in the decomposition of $\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^\ell$ for $\ell > d$ because of the following simple lemma.

4.3.2 Lemma. *Let $\lambda \vdash_d dn$ and $\ell \geq d$. We append $\ell - d$ zeros to λ to obtain $\bar\lambda$. Then*

$$\mathrm{mult}_{\bar\lambda}(\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^\ell) = \mathrm{mult}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^d).$$

Proof. Consider $\mathbb{C}^n \subseteq \mathbb{C}^\ell$ be canonically embedded. According to Claim 4.2.13, we have

$$\begin{aligned}\mathrm{span}\{\pi|\widehat{\lambda}\rangle : \pi \in \mathsf{S}_{dn}\} = \mathsf{HWV}_\lambda(\bigotimes{}^{dn}\mathbb{C}^d) &\overset{(*)}{\subseteq} \mathsf{HWV}_{\bar\lambda}(\bigotimes{}^{dn}\mathbb{C}^\ell) \\ &= \mathrm{span}\{\pi|\widehat{\bar\lambda}\rangle : \pi \in \mathsf{S}_{dn}\} = \mathrm{span}\{\pi|\widehat{\lambda}\rangle : \pi \in \mathsf{S}_{dn}\}\end{aligned}$$

and hence $(*)$ is an equality. Applying the projection $\mathcal{P}_{d[n]}$ one obtains the desired result. ☐

The next lemma gives necessary conditions for the positivity of plethysm coefficients.

4.3.3 Lemma. *If $p_\lambda(d[n]) > 0$, then $\lambda \vdash_d dn$.*

Proof. Since we decompose $\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^d$ as a GL_d-representation, only partitions with at most d rows can appear, see Section 4.1.

We now prove $\lambda \vdash dn$. It suffices to show that for all $v \in \mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^d$ we have $\mathrm{diag}_d(\alpha)v = \alpha^{dn}.v$. This is true, because $\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^d \subseteq \bigotimes^{dn}\mathbb{C}^d$ and $g \in \mathsf{GL}_d$ acts on $\bigotimes^{dn}\mathbb{C}^d$ via $g(v_1 \otimes \cdots \otimes v_{dn}) = gv_1 \otimes \cdots \otimes gv_{dn}$ and for $g = \mathrm{diag}_d(\alpha)$ we have $\mathrm{diag}_d(\alpha)v = \alpha.v_1 \otimes \cdots \otimes \alpha.v_{dn}$. The claim follows by multilinearity. ☐

4.3.4 Proposition. *Fix n. The partitions $\{\lambda \mid \exists d : \lambda \vdash_d nd, \ p_\lambda(d[n]) > 0\}$ form a semigroup under addition, i.e., $p_\lambda(d[n]) > 0$ and $p_{\tilde\lambda}(\tilde d[n]) > 0$ imply $p_{\lambda+\tilde\lambda}((d+\tilde d)[n]) > 0$. This semigroup is finitely generated.*

Proof. Since $\mathsf{Sym}^n\mathbb{C}^d$ is a GL_d-variety with coordinate ring $\mathbb{C}[\mathsf{Sym}^n\mathbb{C}^d] = \bigoplus_{\delta\geq 0}\mathsf{Sym}^\delta\mathsf{Sym}^n\mathbb{C}^d$, the lemma is a special case of the following Theorem 4.3.5. ☐

4.3.5 Theorem. *Let $G = \mathsf{GL}_N$ or $G = \mathsf{GL}_{N_1} \times \mathsf{GL}_{N_2} \times \mathsf{GL}_{N_2}$ Given a G-variety Z and let S denote the set of partitions $\lambda \vdash_{N}$ or partition triples $\lambda \vdash_{N_1,N_2,N_3}$, respectively, that satisfy $\{\lambda\} \subseteq \mathbb{C}[Z]$. Then S is a finitely generated semigroup.*

Proof. We have $\{\lambda\} \subseteq \mathbb{C}[Z]$ iff there exists a highest weight vector f of weight λ in $\mathbb{C}[Z]$. Interpreting highest weight vectors as regular functions, we can multiply two highest weight vectors of weight λ and $\tilde{\lambda}$ to get a highest weight vector of weight $\lambda + \tilde{\lambda}$. Hence S is a semigroup.

Let U denote the maximal unipotent group corresponding to G, so $U = U_N$ or $U = U_{N_1} \times U_{N_2} \times U_{N_3}$. According to [Kra85, III.3.2 Satz, p.190], the algebra $\mathbb{C}[Z]^U$ of U-invariants is finitely generated. Decomposing the generators w.r.t. the weight decomposition yields finitely many weight vectors $f_1, \dots, f_r \in \mathbb{C}[Z]^U$ that generate $\mathbb{C}[Z]^U$ as an algebra. If $f \in \mathbb{C}[Z]$ is a highest weight vector of some weight λ, then we can write

$$f = \sum_{i=1}^{s} \beta_i \prod_{j=1}^{r} f_j^{a_{i,j}},$$

where $s \in \mathbb{N}$, $\beta_i \in \mathbb{C}$ and $a_{i,j} \in \mathbb{N}_{\geq 0}$. We do a weight decomposition of the right hand side and obtain i such that $\prod_{j=1}^{r} f_j^{a_{i,j}}$ has weight λ. Since f was an arbitrary highest weight vector, the weights of $f_1, \dots, f_r$ generate the semigroup S. □

Chen, Garsia, and Remmel [CGR84] found a useful algorithm to compute the whole plethysm expansion $\{p_\lambda(d[n]) \mid \lambda \vdash_d dn\}$ for fixed $d, n \in \mathbb{N}$. Their method is implemented in the SCHUR program, online available at `http://sourceforge.net/projects/schur`. A different approach for computing a single plethysm coefficient is given in [Yan98]. In the following special case, the plethysm coefficients are well-known.

4.3.6 Theorem ([Mac95, I.8, Ex. 6(a), p. 138])**.**

$$\mathsf{Sym}^d \mathsf{Sym}^2 \mathbb{C}^d \simeq \bigoplus_{\lambda \vdash d} \{2\lambda\}.$$

The following Theorem 4.3.7 is the most basic form of the so-called *Foulkes conjecture*, which we will analyze in Section 6.1.

4.3.7 Theorem (Hermite reciprocity, [Her54], [FH91, Ex. 6.18], see also [Man98])**.** *The* GL_2*-representations* $\mathsf{Sym}^a \mathsf{Sym}^b \mathbb{C}^2$ *and* $\mathsf{Sym}^b \mathsf{Sym}^a \mathbb{C}^2$ *are isomorphic.*

Very recently Paget and Wildon [PW11] gave some sufficient criteria for the nonvanishing of plethysm coefficients. In Section 6.2 we also prove a criterion for the nonvanishing of plethysm coefficients, conjectured by Weintraub [Wei90].

4.3 (A) Plethysm Coefficients and Weight Spaces

In this subsection we prove two results of Gay [Gay76] and Howe [How87, Prop. 4.3] using highest weight vector theory.

Let $W := \mathbb{C}^{nd}$. Schur-Weyl duality (Theorem 4.2.7) implies that for $\lambda \vdash nd$ a basis of the highest weight vector space $\mathsf{HWV}_\lambda(\bigotimes^{nd} W) \simeq [\lambda]$ is given by the set $F(\lambda, (nd) \times 1)$ of numberings T of shape λ (see Section 4.1). Recall the action of the group algebra $\mathbb{C}[\mathsf{S}_{nd}]$ on $F(\lambda, (nd) \times 1)$, which is induced by the action of S_{nd}. Consider the projection $\mathcal{P}^{\text{inner}}_{n[d]} \in \mathbb{C}[\mathsf{S}_{nd}]$ defined in (4.2.3) (with n and d interchanged) as an element of the group algebra and interpret it as a linear map

$$\mathcal{P}^{\text{inner}}_{n[d]} \colon \mathbb{C}^{F(\lambda,(nd)\times 1)} \twoheadrightarrow \mathbb{C}^{F(\lambda,(nd)\times 1)}.$$

Moreover, recall that $\mathcal{P}^{\text{inner}}_{n[d]}$ projects to the invariant space of the subgroup $\mathsf{S}_d^n \subseteq \mathsf{S}_{nd}$, where S_d^n independently permutes the sets $\{1, \dots, d\}$, $\{d+1, \dots, 2d\}$, and so on. Consider the following linear surjection

$$\varphi \colon \mathbb{C}^{F(\lambda,(nd)\times 1)} \twoheadrightarrow \mathbb{C}^{F(\lambda,n\times d)},$$

$$\begin{array}{|c|c|c|c|c|}\hline 2&1&3&5&8\\\hline 4&7&9&10\\\cline{1-4} 6&12\\\cline{1-2} 11\\\cline{1-1}\end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2&3&3&4\\\cline{1-4} 2&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

Figure 4.3.i: A filling and its image under φ. Here $n = 4, d = 3$.

where to each numbering T we assign the filling $\varphi(T)$ of shape λ and content $n\times d$ obtained by replacing each entry i in T with $\lfloor \frac{i}{d} \rfloor$, see Figure 4.3.i. Observe that $\varphi = \varphi\circ\sigma$ for each $\sigma \in \mathsf{S}_d^n$ and hence $\varphi = \varphi\circ\mathcal{P}^{\text{inner}}_{n[d]}$. Therefore the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}^{F(\lambda,(nd)\times 1)} & \xrightarrow{\ \varphi\ } & \mathbb{C}^{F(\lambda,n\times d)} \\ {\scriptstyle \mathcal{P}^{\text{inner}}_{n[d]}}\downarrow & \nearrow{\scriptstyle \tilde{\varphi}} & \\ \left(\mathbb{C}^{F(\lambda,(nd)\times 1)}\right)^{\mathsf{S}_d^n} & & \end{array}$$

where $\tilde{\varphi}$ is the restriction of φ.

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array} \quad\mapsto\quad \frac{1}{4}\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array} \right)$$

Figure 4.3.ii: The inverse map $\tilde{\varphi}^{-1}$. Here $n = d = 2$.

The linear map $\tilde{\varphi}$ is an isomorphism with a simple inverse map $\tilde{\varphi}^{-1}$, see Figure 4.3.ii. Recall the embedding of $\mathsf{S}_n \simeq Y_{\text{outer}} \subseteq \mathsf{S}_{dn}$ from Subsection 4.2 (A) (with d and n interchanged). This induces an action of S_n on $\mathbb{C}^{F(\lambda,(nd)\times 1)}$. Note that, by construction of the action of S_n, applying elements of S_n to numberings $T \in F(\lambda,(nd)\times 1)$ commutes with applying $\mathcal{P}^{\text{inner}}_{n[d]}$ and also commutes with applying φ. A first consequence is that the invariant space $\left(\mathbb{C}^{F(\lambda,(nd)\times 1)}\right)^{\mathsf{S}_d^n}$ is an S_n-subrepresentation of $\mathbb{C}^{F(\lambda,(nd)\times 1)}$ and $\mathcal{P}^{\text{inner}}_{n[d]}$ is an S_n-morphism. A second consequence is that φ is an S_n-morphism and hence $\tilde{\varphi}$ is an S_n-isomorphism.

It is readily checked that the tableau kernels in the following diagram are S_n-subrepresentations:

$$\begin{array}{ccc} \left(\mathbb{C}^{F(\lambda,(nd)\times 1)}\right)^{\mathsf{S}_d^n} & \xrightarrow{\ \tilde{\varphi}\ } & \mathbb{C}^{F(\lambda,n\times d)} \\ \cup\!\mid & & \cup\!\mid \\ K(\lambda,(nd)\times 1)^{\mathsf{S}_d^n} & & K(\lambda,n\times d) \end{array}$$

Moreover, the restriction of $\tilde{\varphi}$ to $K(\lambda,(nd)\times 1)^{\mathsf{S}_d^n}$ is an S_n-isomorphism between $K(\lambda,(nd)\times 1)^{\mathsf{S}_d^n}$ and $K(\lambda,n\times d)$, which can be checked by applying the definition of the tableau kernel in Section 4.1. Since $\mathcal{P}^{\text{inner}}_{n[d]}$ is S_n-equivariant, after factoring, we obtain an S_n-isomorphism

$$[\lambda]^{\mathsf{S}_d^n} = \left(\mathbb{C}^{F(\lambda,(nd)\times 1)}/K(\lambda,(nd)\times 1)\right)^{\mathsf{S}_d^n} \simeq \mathbb{C}^{F(\lambda,n\times d)}/K(\lambda,n\times d) = \{\lambda\}^{n\times d}.$$

Using Schur-Weyl duality (Theorem 4.2.7) on $[\lambda]^{\mathsf{S}_d^n}$, we obtain

$$\mathsf{HWV}_\lambda(\textstyle\bigotimes^n \mathsf{Sym}^d W) \simeq \{\lambda\}^{n\times d} \qquad (\dagger)$$

as S_n-representations.

A Theorem of Gay. Taking S_n-invariants in (†) we obtain

$$\mathsf{HWV}_\lambda(\mathsf{Sym}^n\mathsf{Sym}^d W) \simeq \left(\{\lambda\}^{n\times d}\right)^{\mathsf{S}_n}.$$

This immediately implies the following theorem.

4.3.8 Theorem ([Gay76])**.** *Let $\lambda \vdash nd$. Then the plethysm coefficient $p_\lambda(n[d])$ can be characterized as the dimension of the space of S_n-invariants of the weight space $\{\lambda\}^{n\times d}$:*

$$p_\lambda(n[d]) = \dim\left(\{\lambda\}^{n\times d}\right)^{\mathsf{S}_n}.$$

We remark that the result in [Gay76] is slightly more general, but it can also be obtained in the same manner as described above, see also [BLMW11, Thm. 8.4.1 and Cor. 8.4.2].

A Theorem of Howe. Let $\lambda = n\times d$. There is only a single semistandard tableau T of shape $n\times d$ with content $n\times d$, see Figure 4.3.iii.

1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4

Figure 4.3.iii: The single semistandard tableau of shape 4×6 and content 4×6.

Hence $\dim\{\lambda\}^{n\times d} = 1$ in this case. It follows from Theorem 4.3.8 that $p_{n\times d}(n[d]) = 1$ if T is S_n-invariant and $p_{n\times d}(n[d]) = 0$ if it is not. Since T is easy to straighten (see Def. 4.1.2), we can readily see that T is S_n-invariant iff d is even. Hence we have shown the following theorem.

4.3.9 Theorem ([How87, Prop. 4.3])**.** *For all n and d we have*

$$p_{n\times d}(n[d]) = \begin{cases} 1 & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases}.$$

Note that, for $d = 2$, Theorem 4.3.9 is a special case of Theorem 4.3.6.

We now present a second derivation of Theorem 4.3.9 from Theorem 4.3.8. The 1-dimensional GL_n-representation $\{n\times d\}$ corresponds to the dth power of the determinant, so $gw = \det(g)^d w$ for all $w \in \{n\times d\}$. In particular, if g is a permutation matrix we get $gw = \mathrm{sgn}(g)^d w$. We have $\{n\times d\} \simeq \{n\times d\}^{n\times d}$ as S_n-representations. Hence $\dim\left(\{n\times d\}^{n\times d}\right)^{\mathsf{S}_n} = 1$ if d is even and $\dim\left(\{n\times d\}^{n\times d}\right)^{\mathsf{S}_n} = 0$ otherwise. Theorem 4.3.9 now again follows from Theorem 4.3.8.

4.4 Kronecker Coefficients

In this section we give a detailed introduction to Kronecker coefficients.

Let $\lambda, \mu \vdash d$. The tensor product $[\lambda]\otimes[\mu]$ is an S_d-representation via the diagonal embedding $\pi \mapsto (\pi,\pi)$, $\pi \in \mathsf{S}_d$. This S_d-representation decomposes as follows

$$[\lambda]\otimes[\mu] \simeq \bigoplus_{\nu\vdash d} k(\lambda;\mu;\nu)[\nu],$$

where the coefficients $k(\lambda;\mu;\nu)$ are called the *Kronecker coefficients*. Kronecker coefficients are **#P**-hard to compute, see [BI08] and [BOR09]. However, for fixed lengths of λ, μ and ν, the Kronecker coefficient $k(\lambda;\mu;\nu)$ can be computed in polynomial time, which is a result of Christandl, Doran, and Walter [CDW12, Cor. 1.4]. We use a program that multiplies characters of the symmetric group to compute Kronecker coefficients, which was written by Harm Derksen and adjusted by the author and Jesko Hüttenhain. We will refer to this program as DERKSEN. The function $(\lambda,\mu,\nu)\mapsto k(\lambda;\mu;\nu)$ is not known to lie in **#P**, which roughly means that there is no satisfying combinatorial description of objects that are counted by the Kronecker coefficient. In fact, this is a major open problem, stated as *Problem 10* by Stanley in [Sta00]. Only partial results have been obtained so far. Lascoux [Las80], Remmel [Rem89, Rem92], Remmel and Whitehead [RW94], and Rosas [Ros01] gave combinatorial interpretations of the Kronecker coefficients where two of the partitions have only two rows or are of hook shape. Recently, Ballantine and Orellana [BO07] managed to describe $k(\lambda;\mu;\nu)$ in the case where $\mu=(d-i,i)$ has a two row shape and the diagram of λ is not contained inside the $2(i-1)\times 2(i-1)$ square. More properties and formulas for specific cases can be found for example in [CM93, Dvi93, Val00, AAV12].

The *symmetric Kronecker coefficients* $\mathsf{sk}\big(\nu;(\lambda)^2\big)$ are defined in [BLMW11] via

$$\mathsf{Sym}^2[\lambda]\simeq\bigoplus_{\nu\vdash d}\mathsf{sk}\big(\nu;(\lambda)^2\big)[\nu] \tag{4.4.1}$$

The following questions are of interest in Geometric Complexity Theory (see Chapter 5) and are wide open.

4.4.2 Questions.
*(1) Is the function $(\lambda,\mu,\nu)\mapsto k(\lambda;\mu;\nu)$ in **#P**?*
(2) Can positivity of $k(\lambda;\mu;\nu)$ be decided in polynomial time?
*(3) Is the function $(\lambda,n,d)\mapsto\mathsf{sk}\big(\lambda;(n{\times}d)^2\big)$ in **#P**?*
(4) Can positivity of $\mathsf{sk}\big(\lambda;(n{\times}d)^2\big)$ be decided in polynomial time?

For the analysis of Kronecker coefficients we start out with a simple lemma.

4.4.3 Lemma. *For $\lambda,\mu\vdash d$ we know the dimension of the space of S_d-invariants:*

$$\dim([\lambda]\otimes[\mu])^{\mathsf{S}_d}=\begin{cases}1 & \text{if } \lambda=\mu,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Claim 4.1.12 we have $[\lambda]\otimes[\mu]\simeq[\lambda]^*\otimes[\mu]\simeq\mathrm{Hom}([\lambda],[\mu])$, where $\mathrm{Hom}([\lambda],[\mu])$ denotes the vector space of linear maps from $[\lambda]$ to $[\mu]$. It is crucial that $\mathrm{Hom}([\lambda],[\mu])$ is an S_d-representation via

$$(\pi\varphi)(v):=\pi\varphi(\pi^{-1}v),$$

for $\pi\in\mathsf{S}_d$, $\varphi\in\mathrm{Hom}([\lambda],[\mu])$, and $v\in[\lambda]$. Let $\mathrm{Hom}_{\mathsf{S}_d}([\lambda],[\mu])$ denote the vector space of S_d-homomorphisms from $[\lambda]$ to $[\mu]$. By definition, we have $(\mathrm{Hom}([\lambda],[\mu]))^{\mathsf{S}_d}=\mathrm{Hom}_{\mathsf{S}_d}([\lambda],[\mu])$. But $\dim(\mathrm{Hom}_{\mathsf{S}_d}([\lambda],[\mu]))\in\{0,1\}$ by Schur's Lemma 3.3.3. Moreover, Schur's lemma also implies $\dim(\mathrm{Hom}_{\mathsf{S}_d}([\lambda],[\mu]))=1$ iff $\lambda=\mu$, as desired. □

The next lemma gives a more symmetric definition of the Kronecker coefficient.

4.4.4 Proposition. $k(\lambda;\mu;\nu)=\dim([\lambda]\otimes[\mu]\otimes[\nu])^{\mathsf{S}_d}$

Proof. We have

$$[\lambda] \otimes [\mu] \simeq \bigoplus_{\nu \vdash d} k(\lambda;\mu;\nu)[\nu]$$

and hence, tensoring with $[\varrho]$, $\varrho \vdash d$, we get

$$([\lambda] \otimes [\mu] \otimes [\varrho])^{\mathsf{S}_d} \simeq \bigoplus_{\nu \vdash d} k(\lambda;\mu;\nu)\,([\nu] \otimes [\varrho])^{\mathsf{S}_d} \overset{\text{Lem. 4.4.3}}{=} k(\lambda;\mu;\varrho)\cdot\mathbb{C},$$

as claimed. □

Analogously we get the following characterization of the symmetric Kronecker coefficient.

4.4.5 Proposition. $\mathsf{sk}\big(\lambda; (\mu)^2\big) = \dim([\lambda]\otimes[\mu]\otimes[\mu])^{\mathsf{S}_d\times\mathsf{S}_2}$, *where the transposition* $\tau \in \mathsf{S}_2$ *acts via* $\tau(w^{(1)} \otimes w^{(2)} \otimes w^{(3)}) := w^{(1)} \otimes w^{(3)} \otimes w^{(2)}$.

Using Proposition 4.4.4 and the fact that there are canonical S_d-isomorphisms $[\lambda]\otimes[\mu]\otimes[\nu] \simeq [\mu]\otimes[\lambda]\otimes[\nu]$ and $[\lambda]\otimes[\mu]\otimes[\nu] \simeq [\mu]\otimes[\nu]\otimes[\lambda]$ implies that $k(\lambda;\mu;\nu)$ is symmetric in all three parameters.

4.4.6 Corollary. *For all partitions* $\lambda, \mu \vdash d$ *we have that* $k(d\times 1;\lambda;\mu) > 0$ *implies* ${}^t\lambda = \mu$.

Proof. We use Proposition 4.4.4 and the fact that $[d\times 1] \otimes [\lambda] \simeq [{}^t\lambda]$.

$$k(d\times 1;\lambda;\mu) \overset{\text{Prop. 4.4.4}}{=} ([d\times 1]\otimes[\lambda]\otimes[\mu])^{\mathsf{S}_d} = ([{}^t\lambda]\otimes[\mu])^{\mathsf{S}_d} \overset{\text{Lem. 4.4.3}}{=} \begin{cases} \mathbb{C}, & \text{if } {}^t\lambda = \mu \\ 0, & \text{otherwise.} \end{cases}$$

□

4.4.7 Lemma. $k(\lambda;\mu;\nu) = k({}^t\lambda;{}^t\mu;\nu) = k({}^t\lambda;\mu;{}^t\nu) = k(\lambda;{}^t\mu;{}^t\nu)$

Proof. We use Proposition 4.4.4 and the fact that $[d\times 1] \otimes [\lambda] \simeq [{}^t\lambda]$. Recall that $[d\times 1]\otimes[d\times 1]$ is the trivial representation. We calculate $k(\lambda;\mu;\nu) = \dim([\lambda]\otimes[\mu]\otimes[\nu])^{\mathsf{S}_d} = \dim([\lambda]\otimes([d\times 1]\otimes[d\times 1])\otimes[\mu]\otimes[\nu])^{\mathsf{S}_d} = \dim(([\lambda]\otimes[d\times 1])\otimes([d\times 1]\otimes[\mu])\otimes[\nu])^{\mathsf{S}_d} = \dim([{}^t\lambda]\otimes[{}^t\mu]\otimes[\nu])^{\mathsf{S}_d} = k({}^t\lambda;{}^t\mu;\nu)$. □

The Kronecker coefficients also appear when studying the general linear group, as the following proposition shows.

4.4.8 Proposition. *The vector space of symmetric powers* $\mathsf{Sym}^d(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)$ *decomposes as a* $\mathsf{GL}_a\times\mathsf{GL}_b\times\mathsf{GL}_c$*-representation as follows:*

$$\mathsf{Sym}^d(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c) \simeq \bigoplus_{\lambda,\mu,\nu \vdash_{a,b,c} d} k(\lambda;\mu;\nu)\{\lambda\}\otimes\{\mu\}\otimes\{\nu\}.$$

Proof. We make essential use of Schur-Weyl duality (Thm. 4.2.7).

$$\begin{aligned}
\mathsf{Sym}^d(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c) &= \Big(\bigotimes\nolimits^d(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)\Big)^{\mathsf{S}_d} \simeq \Big(\bigotimes\nolimits^d\mathbb{C}^a\otimes\bigotimes\nolimits^d\mathbb{C}^b\otimes\bigotimes\nolimits^d\mathbb{C}^c\Big)^{\mathsf{S}_d} \\
&\overset{\text{Thm. 4.2.7}}{\simeq} \bigoplus_{\lambda,\mu,\nu \vdash_{a,b,c} d} ([\lambda]\otimes\{\lambda\}\otimes[\mu]\otimes\{\mu\}\otimes[\nu]\otimes\{\nu\})^{\mathsf{S}_d} \\
&= \bigoplus_{\lambda,\mu,\nu \vdash_{a,b,c} d} ([\lambda]\otimes[\mu]\otimes[\nu])^{\mathsf{S}_d}\otimes\{\lambda\}\otimes\{\mu\}\otimes\{\nu\} \\
&\overset{\text{Prop. 4.4.4}}{=} \bigoplus_{\lambda,\mu,\nu \vdash_{a,b,c} d} k(\lambda;\mu;\nu)\{\lambda\}\otimes\{\mu\}\otimes\{\nu\}
\end{aligned}$$

□

4.4.9 Corollary. *For partitions $\lambda, \mu, \nu \vdash_{a,b,c} d$, the Kronecker coefficient $k(\lambda;\mu;\nu)$ is the dimension of the highest weight vector space $\mathsf{HWV}_{(\lambda,\mu,\nu)}(\mathsf{Sym}^d(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c))$.*

Proof. According to Proposition 4.4.8 we have

$$\mathsf{HWV}_{(\lambda,\mu,\nu)}(\mathsf{Sym}^d(\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c)) \simeq \bigoplus_{\tilde\lambda,\tilde\mu,\tilde\nu \vdash_{a,b,c} d} k(\tilde\lambda;\tilde\mu;\tilde\nu)\mathsf{HWV}_{(\lambda,\mu,\nu)}(\{\tilde\lambda\}\otimes\{\tilde\mu\}\otimes\{\tilde\nu\}),$$

but $\mathsf{HWV}_{(\lambda,\mu,\nu)}(\{\tilde\lambda\} \otimes \{\tilde\mu\} \otimes \{\tilde\nu\})$ has dimension one, iff $\lambda = \tilde\lambda$, $\mu = \tilde\mu$, and $\nu = \tilde\nu$. Otherwise it has dimension zero. ☐

4.4.10 Proposition. *Fix $a, b, c \in \mathbb{N}$. Partition triples $\lambda, \mu, \nu \vdash_{a,b,c}$ with positive Kronecker coefficients form a semigroup, i.e., $k(\lambda;\mu;\nu) > 0$ and $k(\tilde\lambda;\tilde\mu;\tilde\nu) > 0$ imply $k(\lambda+\tilde\lambda;\mu+\tilde\mu;\nu+\tilde\nu) > 0$. This semigroup is finitely generated.*

Proof. Analogous to Proposition 4.3.4. We use the description of the Kronecker coefficient given by Proposition 4.4.8. Since $\mathbb{C}^a\otimes\mathbb{C}^b\otimes\mathbb{C}^c$ is a $\mathsf{GL}_a\times\mathsf{GL}_b\times\mathsf{GL}_c$-variety with coordinate ring $\bigoplus_{d\geq 0}\mathsf{Sym}^d(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)$, the lemma is a special case of Theorem 4.3.5. ☐

Consider the group homomorphism $\mathsf{GL}_a\times\mathsf{GL}_b \to \mathsf{GL}_{ab}$, $(A, B) \mapsto A\otimes B$. Via this map, each GL_{ab}-representation $\mathscr{V}$ is naturally a $\mathsf{GL}_a\times\mathsf{GL}_b$-representation $\mathscr{V}\downarrow^{\mathsf{GL}_{ab}}_{\mathsf{GL}_a\times\mathsf{GL}_b}$. Using this notation we state the following proposition.

4.4.11 Proposition. *Let $\lambda \vdash_{ab} d$. We have*

$$\{\lambda\}\downarrow^{\mathsf{GL}_{ab}}_{\mathsf{GL}_a\times\mathsf{GL}_b} \simeq \bigoplus_{\mu,\nu\vdash_{a,b} d} k(\lambda;\mu;\nu)\{\mu\}\otimes\{\nu\}.$$

Proof. We make essential use of Schur-Weyl duality (Thm. 4.2.7).

$$\begin{aligned}
\{\lambda\} &\overset{\text{Lem. 4.4.3}}{=} \bigoplus_{\varrho\vdash_{ab} d} \{\varrho\}\otimes([\varrho]\otimes[\lambda])^{\mathsf{S}_d} \overset{\text{Thm. 4.2.7}}{\simeq} \left((\textstyle\bigotimes^n\mathbb{C}^{ab})\otimes[\lambda]\right)^{\mathsf{S}_d} \\
&\simeq \left((\textstyle\bigotimes^n\mathbb{C}^{a})\otimes(\textstyle\bigotimes^n\mathbb{C}^{b})\otimes[\lambda]\right)^{\mathsf{S}_d} \\
&\overset{\substack{\text{Thm. 4.2.7}\\ \text{twice}}}{\simeq} \bigoplus_{\mu,\nu\vdash_{a,b} d} ([\mu]\otimes[\nu]\otimes[\lambda])^{\mathsf{S}_d}\otimes\{\mu\}\otimes\{\nu\} \\
&= \bigoplus_{\mu,\nu\vdash_{a,b} d} k(\lambda;\mu;\nu)(\{\mu\}\otimes\{\nu\}).
\end{aligned}$$ ☐

We draw a quick conclusion.

4.4.12 Corollary. *If $k(\lambda;\mu;\nu) > 0$, then $\ell(\lambda) \leq \ell(\mu)\cdot\ell(\nu)$.*

Proof. Fix $a, b \in \mathbb{N}$. If $\ell(\lambda) > ab$, then the GL_{ab}-representation $\{\lambda\} = 0$ is the zero space. Hence $k(\lambda;\mu;\nu) = 0$ for all $\mu, \nu \vdash_{a,b} n$, according to Proposition 4.4.11. ☐

The 1-dimensional rectangular irreducible GL_{ab}-representation $\{ab{\times}c\}$, which corresponds to the cth power of the determinant, decomposes as follows:

$$\{ab{\times}c\}\downarrow^{\mathsf{GL}_{ab}}_{\mathsf{GL}_a\times\mathsf{GL}_b} \simeq \{a{\times}bc\}\otimes\{b{\times}ac\}. \tag{4.4.13}$$

This can be seen by the fact that the Kronecker product $A\otimes B$ of matrices $A \in \mathbb{C}^{a\times a}$ and $B \in \mathbb{C}^{b\times b}$ satisfies $\det(A\otimes B) = \det(A)^b\cdot\det(B)^a$, because $A\otimes B = (A\otimes \mathrm{id}_b)\cdot(\mathrm{id}_a\otimes B)$.

Furthermore, recall from Section 4.1 that the GL_{ab}-representation $\{\lambda\}\otimes\{ab{\times}c\} \simeq \{\lambda + ab{\times}c\}$ is irreducible, even for negative c. Proposition 4.4.11 and (4.4.13) immediately imply the following result.

4.4.14 Corollary. *For $\lambda \vdash_{ab} d$, $\mu \vdash_{a} d$, $\nu \vdash_{b} d$ we have:*

$$k(\lambda; \mu; \nu) = k(\lambda + ab{\times}c; \mu + a{\times}bc; \nu + b{\times}ac).$$

For more information on the discovery of Corollary 4.4.14, see [Val09].

For $\lambda = (\lambda_1, \ldots, \lambda_D)$ we write $-_D\lambda := (-\lambda_D, \ldots, -\lambda_1)$ and $\mu -_D \lambda := \mu + (-_D\lambda)$. In particular, if $\lambda \subseteq D{\times}a$, then $D{\times}a -_D \lambda$ describes the so-called *complementary partition*, which is the partition obtained by removing λ from the lower right of $D{\times}a$, see Figure 4.4.i. Note that as a GL_D-representation, $\{-_D\lambda\} = \{\lambda\}^*$ is the representation dual to $\{\lambda\}$.

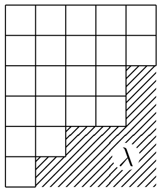


Figure 4.4.i: The complementary partition to $\lambda = (4, 3, 1, 1)$. Here $D = 6$, $a = 5$.

4.4.15 Corollary. *Fix $a, b, c \in \mathbb{N}$. Let $\lambda, \mu, \nu \vdash_{bc,ac,ab} d$ such that $\lambda \subseteq bc{\times}a$, $\mu \subseteq ac{\times}b$, and $\nu \subseteq ab{\times}c$. Then*

$$k(\lambda; \mu; \nu) = k(bc{\times}a -_{bc} \lambda; ac{\times}b -_{ac} \mu; ab{\times}c -_{ab} \nu).$$

Proof.

$$\begin{aligned} k(\lambda; \mu; \nu) &\overset{\text{Lem. 4.4.7}}{=} k(\lambda; {}^t\mu; {}^t\nu) \overset{\text{Prop. 4.4.11}}{=} \mathrm{mult}_{({}^t\mu, {}^t\nu)}\{\lambda\} \overset{\text{dual}}{=} \mathrm{mult}_{(-_b{}^t\mu, -_c{}^t\nu)}\{-_{bc}\lambda\} \\ &\overset{\text{Cor. 4.4.14}}{=} k(bc{\times}a -_{bc} \lambda; b{\times}ac -_b {}^t\mu; c{\times}ab -_c {}^t\nu) \\ &\overset{\text{Lem. 4.4.7}}{=} k(bc{\times}a -_{bc} \lambda; ac{\times}b -_{ac} \mu; ab{\times}c -_{ab} \nu). \end{aligned}$$

□

Corollary 4.4.15 seems to be a result that has not yet been present in the literature. Its discovery is the author's joint work with Peter Bürgisser, Matthias Christandl, Jesko Hüttenhain, and Michael Walter.

4.5 Littlewood-Richardson Coefficients

Fix $n \in \mathbb{N}$ and let $\lambda, \mu \vdash_n a, b$. The tensor product $\{\lambda\} \otimes \{\mu\}$ is a GL_n-representation via the diagonal embedding $g \mapsto (g, g)$, $g \in \mathsf{GL}_n$. This GL_n-representation decomposes as

$$\{\lambda\} \otimes \{\mu\} \simeq \bigoplus_{\nu \vdash_n |\lambda|+|\mu|} c_{\lambda\mu}^{\nu} \{\nu\} \tag{4.5.1}$$

where the coefficients $c_{\lambda\mu}^{\nu}$ are called the *Littlewood-Richardson coefficients*, see e.g. [Sta99, Sec. 7.18 and A.1.3]. Note that a priori $c_{\lambda\mu}^{\nu}$ depends on n. The forthcoming Remark 4.5.5 shows that $c_{\lambda\mu}^{\nu}$ does not change if we append $M - n$ zeros to λ, μ, ν, respectively, and consider the decomposition of the corresponding GL_M-representations ($M \geq n$).

We will see that Littlewood-Richardson are like some additive counterpart of Kronecker coefficients (cp. Prop. 4.4.11 with Prop. 4.5.9), but they also appear as a special case of Kronecker coefficients (see Theorem 7.4.2). The fact that Littlewood-Richardson coefficients are much better understood than Kronecker coefficients makes them an interesting element of study in Geometric Complexity Theory. We start with proving some representation theoretic properties of Littlewood-Richardson coefficients, postponing the involved combinatorial analyses to Part II.

The fact that only partitions with $|\lambda| + |\mu|$ boxes appear on the right hand side of (4.5.1) is easily seen: For $\alpha \in \mathbb{C}^\times$, $v \in \{\lambda\}$, and $w \in \{\mu\}$ we have

$$\begin{aligned}\operatorname{diag}_n(\alpha)(v \otimes w) &= (\operatorname{diag}_n(\alpha)v) \otimes (\operatorname{diag}_n(\alpha)w) = (\alpha^{|\lambda|}.v) \otimes (\alpha^{|\mu|}.w) \\ &= \alpha^{|\lambda|+|\mu|}.(v \otimes w).\end{aligned}$$

Analogously to Lemma 4.4.3 we get the following easy lemma.

4.5.2 Lemma. *For n-generalized partitions λ and μ we know the dimension of the space of GL_n-invariants:*

$$\dim(\{\lambda\} \otimes \{\mu\})^{\mathsf{GL}_n} = \begin{cases} 1 & \text{if } \lambda = \mu^*, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have to take the dual representation here, because unlike S_d-representations, GL_n-representations are not self-dual. We can now describe the Littlewood-Richardson coefficient in terms of invariants, analogous to Proposition 4.4.4, as follows.

4.5.3 Proposition. *For $\lambda, \mu, \nu \vdash_{\overline{n}}$ such that $|\nu| = |\lambda| + |\mu|$ we have*

$$c^\nu_{\lambda\mu} = \dim(\{\lambda\} \otimes \{\mu\} \otimes \{\nu^*\})^{\mathsf{GL}_n}.$$

Proof. We have by definition

$$\{\lambda\} \otimes \{\mu\} \simeq \bigoplus_{\nu \vdash_{\overline{n}} |\lambda|+|\mu|} c^\nu_{\lambda\mu}\{\nu\}$$

and hence, tensoring with the dual representation $\{\varrho\}^*$, $\varrho \vdash_{\overline{n}} |\lambda| + |\mu|$, we get

$$(\{\lambda\} \otimes \{\mu\} \otimes \{\varrho^*\})^{\mathsf{GL}_n} \simeq \bigoplus_{\nu \vdash_{\overline{n}} |\lambda|+|\mu|} c^\nu_{\lambda\mu} \left(\{\nu\} \otimes \{\varrho^*\}\right)^{\mathsf{GL}_n} \overset{\text{Lem. 4.5.2}}{=} c^\varrho_{\lambda\mu} \cdot \mathbb{C},$$

as claimed. □

Proposition 4.5.3 enables us to identify another situation where Littlewood-Richardson coefficients arise, analogously to Proposition 4.4.11.

4.5.4 Proposition. *For $\nu \vdash a + b$ and $n \geq \ell(\nu)$ we have*

$$[\nu]\downarrow^{\mathsf{S}_{a+b}}_{\mathsf{S}_a \times \mathsf{S}_b} \simeq \bigoplus_{\lambda,\mu \vdash_{\overline{n}} a,b} c^\nu_{\lambda\mu}([\lambda] \otimes [\mu]).$$

Proof. We use Schur-Weyl duality (Thm. 4.2.7). Let $n \geq \ell(\nu)$.

$$\begin{aligned}[\nu] &\overset{\text{Lem. 4.5.2}}{\simeq} \bigoplus_{\varrho \vdash_{\overline{n}} a+b} [\varrho] \otimes (\{\varrho\} \otimes \{\nu\}^*)^{\mathsf{GL}_n} \overset{\text{Thm. 4.2.7}}{\simeq} \left((\textstyle\bigotimes^{a+b}\mathbb{C}^n) \otimes \{\nu\}^*\right)^{\mathsf{GL}_n} \\ &= \left((\textstyle\bigotimes^a\mathbb{C}^n) \otimes (\textstyle\bigotimes^b\mathbb{C}^n) \otimes \{\nu\}^*\right)^{\mathsf{GL}_n} \\ &\overset{\substack{\text{Thm. 4.2.7}\\ \text{twice}}}{\simeq} \bigoplus_{\lambda,\mu \vdash_{\overline{n}} a,b} (\{\lambda\} \otimes \{\mu\} \otimes \{\nu\}^*)^{\mathsf{GL}_n} \otimes [\lambda] \otimes [\mu] \\ &\overset{\text{Lem. 4.5.3}}{\simeq} \bigoplus_{\lambda,\mu \vdash_{\overline{n}} a,b} c^\nu_{\lambda\mu}([\lambda] \otimes [\mu]).\end{aligned}$$

□

Note that the symmetry $[\lambda] \otimes [\mu] \simeq [\mu] \otimes [\lambda]$ and Proposition 4.5.4 directly imply $c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}$.

4.5.5 Remark. Recall from (4.5.1) that the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is defined only for $\ell(\nu) \leq n$. From Proposition 4.5.4 we see that for $\ell(\nu) \leq n$ the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is independent of n. This justifies omitting n in the syntax $c^{\nu}_{\lambda\mu}$. ∎

We draw a quick conclusion from Remark 4.5.5.

4.5.6 Corollary. *For $\nu \vdash |\lambda| + |\mu|$, if $c^{\nu}_{\lambda\mu} > 0$, then $\ell(\nu) \geq \max\{\ell(\lambda), \ell(\mu)\}$.*

Proof. W.l.o.g. let $\ell(\lambda) > \ell(\nu)$. If we set $n := \ell(\nu)$, then the left hand side of (4.5.1) vanishes and hence $c^{\nu}_{\lambda\mu} = 0$. □

Corollary 4.5.6 lets us write Proposition 4.5.4 without the dependency on n: For $\nu \vdash a + b$ we have

$$[\nu]\!\downarrow^{\mathsf{S}_{a+b}}_{\mathsf{S}_a \times \mathsf{S}_b} \simeq \bigoplus_{\lambda,\mu \vdash a,b} c^{\nu}_{\lambda\mu}([\lambda] \otimes [\mu]). \tag{4.5.7}$$

4.5.8 Corollary. *For partitions λ, μ, ν satisfying $|\nu| = |\lambda| + |\mu|$ we have $c^{\nu}_{\lambda\mu} = c^{{}^t\nu}_{{}^t\lambda\,{}^t\mu}$.*

Proof. Note that the 1-dimensional alternating representation $[(a+b)\times 1]$ decomposes as $[(a+b)\times 1]\!\downarrow^{\mathsf{S}_{a+b}}_{\mathsf{S}_a \times \mathsf{S}_b} \simeq [a\times 1] \otimes [b\times 1]$. Recall that $[{}^t\nu] \simeq [\nu] \otimes [(a+b)\times 1]$ for $\nu \vdash a + b$. Using (4.5.7) we obtain for $\nu \vdash a + b$

$$\begin{aligned}[{}^t\nu]\!\downarrow^{\mathsf{S}_{a+b}}_{\mathsf{S}_a \times \mathsf{S}_b} &\simeq [(a+b)\times 1]\!\downarrow^{\mathsf{S}_{a+b}}_{\mathsf{S}_a \times \mathsf{S}_b} \otimes [\nu]\!\downarrow^{\mathsf{S}_{a+b}}_{\mathsf{S}_a \times \mathsf{S}_b} \simeq [a\times 1] \otimes [b\times 1] \otimes \bigoplus_{\lambda,\mu \vdash a,b} c^{\nu}_{\lambda\mu}([\lambda] \otimes [\mu]) \\ &\simeq \bigoplus_{\lambda,\mu \vdash a,b} c^{\nu}_{\lambda\mu}([{}^t\lambda] \otimes [{}^t\mu])\end{aligned}$$

and hence $c^{\nu}_{\lambda\mu} = c^{{}^t\nu}_{{}^t\lambda\,{}^t\mu}$, according to (4.5.7). □

One further, pretty prominent example where the Littlewood-Richardson coefficients appear is the following.

4.5.9 Proposition (e.g. [Ful97, §8.3 (20), p. 122])**.** *Consider the embedding $\mathsf{GL}_a \times \mathsf{GL}_b \subseteq \mathsf{GL}_{a+b}$ via block diagonal matrices. Then the irreducible GL_{a+b}-representation $\{\nu\}$, $|\nu| = a + b$, decomposes as follows:*

$$\{\nu\}\!\downarrow^{\mathsf{GL}_{a+b}}_{\mathsf{GL}_a \times \mathsf{GL}_b} \simeq \bigoplus_{\substack{\lambda,\mu \vdash_{\overline{a,b}} \\ |\lambda|+|\mu|=a+b}} c^{\nu}_{\lambda\mu}\{\lambda\} \otimes \{\mu\}.$$

We omit the proof, but deduce the following corollary.

4.5.10 Corollary. $c^{\nu}_{\lambda(i)} = 1$, *if λ can be obtained from ν by removing i boxes, at most one in each column. Otherwise, $c^{\nu}_{\lambda(i)} = 0$.*

Proof. Note that $\{(i)\}$ is the unique irreducible GL_1-representation with i boxes.

$$\{\nu\}\!\downarrow^{\mathsf{GL}_{a+1}}_{\mathsf{GL}_a \times \mathsf{GL}_1} \overset{\text{Prop. 4.5.9}}{\simeq} \bigoplus_{\substack{\lambda,\mu \vdash_{\overline{a,1}} \\ |\lambda|+|\mu|=a+1}} c^{\nu}_{\lambda\mu}\{\lambda\} \otimes \{\mu\} \simeq \bigoplus_{i=0}^{a+1} \bigoplus_{\substack{\lambda \vdash_{\overline{a}} \\ |\lambda|=a+1-i}} c^{\nu}_{\lambda,(i)}\{\lambda\} \otimes \{(i)\}.$$

The claim now follows from Theorem 4.1.9. □

The next corollary treats the special case of hook partitions.

4.5.11 Corollary. *For $a, b \in \mathbb{N}$ we have $[a \ulcorner b]\!\downarrow^{\mathsf{S}_{a+b+1}}_{\mathsf{S}_{a+b}} \simeq [(a-1) \ulcorner b] \oplus [a \ulcorner (b-1)]$.*

Proof. We rewrite the statement as

$$[a \ulcorner b]\downarrow^{\mathsf{S}_{a+b+1}}_{\mathsf{S}_{a+b}\times\mathsf{S}_1} \simeq [(a-1) \ulcorner b] \otimes [(1)] \oplus [a \ulcorner (b-1)] \otimes [(1)].$$

According to Proposition 4.5.4, it remains to show that $c^{a \ulcorner b}_{\lambda,(1)} = 1$ iff $\lambda = (a-1) \ulcorner b$ or $\lambda = a \ulcorner (b-1)$. This is true by Corollary 4.5.10 for $i = 1$. □

In the light of Proposition 4.5.9, Littlewood-Richardson coefficients are like some additive counterpart to the Kronecker coefficients (cp. Prop. 4.4.11): Kronecker coefficients can be defined as the decomposition coefficients arising from the map $\mathsf{GL}_a \times \mathsf{GL}_b \to \mathsf{GL}_{ab}$, while Littlewood-Richardson coefficients arise from the map $\mathsf{GL}_a \times \mathsf{GL}_b \to \mathsf{GL}_{a+b}$. We will carry out an in-depth study of combinatorial properties of Littlewood-Richardson coefficients in Part II.

Chapter 5

Coordinate Rings of Orbits

Recall the scenarios from Section 2.6. We want to prove the existence of obstructions via an orbit-wise upper bound proof as described in Section 3.4 using (3.4.5). To achieve this we need to understand the stabilizer $\mathsf{stab}_G(\mathcal{c}_n)$ (Section 5.1) and understand the dimension of the $\mathsf{stab}_G(\mathcal{c}_n)$-invariant space $\{\lambda\}^{\mathsf{stab}_G(\mathcal{c}_n)}$ of an irreducible G-representation $\{\lambda\}$ (Section 5.2).

5.1 Stabilizers

In this section we describe the stabilizers for both scenarios in Section 2.6, namely for the unit tensor, the matrix multiplication tensor, the determinant polynomial and the permanent polynomial. The main sources are [BI11] (with the more detailed version [BI10]) and [BLMW11] and we closely follow the presentation therein.

Let $G \overset{\iota}{\hookrightarrow} \mathsf{GL}_N$ be a linear algebraic group. Then we can define the group morphism

$$\det\colon G \to \mathbb{C}^\times,\ g \mapsto \det\bigl(\iota(g)\bigr). \tag{5.1.1}$$

Note that $\det \in \mathbb{C}[G]$. We remark that the function det depends on the embedding ι. In the polynomial scenario we have $G = \mathsf{GL}_{n^2}$, $N = n^2$. In the tensor scenario we embed $G = \mathsf{GL}_n^3 \subseteq \mathsf{GL}_{3n}$ as block diagonal matrices, $N = 3n$.

Choose a point $v \in V$ and set $H := \mathsf{stab}_G(v)$. In our case, let $v = \det_n$, $v = \mathrm{per}_n$, $v = \mathcal{E}_n$, or $v = \mathcal{M}_n$. Let $o := o(v) \in \mathbb{N}_{>0} \cup \{\infty\}$ be minimal with the property $\det(h)^o = 1$ for all $h \in H$. In other words, o is the *exponent* of the subgroup $\det(H) \subseteq \mathbb{C}^\times$. Note that $o(v) = 2$ implies $\det(h) = \pm 1$ for all $h \in \mathsf{stab}_G(v)$. In our four cases, we have $o \in \{1, 2\}$, as we will see in Subsections 5.1 (A) and 5.1 (B).

Let $\epsilon \in G$ denote the neutral element. Recall the $G \times G$-action on $\mathbb{C}[G]$ from Subsection 3.4 (A). Note that $(\epsilon, h)\det^o(g) := \det^o(gh) = \det^o(g)\det^o(h) = \det^o(g)$ and hence $\det^o \in \mathbb{C}[G]^{\vec{H}}$. Using Theorem 3.4.3, we obtain a regular function

$$\det_v^o \in \mathbb{C}[Gv],\quad \det_v^o(gv) := \det^o(g). \tag{5.1.2}$$

Caveat: If $o > 1$, then no function $\det_v \in \mathbb{C}[Gv]$ with $\det_v(gv) = \det(g)$ exists. Only $\det_v^o$ is well-defined.

5.1 (A) Tensors

In this short subsection we will list the stabilizers of the unit tensor and the matrix multiplication tensor.

Unit Tensor.

5.1.3 Proposition ([BI11, Prop. 4.1])**.** *The stabilizer* $\mathsf{stab}_{\mathsf{GL}_n^3}(\mathcal{E}_n)$ *is the semidirect product of the normal subgroup* $D_n :=$

$\{(\mathrm{diag}(\alpha_1^{(1)},\dots,\alpha_n^{(1)}),\mathrm{diag}(\alpha_1^{(2)},\dots,\alpha_n^{(2)}),\mathrm{diag}(\alpha_1^{(3)},\dots,\alpha_n^{(3)})) \mid \forall i\colon \alpha_i^{(1)}\alpha_i^{(2)}\alpha_i^{(3)} = 1\}$,

and the symmetric group S_n *diagonally embedded in* GL_n^3 *via triples of equal permutation matrices:*

$$\mathsf{stab}_{\mathsf{GL}_n^3}(\mathcal{E}_n) \simeq D_n \rtimes \mathsf{S}_n.$$

We have $o(\mathcal{E}_n) = 2$ *for* $n > 1$.

Matrix Multiplication Tensor. We consider the following morphism of groups

$$\Phi\colon \mathsf{GL}_n^3 \to \mathsf{GL}_{n^2}^3,\ (A_1, A_2, A_3) \mapsto \big({}^tA_1^{-1} \otimes A_2, {}^tA_2^{-1} \otimes A_3, {}^tA_3^{-1} \otimes A_1\big).$$

The kernel of Φ consists of the multiples of the identity matrix triple $(\mathrm{id},\mathrm{id},\mathrm{id}) \in \mathsf{GL}_n^3$.

5.1.4 Proposition ([BI11, Prop. 5.1])**.** *We have*

$$\mathsf{stab}_{\mathsf{GL}_{n^2}}(\mathcal{M}_n) = \mathrm{im}(\Phi) \simeq \mathsf{GL}_n^3/\mathbb{C}^\times$$

and $o(\mathcal{M}_n) = 1$.

5.1 (B) Polynomials

Define the involution $\tau \in \mathsf{GL}_{n^2}$ by $\tau(e \otimes f) = f \otimes e$. In other words, $\tau\colon \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is the transposition of matrices. After choosing standard bases, τ is a $n^2 \times n^2$ permutation matrix with n entries on the main diagonal. Using the fact that τ is an involution, we get that the number of column switches needed to transform the matrix of τ into the unit matrix is exactly $(n^2 - n)/2$. This number is even, iff $n \bmod 4 \in \{0, 1\}$. Therefore $\det(\tau) = 1$, iff $n \bmod 4 \in \{0, 1\}$. Otherwise $\det(\tau) = -1$.

Let $\Theta_n \subseteq \mathbb{C}^\times$ denote the group of nth roots of unity.

Determinant. The subgroup $Q_{\det} := \{g \otimes h \mid g, h \in \mathsf{SL}_n\}$ of GL_{n^2} is defined as the image of $\mathsf{SL}_n \times \mathsf{SL}_n$ under the morphism $(g, h) \mapsto g \otimes h$. The kernel of this morphism equals $\{(\theta\,\mathrm{id}_n, \theta^{-1}\,\mathrm{id}_n) \mid \theta^n = 1\}$, which is isomorphic Θ_n, so that $Q_{\det} \simeq \big(\mathsf{SL}_n \times \mathsf{SL}_n\big)/\Theta_n$.

We note that $\tau(g \otimes h)\tau = h \otimes g$, so τ acts nontrivially on $Q_{\det}$ by conjugation. Let $Q_{\det}\langle\tau\rangle$ denote the subgroup of GL_{n^2} generated by $Q_{\det}$ and τ. Then $Q_{\det}\langle\tau\rangle \simeq Q_{\det} \rtimes \mathbb{Z}_2$ is a nontrivial semidirect product. The following characterization of the stabilizer dates back to Frobenius in 1897.

5.1.5 Proposition ([Fro97, §7, Satz I], see also [BLMW11, eq. (5.2.1)] and references therein)**.**

$$\mathsf{stab}_{\mathsf{GL}_{n^2}}(\det_n) = Q_{\det}\langle\tau\rangle \simeq \big(\mathsf{SL}_n \times \mathsf{SL}_n\big)/\Theta_n \rtimes \mathbb{Z}_2,$$

and in particular $\det(g) = \pm 1$ *for all* $g \in \mathsf{stab}_{\mathsf{GL}_{n^2}}(\det_n)$. *For* $n \bmod 4 \in \{0, 1\}$ *we have* $o(\det_n) = 1$, *while for* $n \bmod 4 \in \{2, 3\}$ *we have* $o(\det_n) = 2$.

Permanent. Let Q_n denote the set of $n\times n$ matrices that have exactly one nonzero entry in each row and column. If we denote by Q_{per} the image of $Q_n \times Q_n$ under $\mathsf{GL}_n \times \mathsf{GL}_n \to \mathsf{GL}_{n^2}$, $(g,h) \mapsto g \otimes h$, then $Q_{\text{per}} \simeq (Q_n \times Q_n)/\Theta_n$. Consider the subgroup $Q_{\text{per}}\langle\tau\rangle \simeq Q_{\text{per}} \rtimes \mathbb{Z}_2$ of GL_{n^2} generated by Q_{per} and the involution τ. The next proposition is due to Marcus and May, see [MM62]. See also [BLMW11, eq. (5.5.1)] and references therein.

5.1.6 Proposition. *For $n > 2$ we have*

$$\mathsf{stab}_{\mathsf{GL}_{n^2}}(\text{per}_n) = Q_{\text{per}}\langle\tau\rangle \simeq \big(Q_n \times Q_n\big)/\Theta_n \rtimes \mathbb{Z}_2$$

and $o(\text{per}_n) = 2$.

5.1.7 Remark. According to Theorem 3.3.5, all four stabilizers described in this Section 5.1 are linearly reductive. ∎

5.2 Branching Rules

In this section we cite the branching rules for our scenarios that can be used to determine the multiplicity $\text{mult}_{\lambda^*}\mathbb{C}[Gv]$, see (3.4.5). For the unit tensor we present the complete proof, since we will be using the result in Section 8.2.

In the graded coordinate ring $\mathbb{C}[\overline{Gv}]$ only nonnegative degrees appear, because $\mathbb{C}[\overline{Gv}]$ is a factor ring of the polynomial ring $\mathbb{C}[V]$. Since $\mathbb{C}[Gv]$ contains fractions of polynomials, negative degrees can appear there. We write $\mathbb{C}[Gv]_{\geq 0} := \bigoplus_{d\geq 0}\mathbb{C}[Gv]_d$.

5.2 (A) Unit Tensor

Let $\mathsf{Par}_n(d)$ denote the set of partitions of d into at most n parts. The *dominance order* $\preceq$ on $\mathsf{Par}_n(d)$ defines a lattice, in particular two partitions λ, μ have a well defined meet $\lambda \curlywedge \mu$ satisfying $\sum_{i=1}^{j}(\lambda \curlywedge \mu)_i = \min(\sum_{i=1}^{j}\lambda_i, \sum_{i=1}^{j}\mu_i)$, cf. [Sta99, p. 288]. Clearly, this definition generalizes to arbitrary $\lambda \in \mathbb{Z}^n$ and finitely many elements of $\mathbb{Z}^n$ also have a well defined meet.

Recall that for $z \in \mathbb{Z}^n$, $\{\lambda\}^z$ denotes the weight subspace of $\{\lambda\}$ to the weight z, cp. Section 4.1.

The following theorem describes the decomposition of $\mathbb{C}[\mathsf{GL}_n^3\mathcal{E}_n]_{\geq 0}$ into irreducibles.

5.2.1 Theorem ([BI11, Thm. 4.4])**.** $\mathbb{C}[\mathsf{GL}_n^3\mathcal{E}_n]_{\geq 0} =$

$$\bigoplus_{d\geq 0}\bigoplus_{\lambda \vdash_n^* d}\Bigg(\sum_{\substack{\mu\vdash_n d\\ \mu\preceq\lambda^{(1)}\curlywedge\lambda^{(2)}\curlywedge\lambda^{(3)}}} \dim\left(\{\lambda^{(1)}\}^{\mu}\otimes\{\lambda^{(2)}\}^{\mu}\otimes\{\lambda^{(3)}\}^{\mu}\right)^{\mathsf{stab}_{\mathsf{S}_n}(\mu)}\Bigg)\cdot\{\lambda^*\}.$$

Proof. Proposition 5.1.3 states that $H := \mathsf{stab}_{\mathsf{GL}_n^3}(\mathcal{E}_n) \simeq D_n \rtimes \mathsf{S}_n$. We use (3.4.5), which implies

$$\text{mult}_{\lambda^*}(\mathbb{C}[\mathsf{GL}_n\mathcal{E}_n]_{\geq 0}) = \dim\{\lambda\}^H$$

for partition triples λ. The further proof strategy is to first determine the invariant space $\{\lambda\}^{D_n}$ and then analyze its S_n-invariants.

Let $\lambda \vdash_n^* d$. The weight decomposition

$$\{\lambda\} = \{\lambda^{(1)}\}\otimes\{\lambda^{(2)}\}\otimes\{\lambda^{(3)}\} = \bigoplus_{z,z',z''\in\mathbb{Z}^n}\{\lambda^{(1)}\}^{z}\otimes\{\lambda^{(2)}\}^{z'}\otimes\{\lambda^{(3)}\}^{z''}$$

yields

$$\{\lambda\}^{D_n} = \bigoplus_{z,z',z''\in\mathbb{Z}^n} \left(\{\lambda^{(1)}\}^z \otimes \{\lambda^{(2)}\}^{z'} \otimes \{\lambda^{(3)}\}^{z''}\right)^{D_n}.$$

We claim that

$$\left(\{\lambda^{(1)}\}^z \otimes \{\lambda^{(2)}\}^{z'} \otimes \{\lambda^{(3)}\}^{z''}\right)^{D_n} = \begin{cases} \{\lambda^{(1)}\}^z \otimes \{\lambda^{(2)}\}^z \otimes \{\lambda^{(3)}\}^z & \text{if } z = z' = z'' \\ 0 & \text{otherwise.} \end{cases} \qquad (*)$$

Indeed, let $v \in (\{\lambda^{(1)}\}^z \otimes \{\lambda^{(2)}\}^{z'} \otimes \{\lambda^{(3)}\}^{z''})^{D_n}$ be nonzero. Let $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)} \in (\mathbb{C}^\times)^n$. We use multi-index notation and write $(\alpha^{(1)})^z := (\alpha_1^{(1)})^{z_1} \cdots (\alpha_n^{(1)})^{z_n}$. For $t = \left(\mathrm{diag}(\alpha^{(1)}), \mathrm{diag}(\alpha^{(2)}), \mathrm{diag}(\alpha^{(3)})\right) \in D_n$ we obtain $v = tv = (\alpha^{(1)})^z(\alpha^{(2)})^{z'}(\alpha^{(3)})^{z''}v = (\alpha^{(1)})^{z-z''}(\alpha^{(2)})^{z'-z''}v$, using $\alpha_i^{(1)}\alpha_i^{(2)}\alpha_i^{(3)} = 1$. Since $\alpha_i^{(1)}$ and $\alpha_i^{(2)} \in \mathbb{C}^\times$ were arbitrary, we infer $z = z' = z''$. This shows $(*)$.

We put now for $z \in \mathbb{Z}^n$

$$W^z := \{\lambda^{(1)}\}^z \otimes \{\lambda^{(2)}\}^z \otimes \{\lambda^{(3)}\}^z.$$

We note that $\pi W^z = W^{\pi z}$ for $\pi \in \mathsf{S}_n$. According to Proposition 4.1.3, we have $\{\lambda^{(k)}\}^z \neq 0$ iff $|z| = |\lambda^{(k)}|$ and $z \preceq \lambda^{(k)}$. Therefore $W^z \neq 0$ iff $z \in Z$, where

$$Z := \left\{z \in \mathbb{Z}^n \;\middle|\; |z| = d,\, z \preceq \lambda_1 \curlywedge \lambda_2 \curlywedge \lambda_3\right\}.$$

If follows $\{\lambda\}^{D_n} = \oplus_{z\in Z} M^z$.

It is easy to see that the set Z is invariant under the canonical S_n-action, which is the permutation of entries. Each S_n-orbit contains exactly one partition. This partition is the biggest element in the S_n-orbit w.r.t. the dominance order. Let $\mathsf{Par} \subseteq Z$ denote the subset of partitions. For $\mu \in \mathsf{Par}$ we set

$$W_\mu := \bigoplus_{z\in \mathsf{S}_n\mu} W^z.$$

We obtain that $\{\lambda\}^{D_n} = \oplus_{\mu\in\mathsf{Par}} W_\mu$. Proposition 5.1.3 tells us $H = D_n\mathsf{S}_n$ and hence

$$\{\lambda\}^H = (\{\lambda\}^{D_n})^{\mathsf{S}_n} = \bigoplus_{\mu\in\mathsf{Par}} (W_\mu)^{\mathsf{S}_n},$$

using that the W_μ are S_n-invariant. In order to complete the proof it suffices to show that

$$\dim(W_\mu)^{\mathsf{S}_n} = \dim(W^\mu)^{\mathsf{stab}_{\mathsf{S}_n}(\mu)}.$$

For proving this, we fix $\mu \in \mathsf{Par}$ and write $H_\mu := \mathsf{stab}_{\mathsf{S}_n}(\mu)$. Let $\pi_1, \ldots, \pi_r$ be a system of representatives for the left cosets of H_μ in S_n with $\pi_1 = \mathrm{id}$. So we get a decomposition $\mathsf{S}_n = \pi_1 H_\mu \,\dot\cup\, \cdots \,\dot\cup\, \pi_r H_\mu$. Then the S_n-orbit of μ equals $\mathsf{S}_n\mu = \{\pi_1\mu, \ldots, \pi_r\mu\}$. Consider

$$W_\mu = \bigoplus_{j=1}^r \pi_j W^\mu$$

and the corresponding projection $p\colon W_\mu \to W^\mu$. Suppose that $v = \sum_j v_j \in (W_\mu)^{\mathsf{S}_n}$ with $v_j \in \pi_j W^\mu$. Since the spaces $\pi_1 W^\mu, \ldots, \pi_r W^\mu$ are permuted by the action of S_n, we derive from $v = \pi v = \sum_j \pi v_j$ that $v_j = \pi_j v_1$ for all j. Moreover, since every $\sigma \in H_\mu$ fixes W^μ, we obtain $\sigma v_1 = v_1$. Therefore, the map $p\colon (W_\mu)^{\mathsf{S}_n} \to (W^\mu)^{H_\mu}, v \mapsto v_1$ is well defined.

We claim that p is injective: Let $v \in (W_\mu)^{\mathsf{S}_n}$ be an arbitrary element with $p(v) = 0$. We write $v = \sum_j v_j$ with $v_j \in \pi_j W^\mu$, so $v_1 = 0$. But $v_j = \pi_j v_1 = 0$ and hence $v = 0$.

We claim that p is also surjective: For showing this, let $v_1 \in (W^\mu)^{H_\mu}$, set $v_j := \pi_j v_1$, and put $v := \sum_j v_j$. Clearly, $p(v) = v_1$. We need to show that v is S_n-invariant. But

$$v = \sum_{j=1}^{r} \pi_j v_1 = \sum_{j=1}^{r} \frac{1}{|H_\mu|} \sum_{h \in H_\mu} \pi_j h v_1 = \frac{1}{|H_\mu|} \sum_{\pi \in \mathsf{S}_n} \pi v_1,$$

which is obviously S_n-invariant. □

We get the following corollary.

5.2.2 Corollary. *Let $\lambda \vdash^*_n d$. If there is a regular $\mu \vdash_n d$, $\mu \preceq \lambda^{(1)} \curlywedge \lambda^{(2)} \curlywedge \lambda^{(3)}$, then $\{\lambda\}^* \subseteq \mathbb{C}[\mathsf{GL}_n^3 \mathcal{E}_n]$.*

Proof. The corollary follows from Theorem 5.2.1 and the fact that the S_n-stabilizer of a regular partition $\mu \vdash_n$ is trivial. □

5.2 (B) Determinant, Permanent and Matrix Multiplication

For the sake of completeness, in this short subsection we cite the other decomposition rules of interest, namely the ones for the determinant, the permanent, and the matrix multiplication tensor.

5.2.3 Theorem ([BLMW11, eq. (5.2.6)])**.**

$$\mathbb{C}[\mathsf{GL}_{n^2} \det_n]_{\geq 0} = \bigoplus_{d \geq 0} \bigoplus_{\lambda \vdash_{n^2} nd} \mathsf{sk}\big(\lambda; (n \times d)^2\big)\{\lambda^*\}.$$

5.2.4 Theorem ([BLMW11, eq. (5.5.2)])**.**

$$\mathbb{C}[\mathsf{GL}_{n^2} \mathrm{per}_n]_{\geq 0} = \bigoplus_{d \geq 0} \bigoplus_{\lambda \vdash_{n^2} dn} \mathrm{mult}_\lambda \{\lambda^*\}$$

with

$$\mathrm{mult}_\lambda = \frac{1}{2} \sum_{\substack{\mu,\nu \vdash_n dn \\ \mu \neq \nu}} k(\lambda; \mu; \nu) p_\mu(n[d]) p_\nu(n[d]) + \sum_{\mu \vdash_n dn} \mathsf{sk}\big(\lambda; (\mu)^2\big) \binom{p_\mu(n[d]) + 1}{2}.$$

5.2.5 Theorem ([BI11, Thm. 5.3])**.** $\mathbb{C}[\mathsf{GL}_{n^2}^3 \mathcal{M}_n]_{\geq 0} =$

$$\bigoplus_{d \geq 0} \bigoplus_{\lambda \vdash^*_{n^2} d} \left(\sum_{\mu \vdash^*_n d} k(\lambda^{(1)}; \mu^{(2)}; \mu^{(3)}) \cdot k(\mu^{(1)}; \lambda^{(2)}; \mu^{(3)}) \cdot k(\mu^{(1)}; \mu^{(2)}; \lambda^{(3)}) \right) \{\lambda^*\}.$$

5.3 Inheritance

Up to now we analyzed the coordinate rings of $\overline{\mathsf{GL}_{m^2} \mathrm{per}_m}$ and $\overline{\mathsf{GL}_{m^2}^3 \mathcal{M}_m}$, but not the ones of the varieties $\overline{\mathsf{GL}_{n^2} z^{n-m} \mathrm{per}_m}$ and $\overline{\mathsf{GL}_{n^2}^3 \mathcal{M}_m}$. In this section we want carry out a small analysis related to these other coordinate rings. We begin with the tensor scenario, the polynomial scenario being highly analogous.

5.3.1 Proposition (Inheritance for tensors, see [BI10, Prop. 3.3, Sec. 10.5], [LM04, Prop. 4.4])**.** *Let $v \in \bigotimes^3 \mathbb{C}^M \subseteq \bigotimes^3 \mathbb{C}^n$ for $n \geq M$.*

(1) Let $\bar\lambda \vdash^*_n d$ *be obtained from* $\lambda \vdash^*_M d$ *by appending zeros. Then*

$$\mathrm{mult}_{\bar\lambda^*}\left(\mathbb{C}[\overline{\mathsf{GL}_n^3 v}]\right) = \mathrm{mult}_{\lambda^*}\left(\mathbb{C}[\overline{\mathsf{GL}_M^3 v}]\right).$$

(2) If a partition triple μ *of* d *satisfies* $\ell(\mu^{(k)}) > M$ *for some* $k \in \{1,2,3\}$*, then*

$$\mathrm{mult}_{\mu^*}\left(\mathbb{C}[\overline{\mathsf{GL}_n^3 v}]\right) = 0.$$

Proof. (1): For functions $f\colon \mathsf{GL}_M^3 v \to \mathbb{C}$ define $f_v\colon \mathsf{GL}_M^3 \to \mathbb{C}$ via $f_v(g) := f(gv)$. Analogously we define $\bar f_v\colon \mathsf{GL}_n^3 \to \mathbb{C}$ for $\bar f\colon \mathsf{GL}_n^3 v \to \mathbb{C}$. In particular, for highest weight vectors $\bar f := \langle\widehat{\lambda}|\pi\mathcal{P}_d \in \mathsf{Sym}^d\bigotimes^3(\mathbb{C}^n)^*$, $\pi \in \mathsf{S}_d^3$ we get that

$$\bar f_v(g) = \langle\widehat{\lambda}|\pi g|v^{\otimes d}\rangle$$

is a polynomial in the entries of g. Let $q\colon \mathsf{GL}_n^3 \to (\mathbb{C}^{M\times M})^3$ denote the projection to the upper left $M\times M$ matrices. The crucial insight is that since $v \in \bigotimes^3\mathbb{C}^M$, we have by by construction $\bar f_v \circ q = \bar f_v$.

It suffices to show that if we take finitely many highest weight vectors $\bar f^i := \langle\widehat{\lambda}|\pi_i\mathcal{P}_d$ such that their restriction to $\mathsf{GL}_n^3 v$ is linearly independent, then the restrictions to $\mathsf{GL}_M^3 v$ are linearly independent as well. Let f^i denote the restriction of $\bar f^i$ to $\mathsf{GL}_M^3 v$ and assume that for some $\alpha_i \in \mathbb{C}$ we have

$$\forall g \in \mathsf{GL}_M^3 : \sum_i \alpha_i f_v^i(g) = 0.$$

By continuity we get

$$\forall g \in (\mathbb{C}^{M\times M})^3 : \sum_i \alpha_i f_v^i(g) = 0,$$

which implies

$$\forall \bar g \in (\mathbb{C}^{n\times n})^3 : \sum_i \alpha_i \bar f_v^i(\bar g) = \sum_i \alpha_i \bar f_v^i(q(\bar g)) = 0,$$

because $\bar f_v^i \circ q = \bar f_v^i$. Since the $\bar f_v^i$ are linearly independent, all α_i vanish.

(2): Assume $\mathrm{mult}_{\mu^*}\mathbb{C}[\overline{\mathsf{GL}_n^3 v}] > 0$. Choose $\bar f \in \mathbb{C}[\overline{\mathsf{GL}_n^3 v}]$ such that $\bar f(v) \neq 0$. We decompose $\bar f$ according to the weight decomposition of $\mathbb{C}[V]$ and obtain at least one weight vector f not vanishing at v of some weight $-z \in (\mathbb{Z}^n)^3$. The weight satisfies $z^{(k)} \preceq \mu^{(k)}$ for all $k \in \{1,2,3\}$, see Proposition 4.1.3. Let $t \in \mathsf{GL}_n^3$ denote a triple of diagonal matrices that have each a 1 at position (i,i) for all $1 \le i \le M$. Then $tv = v$. Hence we have

$$f(v) = f(tv) = (t^{-1}f)(v) = \prod_{k=1}^{3}\prod_{i=M+1}^{n}(t_i^{(k)})^{z_i^{(k)}} f(v).$$

Since $f(v) \neq 0$ and $t_i^{(k)} \in \mathbb{C}^\times$ for $i > M$ was arbitrary, it follows that $z_i^{(k)} = 0$ for all $i > M$. Since $z^{(k)} \preceq \mu^{(k)}$, it follows that $\ell(\mu^{(k)}) \le M$. □

In the polynomial scenario, a completely analoguous argument can be made, which is stated in the following proposition.

5.3.2 Proposition (Inheritance for polynomials, see [MS08], [BLMW11, Sec. 6])**.** *Let* $v \in \mathsf{Sym}^n\mathbb{C}^M \subseteq \mathsf{Sym}^n\mathbb{C}^{n^2}$ *for* $n^2 \ge M$.
(1) Let $\bar\lambda \vdash_{n^2} nd$ *be obtained from* $\lambda \vdash_M nd$ *by appending zeros. Then*

$$\mathrm{mult}_{\bar\lambda^*}\mathbb{C}[\overline{\mathsf{GL}_{n^2} v}] = \mathrm{mult}_{\lambda^*}\mathbb{C}[\overline{\mathsf{GL}_M v}].$$

(2) If $\mu \vdash nd$ satisfies $\ell(\mu) > M$, then

$$\mathrm{mult}_{\mu^*}\, \mathbb{C}[\overline{\mathsf{GL}_{n^2} v}] = 0.$$

Proof. The proof is completely analogous to the one of Proposition 5.3.1. □

5.3.3 Definition. We say that a statement $A(x)$ *holds for a generic point* $x \in \mathbb{A}^\eta$, if $A(x)$ holds for every point x in a nonempty Zariski-open subset of $\mathbb{A}^\eta$. ■

For a partition triple λ we write $k(()\lambda)$ to denote its Kronecker coefficient.

5.3.4 Corollary. *Let $\lambda \vdash^*_M d$ and let $\bar\lambda \vdash^*_n d$ be obtained by appending zeros, ($n \geq M$). For a generic point $v \in \bigotimes^3 \mathbb{C}^M \subseteq \bigotimes^3 \mathbb{C}^n$ we have $\mathrm{mult}_{\bar\lambda^*}\, \mathbb{C}[\overline{\mathsf{GL}_n^3 v}] > 0$ iff $k(\lambda) > 0$.*

Proof. If $k(\lambda) = 0$, then $k(\bar\lambda) = 0$ and hence $\mathrm{mult}_{\bar\lambda^*}\, \mathbb{C}[\overline{\mathsf{GL}_n^3 v}] = 0$ for all $v \in \bigotimes^3 \mathbb{C}^n$, according to Proposition 4.4.8.

Moreover, according to Proposition 4.4.8, if $k(\lambda) > 0$, then there exists a nonzero highest weight vector polynomial $f \in \mathbb{C}[\bigotimes^3 \mathbb{C}^M]$ of weight λ^*. Hence $f(v) \neq 0$ for a generic $v \in \bigotimes^3 \mathbb{C}^M$, in particular f does not vanish completely at $\mathsf{GL}_M^3 v$. It follows $\mathrm{mult}_{\lambda^*}(\mathbb{C}[\overline{\mathsf{GL}_M^3 v}]) > 0$ and with Proposition 5.3.1 we have $\mathrm{mult}_{\bar\lambda^*}(\mathbb{C}[\overline{\mathsf{GL}_n^3 v}]) > 0$ for a generic $v \in \bigotimes^3 \mathbb{C}^M$. □

5.3.5 Corollary. *Let $\lambda \vdash_M nd$ and let $\bar\lambda \vdash_{n^2} nd$ be obtained by appending zeros ($n^2 \geq M$). For a generic point $v \in \mathsf{Sym}^n \mathbb{C}^M \subseteq \mathsf{Sym}^n \mathbb{C}^{n^2}$ we have $\mathrm{mult}_{\bar\lambda^*}\, \mathbb{C}[\overline{\mathsf{GL}_{n^2} v}]_d > 0$ iff $p_\lambda(d[n]) > 0$.*

Proof. Analogous to the proof of Corollary 5.3.4. □

5.3 (A) Generic Tensors

In this subsection we give an example on how to use Theorem 5.2.1. Let $M = 3$, $n = 4$, $\lambda = (3{\times}3)^3$. We have $k(\lambda) = 1$, verifiable with DERKSEN. This implies $\{\lambda\}^* \subseteq \mathbb{C}[\overline{\mathsf{GL}_n^3 v}]$ for a generic $v \in \bigotimes^3 \mathbb{C}^3$ according to Corollary 5.3.4. If we can show that $\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\mathsf{GL}_n^3 \mathcal{E}_4] = 0$, then $\underline{R}(v) > 4$ for a generic $v \in \bigotimes^3 \mathbb{C}^3$, which is optimal, see [Lic85]. The next claim uses Theorem 5.2.1 to show that indeed $\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\mathsf{GL}_n^3 \mathcal{E}_4] = 0$.

5.3.6 Claim. $\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\mathsf{GL}_4^3 \mathcal{E}_4] = 0$.

Proof. Recall the notation of Theorem 5.2.1. We have $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = 3{\times}3$ and $\{\mu \vdash_4 9 \mid \mu \preceq \lambda^{(1)} \curlywedge \lambda^{(2)} \curlywedge \lambda^{(3)}\} = \{(3,3,3), (3,3,2,1), (3,2,2,2)\}$. The weight spaces $\{3{\times}3\}^\mu$ corresponding to these μ are 1-dimensional and they are spanned by the unique semistandard tableaux

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&3\\\hline\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline 3&3&4\\\hline\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline 3&4&4\\\hline\end{array}\,.$$

of shape $3{\times}3$ and content μ, respectively, see Section 4.1. Since the weight spaces $\{3{\times}3\}^\mu$ are 1-dimensional, the group $\mathsf{stab}_{\mathsf{S}_4}(\mu)$ acts on $\{3{\times}3\}^\mu$ trivially or alternatingly. Since $(-1)^3 = -1$ we have $\bigotimes^3 \{3{\times}3\}^\mu \simeq \{3{\times}3\}^\mu$ for the three μ considered. It remains to check for each μ that $\{3{\times}3\}^\mu$ is not invariant under $\mathsf{stab}_{\mathsf{S}_4}(\mu)$.

The group $\mathsf{stab}_{\mathsf{S}_4}(3,3,3) = \mathsf{S}_3$ acts alternatingly on $\{3{\times}3\}^{(3,3,3)}$, cp. the proof of Theorem 4.3.9. The same holds true for $\mathsf{stab}_{\mathsf{S}_4}(3,3,2,1) = \mathsf{S}_2 \times \mathsf{S}_1 \times \mathsf{S}_1$ acting on $\{3{\times}3\}^{(3,3,2,1)}$. Consider now $\mu = (3,2,2,2)$, where $\mathsf{stab}_{\mathsf{S}_4}(\mu) \simeq \mathsf{S}_1 \times \mathsf{S}_3$. As we see from the identification of semistandard tableaux in Figure 5.3.i, the weight spaces $\{3{\times}3\}^{(3,2,2,2)}$ and $\{2{\times}3\}^{(2,2,2)}$ are isomorphic as S_3-representations, where S_3 permutes $\{2,3,4\}$.

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline 3&4&4\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&3\\\hline\end{array}$$

Figure 5.3.i: The identification of semistandard tableaux.

According to Theorem 4.3.8 we have $\dim\left(\{2{\times}3\}^{3\times 2}\right)^{\mathsf{S}_3} = p_{2\times 3}(3[2])$, which is zero according to Theorem 4.3.6, because $2{\times}3 = (3,3)$ is not an even partition. $\square$

5.4 Stability and Exponent of Regularity

Up to this point we can use the beautiful description of (3.4.5) only to get upper bounds on the multiplicities we are interested in, namely on $\mathrm{mult}_\lambda\, \mathbb{C}[\overline{G\mathfrak{c}_n}]$ and $\mathrm{mult}_\lambda\, \mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]$ by using Claim 3.4.1. While upper bounds for $\mathrm{mult}_\lambda\, \mathbb{C}[\overline{G\mathfrak{c}_n}]$ can be sufficient for constructing representation theoretic obstructions, we need lower bounds for $\mathrm{mult}_\lambda\, \mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]$ as well. Although the state of the art for proving these lower bounds is by explicitly constructing polynomials in $\mathbb{C}[\overline{G\mathfrak{h}_{m,n}}]$, there is hope that the relationship between $\mathbb{C}[Gv]$ and $\mathbb{C}[\overline{Gv}]$ is significantly tighter than just the upper bound on multiplicities. In this subsection we want to make this relationship more explicit in both scenarios of Section 2.6. However, we will only need these results Chapter 9.

Fix a subgroup $\tilde{G} \subseteq G \cap \mathsf{SL}_N$ and let $T \subseteq G$ denote a group satisfying $T\tilde{G} = G$. In the polynomial scenario we choose

$$\tilde{G} = \mathsf{SL}_{n^2} \subseteq \mathsf{GL}_{n^2} \text{ and } T = \mathbb{C}^\times \,\mathrm{id}_{n^2}\,.$$

In the tensor scenario we choose

$$\tilde{G} = \mathsf{SL}_n^3 \subseteq \mathsf{GL}_n^3 \text{ and } T = (\mathbb{C}^\times \,\mathrm{id}_n)^3.$$

We now analyze $f(tv)$, $t \in T$, in both scenarios for $f \in \mathbb{C}[Gv]_d$.

5.4.1 Lemma. *In the polynomial scenario, let $X = Gv$ or $X = \overline{Gv}$. For every $f \in \mathbb{C}[X]_d$ and $t \in T$ we have $f^n(tv) = \det^d(t).f^n(v)$.*

Proof. Let $\eta := \dim(\mathsf{Sym}^n\mathbb{C}^{n^2})$. Let $t_1 \in \mathbb{C}^\times$ denote the element on the diagonal of t. The representation matrix satisfies $\rho(t) = \mathrm{diag}_\eta(t_1^n)$ and $\det(t) = t_1^{n^2}$. Further,

$$f^n(tv) = f^n(\rho(t)v) = f^n((t_1^n.\,\mathrm{id}_\eta)v) = f^n(t_1^n.v) = t_1^{dn^2}.f^n(v) = \det{}^d(t).f(v). \quad \square$$

5.4.2 Lemma. *In the tensor scenario, let $X = Gv$ or $X = \overline{Gv}$. For every $f \in \mathbb{C}[X]_d$ and $t \in T$ we have $f^n(tv) = \det^d(t).f^n(v)$.*

Proof. Let $\eta := \dim(\bigotimes^3\mathbb{C}^n) = n^3$. Let $(t_1,t_2,t_3) \in (\mathbb{C}^\times)^3$ denote the elements on the diagonal of t, so $\det(t) = (t_1t_2t_3)^n$. We see that the representation matrix satisfies $\rho(t) = \mathrm{diag}_n(t_1) \otimes \mathrm{diag}_n(t_2) \otimes \mathrm{diag}_n(t_3) = \mathrm{diag}_{n^3}(t_1t_2t_3)$. Further,

$$\begin{aligned} f^n(tv) &= f^n(\rho(t)v) = f^n((t_1t_2t_3.\,\mathrm{id}_\eta)v) = f^n((t_1t_2t_3).v)\\ &= (t_1t_2t_3)^{dn}.f^n(v) = \det{}^d(t).f^n(v). \end{aligned} \quad \square$$

The following geometric definition is crucial for the forthcoming Proposition 5.4.5.

5.4.3 Definition. A point $v \in V$ is called *$\tilde{G}$-stable*, if the orbit $\tilde{G}v$ is closed in V. ■

5.4.4 Proposition. *The following points are stable.*
(1) $\det_n$ *is* SL_{n^2}*-stable.*
(2) per_n *is* SL_{n^2}*-stable.*
(3) $\mathcal{E}_n$ *is* SL_n^3*-stable.*
(4) $\mathcal{M}_n$ *is* $\mathsf{SL}_{n^2}^3$*-stable.*

Proof. The first two statements follow from the Hilbert-Mumford-Kempf criterion [Kem78]. This is first noted in [MS01, Thm. 4.1] for (1) and in [MS01, Thm. 4.7] for (2). Statement (3) is a special case of [Mey06, Thm. 5.2.1]. Statement (4) is proved in [Mey06, Prop. 5.2.1]. Unfortunately the thesis [Mey06] is available only at ETH Zürich, so we cite another source here: For statement (3) see [BI11, Prop. 4.2] and for statement (4) see [BI11, Prop. 5.2]. □

Recall from (5.1.2) the definition of the exponent $o(v)$ of v.

5.4.5 Proposition. *In both scenarios from Section 2.6, for each highest weight vector* $f \in \mathbb{C}[Gv]_d$ *there exists an integer* $\gamma \in \mathbb{Z}$ *satisfying* $(\det_v^o)^\gamma f^o \in \mathbb{C}[\overline{Gv}]$.

Proof. W.l.o.g. $f \neq 0$. We see that f does not vanish on the whole $\tilde{G}v$, because if $f(\tilde{G}v) = 0$, then, according to Lemma 5.4.1/Lemma 5.4.2, for all $t \in T$ we would have $f(t\tilde{G}v) = \det(t)^d . f^n(\tilde{G}v) = 0$. This would imply $f(Gv) = 0$ and hence $f = 0$.

By assumption, f lies in an irreducible G-representation and hence in an irreducible $\tilde{G}$-representation, see Lemma 4.1.11. Let $\iota\colon \tilde{G}v \hookrightarrow Gv$ denote the inclusion map. Then the restriction $\tilde{f} := f \circ \iota \in \mathbb{C}[\tilde{G}v]$ is nonzero and $\tilde{f}$ lies in an irreducible $\tilde{G}$-representation $\mathscr{W} \subseteq \mathbb{C}[\tilde{G}v]$, because $f \mapsto \tilde{f}$ is $\tilde{G}$-equivariant.

The $\tilde{G}$-equivariant restriction morphism $\mathrm{res}\colon \mathbb{C}[\overline{Gv}] \to \mathbb{C}[\tilde{G}v]$ is surjective since $\tilde{G}v$ is closed due to the stability of v (Proposition 5.4.4) and hence $\mathbb{C}[\tilde{G}v]$ is a factor ring of $\mathbb{C}[\overline{Gv}]$. We decompose each degree δ component $\mathbb{C}[\overline{Gv}]_\delta$ into irreducibles. By Schur's Lemma 3.3.3 there exists an irreducible $\tilde{G}$-representation $\mathscr{V} \subseteq \mathbb{C}[\overline{Gv}]_\delta$ for some δ such that $\mathrm{res}(\mathscr{V}) = \mathscr{W}$. Let $\bar{f}$ be a preimage of $\tilde{f}$ under res. The situation can be illustrated as follows:

$$\begin{array}{ccccc} \mathbb{C}[Gv] & \twoheadrightarrow & \mathbb{C}[\tilde{G}v] & \overset{\mathrm{res}}{\twoheadleftarrow} & \mathbb{C}[\overline{Gv}] \\ \cup & & \cup & & \cup \\ f & \longmapsto & \tilde{f} & \longleftarrow\!\shortmid & \bar{f} \end{array}$$

For each $g \in G$ there exists a factorization $g = t\tilde{g}$ with $\tilde{g} \in \tilde{G}$, $t \in T$. Since $\bar{f}$ is homogeneous of degree δ, we have by Lemma 5.4.1/Lemma 5.4.2

$$\bar{f}^n(gv) = \bar{f}^n(t\tilde{g}v) = \det(t)^\delta \bar{f}^n(\tilde{g}v) = \det(t)^\delta f^n(\tilde{g}v).$$

Analogously $f^n(gv) = \det(t)^d f^n(\tilde{g}v)$. Therefore

$$\bar{f}^n(gv) = \det(t)^{\delta-d} f^n(gv) = \det(g)^{\delta-d} f^n(gv),$$

which implies $\bar{f}^{on}(gv) = \det(g)^{o(\delta-d)} f^{on}(gv)$.
Hence, by (5.1.2), $\bar{f}^{on}(gv) = \det_v^o(gv)^{\delta-d} f^{on}(gv)$ and therefore

$$(\det_v^o)^{\delta-d} f^{on} = \bar{f}^{on} \in \mathbb{C}[\overline{Gv}].$$

Recall that f and $\bar{f}$ are highest weight vectors to weights that have the same $\tilde{G}$-weight. Hence the degrees d and δ in which they appear differ by a multiple of n (tensor case) or n^2 (polynomial case). We write $\delta - d = n\gamma$ and obtain

$$(\det_v^o)^{n\gamma} f^{on} = \bar{f}^{on}.$$

Since $\overline{Gv}$ is connected and $\bar{f}$ is continuous, it follows

$$(\det_v^o)^\gamma f^o = \theta \bar{f}^o \in \mathbb{C}[\overline{Gv}]$$

for some nth root of unity θ. □

We call the smallest integer γ in the situation of Proposition 5.4.5 the *exponent of regularity of f*.

Chapter 6

Representation Theoretic Results

In this chapter we present our main representation theoretic results, namely Theorem 6.2.1 and Theorem 6.3.2. In Section 6.1, we explain our studies regarding the Foulkes-Howe map.

6.1 Kernel of the Foulkes-Howe Map

In 1950, H. O. Foulkes conjectured the following property of plethysms.

6.1.1 Conjecture ([Fou50]). *If* $a \leq b$, *then* $\mathsf{Sym}^a\mathsf{Sym}^b\mathbb{C}^b \subseteq \mathsf{Sym}^b\mathsf{Sym}^a\mathbb{C}^b$ *as a* GL_b*-subrepresentation.*

This is clearly equivalent to saying that if $a \leq b$, then $p_\lambda(a[b]) \leq p_\lambda(b[a])$ for all $\lambda \vdash_{\overline{b}} ab$. The conjecture is still open, but it is known to hold in several special cases, see e.g. [Thr42, Bri93, DS00, MN05, McK08]. [Man98] proves that an asymptotic version is true. The (false) *Foulkes-Howe conjecture* [How87, p. 93], first disproved by Müller and Neunhöffer [MN05], states that the inclusion in Conjecture 6.1.1 is given by the *Foulkes-Howe map* $\Psi_{a,b}$ defined via the following commutative diagram (cp. [Lan11, §8.6]):

$$\begin{array}{ccc} \mathsf{Sym}^a\mathsf{Sym}^b\mathbb{C}^b & \xrightarrow{\Psi_{a,b}} & \mathsf{Sym}^b\mathsf{Sym}^a\mathbb{C}^b \\ \Big\downarrow \iota & & \Big\uparrow \mathcal{P}_{b[a]} \\ \bigotimes^a\bigotimes^b\mathbb{C}^b & \xrightarrow{\mathsf{reorder}_{a,b}} & \bigotimes^b\bigotimes^a\mathbb{C}^b \end{array}$$

where ι denotes the canonical embedding of symmetric tensors in the space of all tensors and $\mathsf{reorder}_{a,b}$ is defined in (4.2.15).

For $a = b = 5$ this conjecture was disproved in [MN05] by showing that the determinant of $\Psi_{5,5}$ is zero. We describe how to obtain the decomposition of the kernel of $\Psi_{5,5}$ into irreducible GL_5-representations with the help of a computer.

We proceed in two steps. First, we compute a basis of the space of highest weight vectors in $\mathsf{Sym}^5\mathsf{Sym}^5\mathbb{C}^5$. Then we apply $\mathsf{reorder}_{a,b}$ to the basis and check the resulting polynomials for linear independence.

Small Example. Let $a = b = 2$. We are going to show that $\Psi_{2,2}$ is an isomorphism. Theorem 4.3.6 provides the decomposition $\mathsf{Sym}^2\mathsf{Sym}^2(\mathbb{C}^2)^* \simeq \{(4)^*\} \oplus \{(2,2)^*\}$ into GL_2-representations. We want to construct highest weight vectors of $\mathsf{Sym}^2\mathsf{Sym}^2(\mathbb{C}^2)^*$ as described in Claim 4.2.13. To achieve this, for both

$\lambda \in \{(4),(2,2)\}$, it is sufficient to choose a permutation π_λ from S_4 such that $\langle\widehat{\lambda}|\pi\mathcal{P}_{2[2]} \neq 0$.

We set $\pi_{(4)} := \mathrm{id}$. Then $\langle\widehat{(4)}|\mathcal{P}_{2[2]}|1111\rangle = \langle 1111|1111\rangle = 1$ and hence $\langle\widehat{(4)}|\mathcal{P}_{2[2]}$ is the nonzero highest weight vector polynomial in $\mathsf{Sym}^2\mathsf{Sym}^2(\mathbb{C}^2)^*$ of weight $(4)^*$, which is unique up to scaling.

For the weight $\lambda = (2,2)$ we have

$$\langle\widehat{(2,2)}| = \langle 1212| - \langle 2112| + \langle 2121| - \langle 1221| \tag{†}$$

and (Remark 4.2.4)

$$\mathcal{P}_{2[2]}|1122\rangle = \tfrac{1}{2}|1122\rangle + \tfrac{1}{2}|2211\rangle. \tag{$*$}$$

So choose the transposition $\pi := \pi_{(2,2)} := \left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right) \in \mathsf{S}_4$ which fixes the first and last entry. Then we obtain

$$\langle\widehat{(2,2)}|\pi\mathcal{P}_{2[2]}|1122\rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1 \neq 0. \tag{6.1.2}$$

Hence $\langle\widehat{(2,2)}|\mathcal{P}_{2[2]}$ is a nonzero highest weight vector polynomial in $\mathsf{Sym}^2\mathsf{Sym}^2(\mathbb{C}^2)^*$ of weight $(2,2)^*$, which is unique up to scaling. Note that for the computation in ($*$) we can make use of Remark 4.2.4. In larger cases this results in a major speedup.

Let $\xi := \mathsf{reorder}_{2,2}$. We now apply ξ to our basis vectors and check whether they vanish or not, i.e., we check whether $\langle\widehat{\lambda}|\mathcal{P}_{2[2]}\xi\mathcal{P}_{2[2]} \neq 0$. The fact $\xi|1111\rangle = |1111\rangle$ implies that $\langle\widehat{(4)}|\mathcal{P}_{2[2]}\xi\mathcal{P}_{2[2]}$ does not vanish. For the second polynomial we write $\mathcal{P} := \mathcal{P}_{2[2]}$ and recall $\mathcal{P}|1212\rangle = \frac{1}{4}(|1212\rangle + |2112\rangle + |1221\rangle + |2121\rangle) = \mathcal{P}|2121\rangle$. Now we compute

$$\begin{aligned}\pi\mathcal{P}\xi\mathcal{P}|1122\rangle &= \pi\mathcal{P}\xi\tfrac{1}{2}(|1122\rangle + |2211\rangle) = \pi\mathcal{P}(\tfrac{1}{2}|1212\rangle + \tfrac{1}{2}|2121\rangle)\\ &= \tfrac{1}{4}\pi(|1212\rangle + |2112\rangle + |2121\rangle + |1221\rangle)\\ &= \tfrac{1}{4}(|1122\rangle + |2112\rangle + |2211\rangle + |1221\rangle).\end{aligned}$$

Hence, using (†), we obtain

$$\langle\widehat{(2,2)}|\pi\mathcal{P}\xi\mathcal{P}|1122\rangle = -\tfrac{1}{4} - \tfrac{1}{4} = -\tfrac{1}{2} \neq 0.$$

Therefore $\langle\widehat{(2,2)}|\pi\mathcal{P}\xi\mathcal{P}$ does not vanish. We are done proving that $\Psi_{2,2}$ is an isomorphism.

The Larger Case. Let $a = b = 5$. We obtained the decomposition of the GL_5-representation $\mathsf{Sym}^5\mathsf{Sym}^5\mathbb{C}^5$ by using the Schur software. For the sake of completeness, the following list gives the decomposition. Note that the plethysm coefficients range from 1 to 7.

$\{25\} \oplus \{23,2\} \oplus \{22,3\} \oplus 2\{21,4\} \oplus \{21,2,2\} \oplus 2\{20,5\} \oplus \{20,4,1\} \oplus \{20,3,2\} \oplus 2\{19,6\} \oplus \{19,5,1\} \oplus 3\{19,4,2\} \oplus$
$\{19,2,2,2\} \oplus 2\{18,7\} \oplus 3\{18,6,1\} \oplus 3\{18,5,2\} \oplus 2\{18,4,3\} \oplus \{18,4,2,1\} \oplus \{18,3,2,2\} \oplus 3\{17,8\} \oplus 2\{17,7,1\} \oplus$
$5\{17,6,2\} \oplus 2\{17,5,3\} \oplus 2\{17,5,2,1\} \oplus 3\{17,4,4\} \oplus \{17,4,3,1\} \oplus 2\{17,4,2,2\} \oplus \{17,2,2,2,2\} \oplus 2\{16,9\} \oplus 3\{16,8,1\} \oplus$
$5\{16,7,2\} \oplus \{16,7,1,1\} \oplus 5\{16,6,3\} \oplus 3\{16,6,2,1\} \oplus 3\{16,5,4\} \oplus 2\{16,5,3,1\} \oplus 3\{16,5,2,2\} \oplus 2\{16,4,4,1\} \oplus$
$\{16,4,3,2\} \oplus \{16,4,2,2,1\} \oplus 2\{15,10\} \oplus 2\{15,9,1\} \oplus 6\{15,8,2\} \oplus \{15,8,1,1\} \oplus 4\{15,7,3\} \oplus 4\{15,7,2,1\} \oplus 6\{15,6,4\} \oplus$
$4\{15,6,3,1\} \oplus 4\{15,6,2,2\} \oplus 3\{15,5,4,1\} \oplus 2\{15,5,3,2\} \oplus \{15,5,3,1,1\} \oplus \{15,5,2,2,1\} \oplus 3\{15,4,4,2\} \oplus \{15,4,2,2,2\} \oplus$
$\{14,11\} \oplus 3\{14,10,1\} \oplus 5\{14,9,2\} \oplus \{14,9,1,1\} \oplus 6\{14,8,3\} \oplus 4\{14,8,2,1\} \oplus 6\{14,7,4\} \oplus 5\{14,7,3,1\} \oplus 4\{14,7,2,2\} \oplus$
$\{14,7,2,1,1\} \oplus 4\{14,6,5\} \oplus 6\{14,6,4,1\} \oplus 4\{14,6,3,2\} \oplus 2\{14,6,2,2,1\} \oplus \{14,5,5,1\} \oplus 4\{14,5,4,2\} \oplus \{14,5,4,1,1\} \oplus$
$\{14,5,3,2,1\} \oplus \{14,5,2,2,2\} \oplus \{14,4,4,3\} \oplus \{14,4,4,2,1\} \oplus \{13,12\} \oplus \{13,11,1\} \oplus 4\{13,10,2\} \oplus \{13,10,1,1\} \oplus$
$4\{13,9,3\} \oplus 4\{13,9,2,1\} \oplus 7\{13,8,4\} \oplus 5\{13,8,3,1\} \oplus 5\{13,8,2,2\} \oplus \{13,8,2,1,1\} \oplus 3\{13,7,5\} \oplus 7\{13,7,4,1\} \oplus$
$4\{13,7,3,2\} \oplus 2\{13,7,3,1,1\} \oplus \{13,7,2,2,1\} \oplus 4\{13,6,6\} \oplus 4\{13,6,5,1\} \oplus 7\{13,6,4,2\} \oplus \{13,6,4,1,1\} \oplus \{13,6,3,3\} \oplus$
$2\{13,6,3,2,1\} \oplus \{13,6,2,2,2\} \oplus \{13,5,5,2\} \oplus \{13,5,5,1,1\} \oplus 2\{13,5,4,3\} \oplus 2\{13,5,4,2,1\} \oplus 2\{13,4,4,4\} \oplus$
$\{13,4,4,2,2\} \oplus \{12,2,1\} \oplus 2\{12,11,2\} \oplus 4\{12,10,3\} \oplus 3\{12,10,2,1\} \oplus 5\{12,9,4\} \oplus 4\{12,9,3,1\} \oplus 3\{12,9,2,2\} \oplus$
$\{12,9,2,1,1\} \oplus 5\{12,8,5\} \oplus 7\{12,8,4,1\} \oplus 5\{12,8,3,2\} \oplus \{12,8,3,1,1\} \oplus 2\{12,8,2,2,1\} \oplus 3\{12,7,6\} \oplus 5\{12,7,5,1\} \oplus$
$7\{12,7,4,2\} \oplus 2\{12,7,4,1,1\} \oplus \{12,7,3,3\} \oplus 2\{12,7,3,2,1\} \oplus \{12,7,2,2,2\} \oplus 3\{12,6,6,1\} \oplus 5\{12,6,5,2\} \oplus$
$\{12,6,5,1,1\} \oplus 4\{12,6,4,3\} \oplus 3\{12,6,4,2,1\} \oplus \{12,6,3,2,2\} \oplus \{12,5,5,2,1\} \oplus 2\{12,5,4,4\} \oplus \{12,5,4,3,1\} \oplus$

$\{12,5,4,2,2\} \oplus \{12,4,4,4,1\} \oplus \{11,1,2,1\} \oplus \{11,1,1,1,1\} \oplus 3\{11,10,4\} \oplus 3\{11,10,3,1\} \oplus 2\{11,10,2,2\} \oplus 2\{11,9,5\} \oplus 5\{11,9,4,1\} \oplus 3\{11,9,3,2\} \oplus 2\{11,9,3,1,1\} \oplus \{11,9,2,2,1\} \oplus 4\{11,8,6\} \oplus 5\{11,8,5,1\} \oplus 7\{11,8,4,2\} \oplus 2\{11,8,4,1,1\} \oplus \{11,8,3,3\} \oplus 2\{11,8,3,2,1\} \oplus \{11,8,2,2,2\} \oplus 4\{11,7,6,1\} \oplus 4\{11,7,5,2\} \oplus 3\{11,7,5,1,1\} \oplus 4\{11,7,4,3\} \oplus 3\{11,7,4,2,1\} \oplus \{11,7,3,3,1\} \oplus 4\{11,6,6,2\} \oplus 3\{11,6,5,3\} \oplus 2\{11,6,5,2,1\} \oplus 3\{11,6,4,4\} \oplus 2\{11,6,4,3,1\} \oplus 2\{11,6,4,2,2\} \oplus \{11,5,5,3,1\} \oplus \{11,5,4,4,1\} \oplus \{11,4,4,4,2\} \oplus 3\{10,10,5\} \oplus 2\{10,10,4,1\} \oplus \{10,10,3,2\} \oplus \{10,10,2,2,1\} \oplus 2\{10,9,6\} \oplus 3\{10,9,5,1\} \oplus 4\{10,9,4,2\} \oplus \{10,9,4,1,1\} \oplus \{10,9,3,3\} \oplus \{10,9,3,2,1\} \oplus \{10,8,7\} \oplus 4\{10,8,6,1\} \oplus 5\{10,8,5,2\} \oplus \{10,8,5,1,1\} \oplus 3\{10,8,4,3\} \oplus 3\{10,8,4,2,1\} \oplus \{10,8,3,2,2\} \oplus \{10,7,7,1\} \oplus 3\{10,7,6,2\} \oplus \{10,7,6,1,1\} \oplus 3\{10,7,5,3\} \oplus 2\{10,7,5,2,1\} \oplus 3\{10,7,4,4\} \oplus 2\{10,7,4,3,1\} \oplus \{10,7,4,2,2\} \oplus 3\{10,6,6,3\} \oplus 2\{10,6,6,2,1\} \oplus 2\{10,6,5,4\} \oplus \{10,6,5,3,1\} \oplus \{10,6,5,2,2\} \oplus 2\{10,6,4,4,1\} \oplus \{10,6,4,3,2\} \oplus \{10,5,4,4,2\} \oplus \{9,9,6,1\} \oplus \{9,9,5,2\} \oplus \{9,9,5,1,1\} \oplus \{9,9,4,3\} \oplus \{9,9,4,2,1\} \oplus \{9,9,3,3,1\} \oplus \{9,8,8\} \oplus \{9,8,7,1\} \oplus 3\{9,8,6,2\} \oplus \{9,8,6,1,1\} \oplus 2\{9,8,5,3\} \oplus 2\{9,8,5,2,1\} \oplus 2\{9,8,4,4\} \oplus \{9,8,4,3,1\} \oplus \{9,8,4,2,2\} \oplus \{9,7,7,1,1\} \oplus 2\{9,7,6,3\} \oplus \{9,7,6,2,1\} \oplus \{9,7,5,4\} \oplus 2\{9,7,5,3,1\} \oplus \{9,7,4,4,1\} \oplus \{9,7,4,3,2\} \oplus 2\{9,6,6,4\} \oplus \{9,6,6,3,1\} \oplus \{9,6,6,2,2\} \oplus \{9,6,5,4,1\} \oplus \{9,6,5,3,2\} \oplus \{9,6,4,4,2\} \oplus \{9,4,4,4,4\} \oplus \{8,8,7,2\} \oplus \{8,8,6,3\} \oplus \{8,8,6,2,1\} \oplus \{8,8,5,4\} \oplus \{8,8,5,2,2\} \oplus \{8,8,4,4,1\} \oplus \{8,7,6,4\} \oplus \{8,7,6,3,1\} \oplus \{8,7,5,4,1\} \oplus \{8,7,4,4,2\} \oplus \{8,6,6,5\} \oplus \{8,6,6,4,1\} \oplus \{8,6,6,3,2\} \oplus \{8,6,4,4,3\} \oplus \{7,7,5,3,3\} \oplus \{7,6,6,4,2\} \oplus \{6,6,6,6,1\}$

Fix $\lambda \vdash_{a} ab$. With the following method one can construct a basis Ω of the highest weight vector space $\mathsf{HWV}_{\lambda^*}(\mathbb{C}[\mathsf{Sym}^a\mathsf{Sym}^b\mathbb{C}^b])$. We put $f_\pi := \langle\widehat{\lambda}|\pi$.

Set $\Omega \leftarrow \emptyset$.
while $|\Omega| < p_\lambda(a[b])$ **do**
  Choose a random permutation $\pi \in \mathsf{S}_{ab}$.
  Evaluate the polynomials in $\Omega \cup \{f_\pi\}$ at $|\Omega|+1$ many random points in $\mathsf{Sym}^b\mathbb{C}^b$.

  The eval. results are written into an $(|\Omega|+1) \times (|\Omega|+1)$ matrix A, where each column corresponds to a polynomial and each row corresponds to a point.
  if A is regular **then**
    Set $\Omega \leftarrow \Omega \cup \{f_\pi\}$.
  end if
end while

It is clear that upon termination the above Las Vegas algorithm yields a desired basis Ω.

Of course, the method can be sped up by evaluating at the same points and reusing evaluation results. A crucial speedup can be obtained when choosing only points of format $w^{\otimes b}$, where $w \in \mathbb{C}^b$, because in this case the evaluation can be done using Remark 4.2.4.

We remark that there is also the possibility to skip permutations π where one can see beforehand that their evaluation must vanish, but in the small case $a = b = 5$ this gives no considerable speedup.

Applying this algorithm for all $\lambda \vdash_5 25$, it takes only a few seconds to compute a basis of the space of highest weight vectors of $\mathsf{Sym}^5\mathsf{Sym}^5(\mathbb{C}^5)^*$, where each basis vector is encoded as a permutation triple π_i.

To compute the kernel of $\Psi_{5,5}$ we again fix $\lambda \vdash_5 25$. Let $\Omega = \{f_1, \dots, f_{p_\lambda(5[5])}\}$ be the basis of $\mathsf{HWV}_{\lambda^*}(\mathbb{C}[\mathsf{Sym}^5\mathsf{Sym}^5\mathbb{C}^5])$ computed by the above algorithm. Next, we randomly choose $w_i \in \mathbb{C}^5$, for $1 \le i \le p_\lambda(5[5])$. Then we define a $p_\lambda(5[5]) \times p_\lambda(5[5])$ matrix B with entries

$$B_{i,j} := f_j(\mathsf{reorder}_{5,5} w_i), \tag{6.1.3}$$

where f_j is the jth basis vector in Ω. If B has full rank, then $\mathrm{mult}_\lambda(\ker \Psi_{5,5}) = 0$. We observed this for most $\lambda \vdash_5 25$. Clearly, B depends on the choice of the w_i. For generic w_i the rank of B is well-defined and denoted by $\mathrm{rank}(B_{\mathrm{gen}})$. We have

$$\mathrm{mult}_\lambda(\ker \Psi_{5,5}) = p_\lambda(5[5]) - \mathrm{rank}(B_{\mathrm{gen}}) \le p_\lambda(5[5]) - \mathrm{rank}(B)$$

for any B depending on the choice of the w_i. We chose random w_i and say that our computations hold "with high probability".

Computational Results. Using the above method for $a = b = 4$, we proved that $\Psi_{4,4}$ is actually an isomorphism, because all $p_\lambda(4[4]) \times p_\lambda(4[4])$ matrices B

that we calculated had full rank. For $\Psi_{5,5}$ we computed "with high probability" the following decomposition of the kernel:

$$\begin{array}{rcllll} \ker\Psi_{5,5} & \simeq & \{(14,7,2,2)\} & \oplus\ \{(13,7,2,2,1)\} & \oplus\ \{(13,4,4,2,2)\} & \\ & \oplus & \{(12,7,3,2,1)\} & \oplus\ \{(12,6,3,2,2)\} & \oplus\ \{(12,5,4,3,1)\} & (6.1.4) \\ & \oplus & \{(11,5,4,4,1)\} & \oplus\ \{(10,8,4,2,1)\} & \oplus\ \{(9,7,6,3)\}. & \end{array}$$

6.1.5 Remark. Unfortunately, for the evaluation $f_j(\mathsf{reorder}_{5,5}w_i)$ in (6.1.3) we cannot make use of Remark 4.2.4, because applying $\mathsf{reorder}_{5,5}$ destroys the symmetries of the w_i. This is the main algorithmical challenge. We used a time-consuming branch-and-bound algorithm to carry out the calculations. We do not go into the details here. ■

Since our result only covers one special case, it is natural to ask the following quesion, which is still wide open.

6.1.6 Question. *How does the above result generalize to arbitrary a and b?*

6.2 Even Partitions in Plethysms

In this section we present our results from [BCI11a] with considerably more constructive proofs. We explicitly construct a specific highest weight vector, which shows that plethysm coefficients of even partitions are always nonzero under the obvious assumptions from Lemma 4.3.3.

6.2.1 Theorem ([BCI11a])**.** *For all $k, n, d \in \mathbb{N}$ with $d \leq k$ and for all partitions $\lambda \vdash_d dn$, the irreducible GL_k-representation $\{2\lambda\}$ occurs in the plethysm $\mathsf{Sym}^d\mathsf{Sym}^{2n}\mathbb{C}^k$, i.e., $p_{2\lambda}(d[2n]) > 0$.*

We remark that Theorem 6.2.1 is consistent with Theorem 4.3.6 (for $n = 1$). Manivel [Man98] proved an asymptotic version of Theorem 6.2.1 using geometric methods. We prove Theorem 6.2.1 by explicitly writing down a highest weight vector of weight 2λ in $\mathsf{Sym}^d\mathsf{Sym}^{2n}\mathbb{C}^k$. This gives an affirmative answer to a conjecture by Weintraub [Wei90]. Very recently, [MM12] gave another proof of Weintraub's conjecture by an explicit construction of highest weight vectors in $\mathsf{Sym}^d\bigwedge^{2n}\mathbb{C}^k$.

Theorem 6.2.1 will be complemented by a new, similar Theorem 6.2.2, which states that $p_{(2\lambda+(d))}(d[2n+1]) > 0$.

Proof of Theorem 6.2.1. We start out by a filling of shape λ and content $d \times n$ that does not contain an entry twice in any column. Such a filling T can easily be found by inserting entries $1, 2, \ldots, d, 1, 2, \ldots, d, 1, 2, \ldots$ into λ columnwise, see the left hand side of Figure 6.2.i. Let T' denote the filling that we obtain by duplicating each column, see Figure 6.2.i. The list of entries in the ith column of T' shall be

1	1	4	2	3	4
2	2	1			
3	3				
4					

$\rightsquigarrow$

1	1	1	1	4	4	2	2	3	3	4	4
2	2	2	2	1	1						
3	3	3	3								
4	4										

Figure 6.2.i: Duplicating columns of a filling T to obtain a filling T'. Here $d = 4$ and $n = 3$.

denoted by $\mathrm{col}_i^{T'}$. It is sufficient (see Claim 4.2.13) to construct $\pi \in \mathsf{S}_{2nd}$ such that $\langle\widehat{2\lambda}|\pi\mathcal{P}_{d[2n]} \neq 0$.

First, we choose a matrix $A \in \mathbb{R}^{d\times d}$ in a way such that for all $1 \le i \le \ell(\lambda)$ all $i \times i$ submatrices formed by any i columns and the first i rows have nonzero determinant. For example, one could choose the entries of A as real numbers algebraically independent over $\mathbb{Q}$. Now we choose $\pi \in \mathsf{S}_{2nd}$ such that

$$\pi(1,1,\dots,1,2,2,\dots,2,\dots,d,d,\dots,d) = (\mathrm{col}_1^{T'}, \mathrm{col}_2^{T'}, \dots, \mathrm{col}_{2\lambda_1}^{T'}).$$

It follows that

$$\begin{aligned}\langle\widehat{2\lambda}|\pi A|11\cdots122\cdots2\cdots dd\cdots d\rangle &= \langle\widehat{2\lambda}|A\Big(\pi|11\cdots122\cdots2\cdots dd\cdots d\rangle\Big)\\ &= \left(\bigotimes_{i=1}^{2\lambda_1}\langle\widehat{{}^t(2\lambda)_i}|\right) A \left(\bigotimes_{i=1}^{2\lambda_1}|\,\mathrm{col}_i^{T'}\rangle\right)\\ &= \prod_{i=1}^{2\lambda_1}\langle\widehat{{}^t(2\lambda)_i}|A|\,\mathrm{col}_i^{T'}\rangle = \prod_{i=1}^{\lambda_1}\langle\widehat{{}^t\lambda_i}|A|\,\mathrm{col}_i^{T}\rangle^2.\end{aligned}$$

Every factor $\langle\widehat{{}^t\lambda_i}|A|\,\mathrm{col}_i^T\rangle$ is the determinant (see (4.2.8)) of a submatrix of A. (Here we use that the entries of T are pairwise distinct.) These determinants are nonzero real numbers by our assumption on A. Hence

$$\langle\widehat{2\lambda}|\pi A|11\cdots122\cdots2\cdots dd\cdots d\rangle > 0.$$

Applying $\sigma \in \mathsf{S}_d$ to our filling T' preserves the property that $\sigma(T')$ has no double entries in a column. Hence, analogously to the above argument, we get

$$\langle\widehat{2\lambda}|\pi A|\sigma(1)\sigma(1)\cdots\sigma(1)\sigma(2)\sigma(2)\cdots\sigma(2)\cdots\sigma(d)\sigma(d)\cdots\sigma(d)\rangle > 0$$

for all $\sigma \in \mathsf{S}_d$. Since (cf. Remark 4.2.4)

$$\begin{aligned}&\mathcal{P}_{d[2n]}|11\cdots122\cdots2dd\cdots d\rangle = \\ &\quad\frac{1}{d!}\sum_{\sigma\in\mathsf{S}_d}|\sigma(1)\sigma(1)\cdots\sigma(1)\sigma(2)\sigma(2)\cdots\sigma(2)\cdots\sigma(d)\sigma(d)\cdots\sigma(d)\rangle,\end{aligned}$$

we have

$$\langle\widehat{2\lambda}|\pi\mathcal{P}_{d[2n]}A|11\cdots122\cdots2\cdots dd\cdots d\rangle > 0$$

and hence

$$\langle\widehat{2\lambda}|\pi\mathcal{P}_{d[2n]} \neq 0,$$

which proves Theorem 6.2.1. □

We can use the same proof technique as above to prove the following similar theorem.

6.2.2 Theorem. *For all $k,n,d \in \mathbb{N}$ with $d \le k$ and for all partitions $\lambda \vdash_d dn$, let $\mu := 2\lambda + (d)$. The irreducible GL_k-representation $\{\mu\}$ occurs in the plethysm $\mathsf{Sym}^d\mathsf{Sym}^{(2n+1)}\mathbb{C}^k$, i.e., $p_\mu(d[(2n+1)]) > 0$.*

Proof. The proof is the same as the proof of Theorem 6.2.1 with minor details changed as follows. The tableau T' is generated from T not just by duplicating the columns, but by additionally adding boxes with entries $1,\dots,d$ to the first row, see Figure 6.2.ii. Now choose the matrix A as before with the additional property that the entries in the first row are positive real numbers. The rest of the proof is analogous to Theorem 6.2.1. □

1	1	4	2	3	4
2	2	1			
3	3				
4					

$\rightsquigarrow$

1	1	1	1	4	4	2	2	3	3	4	4	1	2	3	4
2	2	2	2	1	1										
3	3	3	3												
4	4														

Figure 6.2.ii: Obtaining T' from T. Here $d = 4$ and $n = 3$.

6.3 Nonvanishing of Symmetric Kronecker Coefficients

6.3 (A) Moment Polytopes

In this subsection we describe the *moment polytope*, a basic object used to study asymptotic properties of Kronecker coefficients. This natural object can also easily be defined for other representation theoretic multiplicities, but we only need it in this special case.

Let $\lambda \vdash_N D$. Then $\frac{\lambda}{D} := (\frac{\lambda_1}{D}, \frac{\lambda_2}{D}, \ldots, \frac{\lambda_N}{D})$ defines a probability distribution on $\{1, \ldots, N\}$.

Fix $a, b, c \in \mathbb{N}$. Let $K(a,b,c)$ denote the semigroup of triples (λ, μ, ν), $\lambda, \mu, \nu \vdash_{a,b,c} D$, with positive Kronecker coefficient, cp. Proposition 4.4.10. We define the (rational) *moment polytope* $\mathsf{Kron}(a,b,c)$ as

$$\mathsf{Kron}(a,b,c) := \left\{ \frac{1}{|\lambda|}(\lambda, \mu, \nu) \;\middle|\; (\lambda, \mu, \nu) \in K(a,b,c) \right\}.$$

A rational point $x \in \mathbb{Q}^a \oplus \mathbb{Q}^b \oplus \mathbb{Q}^c$ is contained in the moment polytope iff there exists a stretching factor $\gamma \in \mathbb{N}$ such that γx is a partition triple with positive Kronecker coefficient.

6.3.1 Theorem. *$\mathsf{Kron}(a,b,c)$ is a polytope in $\mathbb{Q}^{a+b+c}$, i.e., $\mathsf{Kron}(a,b,c)$ is the convex hull of finitely many points.*

Proof. Take two rational points $x, y \in K(a,b,c)$ and choose $\gamma \in \mathbb{N}$ such that $k(\gamma x) > 0$ and $k(\gamma y) > 0$. Since $K(a,b,c)$ is a semigroup, for all natural numbers $0 \leq j \leq i$ we have $k(j\gamma x + (i-j)\gamma y) > 0$, which places all rational points between x and y into $\mathsf{Kron}(a,b,c)$ and hence $\mathsf{Kron}(a,b,c)$ is a convex set.

The fact that $K(a,b,c)$ is finitely generated (Proposition 4.4.10) ensures that $\mathsf{Kron}(a,b,c)$ can be described as the convex combination of finitely many rational points. □

Alternative proofs for Theorem 6.3.1 are given in [Man98], [Fra02], [CHM07], and [Kly06].

6.3 (B) Asymptotic Result

In the polynomial scenario from Section 2.6, for the purpose of finding obstructions with orbit-wise upper bound proof, it is required to find partitions λ with $\mathsf{sk}\big(\lambda; (n{\times}d)^2\big) = 0$, see Theorem 5.2.3. In this subsection we prove the following theorem of [BCI11b], which gives a negative answer in many cases.

6.3.2 Theorem ([BCI11b])**.** *Fix n. There exists a stretching factor $\gamma \in \mathbb{N}$ such that for all $d \in \mathbb{N}$, $\lambda \vdash_{n^2} dn$ we have $k(\gamma(n{\times}d); \gamma(n{\times}d); \gamma\lambda) \neq 0$ and $\mathsf{sk}\big(2\gamma\lambda; \big(2\gamma(n{\times}d)\big)^2\big) \neq 0$.*

The proof strategy for Theorem 6.3.2 is as follows. The assertion on the symmetric Kronecker coefficient will follow from the first one with Lemma 6.3.7. For the first part, we will prove that for all λ there exists an individual stretching factor γ_λ such that $k(\gamma_\lambda(n\times d);\gamma_\lambda(n\times d);\gamma_\lambda\lambda)\neq 0$. Since for fixed n the semigroup of partition triples

$$\{(n\times d, n\times d, \lambda) \mid d\in\mathbb{N},\ \lambda\vdash_{n^2} nd\}$$

is finitely generated, we are done.

To show the existence of the individual stretching factor γ_λ it suffices to show that the rational point $\frac{1}{|\lambda|}(n\times d, n\times d,\lambda)$ lies in the moment polytope $\mathsf{Kron}(n,n,n^2)$ as seen in Subsection 6.3 (A). The idea is to use a different interpretation of the moment polytope, coming from quantum information theory. The closure $\overline{\mathsf{Kron}(a,b,ab)}$ has a nice interpretation in the *quantum marginal problem* as described below. To describe this interpretation, we introduce some concepts and notation.

A *density operator* is a Hermitian positive semidefinite matrix with trace 1.

Let $\mathrm{spec}(A)$ denote the list of ordered eigenvalues of $A\in\mathbb{C}^{a\times a}$.

The *partial traces* are the linear maps defined as follows:

$\mathrm{tr}_1\colon \mathbb{C}^{a\times a}\otimes\mathbb{C}^{b\times b}\to\mathbb{C}^{b\times b}$, $A\otimes B\mapsto \mathrm{tr}(A)\cdot B$.

$\mathrm{tr}_2\colon \mathbb{C}^{a\times a}\otimes\mathbb{C}^{b\times b}\to\mathbb{C}^{a\times a}$, $A\otimes B\mapsto \mathrm{tr}(B)\cdot A$.

We identify the spaces $\mathbb{C}^{a\times a}\otimes\mathbb{C}^{b\times b}$ and $\mathbb{C}^{ab\times ab}$ using the canonical isomorphism.

6.3.3 Claim. *The partial traces* $\mathrm{tr}_1\Xi\in\mathbb{C}^{b\times b}$ *and* $\mathrm{tr}_2\Xi\in\mathbb{C}^{a\times a}$ *of a density operator* $\Xi\in\mathbb{C}^{ab\times ab}$ *are density operators.*

Proof. Let $\Xi=\big(\Xi_{i,i',j,j'}\big)_{(i,i'),(j,j')}\in\mathbb{C}^{ab\times ab}$ be a density operator, where rows and columns are indexed by pairs from $\{1,\dots,a\}\times\{1,\dots,b\}$. The crucial fact is that

$$\mathrm{tr}_1\Xi=\Big(\sum_{k=1}^{a}\Xi_{k,i',k,j'}\Big)_{i',j'}.$$

This description implies that $\mathrm{tr}_1\Xi$ is Hermitian, because Ξ is Hermitian. We also see that $\mathrm{tr}(\mathrm{tr}_1\Xi)=\sum_{i'=1}^{b}\sum_{k=1}^{a}\Xi_{k,i',k,i'}=\mathrm{tr}(\Xi)=1$. Since Ξ is positive semidefinite, we have $\langle v|\Xi|v\rangle\geq 0$ for all $|v\rangle\in\mathbb{C}^{ab}$, which means

$$\sum_{\substack{i,j=1,\dots,a\\ i',j'=1,\dots,b}}\overline{v_{i,i'}}v_{j,j'}\Xi_{i,i',j,j'}\geq 0$$

for all $v_{i,i'}\in\mathbb{C}^{ab}$. In particular, if we set $|v\rangle=|k\rangle\otimes|w\rangle$, $1\leq k\leq a$, $|w\rangle\in\mathbb{C}^{b}$, we obtain

$$\sum_{i',j'=1,\dots,b}\overline{w_{i'}}w_{j'}\Xi_{k,i',k,j'}\geq 0 \tag{$*$}$$

for all $w_{i'}\in\mathbb{C}^{b}$. Summing $(*)$ over $1\leq k\leq a$, we get that $\mathrm{tr}_1\Xi$ is positive semidefinite and therefore is a density operator. Analogously, $\mathrm{tr}_2\Xi$ is a density operator. □

Using the above concepts, we can now state the quantum marginal description of the moment polytope.

6.3.4 Theorem (Quantum marginal description)**.**

$$\overline{\mathsf{Kron}(a,b,ab)}=\big\{\big(\mathrm{spec}(\mathrm{tr}_1\Xi),\mathrm{spec}(\mathrm{tr}_2\Xi),\mathrm{spec}(\Xi)\big)\mid \Xi\in\mathbb{C}^{ab\times ab}\text{ density operator}\big\}.$$

For proofs see: [CM06], [Kly06], [CHM07].

6.3.5 Remark. [Kly06] gives a method to determine the linear inequalities of $\overline{\mathsf{Kron}(a,b,ab)}$. See also Ressayre [Res10]. ■

Since $\overline{\mathsf{Kron}(a,b,ab)}$ is defined by finitely many rational linear inequalities, every rational point in $\overline{\mathsf{Kron}(a,b,ab)}$ lies in $\mathsf{Kron}(a,b,ab)$. Therefore Theorem 6.3.4 implies that for proving Theorem 6.3.2 it remains to construct the following.

6.3.6 Proposition. *For any decreasing probability distribution r on $\{1,\dots,n^2\}$ there exists a density operator $\Xi \in \mathbb{C}^{n^2\times n^2}$ with spectrum r such that* $\operatorname{spec}(\operatorname{tr}_1\Xi) = \operatorname{spec}(\operatorname{tr}_2\Xi) = (\frac{1}{n},\dots,\frac{1}{n})$.

Proof. Recall the basis $|\iota\rangle$, $1 \le \iota \le n$ of $\mathbb{C}^n$. Then $|(\iota+1)\rangle$ is defined for $1 \le \iota < n$. We set $|(n+1)\rangle := |1\rangle$.

Let $A, B \in \mathsf{GL}_n$ with $A|\iota\rangle = |(\iota+1)\rangle$ and $B|\iota\rangle = \vartheta^\iota|\iota\rangle$, where $\vartheta \in \mathbb{C}$ denotes a primitive nth root of unity. We note that A and B are unitary matrices and $A^{-1}BA = \vartheta B$. For a fixed decreasing probability distribution r on $\{1,\dots,n^2\}$ we define

$$\Xi := \sum_{i,j=1}^{n} r_{ij}|w_{ij}\rangle\langle w_{ij}|,$$

where

$$|w_{ij}\rangle := \frac{1}{\sqrt{n}}\sum_{\iota=1}^{n}|\iota\rangle \otimes A^iB^j|\iota\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \simeq \mathbb{C}^{n^2}.$$

We have $|w_{ij}\rangle = (\mathrm{id}_n \otimes A^iB^j)|w_{00}\rangle$.

We first show that Ξ has spectrum r. This directly follows from the fact that the vectors $|w_{ij}\rangle$ form an orthonormal basis of $\mathbb{C}^n \otimes \mathbb{C}^n$, which can be seen as follows. Clearly, $\langle w_{ij}|w_{ij}\rangle = 1$ for all $1 \le i,j \le n$. We have, for some nth root of unity θ,

$$\begin{aligned}
\langle w_{ij}|w_{k\ell}\rangle &= \langle w_{00}|(\mathrm{id}_n \otimes B^{-j}A^{-i})(\mathrm{id}_n \otimes A^kB^\ell)|w_{00}\rangle \\
&= \theta\,\langle w_{00}|\,\mathrm{id}_n \otimes A^{k-i}B^{\ell-j}|w_{00}\rangle \\
&= \frac{\theta}{n}\sum_{\iota,\iota'=1}^{n}\langle \iota\iota|\,\mathrm{id}_n \otimes A^{k-i}B^{\ell-j}|\iota'\iota'\rangle \\
&= \frac{\theta}{n}\sum_{\iota=1}^{n}\langle \iota|A^{k-i}B^{\ell-j}|\iota\rangle = \frac{\theta}{n}\operatorname{tr}\bigl(A^{k-i}B^{\ell-j}\bigr).
\end{aligned}$$

It is easy to check that $\operatorname{tr}\bigl(A^{k-i}B^{\ell-j}\bigr) = 0$ if $\ell \ne j$ or $k \ne i$. Hence the vectors $|w_{ij}\rangle$ form an orthonormal basis of $\mathbb{C}^n \otimes \mathbb{C}^n$ and therefore Ξ has spectrum r.

It remains to show that the partial traces $\operatorname{tr}_1\Xi$ and $\operatorname{tr}_2\Xi$ have the uniform spectrum. For fixed $1 \le i,j \le n$ we can write $|w_{ij}\rangle = \sum_\iota \frac{1}{\sqrt{n}}|u_\iota\rangle \otimes |v_\iota\rangle$ for the orthonormal bases $\{|u_\iota\rangle = |\iota\rangle\}$ and $\{|v_\iota\rangle = A^iB^j|\iota\rangle\}$ of $\mathbb{C}^n$ (this is sometimes called the *Schmidt decomposition* or the *singular value decomposition*). Hence

$$|w_{ij}\rangle\langle w_{ij}| = \sum_{k,\ell=1}^{n}\frac{1}{n}|u_k\rangle\langle u_\ell| \otimes |v_k\rangle\langle v_\ell|$$

and taking the first partial trace results in

$$\operatorname{tr}_1\bigl(|w_{ij}\rangle\langle w_{ij}|\bigr) = \sum_{k=1}^{n}\frac{1}{n}|v_k\rangle\langle v_k|,$$

which has eigenvalues $(\frac{1}{n},\dots,\frac{1}{n})$. Analogously we see that $\operatorname{tr}_2|w_{ij}\rangle\langle w_{ij}|$ has eigenvalues $(\frac{1}{n},\dots,\frac{1}{n})$. The eigenvalues of $\operatorname{tr}_1(|w_{ij}\rangle\langle w_{ij}|)$ are $(\frac{1}{n},\dots,\frac{1}{n})$ and by linearity it follows $\operatorname{spec}(\operatorname{tr}_1\Xi) = (\frac{1}{n},\dots,\frac{1}{n})$. Analogously for $\operatorname{spec}(\operatorname{tr}_2\Xi)$. □

6.3.7 Lemma. *Let* $\lambda, \mu \vdash_n D$. *If* $[\lambda]$ *occurs in* $[\mu] \otimes [\mu]$, *then* $[2\lambda]$ *occurs in* $\mathsf{Sym}^2([2\mu])$. *In other words,* $k(\lambda;\mu;\mu) \neq 0$ *implies* $\mathsf{sk}\big(2\lambda; \big(2\mu\big)^2\big) \neq 0$.

Proof. Since $k(\lambda;\mu;\mu) \neq 0$, according to Proposition 4.4.8 there exists a highest weight vector f of weight (λ,μ,μ) in $\mathsf{Sym}^D(\bigotimes^3\mathbb{C}^n)$. Consider the linear map $\sigma\colon \bigotimes^3\mathbb{C}^n \to \bigotimes^3\mathbb{C}^n$, $\sigma(v \otimes w_1 \otimes w_2) = v \otimes w_2 \otimes w_1$. Define the polynomial $\bar{f} \in \mathsf{Sym}^D\bigotimes^3\mathbb{C}^n$ via $\bar{f}(x) := f(\sigma x)$. The polynomial $\bar{f}$ is also a highest weight vector of weight (λ,μ,μ). Hence $f \cdot \bar{f}$ is nonzero and a highest weight vector of weight $(2\lambda, 2\mu, 2\mu)$ which is invariant under σ. Therefore $\big([2\lambda] \otimes [2\mu] \otimes [2\mu]\big)^{\mathsf{S}_{2D}}$ has a nonzero invariant with respect to σ and hence

$$\big([2\lambda] \otimes \mathsf{Sym}^2([2\mu])\big)^{S_{2D}} \neq 0.$$

By the definition of the symmetric Kronecker coefficient (4.4.1) we obtain

$$\Big([2\lambda] \otimes \bigoplus_{\varrho \vdash 2D} \mathsf{sk}\big(\varrho; \big(2\mu\big)^2\big)[\varrho]\Big)^{S_{2D}} \neq 0.$$

Lemma 4.4.3 implies $\mathsf{sk}\big(2\lambda; \big(2\mu\big)^2\big) \neq 0$. □

6.3.8 Corollary. *If* $k(\lambda;\mu;\mu) = 1$ *and* $k(2\lambda; 2\mu; 2\mu) = 1$, *then* $\mathsf{sk}\big(2\lambda; \big(2\mu\big)^2\big) = 1$.

Proof. This follows from Lemma 6.3.7 and the fact that $\mathsf{sk}\big(\varrho; \big(\nu\big)^2\big) \leq k(\varrho;\nu;\nu)$ for all partitions ϱ, ν. □

Chapter 7

Obstruction Designs

In this chapter we want to sharpen our methods from Section 6.3 and aim for non-asymptotic results. We introduce a combinatorial concept, the so-called *obstruction designs*, which enables us to construct highest weight vectors in $\mathsf{Sym}^d \bigotimes^3 (\mathbb{C}^n)^*$. The technique is similar to the one used in Section 6.2. Our main tool is Claim 4.2.17, i.e., the fact that $\{\langle\widehat{\lambda}|\pi\mathcal{P}_d : \pi \in \mathsf{S}_d^3\}$, is a generating set of $\mathsf{HWV}_{\lambda^*}\left(\mathsf{Sym}^d \bigotimes^3 (\mathbb{C}^n)^*\right)$ for $\lambda \vdash^*_n d$.

7.1 Set Partitions

We start with some notation. Let $\wp(S)$ denote the powerset of a finite set S, i.e., the set of all subsets of S. Given a set S, we call a subset $\Lambda \subseteq \wp(S)$ of the powerset $\wp(S)$ a *set partition of* S, if for all $s \in S$ there exists exactly one set $e_s \in \Lambda$ with $s \in e_s$. We call e_s the *hyperedge corresponding to* s. If $|e_s| = 1$, then e_s is called a *singleton hyperedge*. The *type* of a set partition Λ is defined as the partition $\lambda \vdash |S|$ obtained from sorting the multiset $\{|e| : e \in \Lambda\}$. Let $V(\Lambda) := \bigcup_{e\in\Lambda} e = S$ denote the ground set.

For a given partition $\lambda \vdash d$, we can define a canonical set partition Λ of $\{1, \dots, d\}$ as follows. Let $\omega_i := \sum_{j=1}^{i} |{}^t\lambda_j|$ be the number of boxes in the first i columns of λ. We define the disjoint hyperedges

$$e_i := \{\omega_{i-1} + 1, \omega_{i-1} + 2, \dots, \omega_i\}$$

and set $\Lambda := \{e_i \mid 1 \le i \le \lambda_1\}$. For example, if $\lambda = (4, 3, 1)$, then ${}^t\lambda = (3, 2, 2, 1)$ and $e_1 = \{1, 2, 3\}$, $e_2 = \{4, 5\}$, $e_3 = \{6, 7\}$, $e_4 = \{8\}$, corresponding to the *canonical columnwise numbering* $\mathcal{T}_\lambda$ of λ, see Figure 7.1.i. For the rest of this subsection, $\mathcal{T}_\lambda$ will denote the canonical columnwise numbering of λ.

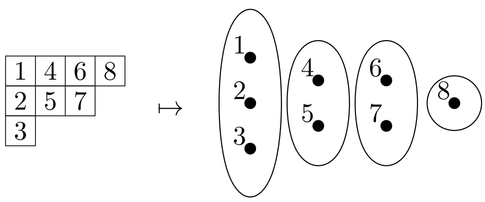


Figure 7.1.i: The canonical columnwise numbering $\mathcal{T}_\lambda$ of $\lambda = (4, 3, 1)$ and its set partition.

Analogously, from any numbering T of λ we obtain a set partition Λ_T of $\{1,\dots,d\}$ with type λ by grouping together each column in a hyperedge.

The map $T \mapsto \Lambda_T$ is not injective in general. Our aim is to classify the fibers. Two numberings T and T' are mapped to the same $\Lambda_T = \Lambda_{T'}$, iff T' can be obtained from T by permuting entries inside of columns and by permuting whole columns of the same length. This observation gives rise to the following definition (see the right hand side of Figure 7.1.ii for an example).

7.1.1 Definition. An *ordered set partition* Λ of a vertex set $V(\Lambda) := \{1,\dots,d\}$ of type λ is a set partition of $V(\Lambda)$ of type λ endowed with (1) linear orderings on each hyperedge $e \in \Lambda$ and (2) for each length $1 \le \ell \le \ell(\lambda)$ a linear ordering on the set $\{e \in \Lambda : |e| = \ell\}$ of hyperedges with the same cardinality ℓ. ■

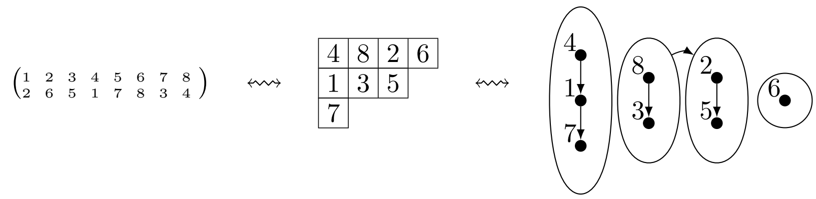


Figure 7.1.ii: The bijections between permutations, numberings, and ordered set partitions of type $\lambda = (4,3,1)$. The orderings are shown with arrows pointing from the smaller element to the bigger.

The above discussion gives an explicit bijection between the set of numberings T of λ and the set of ordered set partitions of type λ by grouping together each column of T in one hyperedge, ordered from top to bottom, and ordering hyperedges of equal length by their appearance in T from left to right, see Figure 7.1.ii for an example. The columns of T are ordered from left to right and this induces an additional linear ordering on the set of hyperedges of an ordered set partition, which is consistent with the single linear orderings of hyperedges of the same length. In particular we can speak of the ith hyperedge of Λ. Additionally, since the hyperedges are linearly ordered, we have a linear order on $V(\Lambda)$ and hence we can write $V(\Lambda)_i$ for the ith element of $V(\Lambda)$.

The permutations $\pi \in \mathsf{S}_d$ are in bijection to the numberings of λ via replacing the entry i in $\mathcal{T}_\lambda$ with the integer $\pi^{-1}(i)$. Hence we get an *explicit bijection* between S_d and the set of ordered set partitions of type λ. For a given ordered set partition Λ of typeλ we denote by π_Λ the corresponding permutation. We can state a first observation:

$$\pi_\Lambda(V(\Lambda)_i) = i. \tag{7.1.2}$$

The crucial property of our bijection is shown in the upcoming Claim 7.1.3, for whose statement we introduce some notation.

The group S_d acts naturally on $(\mathbb{C}^n)^d$ by permuting the positions as follows:

$$\pi(\zeta_1, \zeta_2, \dots, \zeta_d) := (\zeta_{\pi^{-1}(1)}, \zeta_{\pi^{-1}(2)}, \dots, \zeta_{\pi^{-1}(d)}).$$

Given a linearly ordered subset $(e, \prec) \subseteq \{1,\dots,d\}$ with ℓ elements, $\ell \le d$, where the order $\prec$ is not necessarily consistent with the natural order on $\{1,\dots,d\}$, we define the list elements $e^1,\dots,e^\ell$ via $e = \{e^1,\dots,e^\ell\}$ satisfying $e^1 \prec \ldots \prec e^\ell$. For example, for the leftmost hyperedge e in Figure 7.1.ii we have $e^1 = 4$, $e^2 = 1$, and $e^3 = 7$. Given a vector $\zeta \in (\mathbb{C}^n)^d$, we define the *restriction* $\zeta_{|_e}$ of ζ to e as

$$(\zeta_1, \zeta_2, \dots, \zeta_d)_{|_e} := (\zeta_{e^1}, \dots, \zeta_{e^\ell}).$$

7.1.3 Claim. *Fix $\lambda \vdash_n d$ and let e_i denote the ith column of $\mathcal{T}_\lambda$. Let Λ be an ordered set partition. Then, for the ith hyperedge e of Λ, we have*

$$(\zeta_1, \zeta_2, \ldots, \zeta_d)_{|_e} = \big(\pi_\Lambda(\zeta_1, \zeta_2, \ldots, \zeta_d)\big)_{|_{e_i}}$$

for all $\zeta \in (\mathbb{C}^n)^d$.

Proof. Note that $\pi_\Lambda(e^j) = (e_i)^j$ according to (7.1.2). Now the proof is straightforward as follows:

$$\begin{aligned}(\zeta_1, \ldots, \zeta_d)_{|_e} &= (\zeta_{e^1}, \ldots, \zeta_{e^d}) = (\zeta_{\pi_\Lambda^{-1}((e_i)^1)}, \ldots, \zeta_{\pi_\Lambda^{-1}((e_i)^{|e_i|})}) \\ &= (\pi_\Lambda(\zeta_1, \ldots, \zeta_d))_{|_{e_i}}\end{aligned}$$ □

Highest Weight Vectors. Our main motivation for looking at set partitions is the construction of highest weight vectors. For each ordered set partition Λ of type $\lambda \vdash_n d$ we have $\pi_\Lambda \in \mathsf{S}_d$ and hence obtain a nonzero highest weight vector

$$f_\Lambda := \langle\widehat{\lambda}|\pi_\Lambda \in \bigotimes^d (\mathbb{C}^n)^*$$

of weight λ^*, provided $n \geq \ell(\lambda)$.

We next want to determine the *projective stabilizer* $Y_\lambda \subseteq \mathsf{S}_d$ of $\langle\widehat{\lambda}|$, which is defined as

$$Y_\lambda := \{\tau \in \mathsf{S}_d : \langle\widehat{\lambda}|\tau = \pm\langle\widehat{\lambda}|\}. \tag{7.1.4}$$

Consider the Young subgroup $Y_\lambda^{\text{inner}} := \mathsf{S}(e_1) \times \cdots \times \mathsf{S}(e_{\lambda_1})$, where we recall that e_i denotes the ith column of $\mathcal{T}_\lambda$. For $\tau \in Y_\lambda^{\text{inner}}$ we have $\langle\widehat{\lambda}|\tau = \operatorname{sgn}(\tau)\langle\widehat{\lambda}|$, hence $Y_\lambda^{\text{inner}} \subseteq Y_\lambda$. Let Y^{outer} denote the group that interchanges columns of the same length in $\mathcal{T}_\lambda$ while preserving the order in each column. For $\tau \in Y_\lambda^{\text{outer}}$ we have $\langle\widehat{\lambda}|\tau = \langle\widehat{\lambda}|$, hence $Y_\lambda^{\text{outer}} \subseteq Y_\lambda$. One can prove that the projective stabilizer Y_λ is the group generated by Y_λ^{inner} and Y_λ^{outer}.

We are interested in classifying the left cosets of $Y_\lambda \subseteq \mathsf{S}_d$. The ordered set partition corresponding to π and the ordered set partition corresponding to $\tau\pi$ for $\tau \in Y_\lambda^{\text{inner}}$ are the same up to reordering the elements in each hyperedge. For $\tau \in Y_\lambda^{\text{outer}}$ the ordered set partitions corresponding to π and $\tau\pi$ are the same up to reordering the hyperedges. All reorderings can be obtained by applying elements of Y_λ. If we forget about the orderings of ordered set partitions, we obtain the following claim.

7.1.5 Claim. *For a fixed partition $\lambda \vdash d$ there is a bijection between the left cosets of $Y_\lambda \subseteq \mathsf{S}_d$ and the set of set partitions of type λ.*

Hence a set partition Λ of type $\lambda \vdash_n d$ uniquely determines a highest weight vector of weight λ^*

$$f_\Lambda := \pm\langle\widehat{\lambda}|\pi_\Lambda \in \bigotimes^d (\mathbb{C}^n)^*$$

up to a sign, where π_Λ is the permutation corresponding to some ordering of Λ.

Contraction. A finite sequence $\zeta = (\zeta_1, \ldots, \zeta_d)$ of vectors ζ_i in $\mathbb{C}^n$ is called a *list*. A map whose domain is a vertex set is sometimes called a *labeling* of the vertex set. If a vertex set e is linearly ordered, then we can identify lists and labelings with codomain $\mathbb{C}^n$. Given a list $\zeta = (\zeta_1, \ldots, \zeta_d)$, we write $|\zeta\rangle := |\zeta_1 \otimes \cdots \otimes \zeta_d\rangle$. We want to analyze how the scalar product $\langle\widehat{\lambda}|\pi|\zeta\rangle$, for $\pi \in \mathsf{S}_d$ and $|\zeta\rangle \in \bigotimes^d \mathbb{C}^n$, can be interpreted combinatorially using set partitions.

For a fixed $\pi \in \mathsf{S}_d$ and a list ζ we define $\tilde{\zeta} := \pi\zeta$ to obtain

$$\langle\widehat{\lambda}|\pi|\zeta\rangle = \langle\widehat{\lambda}|\tilde{\zeta}\rangle = \prod_{i=1}^{\lambda_1}\langle\widehat{{}^t\lambda_i}|\tilde{\zeta}_{|_{e_i}}\rangle.$$

Note that for $\ell \le n$ and a list $\tilde{\zeta} = (\tilde{\zeta}_1, \dots, \tilde{\zeta}_\ell)$ with $\tilde{\zeta}_i \in \mathbb{C}^n$ the scalar product $\langle\widehat{\ell}|\tilde{\zeta}\rangle$ is just the determinant of the $\ell \times \ell$-matrix $\big(\langle i|\tilde{\zeta}_j\rangle\big)_{i,j}$, see (4.2.8).

Now fix an ordered set partition Λ. Given a hyperedge $e \in \Lambda$ and a hyperedge labeling $\zeta^e\colon e \to \mathbb{C}^n$, we can interpret ζ^e as a list (since e is linearly ordered) and write $|\zeta^e\rangle$. We define the *evaluation*

$$\mathsf{eval}_e(\zeta^e) := \langle\widehat{\ell}|\zeta^e\rangle \in \mathbb{C}.$$

Note that the evaluation $\mathsf{eval}_e(\zeta^e)$ is, up to sign, invariant under changing the linear order of e. For a labeling $\zeta\colon V(\Lambda) \to \mathbb{C}^n$ we define the *evaluation of the ordered set partition* Λ *at the labeling* ζ by

$$\mathsf{eval}_\Lambda(\zeta) := \prod_{e\in\Lambda}\mathsf{eval}_e(\zeta_{|_e}).$$

7.1.6 Proposition. *Let Λ be an ordered set partition. Let $\zeta\colon V(\Lambda) \to \mathbb{C}^n$ be a labeling. We have*

$$\mathsf{eval}_\Lambda(\zeta) = \langle\widehat{\lambda}|\pi_\Lambda|\zeta\rangle.$$

Proof. According to Claim 7.1.3, for the ith hyperedge e of Λ we have

$$\zeta_{|_e} = (\pi_\Lambda\zeta)_{|_{e_i}}.$$

Therefore, if e has size ℓ, then

$$\mathsf{eval}_e(\zeta_{|_e}) = \langle\widehat{\ell}|\zeta_{|_e}\rangle = \langle\widehat{\ell}|(\pi_\Lambda\zeta)_{|_{e_i}}\rangle.$$

The claim follows by definition of $\langle\widehat{\lambda}|$ in (4.2.10). □

7.2 Obstruction Designs

We want to describe the highest weight vectors of $\bigotimes^d\bigotimes^3(\mathbb{C}^n)^*$ with set partitions as we did for $\bigotimes^d(\mathbb{C}^n)^*$. For this we make the following definition, analogously to Definition 7.1.1.

7.2.1 Definition. An *ordered set partition triple* $\mathcal{H}$ consists of a vertex set $V(\mathcal{H}) = \{1, \dots, d\}$ and three ordered set partitions $E^{(k)} := E^{(k)}(\mathcal{H})$, $k \in \{1, 2, 3\}$, of $V(\mathcal{H})$. The elements of each $E^{(k)}$ are called *hyperedges*.

The ordered set partition triple $\mathcal{H}$ is said to have *type* λ, where $\lambda \vdash^{\!*} d$ is a partition triple, if the set partition $E^{(k)}$ has type $\lambda^{(k)}$ for all $1 \le k \le 3$. ■

Via our explicit bijections between S_d and the set of set partitions of a fixed type, we get an explicit bijection between S_d^3 and the set of ordered set partition triples of type λ, see Figure 7.2.i. The permutation triple corresponding to an ordered set partition $\mathcal{H}$ is denoted by $\pi_{\mathcal{H}}$.

Analogously to (7.1.4), for partition triples $\lambda \vdash^{\!*} d$ we define the *projective stabilizer* Y_λ of $\langle\widehat{\lambda}| = \mathsf{reorder}_{3,n}\big(\langle\widehat{\lambda^{(1)}}| \otimes \langle\widehat{\lambda^{(2)}}| \otimes \langle\widehat{\lambda^{(3)}}|\big)$, see (4.2.14), as

$$Y_\lambda := \{\tau \in \mathsf{S}_d^3 : \langle\widehat{\lambda}|\tau = \pm\langle\widehat{\lambda}|\}.$$

One can show that $Y_\lambda = Y_{\lambda^{(1)}} \times Y_{\lambda^{(2)}} \times Y_{\lambda^{(3)}}$, where $Y_{\lambda^{(k)}}$ is the projective stabilizer defined in (7.1.4). Therefore we can again forget about the orderings and arrive at the following definition.

$$\sigma^{(1)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \sigma^{(2)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad \sigma^{(3)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \rightsquigarrow$$

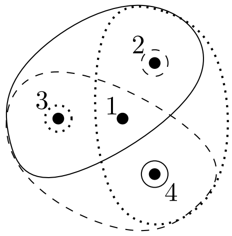


Figure 7.2.i: A set partition triple $\mathcal{H}$ of type $(\lambda, \lambda, \lambda)$, where $\lambda = (2,1,1) = {}^t(3,1)$, and its corresponding permutation triple $\pi_{\mathcal{H}}$ with inverse σ. The solid lines represent the first hyperedge set partition, the dashed lines represent the second one, and the dotted lines represent the third one. To simplify the picture, the hyperedge orderings respect the natural ordering on the natural numbers and are not depicted.

7.2.2 Definition. A *set partition triple* $\mathcal{H}$ consists of a vertex set $V(\mathcal{H}) = \{1, \dots, d\}$ and three set partitions $E^{(k)}$, $k \in \{1,2,3\}$ of $V(\mathcal{H})$. ■

The above discussion implies the following claim, analogously to Claim 7.1.5.

7.2.3 Claim. *For a fixed partition triple $\lambda \vdash^* d$ there is a bijection between the left cosets of $Y_\lambda \subseteq \mathsf{S}_d$ and the set of set partition triples of type λ.*

So each set partition triple $\mathcal{H}$ defines (up to sign) the highest weight vector

$$f_{\mathcal{H}} := \pm\langle\widehat{\lambda}|\pi_{\mathcal{H}} \in \bigotimes^d \bigotimes^3 (\mathbb{C}^n)^*$$

of weight λ^*, where $\lambda \vdash_n^* d$ denotes the type of $\mathcal{H}$.

Triple Contraction. A finite sequence of vectors in $(\mathbb{C}^n)^3$ shall be called a *triple list*. Given a triple list ζ containing d triples, we write

$$\zeta = \begin{pmatrix} \zeta_1^{(1)}, \dots, \zeta_d^{(1)} \\ \zeta_1^{(2)}, \dots, \zeta_d^{(2)} \\ \zeta_1^{(3)}, \dots, \zeta_d^{(3)} \end{pmatrix}.$$

Moreover, we write $\zeta^{(k)} := (\zeta_1^{(k)}, \dots, \zeta_d^{(k)})$ and $\zeta_i := (\zeta_i^{(1)}, \zeta_i^{(2)}, \zeta_i^{(3)})$, and we write $|\zeta\rangle := \mathsf{reorder}_{3,d}(|\zeta^{(1)}\rangle \otimes |\zeta^{(2)}\rangle \otimes |\zeta^{(3)}\rangle) \in \bigotimes^d \bigotimes^3 \mathbb{C}^n$, where $\mathsf{reorder}_{3,d}$ is the linear map defined in (4.2.15). We want to analyze how the scalar product $\langle\widehat{\lambda}|\pi|\zeta\rangle$ can be interpreted combinatorially using set partitions. For an ordered subset $e \subseteq V(\mathcal{H})$ of vertices we identiy triple lists $(\zeta_1, \dots, \zeta_{|e|})$ with labelings on e whose codomain is $(\mathbb{C}^n)^3$.

We define the evaluation function for ordered set partition triples as follows: Given a labeling $\zeta \colon V(\mathcal{H}) \to (\mathbb{C}^n)^3$, we set

$$\mathsf{eval}_{\mathcal{H}}(\zeta) := \mathsf{eval}_{E^{(1)}(\mathcal{H})}(\zeta^{(1)}) \cdot \mathsf{eval}_{E^{(2)}(\mathcal{H})}(\zeta^{(2)}) \cdot \mathsf{eval}_{E^{(3)}(\mathcal{H})}(\zeta^{(3)}).$$

7.2.4 Proposition. *Let $\mathcal{H}$ be an ordered set partition triple of type λ. Let $\zeta \colon V(\mathcal{H}) \to (\mathbb{C}^n)^3$ be a labeling. We have*

$$\mathsf{eval}_{\mathcal{H}}(\zeta) = \langle\widehat{\lambda}|\pi_{\mathcal{H}}|\zeta\rangle.$$

Proof. By (4.2.14) we have $\langle\widehat{\lambda}| = \mathsf{reorder}_{3,d}(\langle\widehat{\lambda}^{(1)}| \otimes \langle\widehat{\lambda}^{(2)}| \otimes \langle\widehat{\lambda}^{(3)}|)$. The claim follows with Proposition 7.1.6. □

Symmetrization. Recall the projection operator $\mathcal{P}_d \colon \bigotimes^d(\mathbb{C}^n)^* \twoheadrightarrow \mathsf{Sym}^d(\mathbb{C}^n)^*$ from (4.2.1). Since $\tau\mathcal{P}_d = \mathcal{P}_d$ for all $\tau \in \mathsf{S}_d$, we get $\langle\widehat{\lambda}|\pi\mathcal{P}_d = \langle\widehat{\lambda}|\pi\tau\mathcal{P}_d$ for all $\tau \in \mathsf{S}_d$. Hence the polynomial described by a set partition triple is independent of the numbering of its vertices. This explains the following definition.

7.2.5 Definition. An *obstruction predesign* is defined to be an equivalence class of set partition triples under renumbering of the vertices. When depicting obstruction predesigns, we omit the vertex numbering of the corresponding set partition triple. ■

So each obstruction predesign describes some polynomial $\langle\widehat{\lambda}|\pi\mathcal{P}_d \in \mathsf{Sym}^d\bigotimes^3(\mathbb{C}^n)^*$ of degree d up to sign. Since we do not care about the sign, we abuse notation in the following way: For every obstruction predesign $\mathcal{H}$ we implicity fix an ordered set partition triple $\mathcal{H}'$ in a way such that $\mathcal{H}$ is obtained from $\mathcal{H}'$ by forgetting about orderings and vertex numbers. Then we define $\mathsf{eval}_{\mathcal{H}}(\zeta) := \mathsf{eval}_{\mathcal{H}'}(\zeta)$.

7.2.6 Corollary. *Let $\mathcal{H}$ be an ordered set partition triple with d vertices. Let $\xi \colon V(\mathcal{H}) \to (\mathbb{C}^n)^3$ be a labeling. We have*

$$\frac{1}{d!}\sum_{\zeta\in\mathsf{S}_d\xi} \mathsf{eval}_{\mathcal{H}}(\zeta) = \langle\widehat{\lambda}|\pi_{\mathcal{H}}\mathcal{P}_d|\xi\rangle.$$

Proof. Follows from Proposition 7.2.4 and the definition of $\mathcal{P}_d$ in (4.2.1). □

Forbidden Patterns in Obstruction Predesigns. Consider the following obstruction predesign $\mathcal{H}$, consisting of four vertices y_1, y_2, y_3, and y_4, depicted in Fig-

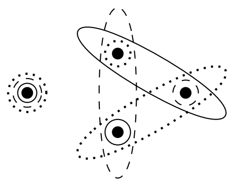


Figure 7.2.ii: An obstruction predesign of type $(3,1)^3 = ({}^t(2,1,1))^3$ corresponding to the zero polynomial. This obstruction predesign is the disjoint union of an obstruction predesign of type $(2,1)^3$ with one of type $(1)^3$.

ure 7.2.ii. The hyperedge sets are defined as $E^{(k)} := \{\{y_1\},\{y_{k+1}\},\{y_{k+2},y_{k+3}\}\}$, $1 \le k \le 3$, where $y_5 := y_2$ and $y_6 := y_3$. The type of $\mathcal{H}$ is $\lambda = (3,1)^3$.

We are going to analyze the polynomial $f_{\mathcal{H}} = \langle\widehat{\lambda}|\pi_{\mathcal{H}}\mathcal{P}_4 \in \mathsf{Sym}^4(\mathbb{C}^2)^*$ corresponding to $\mathcal{H}$. We have

$$\begin{aligned}
&\mathsf{reorder}_{4,3}(\langle\widehat{\lambda}|) = (\langle\widehat{2\otimes 1\otimes 1}|)^{\otimes 3}\\
&\quad = \langle 1211\,1211\,1211| - \langle 2111\,1211\,1211| - \langle 1211\,2111\,1211| + \langle 2111\,2111\,1211|\\
&\quad - \langle 1211\,1211\,2111| + \langle 2111\,1211\,2111| + \langle 1211\,2111\,2111| - \langle 2111\,2111\,2111|
\end{aligned}$$

and applying $\pi := \pi_{\mathcal{H}} = \left(\left(\begin{smallmatrix}1&2&3&4\\1&2&3&4\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3&4\\3&2&1&4\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right)\right) \in \mathsf{S}_4^3$ we obtain

$$\begin{aligned}
&\mathsf{reorder}_{4,3}(\langle\widehat{\lambda}|\pi)\\
&\quad = \langle 1211\,1211\,1121| - \langle 2111\,1211\,1121| - \langle 1211\,1121\,1121| + \langle 2111\,1121\,1121|\\
&\quad - \langle 1211\,1211\,2111| + \langle 2111\,1211\,2111| + \langle 1211\,1121\,2111| - \langle 2111\,1121\,2111|.
\end{aligned}$$

Hence, reordering the tensor factors we get

$$\begin{aligned}
\langle\widehat{\lambda}|\pi &= \langle 111\,221\,112\,111| - \langle 211\,121\,112\,111| - \langle 111\,211\,122\,111| + \langle 211\,111\,122\,111|\\
&- \langle 112\,221\,111\,111| + \langle 212\,121\,111\,111| + \langle 112\,211\,121\,111| - \langle 212\,111\,121\,111|.
\end{aligned}$$

This implies

$$\begin{aligned}\langle\widehat{\lambda}|\pi\mathcal{P}_4 = {} & \langle(111)\odot(221)\odot(112)\odot(111)| - \langle(211)\odot(121)\odot(112)\odot(111)| \\ & - \langle(111)\odot(211)\odot(122)\odot(111)| + \langle(211)\odot(111)\odot(122)\odot(111)| \\ & - \langle(112)\odot(221)\odot(111)\odot(111)| + \langle(212)\odot(121)\odot(111)\odot(111)| \\ & + \langle(112)\odot(211)\odot(121)\odot(111)| - \langle(212)\odot(111)\odot(121)\odot(111)|,\end{aligned}$$

where $\langle t_1 \odot t_2 \odot t_3 \odot t_4| := \frac{1}{24}\sum_{\sigma\in\mathsf{S}_4}\langle t_{\sigma(1)}\otimes t_{\sigma(2)}\otimes t_{\sigma(3)}\otimes t_{\sigma(4)}|$ denotes the symmetric product. One can check that every summand cancels out and obtain $\langle\lambda|\pi_{\mathcal{H}}\mathcal{P}_4 = 0$. The next lemma gives a sufficient criterion for this behaviour.

7.2.7 Lemma. *If an obstruction predesign $\mathcal{H}$ contains two vertices y and y' which lie in the same three hyperedges, then the polynomial $f_{\mathcal{H}} = 0$ is the zero polynomial.*

Note that in the example above, the conditions of Lemma 7.2.7 are *not* satisfied, but nevertheless the corresponding polynomial is the zero polynomial.

Proof of Lemma 7.2.7. We choose an arbitrary triple list $\xi \in \left((\mathbb{C}^n)^3\right)^d$ and show the vanishing of the contraction $\langle\widehat{\lambda}|\pi\mathcal{P}_d|\xi\rangle$, where $\pi \in \mathsf{S}_d^3$ corresponds to $\mathcal{H}$. According to Corollary 7.2.6 we have

$$\langle\widehat{\lambda}|\pi\mathcal{P}_d|\xi\rangle = \frac{1}{d!}\sum_{\zeta\in\mathsf{S}_d\xi}\mathsf{eval}_{\mathcal{H}}(\zeta).$$

We are going to see that switching the label of y and y' pairs summands that add up to zero: Let $\tau\colon V(\mathcal{H}) \to V(\mathcal{H})$ denote the transposition switching y and y'. From a labeling $\zeta\colon V(\mathcal{H}) \to (\mathbb{C}^n)^3$ we get a new labeling $\zeta\circ\tau$ by composition of maps. We show that $\mathsf{eval}_{\mathcal{H}}(\zeta) = -\mathsf{eval}_{\mathcal{H}}(\zeta\circ\tau)$. Let S denote the set of the three hyperedges containing both y and y'. For a hyperedge $e \in E^{(k)}$ we write $\zeta_{|_e} := \zeta^{(k)}{}_{|_e}$. Let

$$\alpha := \prod_{e\notin S}\mathsf{eval}_e(\zeta_{|_e}) = \prod_{e\notin S}\mathsf{eval}_e((\zeta\circ\tau)_{|_e}).$$

We calculate

$$\begin{aligned}\mathsf{eval}_{\mathcal{H}}(\zeta) + \mathsf{eval}_{\mathcal{H}}(\zeta\circ\tau) &= \alpha\prod_{e\in S}\left(\mathsf{eval}_e(\zeta_{|_e}) + \mathsf{eval}_e((\zeta\circ\tau)_{|_e})\right) \\ &= \alpha\prod_{e\in S}\left(\mathsf{eval}_e(\zeta_{|_e}) + (-1)^3\mathsf{eval}_e(\zeta_{|_e})\right) = 0. \qquad \square\end{aligned}$$

7.2.8 Remark. There are other situations in which we can see immediately that $f_{\mathcal{H}} = 0$ for an obstruction predesign $\mathcal{H}$. Consider the case where two vertices y and y' each lie in two singleton hyperedges. If there is a hyperedge containing both y and y' then $f_{\mathcal{H}} = 0$. The proof is analogous to the proof of Lemma 7.2.7. ■

Obstruction predesigns that do *not* satisfy the assumptions of Lemma 7.2.7 are exactly those which satisfy the following crucial definition.

7.2.9 Definition. An *obstruction design* $\mathcal{H}$ is an obstruction predesign $\mathcal{H}$ which satisfies

$$|e_1 \cap e_2 \cap e_3| \leq 1 \text{ for all hyperedge triples } (e_1, e_2, e_3) \in E^{(1)} \times E^{(2)} \times E^{(3)}. \quad ■$$

7.2.10 Proposition. *For a partition triple λ we have*

$$\mathsf{HWV}_{\lambda^*}(\mathsf{Sym}^d\bigotimes{}^3(\mathbb{C}^*)^n) = \operatorname{span}\{f_{\mathcal{H}} : \mathcal{H} \textit{ is an obstruction design of type } \lambda\}.$$

In particular

$$k(\lambda) = \dim \operatorname{span}\{f_{\mathcal{H}} : \mathcal{H} \textit{ is an obstruction design of type } \lambda\}$$

and

$$k(\lambda) \leq |\{\mathcal{H} : \mathcal{H} \textit{ is an obstruction design of type } \lambda\}|.$$

Proof. According to Claim 4.2.17, for $\lambda \vdash^*_n d$, we have that

$$\mathsf{HWV}_{\lambda^*}(\mathsf{Sym}^d \bigotimes^3 (\mathbb{C}^*)^n) = \operatorname{span}\{\langle\widehat{\lambda}|\pi_{\mathcal{H}}\mathcal{P}_d : \pi \in \mathsf{S}_d^3\}.$$

But since obstruction designs $\mathcal{H}$ determine $f_{\mathcal{H}} = \langle\widehat{\lambda}|\pi_{\mathcal{H}}\mathcal{P}_d$ up to a sign, the first assertion follows. The rest of the proposition follows from Corollary 4.4.9. □

7.2.11 Example. Let $\lambda = \big((16,2,1,1),(16,2,1,1),(6,2,12{\times}1)\big) \vdash^*_{14} 20$. Every obstruction design $\mathcal{H}$ of type λ has a hyperedge $e \in E^{(3)}(\mathcal{H})$ of size 14, but at most $4+2+4+2 = 12$ vertices that do not lie in singleton hyperedges from $E^{(2)}$ or $E^{(3)}$. Hence there are two vertices y and y' in e such that $\{y\} \in E^{(1)}(\mathcal{H}) \cap E^{(2)}(\mathcal{H})$ and $\{y'\} \in E^{(1)}(\mathcal{H}) \cap E^{(2)}(\mathcal{H})$ Thus, by Remark 7.2.8 and Proposition 7.2.10, we have $k(\lambda) = 0$. ■

7.2.12 Example. The most trivial obstruction design $\mathcal{H}_0$ is a single vertex with three hyperedges: . It has type $((1),(1),(1))$ and describes the polynomial $f_0 = \langle 111| \in \mathsf{Sym}^1 \bigotimes^3 (\mathbb{C}^n)^*$. We will need f_0 and $\mathcal{H}_0$ in Section 7.4. ■

The following fundamental questions arise when studying obstruction designs.

7.2.13 Questions. *(1) Given an obstruction design $\mathcal{H}$ of type $\lambda \vdash^*_n d$ and a tensor $v \in \bigotimes^3 \mathbb{Z}^n$. What is the complexity of computing the evaluation $f_{\mathcal{H}}(v)$? Is this problem* **#P***-hard under Turing reductions?*

(2) Given an obstruction design $\mathcal{H}$ of type $\lambda \vdash^*_n d$. What is the complexity of deciding whether $f_{\mathcal{H}}(v) = 0$ is the zero polynomial?

(3) For a given partition triple $\lambda \vdash^*_n d$, explicitly describe a maximal linear independent subset of the set of obstruction designs of type λ!

An answer to Question 7.2.13(3) would result in an explicit basis of the invariant space $[\lambda^{(1)}] \otimes [\lambda^{(2)}] \otimes [\lambda^{(3)}]$ and solve one of the most fundamental open questions in the representation theory of the symmetric groups, cf. Remark 4.2.18.

Positivity of Kronecker Coefficients and Discrete Tomography. Fix a finite subset $r \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and identify r with its characteristic function $r \colon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \{0,1\}$. We define the *first marginal distribution* $p^{(1)}(r)$ as $p^{(1)}(r) := \big(\sum_{j,l \in \mathbb{N}} r(i,j,l)\big)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. Analogously we define the *second and third marginal distributions* $p^{(2)}$ and $p^{(3)}$, respectively. For a given partition triple λ let

$$\mathscr{R}_\lambda := \{r \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid p^{(k)}(r) = {}^t\lambda^{(k)} \text{ for all } 1 \leq k \leq 3\}$$

denote the finite set of elements with prescribed marginal distributions. Note that all triples (i,j,l) which are contained in $r \in \mathscr{R}_\lambda$ satisfy $1 \leq i \leq \lambda_1^{(1)}$, $1 \leq j \leq \lambda_1^{(2)}$, and $1 \leq l \leq \lambda_1^{(3)}$. This means that these triples are contained in a box of side lengths $\lambda_1^{(1)} \times \lambda_1^{(2)} \times \lambda_1^{(3)}$. Elements in $\mathscr{R}_\lambda$ shall be called *3-ary relations of format* λ.

We are going to describe a surjective map

$$\varphi \colon \mathscr{R}_\lambda \to \{\mathcal{H} \mid \mathcal{H} \text{ obstruction design of type } \lambda\}.$$

For $r \in \mathscr{R}_\lambda$ let $\mathcal{H} := \varphi(r)$ be defined by setting $V(\mathcal{H}) := r$ and adding hyperedges as follows. The ath hyperedge of $E^{(1)}$ is defined to contain the vertices (a, j, l) for $1 \le j \le \lambda_1^{(2)}$ and $1 \le l \le \lambda_1^{(3)}$. The hyperedge sets $E^{(2)}$ and $E^{(3)}$ are defined analogously. It is easily verifiable that $\mathcal{H}$ is an obstruction design of type λ.

In general, the map φ is not necessarily a bijection. We are going to analyze its fibers. The Young subgroup $\mathsf{S}_{\lambda_1^{(1)}} \times \mathsf{S}_{\lambda_1^{(2)}} \times \mathsf{S}_{\lambda_1^{(3)}}$ acts on $\mathbb{N}^{\lambda_1^{(1)}} \times \mathbb{N}^{\lambda_1^{(2)}} \times \mathbb{N}^{\lambda_1^{(3)}}$ via permutation of positions. The group $\mathsf{stab}_{\mathsf{S}_{\lambda_1^{(1)}} \times \mathsf{S}_{\lambda_1^{(2)}} \times \mathsf{S}_{\lambda_1^{(3)}}}({}^t\lambda^{(1)}, {}^t\lambda^{(2)}, {}^t\lambda^{(3)})$ acts on $\mathscr{R}_\lambda$ via

$$\sigma^{-1} r := \{(i, j, l) \mid \sigma^{(1)}(i), \sigma^{(2)}(j), \sigma^{(3)}(l)\}.$$

Two 3-ary relations of format λ are called *equivalent* if they lie in the same orbit w.r.t. this operation. Let $\mathscr{R}_\lambda/_\sim$ denote the set of equivalence classes. It is readily checked that $\varphi(r_1) = \varphi(r_2)$ iff r_1 is equivalent to r_2. Hence $|\mathscr{R}_\lambda/_\sim|$ equals the number of obstruction designs of type λ. We therefore conclude

$$k(\lambda) \;\le\; |\mathscr{R}_\lambda/_\sim| \;\le\; |\mathscr{R}_\lambda|. \tag{7.2.14}$$

Clearly $|\mathscr{R}_\lambda/_\sim| > 0$ iff $|\mathscr{R}_\lambda| > 0$.

Consider the following decision problem: Given $(\lambda, 1^d)$, where $\lambda \vdash\!\!{}^{*}\, d$ is a partition triple encoded in binary and 1^d is a string of 1s of length d. We define the languages L and ${}^tL$ via $(\lambda, 1^d) \in L$ iff $|\mathscr{R}_\lambda| > 0$ and $(\lambda, 1^d) \in {}^tL$ iff $|\mathscr{R}_{{}^t\lambda}| > 0$. [BDLG01] proved that deciding membership in ${}^tL$ is **NP**-complete. From this it is immediate that deciding membership in L is **NP**-complete as well.

This implies the following proposition.

7.2.15 Proposition. *Given a partition triple λ encoded in unary. Then it is* **NP***-complete to decide whether there exists an obstruction design of type λ.*

We finish this paragraph with some remarks about uniqueness of 3-ary relations of format λ. A set $r \in \mathscr{R}_\lambda$ shall be called *additive*, if there exist three real-valued functions $f^{(k)} \colon \mathbb{N} \to \mathbb{R}$ such that for all i, j, l

$$(i, j, l) \in r \Leftrightarrow f^{(1)}(i) + f^{(2)}(j) + f^{(3)}(l) \ge 0. \tag{†}$$

In [FLRS91, Thms. 1 and 2] it is shown that if $r \in \mathscr{R}_\lambda$ is additive, then $|\mathscr{R}_\lambda| = 1$.

7.2.16 Example. Let $\lambda = ((2n+1) \ulcorner (n+1))^3 \vdash\!\!{}^{*}\, 3n+1$, where we used the hook notation from Section 4.1. The set

$$\begin{aligned} r := &\{(i, 1, 1) \mid 1 \le i \le n+1\} \\ &\cup \{(1, j, 1) \mid 1 \le j \le n+1\} \\ &\cup \{(1, 1, l) \mid 1 \le l \le n+1\} \end{aligned}$$

is easily seen to be a 3-ary relation of format λ. For all $1 \le k \le 3$ we define $f^{(k)} \colon \mathbb{N} \to \mathbb{R}$ via $f^{(k)}(1) = 1$ and $f^{(k)}(i) = -1$ for all $i > 1$. It is easy to check that (†) is satisfied and hence r is additive. We conclude $|\mathscr{R}_\lambda| = 1$. Moreover, if follows that there is exactly one obstruction design of type λ. ■

It was shown in [TCV98, Thm. 1(ii)] that if $|\mathscr{R}_\lambda| = 1$, then r is a *pyramid*, which means that r satisfies the following property.

$$(i, j, l) \in r \implies \Big((i', j', l') \in r \text{ for all } 1 \le i' \le i,\ 1 \le j' \le j,\ 1 \le l' \le l \Big).$$

Observe that r in Example 7.2.16 is indeed a pyramid.

Unique Obstruction Designs. It is rare that partition triples λ have only a single obstruction design. In these cases Proposition 7.2.10 implies $k(\lambda) \leq 1$. We already saw such a triple in Example 7.2.16:

$$\lambda = \big((2n+1) \sqcap (n+1)\big)^3 \overset{*}{\vdash} 3n+1.$$

Figure 7.2.iii shows the obstruction design of type λ for $n = 2$. We will use this triple in Section 8.2.

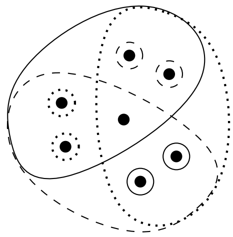


Figure 7.2.iii: The unique obstruction design of type $\lambda = (5 \sqcap 3)^3$. Note that ${}^t\lambda^{(k)} = (5,1,1)$.

The following is also a triple with a single obstruction design $\mathcal{H}$:

$$\lambda = (n^2{\times}n, n^2{\times}n, n^2{\times}n).$$

This obstruction design $\mathcal{H}$ can be best described when embedding the n^3 vertices into three-dimensionsional space such that we have vertices at (i,j,l), $1 \leq i,j,l \leq n$. For each i there is a hyperedge $e_i^{(1)} \in E^{(1)}(\mathcal{H})$ containing the n^2 vertices (i,j,l), $1 \leq j,l \leq n$. Analogously, for each j there is a hyperedge $e_j^{(2)} \in E^{(2)}(\mathcal{H})$ containing the vertices (i,j,l), $1 \leq i,l \leq n$. And in the same manner, for each l there is a hyperedge $e_l^{(3)} \in E^{(3)}(\mathcal{H})$ containing the vertices (i,j,l), $1 \leq i,j \leq n$.

7.2.17 Claim. *$\mathcal{H}$ is the unique obstruction design of type λ.*

Proof. For each $1 \leq k \leq 3$, every obstruction design $\mathcal{H}'$ of type λ has n hyperedges in $E^{(k)}(\mathcal{H}')$ of size n^2.

We show that the intersection $e^{(1)} \cap e^{(2)}$ of any pair $(e^{(1)}, e^{(2)}) \in E^{(1)}(\mathcal{H}') \times E^{(2)}(\mathcal{H}')$ contains at most n vertices. Assume the contrary. Since $E^{(3)}(\mathcal{H}')$ has only n hyperedges, there exists a hyperedge $e^{(3)} \in E^{(3)}(\mathcal{H}')$ containing at least two vertices from $e^{(1)} \cap e^{(2)}$. But this implies $|e^{(1)} \cap e^{(2)} \cap e^{(3)}| > 1$, which is a contradiction to Definition 7.2.9.

Since $|V(\mathcal{H})| = n^3$, we see that the each of the n^2 intersections $e^{(1)} \cap e^{(2)}$ of pairs $(e^{(1)}, e^{(2)}) \in E^{(1)}(\mathcal{H}') \times E^{(2)}(\mathcal{H}')$ not only contains *at most* n vertices, but it contains *exactly* n vertices.

Since $\mathcal{H}$ is an obstruction design, a hyperedge $e^{(3)} \in E^{(3)}(\mathcal{H}')$ contains at most one vertex from each of the n^2 intersections $e^{(1)} \cap e^{(2)}$ of pairs $(e^{(1)}, e^{(2)}) \in E^{(1)}(\mathcal{H}') \times E^{(2)}(\mathcal{H}')$. Since $|V(\mathcal{H})| = n^3$, we have that each $e^{(3)} \in E^{(3)}(\mathcal{H}')$ contains *exactly* one vertex from each of the n^2 intersections. Hence we have $\mathcal{H}' = \mathcal{H}$ as claimed. $\square$

Using Corollary 4.4.15 with $a = b = c = n$ we can conclude $k(\lambda) = 1$. Hence $f_{\mathcal{H}} \neq 0$. Note that, up to scaling, $f_{\mathcal{H}} \in \mathsf{Sym}^n \bigotimes^3 (\mathbb{C}^{n^2})^*$ is the unique polynomial satisfying the following beautiful invariance property:

$$g f_{\mathcal{H}} = \det(g)^{-n} f_{\mathcal{H}} \quad \text{for } g \in \mathsf{GL}_{n^2}^3,$$

using the function $\det\colon \mathsf{GL}_{n^2}^3 \to \mathbb{C}^\times$ defined in (5.1.1).

We will see another family of partition triples that have unique obstruction designs in the proof of Lemma 7.3.1.

Polynomial Evaluation. We now describe how to evaluate the polynomial $f_{\mathcal{H}}$ corresponding to an obstruction design $\mathcal{H}$ at a point $|w\rangle = \sum_{i=1}^{r} |w_i^{(1)}\rangle \otimes |w_i^{(2)}\rangle \otimes |w_i^{(3)}\rangle \in \bigotimes^3 \mathbb{C}^n$. We calculate

$$f_{\mathcal{H}}(w) \overset{\text{Lem. 4.2.2}}{=} \langle\widehat{\lambda}|\pi\mathcal{P}_d|w^{\otimes d}\rangle = \langle\widehat{\lambda}|\pi|w^{\otimes d}\rangle = \sum_{J\in\{1,\dots,r\}^d} \langle\widehat{\lambda}|\pi|w_{J_1}w_{J_2}\cdots w_{J_d}\rangle$$
$$\overset{\text{Prop. 7.2.4}}{=} \sum_{J\in\{1,\dots,r\}^d} \mathsf{eval}_{\mathcal{H}}(w_{J_1}, w_{J_2}, \dots, w_{J_d}). \quad (7.2.18)$$

We can interpret $\zeta := (w_{J_1}, w_{J_2}, \dots, w_{J_d})$ as a vertex labeling $\zeta\colon V(\mathcal{H}) \to (\mathbb{C}^n)^3$ and see that the sum in (7.2.4) is over all vertex labelings $\zeta\colon V(\mathcal{H}) \to (\mathbb{C}^n)^3$ with $\zeta(y) \in \{w_i \mid 1 \le i \le r\}$ for all $y \in V(\mathcal{H})$.

Example: Unit Tensor. In some specific cases we can easily see the vanishing of highest weight vectors on the orbit $\mathsf{GL}_n^3\mathcal{E}_n$ of the unit tensor.

7.2.19 Proposition. *Let $f \in \mathsf{Sym}^{nl}\bigotimes^3(\mathbb{C}^n)^*$ be a highest weight vector of weight λ^* such that $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = n{\times}l$ with l odd and $n > 1$. Then f vanishes on $\mathsf{GL}_n^3\mathcal{E}_n$.*

Proof. According to Proposition 7.2.10, it suffices to show the result for highest weight vectors $f_{\mathcal{H}}$ which correspond to obstruction designs $\mathcal{H}$ of type $\lambda = (n{\times}l)^3$. Let $A \in (\mathbb{C}^{n\times n})^3$ be arbitrary and let $|w_i^{(k)}\rangle := A^{(k)}|i\rangle$ for $1 \le k \le 3$. We have $A\mathcal{E}_n = \sum_{i=1}^{n} |w_i^{(1)}\rangle \otimes |w_i^{(2)}\rangle \otimes |w_i^{(3)}\rangle \in \bigotimes^3\mathbb{C}^n$. Evaluating $f(A\mathcal{E}_n)$ as described in (7.2.18) yields

$$f(A\mathcal{E}_n) = \sum_{J\in\{1,\dots,n\}^{nl}} \mathsf{eval}_{\mathcal{H}}(w_{J_1}, w_{J_2}, \dots, w_{J_{nl}}) \qquad (*)$$

To get a nonzero summand it is necessary that for each hyperedge $e \subseteq \{1,\dots,nl\}$ we have that $J\colon \{1,\dots,nl\} \to \{1,\dots,n\}$ restricted to e is injective. In our special case, since $|e| = n$ for all hyperedges e, this means that J restricted to e is a bijection. Let S denote the set of mappings $J\colon \{1,\dots,nl\} \to \{1,\dots,n\}$ which are bijective upon restriction to each hyperedge e. We define the involution $\varphi\colon S \to S$ by composing $\varphi(J) := (1\ 2) \circ J$, where $(1\ 2) \in \mathsf{S}_n$ switches 1 and 2. Clearly, $J \in S$ iff $\varphi(J) \in S$. Moreover,

$$\mathsf{eval}_{\mathcal{H}}(w_{J_1}, \dots, w_{J_{nl}}) = (-1)^{3l}\mathsf{eval}_{\mathcal{H}}(w_{\varphi(J)_1}, \dots, w_{\varphi(J)_{nl}}),$$

because the switch of 1 and 2 causes a sign change in each of the $3l$ hyperedges. Since $3l$ is an odd number, we paired summands in $(*)$ that add up to zero and hence the sum $f(A\mathcal{E}_n)$ evaluates to zero. □

Latin Squares and the Alon-Tarsi Conjecture. We want to highlight an interesting relation between the evaluation $f_{\mathcal{H}}(v)$ (see Question 7.2.13(1)) and a combinatorial conjecture regarding Latin squares, called the *Alon-Tarsi conjecture*.

7.2.20 Definition. A *Latin square of side length n* is a filling of the shape $n{\times}n$ and content $n{\times}n$ such that in each row and each column each entry appears exactly once. For each row, its entries naturally define an element of S_n. The same holds for each column. The *sign* of a Latin square is defined as the product of the signs of all rows and columns. ■

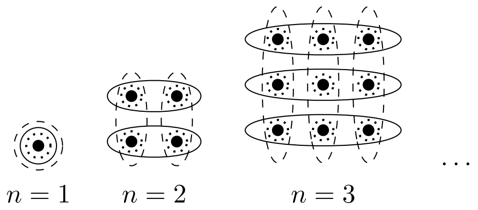


Figure 7.2.iv: A sequence $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots$ of obstruction designs of type $(n{\times}n, n{\times}n, 1{\times}n^2)$.

Consider the sequence $f_{\mathcal{H}_1}, f_{\mathcal{H}_2}, \ldots$ of obstruction designs in Figure 7.2.iv. Define the matrix triple $A_n = (\mathrm{id}_n, \mathrm{id}_n, A_n^{(3)})$ of $n \times n$ matrices via

$$\left(A_n^{(3)}\right)_{i,j} := \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Every nonzero summand of $f_{\mathcal{H}_n}(A_n\mathcal{E}_n)$ in (7.2.18) corresponds to a Latin square of side length n. Moreover, every nonzero summand of $f_{\mathcal{H}_n}(A_n\mathcal{E}_n)$ is either 1 or -1, depending on the sign of its corresponding Latin square. Therefore the evaluation $f_{\mathcal{H}_n}(A_n\mathcal{E}_n)$, up to sign, equals the difference between the even Latin squares and the odd Latin squares of side length n.

It is straightforward to prove that the number of even Latin squares and the number of odd Latin squares are equal for any *odd* number n. Hence $f_{\mathcal{H}_n}(A_n\mathcal{E}_n) = 0$ for odd n.

7.2.21 Conjecture (Alon-Tarsi [AT92])**.** *For even n, the number of even latin squares and the number of odd Latin squares are different.*

Conjecture 7.2.21 is known to be true for all $n \leq 24$, see [Gly10, Cor. 3.4], and for all numbers of format $p-1$ or $p+1$, where p is prime, see [Dri98, Gly10].

We remark that the explicit construction of highest weight vectors in the polynomial scenario leads to questions regarding Latin squares and the Alon-Tarsi conjecture as well, see [Kum12].

The Product. One additional point is worth mentioning. Multiplying two polynomials f_1 and f_2 which are described by obstruction designs $\mathcal{H}_1$ and $\mathcal{H}_2$ is straightforward, as seen in the forthcoming Lemma 7.2.22.

The *disjoint union* $\mathcal{H}_1 \dot\cup \mathcal{H}_2$ of two obstruction designs $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined as follows. As the vertex set we set $V(\mathcal{H}_1 \dot\cup \mathcal{H}_2) := V(\mathcal{H}_1) \dot\cup V(\mathcal{H}_2)$ and for the edges we define $E^{(k)}(\mathcal{H}_1 \dot\cup \mathcal{H}_2) := E^{(k)}(\mathcal{H}_1) \cup E^{(k)}(\mathcal{H}_2)$ for all $1 \leq k \leq 3$. It is easily checked that $\mathcal{H}_1 \dot\cup \mathcal{H}_2$ is an obstruction design. Graphically, this construction means that we draw both obstruction designs next to each other and consider them as one obstruction design, see Figure 7.2.ii for an example.

7.2.22 Lemma. *Let $f_{\mathcal{H}_1} \in \mathsf{Sym}^{d_1} \bigotimes^3 (\mathbb{C}^n)^*$ and $f_{\mathcal{H}_2} \in \mathsf{Sym}^{d_2} \bigotimes^3 (\mathbb{C}^n)^*$ be given by obstruction designs $\mathcal{H}_1$ and $\mathcal{H}_2$. Then*

$$f_{\mathcal{H}_1 \dot\cup \mathcal{H}_2} = f_{\mathcal{H}_1} \cdot f_{\mathcal{H}_2}.$$

Proof. The fact that $\mathcal{H}_1 \dot\cup \mathcal{H}_2$ corresponds to the product $f_{\mathcal{H}_1} \cdot f_{\mathcal{H}_2}$ of polynomials is easily checked by evaluating $\mathcal{H}_1 \dot\cup \mathcal{H}_2$ at an arbitrary point via (7.2.18). □

7.2.23 Remark. A graph construction similar to that of obstruction designs has been made in [LM04, Sec. 6.9]. ■

7.3 Symmetric Obstruction Designs

In this section we show an application of obstruction designs to symmetric Kronecker coefficients by proving the following lemma, which will be used in the proof of Claim 8.1.3(4).

7.3.1 Lemma.

$$\mathsf{sk}\big(n^2\times 1; (n\times n)^2\big) = \begin{cases} 1 & \textit{for} \;\; n \bmod 4 \in \{0,1\} \\ 0 & \textit{for} \;\; n \bmod 4 \in \{2,3\} \end{cases}.$$

Proof. Recall $[n^2\times 1]\otimes[n\times n] \simeq [n\times n]$ and hence $k(n^2\times 1; n\times n; n\times n) = 1$. The crucial fact is that there exists a unique obstruction design $\mathcal{H}$ of type $(n^2\times 1, n\times n, n\times n)$, see Figure 7.3.i, and hence, up to scaling, $f_{\mathcal{H}}$ is the unique nonzero highest weight vector of type $(n^2\times 1, n\times n, n\times n)^*$ in $\mathsf{Sym}^{n^2}\bigotimes^3(\mathbb{C}^{n^2})^*$. Nonzeroness of $f_{\mathcal{H}}$ follows immediately from $k(n^2\times 1; n\times n; n\times n) = 1$, but can also be shown in an elementary way. We omit the details.

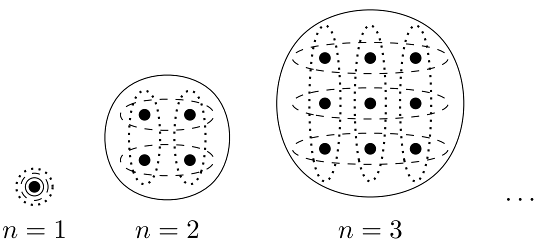


Figure 7.3.i: These obstruction designs are unique for their type. It is clear how this sequence continues for bigger n.

Define the linear map $\tau\colon \bigotimes^3\mathbb{C}^{n^2} \to \bigotimes^3\mathbb{C}^{n^2}$ via $\tau(|w^{(1)}\rangle\otimes|w^{(2)}\rangle\otimes|w^{(3)}\rangle) := |w^{(1)}\rangle\otimes|w^{(3)}\rangle\otimes|w^{(2)}\rangle$. According to Proposition 4.4.5 it remains to show that $\tau f_{\mathcal{H}} = f_{\mathcal{H}}$ for $n \bmod 4 \in \{0,1\}$ and $\tau f_{\mathcal{H}} = -f_{\mathcal{H}}$ otherwise.

Each vertex y has a row index and a column index from 1 to n. We define y' to be the vertex obtained from y when switching the row and column index. Now choose an arbitrary tensor $|w\rangle = \sum_{i=1}^r |w_i^{(1)}\rangle\otimes|w_i^{(2)}\rangle\otimes|w_i^{(3)}\rangle \in \bigotimes^3\mathbb{C}^{n^2}$ such that $f(w)\neq 0$. From (7.2.18) we get

$$\begin{aligned} f(w) + f(\tau(w)) = &\sum_{J\in\{1,\dots,r\}^d} \mathsf{eval}_{\mathcal{H}}(w_{J_1}, w_{J_2}, \dots, w_{J_d}) \\ &+ \sum_{J\in\{1,\dots,r\}^d} \mathsf{eval}_{\mathcal{H}}(\tau(w_{J_1}), \tau(w_{J_2}), \dots, \tau(w_{J_d})) \end{aligned} \qquad (*)$$

It is crucial to see that, using the special structure of $\mathcal{H}$, we have

$$\mathsf{eval}_{\mathcal{H}}(\tau(w_{J_1}), \tau(w_{J_2}), \dots, \tau(w_{J_d})) = \alpha\, \mathsf{eval}_{\mathcal{H}}(w_{J'_1}, w_{J'_2}, \dots, w_{J'_d}),$$

where $J'(y) := J(y')$ and $\alpha = (-1)^{\frac{n^2-n}{2}}$. We have $\alpha = 1$ iff $n \bmod 4 \in \{0,1\}$. Therefore, in the case $n \bmod 4 \in \{2,3\}$, the map τ pairs up summands in $(*)$ that add up to zero and hence $(*)$ vanishes. On the other hand, in the case $n \bmod 4 \in \{0,1\}$, we have $f(w) + f(\tau(w)) = 2f(w) \neq 0$. ∎

7.4 Reduced Kronecker Coefficients

In this section we give a new proof for the following theorem of Murnaghan, which was first proved in [Mur38] via character theory. Our proof method is based on obstruction designs.

7.4.1 Theorem ([Mur38])**.** *Given partitions λ, μ, ν, then the sequence of Kronecker coefficients $(k(\lambda(d);\mu(d);\nu(d)))_d$ stabilizes, where $\lambda(d) := (d-|\lambda|,\lambda)$ denotes the partition of d that equals λ with an additional first row on top. The limit of this sequence is called the* reduced Kronecker coefficient $\bar{k}(\lambda;\mu;\nu)$.

Upper bounds for d from which on the above sequence is stable can be found in [Bri93], [Val99], and [BOR11].

Proof of Theorem 7.4.1. The proof is based on the description of the Kronecker coefficient in Proposition 7.2.10.

Given an obstruction design, a vertex that is contained in three hyperedges of size 1 is called a *singleton vertex*. Recall from Example 7.2.12 the trivial obstruction design $\mathcal{H}_0$ consisting only of a single singleton vertex and recall the corresponding polynomial f_0. If we have an obstruction design $\mathcal{H}$ of type λ corresponding to the polynomial f, then we can obtain a new obstruction design $\mathcal{H} \dot\cup \mathcal{H}_0$ of type $(\lambda^{(1)}+(1),\lambda^{(2)}+(1),\lambda^{(3)}+(1))$ by multiplying $f\cdot f_0$ as described in Lemma 7.2.22. The obstruction design $\mathcal{H} \dot\cup \mathcal{H}_0$ is obtained from $\mathcal{H}$ by adding a new singleton vertex.

Obviously, the dimension of the linear span of a set of polynomials does not change when multiplying each element with a fixed nonzero polynomial f_0.

Fix some partition triple $\lambda \models^{*} d$. It suffices to show that for all i big enough we have

$$\begin{aligned} &\big\{\mathcal{H} \text{ obstr. design of type } \big(\lambda^{(1)}+(i+1),\lambda^{(2)}+(i+1),\lambda^{(3)}+(i+1)\big)\big\} \qquad (*) \\ =&\big\{\mathcal{H}\dot\cup\mathcal{H}_0 \mid \mathcal{H} \text{ obstr. design of type } \big(\lambda^{(1)}+(i),\lambda^{(2)}+(i),\lambda^{(3)}+(i)\big)\big\}. \end{aligned}$$

This can be seen as follows. Define $\psi\colon \mathbb{N}\to\mathbb{N}$, $\psi(1)=0$, $\psi(x)=x$ for $x\neq 1$. Let $j(\lambda) := \sum_{k=1}^{3}\sum_{i=1}^{\lambda_1^{(k)}} \psi({}^t\lambda_i^{(k)})$ denote the number of boxes in columns with more than one box. Note that $j(\lambda) = j(\lambda+((i),(i),(i)))$ for all $i\in\mathbb{N}$. If $d+i+1 > j(\lambda)$, then each obstruction design of type $\lambda \models^{*} d+i+1$ has at least one singleton vertex and hence is of the form $\mathcal{H}\dot\cup\mathcal{H}_0$. Hence $(*)$ follows for $i \geq j(\lambda)-d$. □

In 1958 Littlewood proved the following beautiful relationship between reduced Kronecker coefficients and Littlewood-Richardson coefficients.

7.4.2 Theorem ([Lit58])**.** *Given partitions λ, μ, ν with $|\nu| = |\lambda|+|\mu|$, then $\bar{k}(\lambda;\mu;\nu) = c^{\nu}_{\lambda\mu}$, where $c^{\nu}_{\lambda\mu}$ denotes the* Littlewood-Richardson coefficient.

We focus on the Littlewood-Richardson coefficient in the whole Part II.

Chapter 8

Explicit Obstructions

In this chapter we present various explicit calculations, in particular we present an obstruction family which proves the lower bound $\underline{R}(\mathcal{M}_m) \geq \frac{3}{2}m^2 - 2$ on the border rank of matrix multiplication.

8.1 Some Computations

8.1 (A) Orbit-wise Upper Bounds

Recall the scenarios from Section 2.6. A polynomial serving as an obstruction against $\mathscr{h}_{m,n} \in \overline{G\mathscr{c}_n}$ is a polynomial in the vanishing ideal of $G\mathscr{c}_n$ which does not vanish on some point in $G\mathscr{h}_{m,n}$. Our main concern in this subsection is the first property, i.e., we want to find polynomials in the vanishing ideal of $G\mathscr{c}_n$.

Let $\lambda \vdash_{n^2} dn$. In the polynomial scenario, a sufficient criterion for the existence of a highest weight vector of weight λ^* in the vanishing ideal $I(\mathsf{GL}_{n^2}\det_n)$ is given by

$$p_\lambda(d[n]) > \mathsf{sk}\big(\lambda; \big(n \times d\big)^2\big), \tag{8.1.1}$$

because

$$\begin{aligned}\operatorname{mult}_{\lambda^*}(I(\mathsf{GL}_{n^2}\det_n)) &= \underbrace{\operatorname{mult}_{\lambda^*}(\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^{n^2})}_{\overset{(4.2.5)}{=} p_\lambda(d[n])} - \operatorname{mult}_{\lambda^*}(\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]) \\ &\overset{(3.4.2)}{\geq} p_\lambda(d[n]) - \underbrace{\operatorname{mult}_{\lambda^*}(\mathbb{C}[\mathsf{GL}_{n^2}\det_n])}_{\overset{\text{Thm. } 5.2.3}{=} \mathsf{sk}(\lambda;(n\times d)^2)}.\end{aligned}$$

An abundance of partitions λ satisfying (8.1.1) is given in the Appendix A.1.

Generic Ternary Forms in Seven Variables. In this paragraph, we show how the property (8.1.1) can be used to deduce statements about generic polynomials. Consider $n = 3$, $M = 7$, and $d = 12$ and take the partition

$$\lambda = (13, 13, 2, 2, 2, 2, 2) \vdash_7 36 = 12 \cdot 3.$$

Using Schur we computed $p_\lambda(12[3]) = 1$, which implies $\{\lambda\}^* \subseteq \mathbb{C}[\overline{\mathsf{GL}_9 f}]_{12}$ for a generic $f \in \mathsf{Sym}^3\mathbb{C}^7$, according to Corollary 5.3.5. Moreover, using Derksen we computed $\mathsf{sk}\big(\lambda; \big(3\times 12\big)^2\big) = 0$, and hence Theorem 5.2.3 implies $\{\lambda\}^* \not\subseteq \mathbb{C}[\mathsf{GL}_9\det_3]_{12}$ and in particular $\{\lambda\}^* \not\subseteq \mathbb{C}[\overline{\mathsf{GL}_9\det_3}]_{12}$. Therefore $f \notin \overline{\mathsf{GL}_9\det_3}$ for a generic $f \in \mathsf{Sym}^3\mathbb{C}^7$. Note that λ is one of many partitions listed in the Appendix A.1.

A Conjecture on Symmetric Kronecker Coefficients. Let $\lambda \vdash_{n^2} dn$. A necessary criterion for (8.1.1) is $p_\lambda(d[n]) > 0$. Lemma 4.3.3 implies that if $p_\lambda(d[n]) > 0$, then $\ell(\lambda) \leq d$. Hence for finding irreducible representations in the vanishing ideal of $\mathsf{GL}_{n^2}\det_n$, the only relevant partitions are from the following set $\mathsf{Par}_{d,n}$.

$$\mathsf{Par}_{d,n} := \{\lambda : |\lambda| = dn,\ \ell(\lambda) \leq \max\{d, n^2\}\}.$$

For even n, a sufficient criterion for $p_\lambda(d[n]) > 0$ is given by Theorem 6.2.1: it suffices that $\lambda \in \mathsf{Par}_{d,n}$ is even. To prove (8.1.1) for those $\lambda \vdash dn$ it suffices to show $\mathsf{sk}\big(\lambda; (n\times d)^2\big) = 0$. But it seems that for all even $\lambda \in \mathsf{Par}_{d,n}$ we have $\mathsf{sk}\big(\lambda; (n\times d)^2\big) > 0$. Using Derksen we verified $\mathsf{sk}\big(\lambda; (n\times d)^2\big) > 0$ for all even $\lambda \in \mathsf{Par}_{d,n}$ in the following cases:

n	d
2	$1,\ldots,30$
4	$1,\ldots,12$
6	$1,\ldots,9$
8	$1,\ldots,7$
10	$1,\ldots,5$

This gives some computational evidence to the following conjecture.

8.1.2 Conjecture. *For all even n and for all even partitions $\lambda \vdash nd$ of length at most $\max\{d, n^2\}$ we have $\mathsf{sk}\big(\lambda; (n\times d)^2\big) > 0$.*

It is easy to see (for example using obstruction designs) that the conjecture holds for $d = 1$. Using results on two-row Kronecker coefficients [RW94, Ros01], we can prove for the cases where $n = 2$ or $d = 2$ the weaker form of the conjecture, for Kronecker coefficients instead of symmetric Kronecker coefficients.

8.1 (B) Regular Determinant Function

Recall from Section 5.1 the exponents $o(\mathcal{E}_n) = 2$, $o(\mathcal{M}_m) = 1$, $o(\mathrm{per}_m) = 2$, $o(\det_n) = 1$ for $n \bmod 4 \in \{0,1\}$, and $o(\det_n) = 2$ for $n \bmod 4 \in \{2,3\}$. In these cases we want to analyze the function $\det_v^o \colon Gv \to \mathbb{C}$ introduced in (5.1.2). First of all, we see in the following claim that $\det_v^o$ can be characterized as the unique highest weight vector in $\mathbb{C}[Gv]$ with a certain weight (up to a scaling factor).

8.1.3 Claim. *The functions $\det^2_{\mathcal{E}_n}$, $\det_{\mathcal{M}_m}$, $\det^2_{\mathrm{per}_m}$, and $\det^o_{\det_n}$ are highest weight vectors of weight $((n\times 2)^3)^*$, $((m^2\times 1)^3)^*$, $(m^2\times 2)^*$, and $(n^2\times o)^*$, respectively, where $o = 1$ for $n \bmod 4 \in \{0,1\}$, and $o = 2$ otherwise. Moreover,*
(1) $\mathrm{mult}_{(n\times 2, n\times 2, n\times 2)^*}(\mathbb{C}[\mathsf{GL}_n^3 \mathcal{E}_n]) = 1$ *for all n.*
(2) $\mathrm{mult}_{(m^2\times 1, m^2\times 1, m^2\times 1)^*}(\mathbb{C}[\mathsf{GL}_{m^2}^3 \mathcal{M}_m]) = 1$ *for all m.*
(3) $\mathrm{mult}_{(m^2\times 2)^*}(\mathbb{C}[\mathsf{GL}_{m^2}\mathrm{per}_m]) = 1$ *for all m.*
(4) $\mathrm{mult}_{(n^2\times 1)^*}(\mathbb{C}[\mathsf{GL}_{n^2}\det_n]) = 1$ *for all $n \bmod 4 \in \{0,1\}$.*
$\mathrm{mult}_{(n^2\times 2)^*}(\mathbb{C}[\mathsf{GL}_{n^2}\det_n]) = 1$ *for all $n \bmod 4 \in \{2,3\}$.*

Proof. We first show that $\det_v^o$ is a highest weight vector in all four cases. Let $U \subseteq G$ denote the maximal unipotent group. For $u \in U$ and $g \in G$ we have $u\det_v^o(gv) = \det_v^o(u^{-1}gv) = \det^o(u^{-1}g) = \det^o(g) = \det_v^o(gv)$, because $\det(u^{-1}) = 1$. Henec $\det_v^o$ is fixed by U. We determine the weight in all four cases separately.
(1) Let $v = \mathcal{E}_n$ and $g \in \mathsf{GL}_n^3$. For a triple $t = (t^{(1)}, t^{(2)}, t^{(3)}) \in \mathsf{GL}_n^3$ of diagonal matrices we have $(t\det_v^2)(gv) = \det_v^2(t^{-1}gv) = \det(t^{-2}g^2) = \det(t^{-2})\det_v^2(gv) = {t_1^{(1)}}^{-2} \cdots {t_n^{(1)}}^{-2} \cdots {t_n^{(3)}}^{-2} . \det_v^2(gv)$. Hence $\det_v^2$ has weight $(n\times 2, n\times 2, n\times 2)^*$.
(2) is completely analogous to (1). We obtain $\det_{\mathcal{M}_m}$ has weight $(n\times 1, n\times 1, n\times 1)^*$.

(3) Let $v = \mathrm{per}_m$ and $g \in \mathsf{GL}_{m^2}$. For a diagonal matrix $t \in \mathsf{GL}_{m^2}$ we have

$$\begin{aligned}(t\det_v^2)(gv) &= \det_v^2(t^{-1}gv) = \det(t^{-2}g^2) = \det(t^{-2})\det_v^2(gv)\\ &= {t_1}^{-2}\cdots {t_{m^2}}^{-2}.\det_v^2(gv).\end{aligned}$$

Hence $\det_v^2$ has weight $(m^2{\times}2)^*$.
(4) is completely analogous to (3). We conclude that $\det^o_{\det_n}$ has weight $(n^2{\times}o)^*$.

It remains to show the uniqueness of the highest weight vector $\det_v^o$.
(1). Let $\lambda = (n{\times}2)^3$. The sum in Theorem 5.2.1 contains only a single summand, namely for $\mu = n{\times}2$. We have $\mathsf{stab}_{\mathsf{S}_n}(\mu) = \mathsf{S}_n$. There is only a single semistandard tableau of shape $n{\times}2$ and content $n{\times}2$ and it has two entries i in row i. Hence the weight space $\{n{\times}2\}^{n\times 2}$ is 1-dimensional, see Section 4.1. Therefore $\{n{\times}2\}^{n\times 2}$ and $\bigotimes^3\{n{\times}2\}^{n\times 2}$ are isomorphic S_n-representations. According to Theorem 4.3.8, $\dim\left(\{n{\times}2\}^{n\times 2}\right)^{\mathsf{S}_n} = p_{n\times 2}(n[2])$, which is 1 by Theorem 4.3.9.
(2). We use Theorem 5.2.5. Recall the fact that for $\lambda\vdash m^2$ we have $[m^2{\times}1]\otimes[\lambda] \simeq [{}^t\lambda]$. Since $[m{\times}m]\otimes[m^2{\times}1] \simeq [{}^t(m{\times}m)] \simeq [(m{\times}m)]$, we have $k(m^2{\times}1; m{\times}m; m{\times}m) = 1$. Hence if we choose $\mu^{(1)} = \mu^{(2)} = \mu^{(3)} = m{\times}m$, then we see

$$\mathrm{mult}_{(m^2\times 1, m^2\times 1, m^2\times 1)}(\mathbb{C}[\mathsf{GL}^3_{m^2}\mathcal{M}_m]) \geq 1.$$

Additionally, if for $\mu \vdash^{*}_{m} m^2$ we have

$$k(m^2{\times}1;\mu^{(2)};\mu^{(3)})k(m^2{\times}1;\mu^{(1)};\mu^{(3)})k(m^2{\times}1;\mu^{(1)};\mu^{(2)}) > 0,$$

then, according to (4.4.6), we have $\mu^{(1)} = {}^t\mu^{(2)} = \mu^{(3)} = {}^t\mu^{(1)}$, and analogously $\mu^{(2)} = {}^t\mu^{(2)}$ and $\mu^{(3)} = {}^t\mu^{(3)}$. The only partitions with m^2 boxes and at most m rows that are invariant under transposition are the rectangular partitions $m{\times}m$. It follows $\mathrm{mult}_{(m^2\times 1,m^2\times 1,m^2\times 1)}(\mathbb{C}[\mathsf{GL}^3_{m^2}\mathcal{M}_m]) = 1$.
(3). We use Theorem 5.2.4. We have $\lambda = m^2{\times}2$ and $d = 2m$. Since $\ell(\lambda) = m^2$, for a nonzero summand we must have $\ell(\mu) = \ell(\nu) = m$ (Cor. 4.4.12). Since the GL_{m^2}-representation $\{\lambda\}$ is 1-dimensional, Proposition 4.4.11 tells us that there is only one choice of $\mu,\nu \vdash_{m} 2m^2$ satisfying $k(\lambda;\mu;\nu) > 0$. In fact, we know that $\{\lambda\}\downarrow^{\mathsf{GL}_{m^2}}_{\mathsf{GL}_m\times\mathsf{GL}_m} \simeq \{m{\times}2m\}\otimes\{m{\times}2m\}$, see (4.4.13). Hence $k(m^2{\times}2; m{\times}2m; m{\times}2m) = 1$. But since $k(m^2{\times}1; m{\times}m; m{\times}m) = 1$, it follows that $\mathsf{sk}\big(m^2{\times}1; (m{\times}m)^2\big) = 1$ (see Corollary 6.3.8). The fact $p_{m\times 2m}(m[2m]) = 1$ follows from Theorem 4.3.9.
(4). We use Theorem 5.2.3 for $\lambda = n^2{\times}2$ and $d = 2n$. Similar to what we have seen in (3), we have $k(n^2{\times}1; n{\times}n; n{\times}n) = k(n^2{\times}2; n{\times}2n; n{\times}2n) = 1$. Hence $\mathsf{sk}\big(n^2{\times}1; (n{\times}2n)^2\big) = 1$ according to Corollary 6.3.8, which proves the second part of the claim. The fact that $\mathsf{sk}\big(n^2{\times}1; (n{\times}n)^2\big) = 1$ for all $n \bmod 4 \in \{0,1\}$ follows from Lemma 7.3.1. □

The Unit Tensor. For the rest of this subsection we focus on the unit tensor case $v = \mathcal{E}_n$, $G = \mathsf{GL}_n^3$. We want to describe the regular function $\det^2_{\mathcal{E}_n} \in \mathbb{C}[G\mathcal{E}_n]$ as a quotient of polynomials. For $n \leq 4$ we have $k(n{\times}2; n{\times}2; n{\times}2) = 1$ and $\det^2_{\mathcal{E}_n}$ is the restriction of the (essentially unique) highest weight vector polynomial in $\mathbb{C}[V]$. But for $n > 4$ we have

$$k(n{\times}2;n{\times}2;n{\times}2) \overset{\text{Lem. 4.4.7}}{=} k(2{\times}n;2{\times}n;n{\times}2) \overset{\text{Cor. 4.4.12}}{=} 0.$$

Hence for $n > 4$, the regular function $\det^2_{\mathcal{E}_n}$ is not a restriction of a polynomial on V.

However, in small cases we can describe $\det^2_{\mathcal{E}_n}$ as a quotient of two polynomials as follows. First note that $gf = \det(g)^d f$ for $g \in G$ and for all highest weight vectors f of weight $(n{\times}d, n{\times}d, n{\times}d)$. Therefore f vanishes at a point in $G\mathcal{E}_n$ iff f vanishes completely on $G\mathcal{E}_n$. Note that if there exists a highest weight vector $f_{n,d}$ in $\mathbb{C}[V]$ of weight $(n{\times}d, n{\times}d, n{\times}d)$ that does not vanish on $G\mathcal{E}_n$, then $k(n{\times}d; n{\times}d; n{\times}d) > 0$. According to Proposition 7.2.19, d must be an even number.

In the cases $(n, d) \in \{(5, 4), (5, 6), (6, 4), (6, 6)\}$ we showed the existence of a highest weight vector $f_{n,d} \in \mathbb{C}[V]$ of weight $(n{\times}d, n{\times}d, n{\times}d)$ by explicit constructions as follows: First we chose random obstruction designs by choosing random permutation triples and testing the obstruction design definition. Then we calculated their evaluation at random points in $G\mathcal{E}_n$ and obtained that the evaluation result was nonzero. Hence we can explicitly write

$$\det{}^2_{\mathcal{E}_5} = \frac{f_{5,6}}{f_{5,4}} \quad \text{and} \quad \det{}^2_{\mathcal{E}_6} = \frac{f_{6,6}}{f_{6,4}},$$

where the choices of $f_{n,d}$ are not necessarily unique. This leads to the following conjecture.

8.1.4 Conjecture. *The regular function* $\det^2_{\mathcal{E}_n} \in \mathbb{C}[\mathsf{GL}_n^3 \mathcal{E}_n]$ *can be written as a quotient* $\frac{f_{n,d+2}}{f_{n,d}}$ *of two polynomials, where both* $f_{n,j}$ *are highest weight vectors in* $\mathbb{C}[\bigotimes^3 \mathbb{C}^n]$ *of weight* $(n{\times}j, n{\times}j, n{\times}j)$ *with* $j \in \{d, d+2\}$.

It is also natural to ask the analogous question about representing $\det_v^o \in \mathbb{C}[Gv]$ as a quotient of polynomials for $v = \mathcal{M}_m$, $\det_n$, and per_m (Here $o = o(v)$ denotes the exponent of the stabilizer of v, cf. Sec. 5.1).

8.2 $m \times m$ Matrix Multiplication

In this section, we show the following lower bound (Theorem 8.2.1) on the border rank of $m \times m$ matrix multiplication via an *explicit construction of an obstruction family*, see Definition 8.2.2. The lower bound on the border rank that we obtain is promising, but it does not reach the best bound that is known.

8.2.1 Theorem.

$$\underline{R}(\mathcal{M}_m) \geq \begin{cases} \frac{3}{2}m^2 - 2 & \text{for } m \text{ even} \\ \frac{3}{2}m^2 - \frac{1}{2} & \text{for } m \text{ odd} \end{cases}.$$

Our partition triple will consist of three hooks.

8.2.2 Definition. For the rest of this section, for $\kappa \in \mathbb{N}$, define the hook triple

$$\lambda := \big((2\kappa + 1) \ulcorner (\kappa + 1)\big)^3$$

and set $d := 3\kappa + 1$. The obstruction design $\mathcal{H} := \mathcal{H}_\kappa$ consists of d vertices divided into disjoint sets $V^{(1)} \mathbin{\dot\cup} V^{(2)} \mathbin{\dot\cup} V^{(3)} \mathbin{\dot\cup} \{y^0\}$, where $|V^{(k)}| = \kappa$ for all $1 \leq k \leq 3$. There are only three hyperedges of size larger than 1, called $e^{(k)}$ for $1 \leq k \leq 3$. We set $e^{(k)} := V^{(k+1)} \cup V^{(k+2)} \cup \{y^0\}$, where $V^{(k)} := V^{(k-3)}$ for $k > 3$. The obstruction design $\mathcal{H}$ is depicted in Figure 8.2.i. ∎

8.2.3 Remark. Example 7.2.16 shows that $\mathcal{H}$ is the *unique* obstruction design of type λ. Therefore $k(\lambda) \leq 1$. In fact, by a formula of Remmel [Rem89, Thm. 2.1] we know $k(\lambda) = 1$ (see Theorem 8.3.2). Using this result, we know that $f_{\mathcal{H}} \neq 0$. We will also prove this later on by explicitly evaluating $f_{\mathcal{H}}$ in Subsection 8.2 (B). ∎

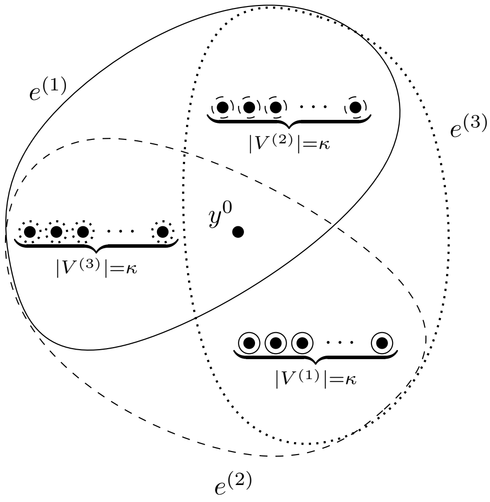


Figure 8.2.i: The unique family of obstruction designs corresponding to the hook partition triple in Definition 8.2.2.

8.2.4 Proposition. *Fix an odd number $m \in \mathbb{N}$ and set $\kappa := \frac{m^2-1}{2}$ and $d := 3\kappa + 1$.*

(1) For all matrix triples $A \in (\mathbb{C}^{3\kappa\times 3\kappa})^3$ we have $f_{\mathcal{H}}(A\mathcal{E}_{3\kappa}) = 0$.

(2) There exists a matrix triple $A \in (\mathbb{C}^{3\kappa\times 3\kappa})^3$ such that $f_{\mathcal{H}}(A\mathcal{M}_m) \neq 0$.

Proposition 8.2.4 explicitly exhibits an obstruction family and directly implies Theorem 8.2.1 for odd numbers. Restricting m to odd numbers is just to avoid some tedious technicalities in Subsection 8.2 (B). We omit to handle the case where m is even.

8.2 (A) Vanishing on the Unit Tensor Orbit

In this subsection we prove Proposition 8.2.4(1) directly. An orbit-wise upper bound proof can be found in Subsection 8.3 (A).

Let $A \in (\mathbb{C}^{3\kappa\times 3\kappa})^3$ be arbitrary. We define the triple list

$$w := \big((A^{(1)}|1\rangle, A^{(2)}|1\rangle, A^{(3)}|1\rangle), \ldots, (A^{(1)}|3\kappa\rangle, A^{(2)}|3\kappa\rangle, A^{(3)}|3\kappa\rangle)\big).$$

According to (7.2.18) we have

$$f_{\mathcal{H}}(\mathcal{E}_{3\kappa}) = \sum_{J \in \{1,\ldots,3\kappa\}^{3\kappa+1}} \mathsf{eval}_{\mathcal{H}}(Aw_{J_1}, \ldots, Aw_{J_{3\kappa+1}}). \qquad (*)$$

The crucial property of $\mathcal{H}$ is that for each pair of vertices $\{y_1, y_2\}$ there exists a hyperedge e of $\mathcal{H}$ containing both y_1 and y_2. By the pidgeon-hole principle, for each labeling $J\colon V(\mathcal{H}) \to \{1, \ldots, 3\kappa\}$ there exists a pair of vertices $\{y_1, y_2\}$ such that $J(y_1) = J(y_2)$. The crucial property of $\mathcal{H}$ implies that y_1 and y_2 lie in a common hyperedge e. Hence $\mathsf{eval}_e((Aw_{J_1}, \ldots, Aw_{J_{3\kappa+1}})_{|e}) = 0$, because it is the determinant of a matrix with two equal columns. Therefore, each summand in $(*)$ vanishes, which proves Proposition 8.2.4(1).

8.2 (B) Evaluation at the Matrix Multiplication Tensor

In this subsection we prove Proposition 8.2.4(2) by an explicit construction of a matrix triple $A = (A^{(1)}, A^{(2)}, A^{(3)})$ consisting of maps $A^{(k)} : \mathbb{C}^{m\times m} \to \mathbb{C}^{m^2}$. We make use of the fact that m is odd.

For notational convenience, we define the triples

$$t_{ijl} := \big(|(ij)\rangle, |(jl)\rangle, |(li)\rangle\big) \in (\mathbb{C}^{m\times m})^3 \tag{8.2.5}$$

and the triple list w of length m^3 obtained by concatenating all t_{ijl} for $1 \le i, j, l \le m$ in any order. Recall that

$$\mathcal{M}_m = \sum_{i,j,l} t^{(1)}_{ijl} \otimes t^{(2)}_{ijl} \otimes t^{(3)}_{ijl}.$$

We put

$$\mathscr{T} := \{t_{ijl} \mid 1 \le i, j, l \le m\}.$$

According to (7.2.18) we have

$$f_{\mathcal{H}}(A\mathcal{M}_m) = \sum_{J\in\{1,\ldots,m^3\}^d} \mathsf{eval}_{\mathcal{H}}(Aw_{J_1}, \ldots, Aw_{J_d}). \tag{$*$}$$

Consider the polynomial ring $\Gamma = \mathbb{C}[X_1, \ldots, X_N]$, where X_i are indeterminates. According to Lemma 2.1.5, if a function $f \in \Gamma$ is nonzero, then there exist values $\alpha_i \in \mathbb{C}$, $1 \le i \le N$, such that $f(\alpha_1, \ldots, \alpha_N) \ne 0$. We will define the $m^2 \times m^2$ matrix triple A with matrix entries being affine linear in the indeterminates X_i. Hence we write the sum $f_{\mathcal{H}}(A\mathcal{M}_m)$ as an element of Γ. We will provide a monomial of $f_{\mathcal{H}}(A\mathcal{M}_m)$ in the X_i with nonzero coefficient in (8.2.7).

Invariance in each $V^{(k)}$. We use the short notation $\mathsf{eval}_e(\zeta) := \mathsf{eval}_e(\zeta^{(k)}_{|e})$ for a hyperedge $e \in E^{(k)}$ and a triple labeling ζ. We start out with the following easy claim.

8.2.6 Claim. *Let $\sigma : V(\mathcal{H}) \to V(\mathcal{H})$ be a bijection satisfying $\sigma(V^{(k)}) = V^{(k)}$ for all $1 \le k \le 3$. For every triple labeling $\zeta : V(\mathcal{H}) \to (\mathbb{C}^{m^2})^3$ we have*

$$\mathsf{eval}_{\mathcal{H}}(\zeta) = \mathsf{eval}_{\mathcal{H}}(\zeta \circ \sigma).$$

Proof. It suffices to show the claim for a transposition $\tau = \sigma$ exchanging two elements of $V^{(1)}$, because the situation for $V^{(2)}$ and $V^{(3)}$ is completely symmetric. We have $\prod_{e\in E^{(1)}} \mathsf{eval}_e(\zeta) = \prod_{e\in E^{(1)}} \mathsf{eval}_e(\zeta \circ \tau)$, because up to reordering both products have the same factors. For $2 \le k \le 3$ we have $\mathsf{eval}_e(\zeta) = \mathsf{eval}_e(\zeta \circ \tau)$ for every singleton hyperedge $e \in E^{(k)}$ and $\mathsf{eval}_{e^{(k)}}(\zeta) = -\mathsf{eval}_{e^{(k)}}(\zeta \circ \tau)$. Therefore $\prod_{e\in E^{(k)}} \mathsf{eval}_e(\zeta) = -\prod_{e\in E^{(k)}} \mathsf{eval}_e(\zeta \circ \tau)$. As a result we get $\mathsf{eval}_{\mathcal{H}}(\zeta) = (-1)^2 \mathsf{eval}_{\mathcal{H}}(\zeta \circ \tau)$. □

Special structure of the matrix triple. Recall that m is odd and $\kappa = \frac{m^2-1}{2}$. Let $a := \frac{m+1}{2}$. Define the set $O_m := \{1, \ldots, m\} \times \{1, \ldots, m\} \setminus \{(a, a)\}$ consisting of $m^2 - 1$ pairs. Fix an arbitrary bijection

$$\varphi : O_m \to \{2, \ldots, m^2\}.$$

Let $\bar{i} := m + 1 - i$ for $1 \le i \le m$. (We may think of the map $i \mapsto \bar{i}$ as a reflection at a; note $\bar{a} = a$.) Let

$$\Gamma := \mathbb{C}[\{X^{(k)}_i : 1 \le k \le 3,\, 1 \le i \le m\}]$$

denote the polynomial ring in $3m$ variables. For each $1 \le k \le 3$ we define the linear map $A^{(k)} \colon \mathbb{C}^{m \times m} \to \mathbb{C}^{m^2}$ by

$$A^{(k)}|(ij)\rangle := \begin{cases} X_a^{(k)}|1\rangle & \text{if } i = j = a \\ |\varphi(i,\bar{i})\rangle + X_i^{(k)}|1\rangle & \text{if } i \neq j \text{ and } j = \bar{i} \\ |\varphi(i,j)\rangle & \text{if } j \neq \bar{i} \end{cases} .$$

Hence $A^{(k)}$ looks as follows:

$$\left(\begin{array}{ccccccccc|c} X_a^{(k)} & X_1^{(k)} & X_2^{(k)} & \cdots & X_{a-1}^{(k)} & X_{a+1}^{(k)} & \cdots & X_{m-1}^{(k)} & X_m^{(k)} & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & \ddots & & & & & & 0 \\ & & & & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ \hline & & & & 0 & & & & & \mathrm{id}_{m^2-m} \end{array}\right), \quad (**)$$

where we arranged the rows and columns as follows: The left m columns correspond to the vectors $|(i\bar{i})\rangle$, where the leftmost one corresponds to $|(a,a)\rangle$. The top row corresponds to the vector $|1\rangle$ and the following $m-1$ rows correspond to the vectors $|\varphi(i,\bar{i})\rangle$. Recall that $f_{\mathcal{H}}(A\mathcal{M}_m)$ is a sum of products of determinants of submatrices of $A^{(k)}$.

The sum $f_{\mathcal{H}}(A\mathcal{M}_m)$ is an element of Γ and we are interested in its coefficient of the monomial $\mathcal{X}$, where

$$\mathcal{X} := \prod_{k=1}^{3} X_a^{(k)} \prod_{i=1}^{m} \big(X_i^{(k)}\big)^{|i-\bar{i}|}. \tag{8.2.7}$$

We remark that the degree of $\mathcal{X}$ is $3(1 + \sum_{i=1}^{m} |i - \bar{i}|)$. It is readily checked that $\sum_{i=1}^{m} |i - \bar{i}| = \kappa$.

Fix any numbering of the vertices of $\mathcal{H}$. For $J \in \{1, \dots, m^3\}^d$ we abuse notation and define the map $J \colon V(\mathcal{H}) \to \mathscr{T}$ via $J(y) := w_{J_y}$. With this notation, $(*)$ becomes

$$\sum_J \mathsf{eval}_{\mathcal{H}}\big(AJ(1), \dots, AJ(d)\big),$$

or $\sum_J \mathsf{eval}_{\mathcal{H}}(AJ)$ in short notation. We call a triple labeling $J \colon V(\mathcal{H}) \to \mathscr{T}$ *nonzero*, if the coefficient of $\mathcal{X}$ in the polynomial $\mathsf{eval}_{\mathcal{H}}(AJ)$ is nonzero. Note that the sum of the evaluations of all nonzero triple labelings is the coefficient of $\mathcal{X}$ in the polynomial $f_{\mathcal{H}}(A\mathcal{M}_m)$. We will count and classify all nonzero triple labelings and show that they evaluate to the same nonzero value. This implies that the coefficient of $\mathcal{X}$ in $f_{\mathcal{H}}(A\mathcal{M}_m)$ is a sum without cancellations and is hence nonzero.

Separate Analysis of the Three Layers. Given a triple labeling $J \colon V(\mathcal{H}) \to \mathscr{T}$, we define $J^{(k)} \colon V(\mathcal{H}) \to \{|(ij)\rangle \mid 1 \le i, j \le m\}$ by composing J with the projection to the kth component.

8.2.8 Claim. *Fix a nonzero triple labeling J and fix $1 \le k \le 3$. For all $y \in V^{(k)}$ we have $J^{(k)}(y) = |(i\bar{i})\rangle$ for some $1 \le i \le m$.*

Proof. Let $y \in V^{(k)}$. Since $\{y\} \in E^{(k)}$ we have $\langle 1|A^{(k)}|J^{(k)}(y)\rangle \neq 0$. From the definition of A it follows that $J^{(k)}(y) = |(ij)\rangle$ and the third case $j \neq \bar{i}$ is excluded. Hence $j = \bar{i}$. □

8.2.9 Claim. *For every nonzero triple labeling J we have $J(y^0) = (|(aa)\rangle, |(aa)\rangle, |(aa)\rangle)$.*

Proof. Let J be a nonzero triple labeling. Hence the coefficient of $\mathcal{X}$ in $\mathsf{eval}_{\mathcal{H}}(AJ(1), \ldots, AJ(d))$ is nonzero. For the following argument it is important to keep the structure of the matrix $A^{(k)}$ in mind, cf. ($**$). Recall that $f_{\mathcal{H}}(A\mathcal{M}_m)$ is a sum of products of certain subdeterminants of $A^{(k)}$ that are determined by the hyperedges in $E^{(k)}(\mathcal{H})$. Since the degree of $X_a^{(k)}$ in $\mathcal{X}$ is 1, we have that for all $1 \leq k \leq 3$ there is exactly one vertex $y_k \in V(\mathcal{H})$ with $J^{(k)}(y_k) = |(aa)\rangle$. Recall that the hyperedge $e^{(k)}$ has size $2\kappa + 1 = m^2$. Since J is a nonzero triple labeling, $J^{(k)}$ is injective on hyperedges and hence $|\{J^{(k)}(y) : y \in e^{(k)}\}| = m^2$. But since the image $J^{(k)}(V(\mathcal{H}))$ has cardinality at most m^2, $J^{(k)}$ is actually bijective on $e^{(k)}$. Since there is only one vertex y satisfying $J^{(k)}(y) = |(aa)\rangle$, namely the vertex $y = y_k$, it follows $y_k \in e^{(k)}$. Since $e^{(1)} \cap e^{(2)} \cap e^{(3)} = \{y^0\}$, it remains to show that $y_1 = y_2 = y_3$.

The structure of the matrix multiplication tensor implies that $J(y_1) = (|(aa)\rangle, |(ai)\rangle, |(ia)\rangle)$ for some $1 \leq i \leq m$. If $a = i$, then, by definition of y_2 and y_3 and uniqueness, we have $y_1 = y_2 = y_3$ and we are done.

Now assume $a \neq i$ and $y_1 \neq y^0$. W.l.o.g. $y_1 \in V^{(3)}$. Using Claim 8.2.8 we conclude that $J^{(3)}(y_1) = |i\bar{i}\rangle$ for some $1 \leq i \leq m$. Hence $\bar{i} = a$ contradicting $i \neq a$. Thus we have shown that $y_1 = y^0$. Similarly, we show that $y_2 = y_3 = y^0$. □

8.2.10 Claim. *For each nonzero triple labeling J we have $J^{(k)}(V^{(k)}) = \{|(i\bar{i})\rangle \mid 1 \leq i \leq m\} \setminus \{|(aa)\rangle\}$, where the preimage of each $|(i\bar{i})\rangle$ under $J^{(k)}$ has size $|i - \bar{i}|$.*

Proof. According to Claim 8.2.9 we have $J(y^0) = |(aa)(aa)(aa)\rangle$. For the following look again at the structure of $A^{(k)}$, cf. ($**$). Since $A^{(k)}|(aa)\rangle$ is a multiple of $|1\rangle$, we have that $\mathsf{eval}_{e^{(k)}}(J)$ is a multiple of $X_a^{(k)}$. Moreover, for $i \neq a$, the variable $X_i^{(k)}$ does not appear in the expansion of $\mathsf{eval}_{e^{(k)}}(J^{(k)})$. Since for a fixed $1 \leq k \leq 3$ there are $\kappa = \sum_{i=1}^{m} |i - \bar{i}|$ many contributions of a factor $X_i^{(k)}$ in the monomial $\mathcal{X}$, these factors must be contributed at vertices in $V^{(k)}$. Since $|V^{(k)}| = \kappa$, the only possibility is that all $y \in V^{(k)}$ satisfy $J^{(k)}(y) = |i\bar{i}\rangle$ for some $1 \leq i \leq m$, $i \neq a$. The specific requirement for the number of factors $X_i^{(k)}$ which are encoded in $\mathcal{X}$ in (8.2.7) finishes the proof. □

Coupling the Analysis of the Three Layers. Define the bijective map

$$\tau \colon O_m \to O_m, \quad \tau(ij) = (j\bar{i}),$$

which corresponds to the rotation by 90°. Clearly, $\tau^4 = \mathrm{id}$. The map τ induces a map $\wp(O_m) \to \wp(O_m)$ on the powerset, which we also call τ. Define the involution (taking the complement)

$$\iota \colon \wp(O_m) \to \wp(O_m), \quad S \mapsto O_m \setminus S.$$

Clearly, we have $\tau \circ \iota = \iota \circ \tau$. We will only be interested in subsets $S \subseteq O_m$ with exactly $|O_m|/2 = \kappa$ many elements and their images under τ and ι. The subsets $S \subseteq O_m$ that satisfy $\iota(S) = \tau(S)$ will be of special interest. Geometrically, these are the sets that get inverted when rotating by 90°, see Figure 8.2.ii for examples.

In the following we identify the sets $J^{(k)}(V^{(k')})$, for $1 \leq k, k' \leq 3$, with their corresponding subsets of O_m.

In Claim 8.2.10 we analyzed the labels $J^{(k)}(V^k)$. In the next claim we turn to $J^{(k)}(V^{k'})$, where $k \neq k'$.

8.2.11 Claim. *Every nonzero triple labeling J is completely determined by the image $J^{(1)}(V^{(3)})$ (up to permutations in the $V^{(k)}$, see Claim 8.2.6) as follows.*

- $J^{(2)}(V^{(3)}) = \tau(J^{(1)}(V^{(3)}))$,
- $J^{(2)}(V^{(1)}) = \iota(J^{(2)}(V^{(3)}))$,
- $J^{(3)}(V^{(1)}) = \tau(J^{(2)}(V^{(1)}))$,
- $J^{(3)}(V^{(2)}) = \iota(J^{(3)}(V^{(1)}))$,
- $J^{(1)}(V^{(2)}) = \tau(J^{(3)}(V^{(2)}))$.

Moreover, $\tau(J^{(1)}(V^{(3)})) = \iota(J^{(1)}(V^{(3)}))$.

Proof. According to Claim 8.2.10 we have that each vertex $y \in V^{(3)}$ satisfies

$$J(y) = \big(|(ij)\rangle, |(\tau(ij))\rangle, |(\bar{i}i)\rangle\big)$$

for some $1 \le i, j \le m$, $i \ne a$. In particular, using that τ is injective, we have

$$\tau(J^{(1)}(V^{(3)})) = J^{(2)}(V^{(3)}).$$

Since J is nonzero, $J^{(2)}$ is injective on $e^{(2)}$. We even have that $J^{(2)}$ is bijective on $e^{(2)}$, because $|e^{(2)}| = m^2$. Using that $e^{(2)} = V^{(1)} \dot\cup V^{(3)} \dot\cup \{y^0\}$ we see that

$$J^{(2)}(V^{(1)}) = O_m \setminus J^{(2)}(V^{(3)}) = \iota(J^{(2)}(V^{(3)})).$$

For the same reason, we can deduce $J^{(3)}(V^{(1)}) = \tau(J^{(2)}(V^{(1)}))$ and $J^{(3)}(V^{(2)}) = \iota(J^{(3)}(V^{(1)}))$. And applying these arguments one more time we get $J^{(1)}(V^{(2)}) = \tau(J^{(3)}(V^{(2)}))$ and $J^{(1)}(V^{(3)}) = \tau(J^{(1)}(V^{(2)}))$. Summarizing (recall $\tau \circ \iota = \iota \circ \tau$) we have

$$J^{(1)}(V^{(3)}) = \tau^3\iota^3(J^{(1)}(V^{(3)})) = \tau^{-1}\iota(J^{(1)}(V^{(3)})),$$

which is equivalent to $\tau(J^{(1)}(V^{(3)})) = \iota(J^{(1)}(V^{(3)}))$. □

Additionally to the constraint $\tau(J^{(1)}(V^{(3)})) = \iota(J^{(1)}(V^{(3)}))$ given in Claim 8.2.11, Claim 8.2.10 implies that in $J^{(1)}(V^{(3)})$ there are $|i - \bar{i}|$ many elements of the form $|(i\bar{i})\rangle$ for each $1 \le i \le m$.

This motivates the following definition.

8.2.12 Definition. A subset $S \subseteq O_m$ is called *valid*, if

(1) $|S| = \frac{m^2-1}{2} = \kappa$,

(2) $\tau(S) = \iota(S)$,

(3) $|p^{-1}(i)| = |i - \bar{i}|$ for all $1 \le i \le m$

where $p\colon S \to \{1, \dots, m\}$ is the projection to the first component, see Figure 8.2.ii for an example. ■

8.2.13 Proposition. *For all nonzero triple labelings J we have that $J^{(1)}(V^{(3)})$ is a valid set. On the other hand, for every valid set S there exists exactly one nonzero triple labeling J with $J^{(1)}(V^{(3)}) = S$ up to permutations in the $V^{(k)}$.*

Proof. For the first statement, property (2) of Definition 8.2.12 follows from Claim 8.2.11 and property (3) of Definition 8.2.12 follows from Claim 8.2.10. The second statement can be readily checked with Claim 8.2.9 and Claim 8.2.11. □

The next claim classifies all valid sets.

8.2.14 Lemma. *A set $S \subseteq O_m$ is valid iff the following conditions are all satisfied (see Figure 8.2.iii for an illustration):*

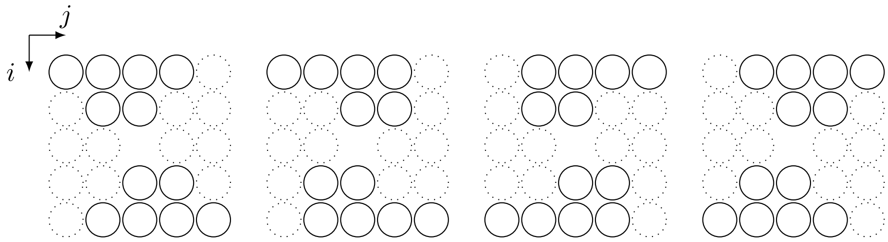


Figure 8.2.ii: In each of the four pictures the vertices with solid border form a valid set for $m = 5$. The vertex in row i and column j represents the tuple (ij). The dotted vertices do *not* belong to the valid sets. Note that each vertex that does not lie on one of the two diagonals either lies in all valid sets or in no one. According to Lemma 8.2.14, there are no other valid sets for $m = 5$.

(1) $\Big\{(ij) \mid (i < j \text{ and } i < \bar{j}) \text{ or } (i > j \text{ and } i > \bar{j})\Big\} \subseteq S$, *represented by solid vertices in Figure 8.2.iii.*

(2) $\Big\{(ij) \mid (i > j \text{ and } i < \bar{j}) \text{ or } (i < j \text{ and } i > \bar{j})\Big\} \cap S = \emptyset$, *represented by dotted vertices in Figure 8.2.iii.*

(3) For all $1 \leq i \leq \frac{m-1}{2}$ *there are two mutually exclusive cases, (a) and (b), represented by the two vertices* x_i *and the two vertices* $\overline{x_i}$*, respectively, in Figure 8.2.iii.*

(a) $\{(ii), (\bar{i}\,\bar{i})\} \subseteq S$ *and* $\{(i\bar{i}), (\bar{i}i)\} \cap S = \emptyset$,

(b) $\{(i\bar{i}), (\bar{i}i)\} \subseteq S$ *and* $\{(ii), (\bar{i}\,\bar{i})\} \cap S = \emptyset$.

These choices results in $2^{\frac{m-1}{2}}$ *valid sets.*

Proof. As indicated in Figure 8.2.iii, for each tuple (ij) we call i the *row* of (ij). For S to be valid, according to Definition 8.2.12(3), S must contain $|i - \bar{i}|$ elements in row i and according to Definition 8.2.12(2), $\tau(s) \notin S$ for all $s \in S$.

In particular, S must contain $m - 1$ elements in row 1. If $(11) \in S$, then $(1m) \notin S$, because $\tau(11) = (1m)$. Hence there are only two possibilities: **(a):** $\{(1j) \mid 1 \leq j < m\} \subseteq S$ or **(b):** $\{(1j) \mid 1 < j \leq m\} \subseteq S$. By symmetry, for row m we get **(a'):** $\{(mj) \mid 1 \leq j < m\} \subseteq S$ or **(b'):** $\{(mj) \mid 1 < j \leq m\} \subseteq S$. But since $\tau(1m) = (mm)$ and $\tau(m1) = (11)$, the fact $\tau(S) = \iota(S)$ implies that **(a)** iff **(b')** and that **(a')** iff **(b)**. We are left with the two possibilities $\big($**(a)** and **(b')**$\big)$ or $\big($**(a')** and **(b)**$\big)$.

Now consider row 2. We have $\tau(21) = (1, m-1) \in S$ and hence $(21) \notin S$. In the same manner we see $(2m) \notin S$. We are left to choose $m - 3$ elements from the $m - 2$ remaining elements in row 2. The same argument as for row 1 gives two possibilities: **(a):** $\{(2j) \mid 2 \leq j < m-1\} \subseteq S$ or **(b'):** $\{(2j) \mid 2 < j \leq m-1\} \subseteq S$. Analogously for row $m - 1$ we have **(a):** $\{((m-1), j) \mid 2 \leq j < m-1\} \subseteq S$ or **(b'):** $\{((m-1), j) \mid 2 < j \leq m-1\} \subseteq S$. With the same reasoning as for the rows 1 and m we get **(a)** iff **(b')** and that **(a')** iff **(b)**. Again we are left with the two possibilities $\big($**(a)** and **(b')**$\big)$ or $\big($**(a')** and **(b)**$\big)$.

Continuing these arguments we end up with $2^{\frac{m-1}{2}}$ possibilities. It is easy to see that each of these possibilities gives a valid set. □

The following claim finishes the proof of Proposition 8.2.4(2).

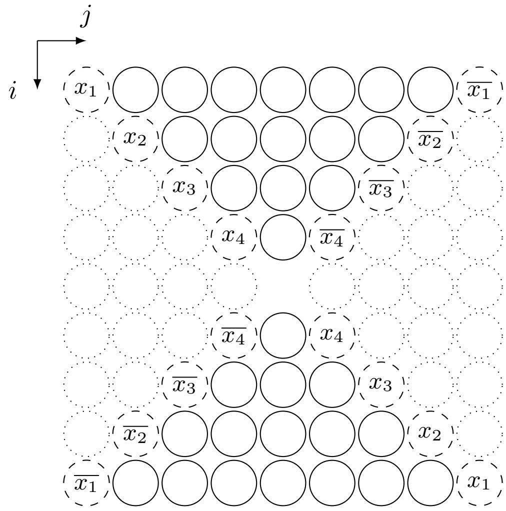


Figure 8.2.iii: The case $m = 9$. Vertices that appear in all valid subsets are drawn with a solid border. Vertices that appear in no valid subset are drawn with a dotted border. Vertices that appear in half of all valid subsets are drawn with a dashed border. These contain a vertex label x_i or $\overline{x_i}$. Each valid set corresponds to a choice vector $x \in \{\text{true}, \text{false}\}^4$ determining whether the x_i or the $\overline{x_i}$ are contained in S. This results in $2^4 = 16$ valid sets $S \subseteq O_m$.

8.2.15 Claim. *All nonzero triple labelings J have the same coefficient of $\mathcal{X}$ in the polynomial* $\mathsf{eval}_{\mathcal{H}}(AJ)$.

Proof. Take two nonzero triple labelings J and J'. According to Proposition 8.2.13, both sets $J^{(1)}(V^{(3)})$ and $J'^{(1)}(V^{(3)})$ are valid sets. Because of Lemma 8.2.14, it suffices to consider only the case where $J^{(1)}(V^{(3)})$ and $J'^{(1)}(V^{(3)})$ differ by a single involution $\sigma\colon O_m \to O_m$, where for some fixed $1 \le i \le \frac{m-1}{2}$ we have $\sigma(ii) = (i\bar{i})$ and $\sigma(\bar{i}\,\bar{i}) = (\bar{i}i)$, and σ is constant on all other pairs. We remark that σ restricted to the four pairs $\{(ii), (\bar{i}i), (i\bar{i}), (\bar{i}\,\bar{i})\}$ corresponds to a reflection in the second component.

We analyze the labels that are affected by this reflection. We only perform the analysis for one of the two symmetric cases, namely for $\{|(ii)\rangle, |(\bar{i}\,\bar{i})\rangle\} \subseteq J^{(1)}(V^{(3)})$. Note that this implies

$$\big\{ \big(|(ii)\rangle, |(i\bar{i})\rangle, |(\bar{i}i)\rangle\big), \big(|(\bar{i}\,\bar{i})\rangle, |(\bar{i}i)\rangle, |(i\bar{i})\rangle\big) \big\} \subseteq J(V^{(3)}), \qquad (\dagger)$$

according to Claim 8.2.10. We adapt the notation from (8.2.5) to our special situation and write $t_{000} := t_{\bar{i}\,\bar{i}\,\bar{i}}$, $t_{001} := t_{\bar{i}\,\bar{i}\,i}$, $\ldots$, $t_{111} := t_{iii}$. Using this notation, ($\dagger$) reads as follows: $\{t_{110}, t_{001}\} \subseteq J(V^{(3)})$. Using Claim 8.2.11 we get

$$\{t_{110}, t_{001}\} \subseteq J(V^{(3)}), \quad \{t_{101}, t_{010}\} \subseteq J(V^{(2)}), \quad \{t_{011}, t_{100}\} \subseteq J(V^{(1)}).$$

Applying the involution σ to $J^{(1)}(V^{(3)})$, we can use Claim 8.2.10 again to get

$$\big\{ \big(|(i\bar{i})\rangle, |(\bar{i}\,\bar{i})\rangle, |(\bar{i}i)\rangle\big), \big(|(\bar{i}i)\rangle, |(ii)\rangle, |(i\bar{i})\rangle\big) \big\} \subseteq J'(V^{(3)}).$$

Applying Claim 8.2.11 and using our short syntax, we get:

$$\{t_{100}, t_{011}\} \subseteq J'(V^{(3)}), \quad \{t_{001}, t_{110}\} \subseteq J'(V^{(2)}), \quad \{t_{010}, t_{101}\} \subseteq J'(V^{(1)}).$$

We see that exactly the same triples occur in $J(V(\mathcal{H}))$ as in $J'(V(\mathcal{H}))$. We focus now on $J^{(1)}$ and $J'^{(1)}$ and see the following:

$$\{(ii), (\bar{i}\,\bar{i})\} \subseteq J^{(1)}(V^{(3)}) \text{ and } \{(i\bar{i}), (\bar{i}i)\} \subseteq J^{(1)}(V^{(2)})$$

and

$$\{(i\bar{i}), (\bar{i}i)\} \subseteq {J'}^{(1)}(V^{(3)}) \text{ and } \{(\bar{i}\,\bar{i}), (ii)\} \subseteq {J'}^{(1)}(V^{(2)}).$$

This gives exactly two switches of positions in $e^{(1)} = V^{(2)} \dot\cup V^{(3)} \dot\cup \{y^0\}$, hence

$$\mathsf{eval}_{e^{(1)}}(AJ) = (-1)^2 \mathsf{eval}_{e^{(1)}}(AJ') = \mathsf{eval}_{e^{(1)}}(AJ').$$

Analogously we can prove that $\mathsf{eval}_{e^{(k)}}(AJ) = \mathsf{eval}_{e^{(k)}}(AJ')$ for all $2 \le k \le 3$ and therefore $\mathsf{eval}_{\mathcal{H}}(AJ) = \mathsf{eval}_{\mathcal{H}}(AJ')$. □

Proposition 8.2.4 is completely proved.

We finish this subsection with the remark that the following strengthening of Proposition 8.2.4 holds, which can be proved completely analogously. For $\alpha \in (\mathbb{C}^\times)^{m\times m\times m}$ let

$$\mathcal{M}_m^\alpha := \sum_{i,j,l=1}^{m} \alpha_{ijl} |(ij)\rangle \otimes |(jl)\rangle \otimes |(li)\rangle.$$

8.2.16 Proposition. *For the obstruction design $\mathcal{H}$ from Proposition 8.2.4 we have*

$$f_{\mathcal{H}}(A\mathcal{M}_m^\alpha) \neq 0,$$

where A is the same matrix triple as in the proof of Proposition 8.2.4. It follows

$$\underline{R}(\mathcal{M}_m^\alpha) \ge \begin{cases} \frac{3}{2}m^2 - 2 & \text{for } m \text{ even} \\ \frac{3}{2}m^2 - \frac{1}{2} & \text{for } m \text{ odd} \end{cases}$$

for all $\alpha \in (\mathbb{C}^\times)^{m\times m\times m}$.

8.2.17 Remark. The lower bound given in Proposition 8.2.16 can be interpreted as a lower bound on the so-called *border s-rank* of the matrix multiplication tensor, see [CU12]. ■

8.3 Further Results

This section contains two independent results. In Subsection 8.3 (A) we improve the result $\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\overline{\mathsf{GL}_{3\kappa}^3 \mathcal{E}_{3\kappa}}] = 0$ from Subsection 8.2 (A) by showing that even $\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\mathsf{GL}_{3\kappa}^3 \mathcal{E}_{3\kappa}] = 0$. In Subsection 8.3 (B) we explain how to improve the lower bound $\underline{R}(\mathcal{M}_2) \ge 4$ from Section 8.2 for $m = 2$ to its optimal value $\underline{R}(\mathcal{M}_2) \ge 7$.

8.3 (A) Orbit-wise Upper Bound Proof

In this subsection we prove the following proposition.

8.3.1 Proposition. *Let $\kappa \in \mathbb{N}$ and $d := 3\kappa + 1$. Let $\nu := (2\kappa+1) \ulcorner (\kappa+1)$ and put $\lambda := (\nu, \nu, \nu) \vdash^* d$. We have*

$$\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\mathsf{GL}_{3\kappa}^3 \mathcal{E}_{3\kappa}] = 0.$$

To analyze hook partitions, we use the following Theorem 8.3.2, proved by Remmel [Rem89, Thm. 2.1], see also [Ros01]. For example, using Theorem 8.3.2, it is easy to check that $k(\lambda) = 1$ in Proposition 8.3.1.

8.3.2 Theorem ([Ros01, Pf. of Thm. 3(4.)])**.** *Three hooks partitions $(a_1+1) \ulcorner (d-a_1)$, $(a_2+1) \ulcorner (d-a_2)$, and $(a_3+1) \ulcorner (d-a_3)$ of d have Kronecker coefficient 1, iff*

$$|a_1 - a_2| \le a_3 \quad \text{and} \quad a_3 \le a_1 + a_2 \quad \text{and} \quad 2d - a_1 - a_2 - a_3 - 2 \ge 0.$$

Otherwise their Kronecker coefficient is zero.

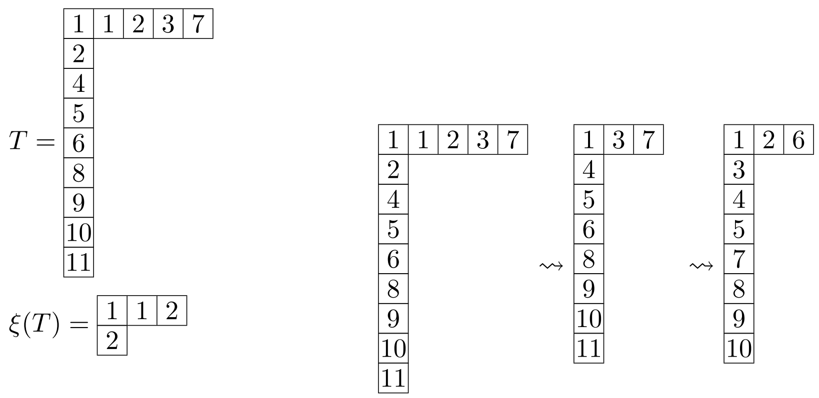
At the first glance, the symmetry in a_1, a_2, and a_3 may not be obvious in the above theorem, but we remark that the first two conditions $|a_1 - a_2| \leq a_3$ and $a_3 \leq a_1 + a_2$ together are exactly the conditions that a triangle with side lengths a_1, a_2, and a_3 exists.

Proof of Proposition 8.3.1. Theorem 5.2.1 states that

$$\operatorname{mult}_{\lambda^*}(\mathbb{C}[\mathsf{GL}_{3\kappa}^3 \mathcal{E}_{3\kappa}]) = \sum_{\substack{\mu \vdash_{3\kappa} d \\ \mu \preceq \nu}} \dim \left(\{\nu\}^\mu \otimes \{\nu\}^\mu \otimes \{\nu\}^\mu\right)^{\mathsf{stab}_{\mathsf{S}_{3\kappa}}(\mu)}.$$

We are going to prove that each summand vanishes. So fix an arbitrary $\mu \vdash_{3\kappa} d$. Let i be defined as the sum over all components μ_j of μ that satisfy $\mu_j \neq 1$, e.g. if $\kappa = 4$ and $\mu = (4,2,2,2,1,1,1) \vdash 13$, then $i = 4+2+2+2 = 10$. By definition $i \neq 1$. Additionally, $i \neq 0$, because μ has $3\kappa + 1$ boxes, but at most 3κ rows. Let $d' := 3\kappa + 1 - i$ be the number of 1s in μ. The group $\mathsf{S}_{d'}$ is a subgroup of $\mathsf{stab}_{\mathsf{S}_{3\kappa}}(\mu)$, acting on the last d' entries and letting the first $3\kappa - d'$ entries fixed. Let $W := \{\nu\}^\mu$ denote the μ-weight space of $\{\nu\}$. It suffices to show that $\bigotimes^3 W$ has no $\mathsf{S}_{d'}$ invariant, because then $\bigotimes^3 W$ has no $\mathsf{stab}_{\mathsf{S}_{3\kappa}}(\mu)$ invariant as well. The strategy is as follows. We will decompose W as a sum of irreducible $\mathsf{S}_{d'}$-representations. The invariant space $(\bigotimes^3 W)^{\mathsf{S}_{d'}}$ is given by the Kronecker coefficients, see Proposition 4.4.4. Then we are going to show that all these Kronecker coefficients vanish.

Recalling Section 4.1, a basis of W is given by the semistandard tableaux of shape ν and content μ. For a semistandard tableau T of shape ν and content μ we define $\xi(T)$ to be the semistandard tableau consisting of exactly those i boxes which contain entries that appear at least twice in T, see Figure 8.3.i(a). Let $\mathsf{sh}(\xi) \vdash i$ denote the shape of ξ. We note that $\ell(\mathsf{sh}(\xi)) \leq \lfloor \frac{i}{2} \rfloor$, because all entries in a column of a semistandard tableau differ and each entry appears at least twice. Observe that the linear subspace



(a) A tableau T with its tableau $\xi(T)$, $i = 4$ and $\mu = (2,2,9\times 1)$.

(b) The tableau $\xi(T)$ in T is retracted to a single 1. Then all other entries are lowered so that each number appears once.

Figure 8.3.i: The action of $\mathsf{S}_{d'}$ on $\mathscr{S}_\xi$. Here $\kappa = 4$.

$$\mathscr{S}_\xi := \operatorname{span}\{T \mid T \text{ semistd of shape } \nu \text{ and content } \mu \text{ such that } \xi(T) = \xi\} \subseteq W$$

is closed under the action of $\mathsf{S}_{d'}$ permuting the last d' elements, which is readily seen using the fact that for a semistandard tableau $T \in \mathscr{S}_\xi$ the filling $\pi(T)$ for $\pi \in \mathsf{S}_{d'}$ is easy to straighten, see Definition 4.1.2. We get a decomposition

$$W \simeq \bigoplus_{\substack{\xi \text{ semistd} \\ \mathsf{sh}(\xi) \vdash i \\ \mathsf{sh}(\xi) \subseteq \nu}} \mathscr{S}_\xi.$$

into $\mathsf{S}_{d'}$-representations. We are going to decompose each $\mathscr{S}_\xi$ as an $\mathsf{S}_{d'}$-representation. To each semistandard tableau T in $\mathscr{S}_\xi$ we assign a standard tableau T' of shape $(2\kappa + 2 - \ell(\mathsf{sh}(\xi))) \ulcorner (\kappa + 2 - \mathsf{sh}(\xi)_1)$ and content $(d'+1)\times 1$ by retracting ξ to a single box labeled with 1 and lowering the entries in T, see Figure 8.3.i(b). This retraction map shows

$$\mathscr{S}_\xi \simeq [(2\kappa + 2 - \ell(\mathsf{sh}(\xi))) \ulcorner (\kappa + 2 - \mathsf{sh}(\xi)_1)]\downarrow^{\mathsf{S}_{d'+1}}_{\mathsf{S}_{d'}}$$

as $\mathsf{S}_{d'}$-representations, where on the right hand side $\mathsf{S}_{d'}$ acts on all but the first component. Corollary 4.5.11 shows that, as $\mathsf{S}_{d'}$-representations,

$$\mathscr{S}_\xi \simeq [(2\kappa + 1 - \ell(\mathsf{sh}(\xi))) \ulcorner (\kappa + 2 - \mathsf{sh}(\xi)_1)] \oplus [(2\kappa + 2 - \ell(\mathsf{sh}(\xi))) \ulcorner (\kappa + 1 - \mathsf{sh}(\xi)_1)].$$

We see that the decomposition of W consists of hook partitions only, with length at least $2\kappa + 1 - \lfloor \frac{i}{2} \rfloor$, because $\ell(\mathsf{sh}(\xi)) \le \lfloor \frac{i}{2} \rfloor$.

If the Kronecker coefficient $k(\varrho^{(1)}; \varrho^{(2)}; \varrho^{(3)})$ vanishes for all hook partitions $\varrho^{(k)}$ of d' with length at least $2\kappa + 1 - \lfloor \frac{i}{2} \rfloor$, then the invariant space $(\bigotimes^3 W)^{\mathsf{S}_{d'}}$ is zero, see Proposition 4.4.4. To show $k(\varrho^{(1)}; \varrho^{(2)}; \varrho^{(3)}) = 0$ we use Theorem 8.3.2. In our case we have $d' = 3\kappa + 1 - i$ and $a_1, a_2, a_3 \ge 2\kappa - \lfloor \frac{i}{2} \rfloor$. To get a positive Kronecker coefficients we need to have

$$2\kappa - \lfloor \tfrac{i}{2} \rfloor \le a_3 \le 2d' - a_1 - a_2 - 2 \le 2(3\kappa + 1 - i) - 4\kappa + 2\lfloor \tfrac{i}{2} \rfloor - 2 = 2\kappa - 2i + 2\lfloor \tfrac{i}{2} \rfloor.$$

Therefore $0 \le 3\lfloor \frac{i}{2} \rfloor - 2i$. This is a contradiction to $i > 0$. □

8.3 (B) 2×2 Matrix Multiplication

In this subsection we exhibit obstructions that prove

$$\underline{R}(\mathcal{M}_2) = 7,$$

which is far better than $R(\mathcal{M}_2) \ge 4$, which we obtain from Theorem 8.2.1. The border rank of $\mathcal{M}_2$ was first determined by Landsberg using geometric methods, see [Lan05]. We start out by mentioning that it is easier to find obstructions that prove $\underline{R}(\mathcal{M}_2) \ge 6$: Indeed, consider the following two partition triples $\lambda = \big((5,1,1,1), 4\times 2, 4\times 2\big)$ and $\mu = \big((3,3,1,1), 4\times 2, 4\times 2\big)$. We have $k(\lambda) = k(\mu) = 1$ as verified by Derksen. In [LM04, Prop. 6.5] it is shown that

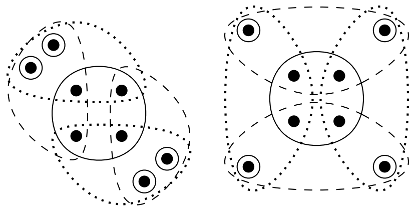
$$\mathrm{mult}_{\lambda^*}(\mathbb{C}[\overline{\mathsf{GL}_5^3 \mathcal{E}_5}]) = \mathrm{mult}_{\mu^*}(\mathbb{C}[\overline{\mathsf{GL}_5^3 \mathcal{E}_5}]) = 0$$

by combinatorial considerations. We can even give an orbit-wise upper bound proof, i.e., we can show that

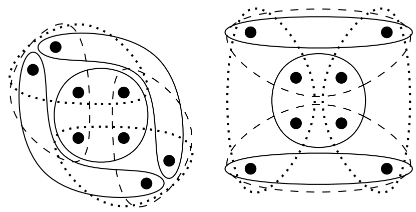
$$\mathrm{mult}_{\lambda^*}(\mathbb{C}[\mathsf{GL}_5^3 \mathcal{E}_5]) = \mathrm{mult}_{\mu^*}(\mathbb{C}[\mathsf{GL}_5^3 \mathcal{E}_5]) = 0$$

using Theorem 5.2.1, similarly to the proof of Claim 5.3.6.

To show that λ and μ are obstructions we construct an obstruction design $\mathcal{H}$ of type λ and μ, respectively, and show that $f_{\mathcal{H}}(v) \neq 0$ for some $v \in \mathsf{GL}_5^3\mathcal{M}_2$. The corresponding obstruction designs are depicted in Figure 8.3.ii. One can check that the depicted obstruction designs are the only ones of this type, for example one can use the bijection between obstruction designs and equivalence classes of 3-ary relations of format λ from Section 7.2. Calculating the evaluation $f_{\mathcal{H}}(v)$ can be done by hand, but is slightly laborious.



(a) Two obstruction designs of type λ that describe the same polynomial.



(b) Two obstruction designs of type μ that describe the same polynomial. A third one can be obtained by exchanging in the right picture the dashed and dotted hyperedges.

Figure 8.3.ii: Obstruction designs of type $\lambda = \big((5,1,1,1), 4{\times}2, 4{\times}2\big)$ and $\mu = \big((3,3,1,1), 4{\times}2, 4{\times}2\big)$.

Using Machine Learning techniques, Jon Hauenstein and Joseph Landsberg proposed the partition triple $\lambda = (5,5,5,4)^3$ as an obstruction to show $\underline{R}(\mathcal{M}_2) = 7$, in particular, they conjectured the existence of a highest weight vector $0 \neq f \in I(\mathsf{GL}_6^3\mathcal{E}_6)$ of weight λ^*. This is equivalent to $\mathrm{mult}_{\lambda^*}\, \mathbb{C}[\overline{\mathsf{GL}_6^3\mathcal{E}_6}] < k(\lambda)$.

With the following Las Vegas algorithm one can construct a basis Ω of the highest weight vector space $\mathsf{HWV}_{\lambda^*}(\mathbb{C}[\bigotimes^3\mathbb{C}^4])$, analogously to Section 6.1.

Set $\Omega \leftarrow \emptyset$.
while $|\Omega| < k(\lambda)$ **do**
  Choose a random obstruction design $\mathcal{H}$ of type λ.
  Evaluate the polyn. in $\Omega \cup \{f_{\mathcal{H}}\}$ at $|\Omega|+1$ many random points in $\bigotimes^3\mathbb{C}^4$.
  The eval. results are written into an $(|\Omega|+1) \times (|\Omega|+1)$ matrix A, where each column corresponds to a polynomial and each row corresponds to a point.
  if A is regular **then**
    Set $\Omega \leftarrow \Omega \cup \{f_{\mathcal{H}}\}$.
  end if
end while

During a run of the above method, we always have that $\Omega \subseteq \mathsf{HWV}_{\lambda^*}(\mathbb{C}[\bigotimes^3(\mathbb{C}^4)^*])$ is linearly independent. Upon termination we end up with a desired basis Ω.

Random obstruction designs can be chosen by simply choosing a random permutation triple π and checking the obstruction design properties of the corresponding triple set partition. While π does not satisfy the properties, a new π is chosen randomly.

Of course, the method can be sped up by evaluating at the same points and reusing evaluation results. The choice of points matters, because evaluation at low rank tensors is much faster than evaluation at high rank tensors.

Derksen yields $k(\lambda) = 31$ for $\lambda = (5,5,5,4)^3$. We computed a basis $\Omega = \{f_1, \ldots, f_{31}\}$ of $\mathsf{HWV}_{\lambda^*}(\mathbb{C}[\bigotimes^3\mathbb{C}^4])$. Then we evaluated each f_i at 31 random points $w_1, \ldots, w_{31} \in \mathsf{GL}_6^3\mathcal{E}_6$ and obtained the 31×31-matrix $A = \big(f_i(w_j)\big)$. It turned out that $\operatorname{rank}(A) = 30$. Hence we could compute $\alpha_i \in \mathbb{C}$ such that $h := \sum_{i=1}^{31} \alpha_i f_i$ vanishes at all the points w_j. *It is likely* that h vanishes on all of $\mathsf{GL}_6 \mathcal{E}_6$.

We evaluated $h(g\mathcal{M}_2) \neq 0$ for some $g \in \mathsf{GL}_6^3$ and hence we proved $\underline{R}(\mathcal{M}_2) > 6$, provided that indeed $h \in I(\mathsf{GL}_6^3 \mathcal{E}_6)$.

To turn this calculation into a proof, we turned to symbolic evaluation methods. It remained to show with a symbolic algorithm that $h \in I(\mathsf{GL}_6^3 \mathcal{E}_6)$. To achieve this, we switched to the partition triple $\mu = (5, 5, 5, 5)^3$, because of the higher symmetry and the fact that DERKSEN yields $k(\mu) = 4$, which is quite small. We performed the same steps as above, but additionally we evaluated h symbolically at $\mathsf{GL}_6^3 \mathcal{E}_6$, which yielded $h \in I(\mathsf{GL}_6^3 \mathcal{E}_6)$. Therefore $\underline{R}(\mathcal{M}_2) = 7$.

To be more specific, let $f_1, f_2, f_3, f_4 \in \mathbb{C}[\bigotimes^3 \mathbb{C}^6]_{20}$ denote the highest weight vectors of weight $((5, 5, 5, 5)^3)^*$ corresponding to the triples of numberings listed below, where the first row of triples describes f_1 and so on.

1	15	10	13	17	12	13	19	2	11	12	8	9	5	17
2	20	12	5	14	9	4	6	16	14	7	10	11	18	3
3	19	7	9	18	15	17	18	8	20	20	4	13	19	16
4	8	16	11	6	10	5	7	3	1	2	6	14	15	1

1	19	17	13	18	15	9	6	7	20	8	3	4	17	15
2	16	8	20	5	8	14	16	11	10	19	6	5	2	9
3	9	7	12	15	1	2	5	3	12	13	7	12	10	20
4	10	14	6	11	19	13	17	18	4	16	1	18	14	11

1	18	16	14	17	6	15	7	9	5	1	18	19	9	11
2	11	10	9	19	8	1	17	3	13	8	6	14	10	16
3	12	7	13	20	16	4	12	11	19	7	17	12	15	3
4	15	5	6	8	14	20	2	18	10	20	2	5	13	4

1	19	15	20	18	9	8	18	20	13	1	12	7	4	10
2	11	6	7	17	19	6	10	15	11	11	3	2	18	8
3	16	9	5	10	2	12	14	7	5	5	17	15	14	16
4	13	14	8	12	1	3	17	4	16	20	9	19	13	6

Using the symbolic evaluation method we showed that the following linear combination h vanishes on $\mathsf{GL}_6 \mathcal{E}_6$:

$$h := -266054\, f_1 + 421593\, f_2 + 755438\, f_3 + 374660\, f_4.$$

Moreover, the evaluation of h on some random point in $\mathsf{GL}_6 \mathcal{M}_2$ does not vanish. This proves $\underline{R}(\mathcal{M}_2) = 7$ by constructing an explicit polynomial in the vanishing ideal $I(\mathsf{GL}_6^3 \mathcal{E}_6)$.

Symbolic evaluation of a polynomial and comparing the result with the zero polynomial is known under the term *polynomial identity testing.* In our case, the polynomials are not given by small arithmetic circuits, but they are succinctly represented as an obstruction design. Our procedure of evaluation uses the special structure of the polynomials, but is still rather time-consuming. Internally, we do not operate with polynomials, but with semistandard tableaux and the straightening algorithm.

8.3.3 Questions.

(1) Is there a single obstruction design $\mathcal{H}$ whose polynomial $f_{\mathcal{H}}$ vanishes on $\mathsf{GL}_6^3 \mathcal{E}_6$, but not on $\mathsf{GL}_6^3 \mathcal{M}_2$?

(2) How does the above construction generalize to $n \times n$ matrix multiplication?

Chapter 9

Some Negative Results

In this chapter we present results which show that some very optimistic approaches, which would significantly simplify the search for obstructions, are doomed to fail. However, none of our results rules out Geometric Complexity Theory as a whole.

9.1 SL-obstructions

We are in the tensor scenario from Section 2.6. Recall from Section 3.3 that a representation theoretic occurence obstruction is an irreducible GL_n^3-representation occuring in $\mathbb{C}[\overline{\mathsf{GL}_n^3\mathcal{M}_m}]$ and not in $\mathbb{C}[\overline{\mathsf{GL}_n^3\mathcal{E}_n}]$. In this section we see that this is not possible for SL_n^3-representations if $n > m^2+1$: All types of irreducible SL_n^3-representations that occur in $\mathbb{C}[\overline{\mathsf{GL}_n^3\mathcal{M}_m}]$ also occur in $\mathbb{C}[\overline{\mathsf{GL}_n^3\mathcal{E}_n}]$. In this section we denote by $\{\lambda\}_{\mathsf{GL}_M^3}$ the irreducible GL_M^3-representation $\{\lambda\}$, emphasizing the associated group. Moreover, for $\lambda \vdash_{\overline{M-1}}$ we write $\{\lambda\}_{\mathsf{SL}_M^3}$ for the irreducible SL_M^3-representation $\{\lambda\}$.

Our main result is the following theorem.

9.1.1 Theorem ([BI11, Thm. 4.6])**.** *For any $\lambda \vdash_{\overline{M}}^* d$ there exists $i \in \mathbb{N}$ such that as GL_{M+1}^3-representations we have $\{\lambda + (M\times i)^3\}^*_{\mathsf{GL}_{M+1}^3} \subseteq \mathbb{C}[\mathsf{GL}_{M+1}^3\mathcal{E}_{M+1}]$.*

We note that passing from $\{\lambda\}^*_{\mathsf{GL}_{M+1}^3}$ to $\{\lambda + (M\times i)^3\}^*_{\mathsf{GL}_{M+1}^3}$ is *not* just tensoring with a power of the determinant. We now explain the consequences of Theorem 9.1.1.

For a partition triple $\lambda \vdash_{\overline{M}}^*$ let $\lambda' \vdash_{\overline{M-1}}^*$ be the SL_M^3-isomorphy type corresponding to λ: The partition triple λ' is obtained from λ by deleting every column that contains M boxes.

Take a partition triple $\lambda \vDash^* d$ with $\lambda \in \mathbb{C}[\overline{\mathsf{GL}_{m^2+1}^3\mathcal{M}_m}]$. Then λ has at most $M := m^2$ rows, which follows from Proposition 5.3.1(2). The following result shows that λ' is not an SL_M^3-obstruction against $\mathcal{M}_m \in \overline{\mathsf{GL}_{M+1}^3\mathcal{E}_{M+1}}$.

9.1.2 Proposition. *For each $\lambda \vdash_{\overline{M}}^* d$ we have that $\{\lambda'\}^*_{\mathsf{SL}_M^3} \subseteq \mathbb{C}[\overline{\mathsf{GL}_{M+1}^3\mathcal{E}_{M+1}}]$ as SL_M^3-representations.*

Proof. Theorem 9.1.1 implies that there exists $i \in \mathbb{N}$ such that

$$\{\lambda + (M\times i)^3\}^*_{\mathsf{GL}_{M+1}^3} \subseteq \mathbb{C}[\mathsf{GL}_{M+1}^3\mathcal{E}_{M+1}].$$

Propostion 5.4.4 tells us that $\mathcal{E}_{M+1}$ is SL_{M+1}^3-stable. We apply Proposition 5.4.5 to $v = \mathcal{E}_{M+1}$ (recall $o(\mathcal{E}_{M+1}) = 1$). This gives $\gamma \in \mathbb{Z}$ such that

$$\{\lambda + (M\times i)^3 + ((M+1)\times\gamma)^3\}^*_{\mathsf{GL}_{M+1}^3} \subseteq \mathbb{C}[\overline{\mathsf{GL}_{M+1}^3\mathcal{E}_{M+1}}].$$

Note that since $\lambda + (M\times i)^3$ has at most M rows in each of its three components, we have $\gamma \geq 0$. Decomposing $\{\lambda + (M\times i)^3 + ((M+1)\times\gamma)^3\}^*_{\mathsf{GL}^3_{M+1}}$ as a GL^3_M-representation with the Pieri rule (Theorem 4.1.9) implies that

$$\{\lambda + (M\times(i+\gamma))^3\}^*_{\mathsf{GL}^3_M} \subseteq \mathbb{C}[\overline{\mathsf{GL}^3_{M+1}\mathcal{E}_{M+1}}].$$

This implies that $\{\lambda'\}^*$ occurs as a SL^3_M-representation in the right-hand side. □

The proof of Theorem 9.1.1 uses Corollary 5.2.2. In our situation we can make explicit use of the fact that $\lambda^{(k)}$ has at most M rows. In the next lemma we see that after replacing $\lambda^{(k)}$ with $\lambda^{(k)} + M\times i$, for sufficiently large i, a regular μ as requested in Corollary 5.2.2 exists. More precisely, we show that there exists a regular partition $\perp_{M+1}(d+iM) \vdash_{\overline{M+1}} d+iM$ such that

$$\perp_{M+1}(d+iM) \preceq (\lambda^{(1)} + M\times i) \curlywedge (\lambda^{(2)} + M\times i) \curlywedge (\lambda^{(3)} + M\times i)$$

for sufficiently large i, which, together with Corollary 5.2.2, finishes the proof of Theorem 9.1.1.

9.1.3 Lemma. *(1) The set of regular partitions in* $\mathsf{Par}_M(d) = \{\lambda \vdash_{\overline{M}} d\}$ *has a unique smallest element* $\perp_M(d)$ *with respect to the dominance order.*

(2) For any $\lambda \in \mathsf{Par}_M(d)$ *we have* $\perp_{M+1}(d+iM) \preceq \lambda + M\times i$ *for sufficiently large* i.

Proof. The proof is fairly standard and translates what is already known about the smallest element of $\mathsf{Par}_M(d)$ into the regular situation by adding/subtracting a staircase partition.

Note that $\lambda \preceq \mu$ implies $\lambda + \nu \preceq \mu + \nu$ for any partitions λ, μ, ν. We call $s(j) := (j, j-1, \ldots, 1)$ the *symmetric staircase partition* with j rows and j columns. Note that if λ is a regular partition with at least j nonzero rows, then $\lambda - s(j)$ is again a partition, since λ has at least $j - i + 1$ boxes in row i.

(1) Let $d = qM + r$ with $0 \leq r < M$. It is obvious that $\square_M(d) := (M\times q) + (r\times 1)$ is the unique smallest element of $\mathsf{Par}_M(d)$. The corresponding diagram has q columns of length M plus one additional column of length r.

For given $d, M \in \mathbb{N}$ we set $\ell := \ell(M,d) := \max\{j \leq M \mid j(j+1)/2 \leq d\}$ and we define the *staircase partition* $\perp_M(d) := s(\ell) + \square_\ell(d - |s(\ell)|)$, see Figure 9.1.i. We observe that $\ell(\perp_M(d)) = \ell$ and moreover $\perp_M(d) = \perp_\ell(d)$.

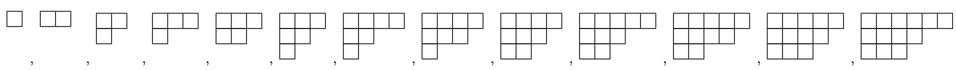


Figure 9.1.i: The staircase partitions $\perp_3(d)$ for $d = 1, \ldots, 13$.

Part (1) of Lemma 9.1.3 claims that $\perp_M(d) \preceq \beta$ for any regular partition $\beta \in \mathsf{Par}_M(d)$.

For showing this, put $\tilde\ell := \ell(\beta)$ and note that $\beta - s(\tilde\ell)$ is a partition by the previous observation. If we had $\ell + 1 \leq \tilde\ell$, then $\beta - s(\ell+1)$ would be a partition as well and hence $d = |\beta| \geq \frac{(\ell+1)(\ell+2)}{2}$ contradicting the maximality of ℓ. So we have $\ell(\perp_M(d)) = \ell \geq \tilde\ell$ and hence $\perp_M(d) - s(\tilde\ell)$ is a partition.

We note that the subpartition consisting of the first $\tilde\ell$ rows of $\perp_M(d) - s(\tilde\ell)$ equals $\square_{\tilde\ell}(d')$ for some $d' \leq d - |s(\tilde\ell)|$. Moreover, $\square_{\tilde\ell}(d') \preceq \square_{\tilde\ell}(d - |s(\tilde\ell)|) \preceq \beta - s(\tilde\ell)$, where the last inequality is due to the minimality of $\square_{\tilde\ell}(d - |s(\tilde\ell)|)$ in $\mathsf{Par}_{\tilde\ell}(d - |s(\tilde\ell)|)$. It follows that $\perp_M(d) - s(\tilde\ell) \preceq \beta - s(\tilde\ell)$, which completes the proof of part (1).

(2). For $i > \frac{M(M+1)}{2} + M$ we define the following regular partition in $\mathsf{Par}_{M+1}(d + iM)$:

$$\lambda^{i,\mathsf{reg}} := (\lambda_1 + i - 1, \lambda_2 + i - 2, \ldots, \lambda_M + i - M, \frac{M(M+1)}{2}).$$

Part (1) yields $\perp_{M+1}(d+iM) \preceq \lambda^{i,\mathsf{reg}}$. Since $\lambda^{i,\mathsf{reg}} \preceq (\lambda_1+i,\ldots,\lambda_M+i,0)$ the claim follows. □

9.2 Cones and Saturated Semigroups

In this section we want to use our insights from Chapters 4, 6, and 5 to analyze the cone of partitions occuring in the coordinate ring of orbits. Recall the scenarios from Section 2.6.

Let S_{Gv} denote the semigroup of partitions $\lambda \vdash_{n^2}$ or partition triples $\lambda \vdash^*_n$ corresponding to irreducible representations $\{\lambda^*\}$ which occur in $\mathbb{C}[Gv]$, where Gv is one of the four cases $\mathsf{GL}_{n^2}\mathrm{det}_n$, $\mathsf{GL}_{m^2}\mathrm{per}_m$, $\mathsf{GL}_n^3\mathcal{E}_n$, or $\mathsf{GL}_{m^2}^3\mathcal{M}_m$. Let $\mathsf{cone}(S_{Gv})$ denote the real cone generated by S. Then $\mathsf{cone}(S_{Gv})$ is a polyhedral cone, because S_{Gv} is finitely generated, see Theorem 4.3.5. In order to prove $\{\lambda\}^* \not\subseteq \mathbb{C}[\overline{G\mathfrak{c}_n}]$ it suffices to show that $\lambda \notin \mathsf{cone}(S_{G\mathfrak{c}_n})$, because of (3.4.2). Unfortunately, this strategy for finding obstructions is too coarse, because the following proposition shows that no candidates for obstructions can lie outside the cone.

9.2.1 Proposition. *Let $C_n := \{x \in \mathbb{R}^n \mid x_1 \geq \ldots \geq x_n \geq 0\}$. Then we have:*
(1) $\mathsf{cone}(S_{\mathsf{GL}_{n^2}\mathrm{det}_n}) = C_{n^2}$.
(2) $\mathsf{cone}(S_{\mathsf{GL}_{m^2}\mathrm{per}_m}) = C_{m^2}$.
(3) $\mathsf{cone}(S_{\mathsf{GL}_n^3\mathcal{E}_n}) = C_n^3$.
(4) $\mathsf{cone}(S_{\mathsf{GL}_{m^2}^3\mathcal{M}_m}) = C_{m^2}^3$.

Proof. (1) Given $\lambda \vdash_{n^2} nd$. Theorem 6.3.2 provides $\gamma \in \mathbb{N}$ such that we have $\mathsf{sk}\big(\gamma\lambda; (n\times\gamma d)^2\big) > 0$. According to Theorem 5.2.3, it follows $\{\gamma\lambda\}^* \subseteq \mathbb{C}[G\mathrm{det}_n]$ and hence $\gamma\lambda \in S_{\mathrm{det}_n}$, which implies $\lambda \in \mathsf{cone}(S_{\mathrm{det}_n})$.

(2) We use Theorem 5.2.4 as follows. Given $\lambda \vdash_{m^2} dm$ We choose $\mu := m\times d$. If we stretch μ by an even number γ, then Theorem 6.2.1 implies $p_{\gamma\mu}(m[\gamma d]) \neq 0$. If we choose γ big enough, then Theorem 6.3.2 implies $\mathsf{sk}\big(\gamma\lambda; (\gamma\mu)^2\big) > 0$. Hence Theorem 5.2.4 implies that $\gamma\lambda \in S_{\mathrm{per}_m}$ and therefore $\lambda \in \mathsf{cone}(S_{\mathrm{per}_m})$.

(3) We use Theorem 5.2.1 as follows. Given $\lambda \vdash^*_n d$, we choose $\mu = n\times 2d$. Note that $\mathsf{stabs}_{\mathsf{S}_n}(\mu) = \mathsf{S}_n$. Theorem 4.3.8 gives the decomposition of the space of S_n-invariants in the weight space $\{2\lambda^{(k)}\}^\mu$:

$$\dim\big(\{2\lambda^{(k)}\}^\mu\big)^{\mathsf{S}_n} = p_{2\lambda}(n[2d]),$$

which is positive by Theorem 6.2.1. Given three S_n-invariant vectors $v^{(k)} \in \{2\lambda^{(k)}\}^{\mathsf{S}_n}$, we construct the desired S_n-invariant vector $v^{(1)}\otimes v^{(2)}\otimes v^{(3)} \in \{2\lambda\}$. Hence Theorem 5.2.1 implies $2\lambda \in S_{\mathcal{E}_n}$ and thus $\lambda \in \mathsf{cone}(S_{\mathcal{E}_n})$.

(4) We use Theorem 5.2.5 as follows. Given $\lambda \vdash^*_{m^2} d$, stretch with m to obtain $m\lambda \vdash^*_{m^2} md$. Now choose $\mu^{(1)} = \mu^{(2)} = \mu^{(3)} = m\times d$. Theorem 6.3.2 implies that there exists a stretching factor $\gamma \in \mathbb{N}$ such that $k(\gamma\lambda^{(1)};\gamma\mu^{(2)};\gamma\mu^{(3)}) > 0$, $k(\gamma\mu^{(1)};\gamma\lambda^{(2)};\gamma\mu^{(3)}) > 0$, and $k(\gamma\mu^{(1)};\gamma\mu^{(2)};\gamma\lambda^{(3)}) > 0$. Theorem 5.2.5 implies $\gamma\lambda \in S_{\mathcal{M}_m}$ and hence λ in $\mathsf{cone}(S_{\mathcal{M}_m})$. □

Instead of just looking at the cones of semigroups, we could make a finer analysis and look at the *saturation of semigroups*. We present only a sketch, leaving much work to be done.

We define the saturation of a semigroup $S \subseteq \mathbb{Z}^N$ as $\mathsf{cone}(S) \cap \mathsf{lattice}(S)$, where $\mathsf{lattice}(S) := S - S := \{s - s' \mid s, s' \in S\}$ is the subgroup generated by S. We shall study only the case of the determinant, hence we set $S_n := S_{\mathsf{GL}_{n^2}\mathrm{det}_n}$. So, by Proposition 9.2.1, $\mathsf{cone}(S_n) = C_{n^2}$ is the trivial cone of partitions. It remains to determine $\mathsf{lattice}(S_n)$. It is clear that for $\lambda \in S_n$ it is necessary that $\lambda_1 + \cdots + \lambda_{n^2}$

is divisible by n. bc In the case of $\det_2$, we actually get a nontrivial lattice which gives additional information:

$$\mathsf{lattice}(S_2) = \{s \in \mathbb{Z}^{n^2} \mid \text{all } s_i \text{ are even}\}$$

This can be proved by the explicit formulas provided in [RW94]. (We omit the details.)

The following computations suggest that $n = 2$ is special and that such parity constraints might not occur for higher n.

9.2.2 Proposition.

$$\mathsf{lattice}(S_3) = \mathscr{L}_3 := \{s \in \mathbb{Z}^9 \mid \sum_i s_i \equiv 0 \mod 3\}.$$

Proof. The proof is based on Kronecker coefficient computation, which was done with Derksen.

Recall from Theorem 5.2.3 that $S_3 = \{\lambda \vdash_9 3d \mid d \in \mathbb{N},\ \mathsf{sk}(\lambda; (3\times d)^2) > 0\}$. We can check that the following partitions are contained in S_3: (3), $(6,3)$, $(6,6,3)$, $(6,6,3,3)$, $(3,3,3,3,3)$, $(3,3,3,3,3,3)$, $(6,3,3,3,3,3,3)$, $(6,6,3,3,3,3,3,3)$, $(6,6,6,3,3,3,3,3,3)$. This easily implies that $3 \cdot |i\rangle \in \mathsf{lattice}(S_n)$ for all $1 \le i \le 9$ and hence $(3\mathbb{Z})^9 \subseteq \mathsf{lattice}(S_3) \subseteq \mathscr{L}_3$. We apply the canonical map $p\colon \mathbb{Z}^9 \twoheadrightarrow (\mathbb{Z}/3\mathbb{Z})^9$ and obtain

$$\mathrm{span}_{\mathbb{Z}/3\mathbb{Z}}\, p(S_3) = p(\mathsf{lattice}(S_3)) \subseteq p(\mathscr{L}_3) = \{s \in (\mathbb{Z}/3\mathbb{Z})^9 \mid \sum_i s_i = 0\}. \qquad (*)$$

We can calculate that the rows of the following 8×9 matrix are all in S_3:

$$\begin{pmatrix} 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 4 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Since this matrix has the rank 8 over $\mathbb{Z}/3\mathbb{Z}$, it follows that equality holds in (∗). Hence the proposition follows. □

Using Derksen, we calculated analogous $(n^2 - 1) \times n^2$ matrices with rows from S_n that have full rank over $\mathbb{Z}/n\mathbb{Z}$ in the cases $n = 4, 5, 6$. (For well-chosen matrix rows the zero divisors in $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ do no harm.) This leads to the following conjecture.

9.2.3 Conjecture. *For all $n > 2$ we have*

$$\mathsf{lattice}(S_{\mathsf{GL}_{n^2}\det_n}) = \{s \in \mathbb{Z}^{n^2} \mid \sum_i s_i \equiv 0 \mod n\}.$$

Clearly, in Geometric Complexity Theory, the object $\mathbb{C}[\overline{Gv}]$ is of higher interest than the object $\mathbb{C}[Gv]$. Therefore we pose the following questions.

9.2.4 Questions.

(1) If in the definition of $S_{\mathsf{GL}_{n^2}\det_n}$ we replace $\mathbb{C}[\mathsf{GL}_{n^2}\det_n]$ by $\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]$, i.e., consider the semigroup of partitions $\lambda \vdash_{n^2}$ corresponding to irreducible representations $\{\lambda^*\}$ which occur in $\mathbb{C}[\overline{\mathsf{GL}_{n^2}\det_n}]$, what can be said about the lattice, the cone and the saturation of the resulting semigroup?

(2) Similarly, what can be said in the cases per_m, $\mathcal{E}_n$, or $\mathcal{M}_m$?

Part II

Littlewood-Richardson Coefficients

"...no apology is intended. We have simply written about mathematics which has interested us, pure or applied."
— L. R. Ford, Jr. and D. R. Fulkerson, [FF62, Preface]

Introduction

Recall from Section 4.5 the definition of the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ as the multiplicity in the decomposition of the tensor product of irreducible GL_n-representations

$$\{\lambda\} \otimes \{\mu\} \simeq \bigoplus_{\nu \vdash_n |\lambda|+|\mu|} c_{\lambda\mu}^{\nu} \{\nu\}.$$

In contrast to the Kronecker coefficients, different combinatorial characterizations of the Littlewood-Richardson coefficients are known. The classic Littlewood-Richardson rule (cf. [Ful97, Ch. I.5]) counts certain skew tableaux, while in Berenstein and Zelevinsky [BZ92], the number of integer points of certain polytopes are counted. A beautiful characterization was given by Knutson and Tao [KT99], who characterized Littlewood-Richardson coefficients either as the number of honeycombs or hives with prescribed boundary conditions. We add to this the following new description, which is based on flows in networks and which serves several algorithmic purposes: We characterize $c_{\lambda\mu}^{\nu}$ as the number of *capacity achieving hive flows* on the honeycomb graph G. In this Part II of the thesis we carry out an in-depth study of hive flows using purely combinatorial and algorithmic means.

All algorithms that we present only use additions, multiplications, and comparisons and the running time is defined to be the number of these operations. Moreover, the occuring numbers are all polynomially bounded in bitsize, so our algorithms are efficient in the bit model as well.

Chapter 10

Hive Flows

In this chapter we introduce the notion of *hive flows*, analyze cycles on the honeycomb graph G, and define the residual network R_f with respect to a hive flow f. We also state the Rerouting Theorem 10.3.22, which is a crucial ingredient in our Algorithms 1 and 2 for deciding positivity of the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$.

10.1 Flow Description of LR Coefficients

10.1 (A) Flows on Digraphs

We fix some terminology regarding flows on directed graphs, compare [AMO93]. Let D be a digraph with vertex set $V(D)$ and edge set $E(D)$. We assume that $s, t \in V(D)$ are two different distinguished vertices, called *source* and *target*, respectively. Let e_{start} denote the vertex where the edge e starts and e_{end} the vertex where e ends. The *inflow* and *outflow* of a map $f\colon E(D) \to \mathbb{R}$ at a vertex $v \in V(D)$ are defined as

$$\mathsf{inflow}(v, f) := \sum_{e_{\mathsf{end}}=v} f(e), \quad \mathsf{outflow}(v, f) := \sum_{e_{\mathsf{start}}=v} f(e),$$

respectively. A *flow on* D is defined as a map $f\colon E(D) \to \mathbb{R}$ that satisfies Kirchhoff's conservation laws: $\mathsf{inflow}(v, f) = \mathsf{outflow}(v, f)$ for all $v \in V(D) \setminus \{s, t\}$.

The set of flows on D is a vector space that we denote by $F(D)$. A flow is called *integral* if it takes only integer values and we denote by $F(D)_{\mathbb{Z}}$ the group of integral flows on D. The quantity $\delta(f) := \sum_{e_{\mathsf{start}}=s} f(e) - \sum_{e_{\mathsf{end}}=s} f(e)$ is called the *overall throughput* of the flow f.

By a *walk* p in D we understand a sequence $x_0, \ldots, x_\ell$ of vertices of D such that $(x_{i-1}, x_i) \in E$ for all $1 \le i \le \ell$. A *path* p in D is defined as a walk such that the vertices $x_0, \ldots, x_\ell$ are pairwise distinct. We will say that $x_0, \ldots, x_\ell$ are the *vertices used by* p. The path p is called an *s-t-path* if $x_0 = s$ and $x_\ell = t$; p is called a *t-s-path* if $x_0 = t$ and $x_\ell = s$. A path p is called *proper*, if p neither uses s nor t. A sequence $x_0, \ldots, x_\ell$ of vertices of D is called a *cycle* c if $x_0, \ldots, x_{\ell-1}$ are pairwise distinct, $x_\ell = x_0$, and $(x_{i-1}, x_i) \in E$ for all $1 \le i \le \ell$. Again we say that $x_0, \ldots, x_\ell$ are the vertices used by c. The set $C(D)$ is defined to be the set of cycles in D. We call c a *proper cycle* if c neither uses s nor t. It will be sometimes useful to identify a path or a cycle with the set of its edges $\{(x_0, x_1), \ldots, (x_{\ell-1}, x_\ell)\}$. Since the starting vertex x_0 of a cycle is not relevant, this does not harm. By a *complete path p in D* we understand an s-t-path, t-s-path, or a cycle in D. (It is not excluded that the cycle passes through s or t.)

A complete path p in D defines a flow f on D by setting $f(e) := 1$ if $e \in p$ and $f(e) := 0$ otherwise. It will be convenient to denote this flow with p as well. We note that $\delta(p) = 1$ for an s-t-path p, $\delta(p) = -1$ for a t-s-path p, and $\delta(c) = 0$ for a cycle c.

A flow is called *nonnegative* if $f(e) \geq 0$ for all edges $e \in E$. We call $\mathrm{supp}(f) := \{e \in E(D) \mid f(e) \neq 0\}$ the *support* of f.

An important method for analyzing flows is the fact that they can be decomposed into complete paths [AMO93, Thm. 3.5].

10.1.1 Lemma. *For any nonnegative flow $f \in F(D)$ there exists a family $p_1, \dots, p_m$ of complete paths in D contained in $\mathrm{supp}(f)$, and positive real numbers $\alpha_1, \dots, \alpha_m$ such that $f = \sum_{i=1}^{m} \alpha_i p_i$. Moreover, if the flow f is integral, then the α_i may be assumed to be natural numbers.* □

We will study flows in two rather different situations. The residual digraph R introduced in Section 10.3 has the property that it never contains an edge (u, v) and its reverse edge (v, u). Only nonnegative flows on R will be of interest.

On the other hand, we also need to look at flows on digraphs resulting from an undirected graph G by replacing each of its undirected edges $\{u, v\}$ by the directed edge $e = (u, v)$ and its reverse $-e := (v, u)$. We shall denote the resulting digraph also by G. To a flow f on G we assign its *reduced representative* $\tilde{f}$ defined by $\tilde{f}(e) := f(e) - f(-e) \geq 0$ and $\tilde{f}(-e) = 0$ if $f(e) \geq f(-e)$, and setting $\tilde{f}(e) := 0$ and $\tilde{f}(-e) = f(-e) - f(e)$ if $f(e) < f(-e)$. It will be convenient to interpret f and $\tilde{f}$ as manifestations of the same flow. Formally, we consider the linear subspace $N(G) := \{f \in \mathbb{R}^{E(G)} \mid \forall e \in E(G) : f(e) = f(-e)\}$ of "null flows" and the factor space

$$\overline{F}(G) := F(G)/N(G). \tag{10.1.2}$$

We call the elements of $\overline{F}(G)$ *flow classes on* G (or simply flows) and identify them with their reduced representative. We note that the overall throughput function factors to a linear function $\delta \colon \overline{F}(G) \to \mathbb{R}$. A flow class is called *integral* if its reduced representative is integral and we denote by $\overline{F}(G)_{\mathbb{Z}}$ the group of integral flow classes on G.

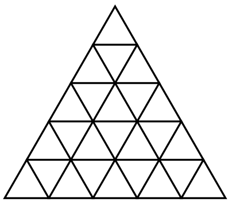
We remark that in the literature on flows, the subtle distinction between flows and their classes is not relevant, as the goal usually is to optimize the throughput of a flow subject to certain capacity constraints. But in the context of Littlewood-Richardson coefficients, we are interested in *counting* the number of capacity achieving flow classes, so that this distinction is necessary.

10.1 (B) Flows on the Honeycomb Graph G

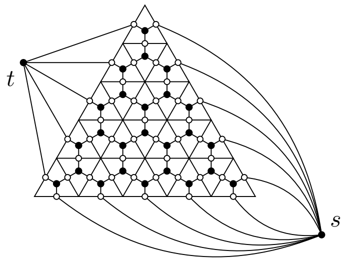
For the rest of this thesis, we fix

$$n := \ell(\nu).$$

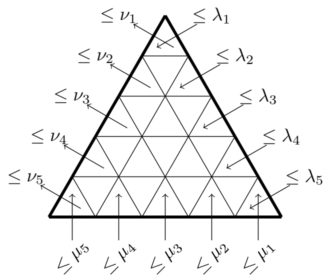
We start with a triangular array of vertices, $n + 1$ on each side, as seen in Figure 10.1.i(a). The resulting planar graph Δ shall be called the *triangular graph* with parameter n, we denote its vertex set with $V(\Delta)$ and its edge set with $E(\Delta)$. A triangle consisting of three edges in Δ is called a *hive triangle*. Note that there are two types of hive triangles: upright and downright oriented ones. A *rhombus* is defined to be the union of an upright and a downright hive triangle which share a common side. In contrast to the usual geometric definition of the term *rhombus* we use this term here in this very restricted sense only. Note that the angles at the corners of a rhombus are either acute of 60° or obtuse of 120°. Two distinct rhombi are called *overlapping* if they share a hive triangle.



(a) The triangular graph Δ for $n = 5$.



(b) The honeycomb graph G.



(c) The throughput capacities.

Figure 10.1.i: Graph constructions.

To realize the dual graph of Δ, as in [Buc00], we introduce a black vertex in the middle of each hive triangle and a white vertex on each hive triangle side, see Figure 10.1.i(b). Moreover, in each hive triangle T, we introduce edges connecting the three white vertices of T with the black vertex. Additionally, we introduce a source vertex s and a target vertex t. The source s is connected by an edge with each white vertex v on the right or on the bottom border of Δ, and the target t is connected by an edge with each white vertex v on the left border of Δ. Clearly, the resulting (undirected) graph G is bipartite and planar. We shall call G the *honeycomb graph* with parameter n.

We study now the vector space $\overline{F}(G)$ of flow classes on G introduced in Subsection 10.1 (A). Recall that for this, we have to replace each edge of G by the corresponding two directed edges. Correspondingly, we will consider G as a directed graph. Any complete path p in the digraph G defines a flow and thus a flow class on G, that we denote by p as well. According to Lemma 10.1.1 we can write each flow class $f \in \overline{F}(G)$ as a nonnegative linear combination of complete paths. (Note that the reduced representative of any flow on G is nonnegative.)

In order to characterize the flow class $f \in \overline{F}(G)$ in a concise way, we introduce the notion of the throughput of f through edges of Δ.

10.1.3 Definition (Throughput)**.** For an edge $k \in E(\Delta)$ let $e \in E(G)$ denote the edge pointing from the white vertex on k into the upright triangle. The throughput $\delta(k, f)$ w.r.t. a flow $f \in \overline{F}(G)$ is defined as $f(e) - f(-e)$. ■

Note that $\overline{F}(G) \to \mathbb{R}, f \mapsto \delta(k, f)$ is a linear form.

It is obvious that a flow class f on G is completely determined by the throughput function $\delta\colon E(\Delta) \to \mathbb{R}, k \mapsto \delta(k, f)$. Furthermore, Kirchhoff's conservation laws translate to the closedness condition

$$\delta(k_1, f) + \delta(k_2, f) + \delta(k_3, f) = 0 \tag{10.1.4}$$

holding for each hive triangle (upright or downright) with sides denoted by k_1, k_2, k_3. Let $Z \subseteq \mathbb{R}^{E(\Delta)}$ denote the linear subspace consisting of the functions δ satisfying (10.1.4) for all hive triangles. We see that the vector space $\overline{F}(G)$ of flow classes on G can be identified with Z via the linear isomorphism

$$\overline{F}(G) \xrightarrow{\sim} Z, \; f \mapsto \delta(.,f). \tag{10.1.5}$$

Moreover, under this identification, integral flow classes f correspond to functions in the subgroup $Z_{\mathbb{Z}}$ consisting of functions δ taking integer values.

By adding up (10.1.4) for all upright hive triangles and subtracting (10.1.4) for all downright hive triangles, taking into account the cancelling of throughputs on all inner sides k, we see that the sum of $\delta(k,f)$ over all border edges k of Δ vanishes. Therefore, we can express the overall throughput $\delta(f)$ as

$$\delta(f) \;=\; \sum_{k \in E_r \cup E_b} \delta(k,f) \;=\; -\sum_{k' \in E_\ell} \delta(k',f), \tag{10.1.6}$$

where E_ℓ, E_r, and E_b denote the sets of edges of Δ on the left side, right side, and bottom side, respectively.

The flow classes on G can be characterized in yet another way. Let x_0 be the top vertex of Δ and define the vector space H of functions $h\colon V(\Delta) \to \mathbb{R}$ satisfying $h(x_0) = 0$. We denote by $H_{\mathbb{Z}}$ the subgroup of functions $h \in H$ taking integer values.

We have the linear isomorphism $\eta\colon H \to Z,\; h \mapsto \delta$ defined as follows: Let x_1 and x_2 denote two vertices of an upright hive triangle, where x_2 is the successor of x_1 in counterclockwise direction. Then $\delta((x_1,x_2)) := h(x_2) - h(x_1)$. The map η is easily seen to be an isomorphism of vector spaces. Clearly this isomorphism induces an isomorphism of groups $H_{\mathbb{Z}} \to Z_{\mathbb{Z}}$.

10.1 (C) Hives and Hive Flows

Following [KT99, Buc00] we define a *hive* on Δ as a function $h \in H$ such that for all rhombi ϱ, the sum of the values of h at the two obtuse vertices of ϱ is greater than or equal to the sum of the values of h at the two acute vertices of ϱ. These inequalities are known as the *hive inequalities*. In pictorial notation,

$$\sigma(\Diamond, h) := h(\bullet\!\Diamond) + h(\Diamond\!\bullet) - h(\overset{\bullet}{\Diamond}) - h(\underset{\bullet}{\Diamond}) \;\geq\; 0$$

where $\bullet\!\Diamond, \Diamond\!\bullet, \overset{\bullet}{\Diamond}, \underset{\bullet}{\Diamond} \in V(\Delta)$ denote the corner vertices of $\Diamond$. We call the $\sigma(\varrho,h)$ the *slack* of the rhombus ϱ with respect to the hive h.

If one interprets $h(v)$ as the height of a point over $v \in V(\Delta)$ and interpolates these points linearly over each hive triangle of Δ one gets a continuous function $h\colon \Delta \to \mathbb{R}$. (Here the triangle Δ is to be interpreted as a convex subset of $\mathbb{R}^2$.) Then the hive inequalities mean that h is a concave function. The function h is linear over a rhombus ϱ iff $\sigma(\varrho,h) = 0$, in which case we call the rhombus ϱ *h-flat*.

10.1.7 Lemma. *For $h \in H$ and $x \in V(\Delta)$ we have $\min_{\partial\Delta} h \leq h(x) \leq n \max_{\partial\Delta} h$, where $\partial\Delta$ denotes the boundary of the convex set $\Delta \subseteq \mathbb{R}^2$.*

Proof. Let $x(m,i)$ denote the vertex of Δ in the mth line parallel to the ground side (counting from the top) and on the ith side parallel to the left side (counting from the left), for $0 \leq i \leq m \leq n$. So $x(0,0)$ is the top vertex and $h(x(0,0)) = 0$ for $h \in H$. Put $a := h(x(1,0))$ and $b := h(x(1,1))$.

Since h is a concave function, its subgraph $S := \{(x,y) \in \Delta \times \mathbb{R} \mid y \leq h(x)\}$ is convex. This implies that $h(x(m,i)) \leq am + (b-a)i$ for all $h \in H$. Therefore, $h(x(m,i)) \leq m \max\{a,b\}$, proving the upper bound.

The lower bound follows easily from the convexity of S. $\square$

In this work, it will be extremely helpful to have some graphical way of describing rhombi and throughputs. We shall denote a rhombus ϱ of Δ by the pictogram $\lozenge$, even though ϱ may lie in any of the three positions "$\lozenge$", "$\sqsupset$" or "$\sqsubset$" obtained by rotating with a multiple of $60°$. Let $\ominus$ denote the edge of Δ given by the *diagonal* of ϱ connecting its two obtuse angles. Then we denote by $\overset{\uparrow}{\ominus}(f) := \delta(\ominus, f)$ the throughput of f through $\ominus$ (going into the upright hive triangle). Similarly, we define the throughput $\overset{\downarrow}{\ominus}(f) := -\delta(\ominus, f)$. The advantage of this notation is that if $\overset{\uparrow}{\ominus}(f)$ is positive, then the flow goes in the direction of the arrow. For instance, using the symbolic notation, we note the following consequence of the flow conservation laws:

$$\lozenge_{\swarrow}(f) + \lozenge_{\searrow}(f) \;=\; \lozenge^{\nwarrow}(f) + \lozenge^{\nearrow}(f).$$

If f is the flow corresponding to the hive $h \in H$ under the isomorphisms $H \simeq Z \simeq \overline{F}(G)$, then the hive inequalities and the definition the isomorphism η imply that

$$\sigma(\lozenge, h) \;=\; \Big(h(\lozenge_{\bullet\text{left}}) - h(\lozenge_{\bullet\text{top}})\Big) + \Big(h(\lozenge_{\bullet\text{right}}) - h(\lozenge_{\bullet\text{bottom}})\Big) \;=\; \lozenge^{\nwarrow}(f) + \lozenge_{\searrow}(f).$$

We define now the slack of a rhombus with respect to a flow f as the slack with respect the corresponding hive h.

10.1.8 Definition. The *slack* of the rhombus $\lozenge$ with respect to $f \in \overline{F}(G)$ is defined as

$$\sigma(\lozenge, f) \;:=\; \lozenge^{\nwarrow}(f) + \lozenge_{\searrow}(f).$$

The rhombus $\lozenge$ is called *f-flat* if $\sigma(\lozenge, f) = 0$. ■

It is clear that $\overline{F}(G) \to \mathbb{R}, f \mapsto \sigma(\varrho, f)$ is a linear form. Note also that using the flow conservation law, the slack can be written in varous different ways:

$$\sigma(\lozenge, f) \;=\; \lozenge^{\nwarrow}(f) + \lozenge_{\searrow}(f) \;=\; \lozenge^{\nwarrow}(f) - \lozenge_{\nwarrow}(f) \;=\; \lozenge_{\swarrow}(f) - \lozenge^{\searrow}(f) \;=\; \lozenge_{\swarrow}(f) + \lozenge^{\nearrow}(f).$$

10.1.9 Definition. A flow $f \in \overline{F}(G)$ is called a *hive flow* iff $\sigma(\varrho, f) \geq 0$ for all rhombi ϱ in Δ. ■

By definition, the hives correspond to the hive flows under the isomorphism $H \simeq \overline{F}(G)$. Note that the set of hive flows is a cone in $\overline{F}(G)$. Figure 10.1.ii provides an example of a hive flow. We encourage the reader to verify the slack inequalities there to get some idea of the nature of these constraints.

We formulate now capacity constraints for the throughputs of hive flows at the border of Δ, depending on a chosen triple $\lambda, \mu, \nu \in \mathbb{N}^n$ of partitions satisfying $|\nu| = |\lambda| + |\mu|$. Hereby, we treat the left border of Δ differently from the right and bottom border of Δ with regard to orientations. To the ith border edge k of Δ on the right border of Δ, counted from top to bottom, we assign the *throughput capacity* $b(k) := \lambda_i$, see Figure 10.1.i(c). Further, we set $b(k) := \mu_i$ for the ith edge k on the bottom border of Δ, counted from right to left. Finally, we set $b(k') := \nu_i$ for the ith edge k' on the left border of Δ, counted from top to bottom. Recall that $\delta(k, f)$ denotes the throughput of a flow f into Δ, while $-\delta(k', f)$ denotes the throughput of f out of Δ.

10.1.10 Definition. Let $\lambda, \mu, \nu \in \mathbb{N}^n$ be a triple of partitions satisfying $|\nu| = |\lambda| + |\mu|$. The *polytope of bounded hive flows* $B := B(\lambda, \mu, \nu) \subseteq \overline{F}(G)$ is defined to be the set of hive flows $f \in \overline{F}(G)$ satisfying

$$0 \leq \delta(k, f) \leq b(k) \quad \text{and} \quad 0 \leq -\delta(k', f) \leq b(k')$$

for all border edges k on the right or bottom border of Δ, and for all border edges k' on the left border of Δ. The *polytope of capacity achieving hive flows* $P := P(\lambda, \mu, \nu)$ consists of those $f \in B(\lambda, \mu, \nu)$ for which $\delta(k, f) = b(k)$ and $-\delta(k', f) = b(k')$ for all k and k' as above. We also set $B_{\mathbb{Z}} := B \cap \overline{F}(G)_{\mathbb{Z}}$ and $P_{\mathbb{Z}} := P \cap \overline{F}(G)_{\mathbb{Z}}$. ■

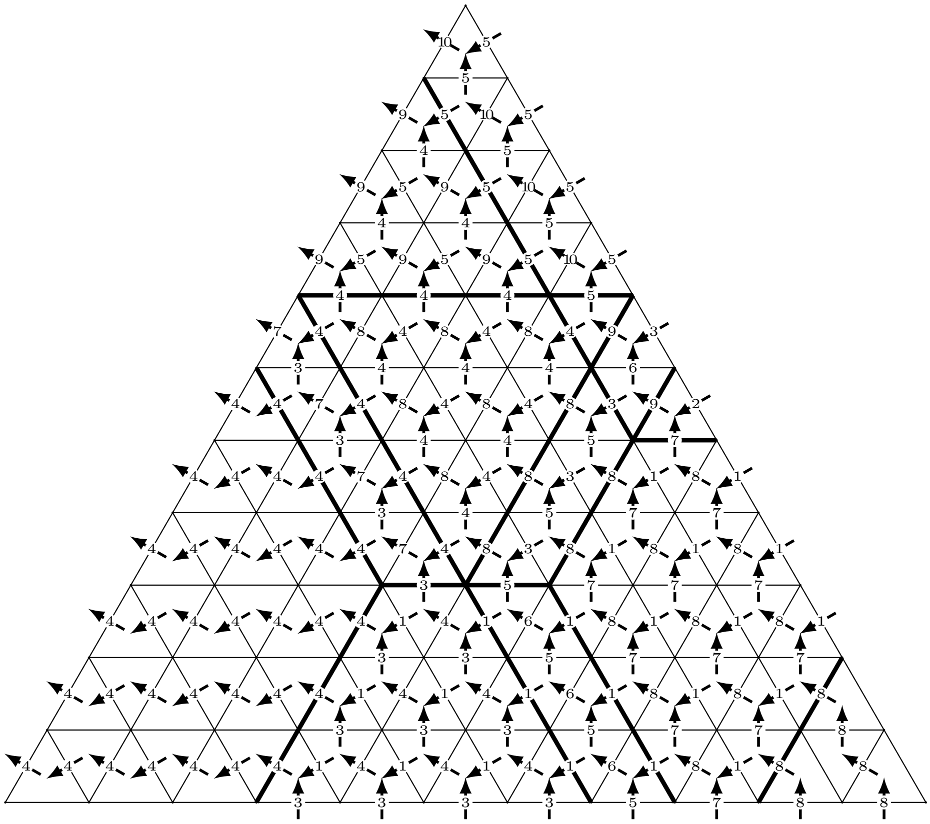


Figure 10.1.ii: A hive flow for $n = 11$, $\lambda = (5,5,5,5,3,2,1,1,1,0,0)$, $\mu = (8,8,7,5,3,3,3,3,0,0,0)$, $\nu = (10,9,9,9,7,4,4,4,4,4,4)$, and its corresponding partition of Δ into flatspaces. The numbers give the throughputs through edges of Δ in the directions of the arrows. The properties of Lemma 10.3.12 and Lemma 10.3.14 are readily verified.

Lemma 10.1.7 and the isomorphism $\overline{F}(G)_{\mathbb{Z}} \simeq H_{\mathbb{Z}}$ imply that B is bounded and thus B and P are indeed polytopes.

We note that by (10.1.6), we have $\delta(f) \leq |\nu|$ for any $f \in B(\lambda, \mu, \nu)$. Moreover, $f \in B(\lambda, \mu, \nu)$ is capacity achieving iff $\delta(f) = |\nu|$.

Knutson and Tao [KT99] (see also [Buc00]) characterized the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ as the number of integral hives taking fixed values on the border vertices of Δ, prescribed by the partitions λ, μ, ν. Their description via the isomorphism $\overline{F}(G)_{\mathbb{Z}} \simeq H_{\mathbb{Z}}$ immediately translates to the following fundamental result.

10.1.11 Proposition. *The Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ equals the number of capacity achieving integral hive flows, i.e., $c_{\lambda\mu}^{\nu} = |P(\lambda, \mu, \nu)_{\mathbb{Z}}|$.* □

To advocate the advantage of the flow interpretation of Littlewood-Richardson coefficients, we show in the next section that $P_{\mathbb{Z}} := P(\lambda, \mu, \nu)_{\mathbb{Z}}$ can be interpreted as the set of vertices of a graph in a natural way. This will be important for searching and enumerating $P_{\mathbb{Z}}$ in an efficient way.

10.2 Properties of Hive Flows

We recall that any complete path p defines a flow on G, denoted by the same symbol. In order to describe the slack of a rhombus with respect to p, we introduce some further terminology.

10.2.1 Definition. A *turn* is defined to be a path in G of length 2 that lies inside Δ, starts at a white vertex and ends with a different white vertex, cp. Figure 10.1.i(b). ∎

Note that there are six turns in each hive triangle. We shall denote turns pictorially by $\lozenge$, $\lozenge$ etc. with the obvious interpretation. Similarly, $\lozenge$ and $\lozenge$ stand for paths consisting of four edges.

In order to describe the different ways a complete path p may pass a rhombus ϱ, we consider the following sets of paths in ϱ.

10.2.2 Definition. The sets of paths, interpreted as subsets of $E(G)$,

$$\Psi_+(\lozenge) := \{\lozenge,\lozenge,\lozenge,\lozenge\}, \quad \Psi_-(\lozenge) := \{\lozenge,\lozenge,\lozenge,\lozenge\}, \text{ and } \Psi_0(\lozenge) := \{\lozenge,\lozenge,\lozenge,\lozenge\}$$

are called the sets of of *positive*, *negative*, and *neutral slack contributions* of the rhombus $\lozenge$, respectively. ∎

For later use the reader should remember that the turns in $\Psi_+(\lozenge)$ at the acute angles are clockwise, while the concatenations of two turns at the obtuse angles are counterclockwise.

The verification of the following is immediate, using Definition 10.1.8 of the slack.

10.2.3 Observation. Let p be a complete path in G and E_ϱ be the set of edges of G contained in a rhombus ϱ. Then $p \cap E_\varrho$ is either empty, or it is a union of one or two slack contributions q. The slack $\sigma(\varrho, p)$ is obtained by adding 1, 0, or -1 over the contributions q contained in p, according to whether q is is positive, negative, or neutral.

10.2.4 Remark. The only situations, in which $p \cap E_\varrho$ is a union of two slack contributions, is when p uses both counterclockwise turns $\lozenge$ and $\lozenge$ at acute angles, or both clockwise turns $\lozenge$ and $\lozenge$ at acute angles, in which case $\sigma(\rho, p) = -2$ or $\sigma(\rho, p) = 2$, respectively. It is not possible that c uses both $\lozenge$ and $\lozenge$ since otherwise, due to the planarity of Δ, c would have to intersect itself. ∎

10.2 (A) The Support of Flows on G

Recall the definition of the support $\mathrm{supp}(d)$ of a flow class $d \in \overline{F}(G)$. By the definition, $\mathrm{supp}(d)$ cannot contain an edge and its reverse. We note the following:

$$\big(\lozenge \subseteq \mathrm{supp}(d) \;\text{ or }\; \lozenge \subseteq \mathrm{supp}(d)\big) \iff \lozenge(d) > 0.$$

Recall from Definition 10.2.2 the sets $\Psi_+(\varrho)$, $\Psi_-(\varrho)$, and $\Psi_0(\varrho)$ of positive, negative, and neutral slack contributions of a rhombus ϱ, respectively, interpreted as sets of directed edges of G. We assign to any slack contribution $p \in \Psi_+(\varrho) \cup \Psi_-(\varrho)$ of a rhombus ϱ its *antipodal contribution* $p' \in \Psi_+(\varrho) \cup \Psi_-(\varrho)$, which is defined by reversing p and then applying a rotation of 180°. For instance, $\lozenge$ is the antipodal contribution of $\lozenge$ and $\lozenge$ is the antipodal contribution of $\lozenge$. Clearly, $p \mapsto p'$ is an involution.

The following lemma on antipodal contributions will be of great use.

10.2.5 Lemma. *Let $d \in \overline{F}(G)$ such that $\sigma(\varrho, d) \geq 0$ for a rhombus ϱ. If $p \subseteq \mathrm{supp}(d)$ for a negative slack contribution p of ϱ, then $p' \subseteq \mathrm{supp}(d)$ for its antipodal contribution p'.*

Proof. 1. Suppose that ◇ $\subseteq \mathrm{supp}(d)$, which means $\delta_1 :=$ ◇$(d) > 0$ and $\delta_2 :=$ ◇$(d) > 0$. Since $\delta_3 :=$ ◇$(d) -$ ◇$(d) = \sigma($◇$, d) \geq 0$ we get ◇$(d) = \delta_1 + \delta_3 > 0$. Moreover, ◇$(d) =$ ◇$(d) -$ ◇$(d) = (\delta_1 + \delta_3) - (\delta_1 - \delta_2) = \delta_3 + \delta_2 > 0$. Altogether, ◇ $\subseteq \mathrm{supp}(d)$.

2. Suppose that ◇ $\subseteq \mathrm{supp}(d)$, which means $\delta_1 :=$ ◇$(d) > 0$, $\delta_2 :=$ ◇$(d) > 0$, and $\delta_3 :=$ ◇$(d) > 0$. Hence ◇$(d) = \delta_3 - \delta_1$. We have $\delta_4 :=$ ◇$(d) -$ ◇$(d) = \sigma($◇$, d) \geq 0$ and thus ◇$(d) = \delta_1 + \delta_4 > 0$. Therefore $\delta_3 = \delta_2 + (\delta_1 + \delta_4)$ and thus ◇$(d) = \delta_3 - \delta_1 = (\delta_2 + \delta_1 + \delta_4) - \delta_1 = \delta_2 + \delta_4 > 0$. Altogether, ◇ $\subseteq \mathrm{supp}(d)$. □

Applying Lemma 10.2.5 successively can provide important information about the support of a flow class d. This is stated in the following lemma on "flow propagation".

It will be convenient to use symbols like ◇, ◇ etc., which stand for the rhombi in the positions relative to ◇ as indicated by the shaded regions. E.g. ◇ stands for the rhombus with diagonal ◇.

10.2.6 Lemma. *Given $d \in \overline{F}(G)$ such that $\sigma($◇$, d) \geq 0$ and ◇ $\subseteq \mathrm{supp}(d)$. Then ◇ $\subseteq \mathrm{supp}(d)$. If additionally $\sigma($◇$, d) \geq 0$, then ◇ $\subseteq \mathrm{supp}(d)$. Similarly, if additionally $\sigma($◇$, d) \geq 0$, then ◇ $\subseteq \mathrm{supp}(d)$.*

For an example on how Lemma 10.2.6 can be used, see Figure 10.2.i.

Figure 10.2.i: Only the hive triangles of a pentagon contained in Δ are drawn. The rhombi of the pentagon have nonnegative slack with respect to the flow d. If the turns in the left picture are in $\mathrm{supp}(d)$, then, by applying Lemma 10.2.6 several times, we see that all the turns in the right picture are in $\mathrm{supp}(d)$.

Proof of Lemma 10.2.6. The first assertion is a direct application of Lemma 10.2.5. Suppose that $\sigma($◇$, d) \geq 0$. Since ◇ $\subseteq \mathrm{supp}(d)$, flow conservation implies that ◇ $\subseteq \mathrm{supp}(d)$ or ◇ $\subseteq \mathrm{supp}(d)$. We want to show ◇ $\subseteq \mathrm{supp}(d)$. If ◇ $\subseteq \mathrm{supp}(d)$, then ◇ $\subseteq \mathrm{supp}(d)$ and ◇ is a negative contribution in ◇. Hence by Lemma 10.2.5, we have ◇ $\subseteq \mathrm{supp}(d)$. The other assertion is proved analogously. □

10.2 (B) The Graph of Capacity Achieving Integral Hive Flows

Fix λ, μ, ν and recall the polytopes B and P from Definition 10.1.10. We show now that $P_{\mathbb{Z}}$ can be naturally seen as the vertex set of a graph.

10.2.7 Definition. We say that $f, g \in P_{\mathbb{Z}}$ are *neighbors* iff the flow $g - f$ is induced by a cycle in G. The resulting graph with the set of vertices $P_{\mathbb{Z}}$ is also denoted by $P_{\mathbb{Z}}$ and it is called the *Littlewood-Richardson graph* or *LR graph* for short. The *neighborhood* of f consists of all neighbors of f and is denoted by $\Gamma(f)$. ■

Clearly, the neighbor relation is symmetric.

For an explicit characterization of the neighbor relation we need the following concepts.

10.2.8 Definition. Let $f \in B$ and c be a proper cycle in G.
(1) We call a rhombus ϱ *nearly f-flat* iff $\sigma(\varrho, f) = 1$.
(2) c is called *f-hive preserving* iff c does not use negative contributions in f-flat rhombi.
(3) c is called *f-secure* iff c is f-hive preserving and c does not use both counterclockwise turns at acute angles in nearly f-flat rhombi (◇). ∎

We remark that c is f-hive preserving iff $f + \varepsilon c \in B$ for sufficiently small $\varepsilon > 0$.

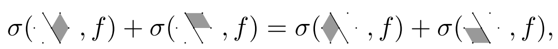
10.2.9 Proposition. *Assume $f \in P_{\mathbb{Z}}$. If $g \in P_{\mathbb{Z}}$ is a neighbor of f, then $g - f$ is an f-secure proper cycle. Conversely, if c is an f-secure proper cycle, then $f + c \in P_{\mathbb{Z}}$ is a neighbor of f.*

Proof. Assume that f and g are neighbors in $P_{\mathbb{Z}}$, so $c := g - f$ is a proper cycle in G. Hence $\sigma(\varrho, f + c) \geq 0$ for each rhombus ϱ. This implies that $\sigma(\varrho, c) \geq 0$ for each f-flat rhombus, that is, c is f-hive preserving. Moreover, if $\sigma(\varrho, f) = 1$, then $\sigma(\varrho, c) \geq -1$. Hence c is f-secure by Observation 10.2.3. The argument can be reversed. □

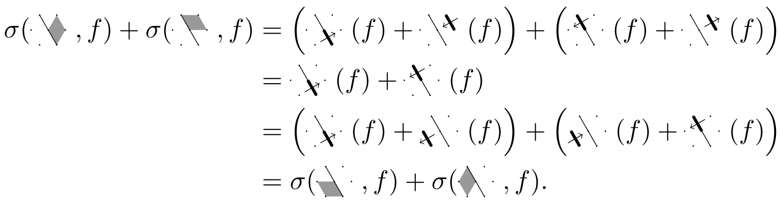
Apparently, the symmetry of the neighbor relation in $P_{\mathbb{Z}}$ does not seem to be obvious from the characterization in Proposition 10.2.9.

Before continuing, we state a useful observation.

10.2.10 Claim. *The union of two overlapping rhombi ϱ_1 and ϱ_2 forms a trapezoid. Glueing together two such trapezoids (ϱ_1, ϱ_2) and (ϱ_1', ϱ_2') at their longer side, we get a hexagon. In this situation, the following* hexagon equality *holds: $\sigma(\varrho_1, f) + \sigma(\varrho_2, f) = \sigma(\varrho_1', f) + \sigma(\varrho_2', f)$ for any flow $f \in \overline{F}(G)$. In pictorial notation, the hexagon equality can be succinctly expressed as*



Proof. We calculate



□

As an immediate consequence we obtain the following.

10.2.11 Corollary. *For all hive flows f, if* ◇ *and* ◇ *are f-flat, then also* ◇ *and* ◇ *are f-flat.* □

10.2.12 Remark. The slacks of rhombi are exactly the numbers in Berenstein-Zelevinsky triangles [PV05] and the hexagon equality (Claim 10.2.10) is just the condition for their validity. ∎

The next results tells us how f-secure cycles may arise.

10.2.13 Theorem. *Let $f \in B_{\mathbb{Z}}$ and c be an f-hive preserving cycle in G of minimal length. Then c is f-secure.*

Proof. We argue by contradiction. Suppose that c is an f-hive preserving cycle in G of minimal length, but not f-secure. So there is a nearly f-flat rhombus ◇ in which c uses both turns ◇ and ◇. Let us call such rhombi *bad*.

Since c has minimal length, it cannot be rerouted via . A moment's thought reveals that as a consequence, we are in one of the following two cases:

(**a**) is f-flat and c uses or (**b**) is f-flat and c uses .

Let us assume that we are in the situation (**a**). So we have the bad, nearly f-flat rhombus and the shaded f-flat rhombus. The hexagon equality (Claim 10.2.10) implies that either is f-flat and is nearly f-flat, or is nearly f-flat and is f-flat. These two possibilities are indicated on the left and right side of the following picture, respectively, where the shaded rhombi are f-flat and the diagonals of nearly f-flat rhombi are drawn thick. Further, parts of c which run in f-flat rhombi, are drawn with straight arrows:

The fact that c uses no negative contributions in f-flat rhombi (and no vertex of G twice) forces c to run exactly as depicted in the following picture:

Hence the second nearly f-flat rhombus, which is framed in the left and right picture, respectively, is bad as well.

So we see that the diagonal of the bad rhombus shares a vertex with the diagonal of another bad rhombus and that both diagonals either lie on the same line or include an angle of 120°. By symmetry, the same conclusion can be drawn in the case (**b**).

By induction, this implies that there is a region bounded by diagonals of bad rhombi. This is impossible, because c would have to run both inside and outside of this region. □

The following is an important insight into the structure of $P_{\mathbb{Z}}$. We postpone the proof to Section 12.3.

10.2.14 Theorem (Connectedness Theorem)**.** *The LR-graph $P_{\mathbb{Z}}$ is connected.*

As an application of our insights, we obtain the following characterization of multiplicity freeness. Recall that $c^{\nu}_{\lambda\mu} = |P(\lambda,\mu,\nu)_{\mathbb{Z}}|$.

10.2.15 Proposition. *Suppose that $f \in P(\lambda,\mu,\nu)_{\mathbb{Z}}$. Then we have $c^{\nu}_{\lambda\mu} > 1$ iff there exists an f-hive preserving cycle in G.*

Proof. If there exists an f-hive preserving cycle in G, then there is also one of minimal length, call it c. Theorem 10.2.13 implies that c is f-secure. Proposition 10.2.9 tells us that $f + c \in P_{\mathbb{Z}}$. It follows that $|P_{\mathbb{Z}}| \geq 2$.

Conversely, assume that $|P_{\mathbb{Z}}| \geq 2$. Since $f \in P_{\mathbb{Z}}$ and $P_{\mathbb{Z}}$ is connected by Theorem 10.2.14, there exists a neighbor $g \in P_{\mathbb{Z}}$ of f. Proposition 10.2.9 tells us that $g - f$ is an f-secure cycle. □

A proof of Fulton's conjecture, first shown in [KTW04] by different methods, is obtained as an easy consequence.

10.2.16 Corollary. *If $c^{\nu}_{\lambda\mu} = 1$, then $c^{N\nu}_{N\lambda N\mu} = 1$ for all $N \geq 1$.*

Proof. By definition, c is an f-hive preserving cycle in G iff c is a Nf-hive preserving cycle in G. Now apply Proposition 10.2.15. □

The characterization in Proposition 10.2.15 points to a way of algorithmically deciding whether $c_{\lambda\mu}^{\nu} > 1$. However, it is not obvious how to efficiently search for f-hive preserving cycles in the graph G. For this, and even for the simpler task of deciding $c_{\lambda\mu}^{\nu} > 0$, we have to construct suitable "residual digraphs", which brings us to the topic of the next section.

10.3 The Residual Digraph R_f

Fix partitions λ, μ, ν such that $|\nu| = |\lambda| + |\mu|$. Recall from Definition 10.1.10 the polytope $B = B(\lambda, \mu, \nu)$ of bounded hive flows and the linear overall throughput function $\delta\colon B \to \mathbb{R}$. Moreover, recall that $\delta(f) \leq |\nu|$ for all $f \in B$ and that a hive flow $f \in B$ is capacity achieving iff $\delta(f) = |\nu|$. According to Proposition 10.1.11, the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ counts the number of integral capacity achieving hive flows. Therefore, in order to decide positivity of $c_{\lambda\mu}^{\nu}$ we shall maximize the overall throughput function δ on the polytope B. (We will later see that we can always find an integral optimum.)

We follow a Ford-Fulkerson approach and try to construct for a given integral hive flow $f \in B_{\mathbb{Z}}$ a digraph R_f, such that adding an s-t-path p in R_f to f leads to a bounded hive flow. We have to guarantee that $f + p$ does not lead to negative slacks of rhombi so that $f + p$ is a hive flow. On the other hand, we want to make sure that f is optimal, when there is no s-t-path in R_f.

10.3 (A) Turnpaths and Turncycles

The intuition is to consider paths in G in which each node remembers its predecessor. This can be formally achieved by studying paths in an auxiliary digraph that we define next.

Recall that a turn is a path in G of length 2 that lies inside Δ, starts at a white vertex and ends with a different white vertex.

10.3.1 Definition. A *turnedge* is an ordered pair of turns that can be concatenated to a path in G. ∎

Note that a turnedge defines a path in G of length 4. We write turnedges pictorially like $\Diamond := (\Diamond, \Diamond)$ etc. We construct now the auxiliary digraph R.

10.3.2 Definition. The *digraph* R has as vertices the turns, henceforth called *turnvertices*, and the source and target of G. The edges of R are the turnedges and the following additional edges: the digraph R contains an edge (s, ϑ) from the source s to any turnvertex ϑ pointing from the right or bottom border of Δ into Δ. Vice versa, for any turnvertex ϑ' pointing from the right or bottom border out of Δ, there is a turnedge (ϑ', s) in R. Similarly, for the target t, there are edges (ϑ, t) for each turnvertex ϑ pointing from the left border of Δ out of Δ and vice versa, there are edges (t, ϑ') for each turnvertex ϑ' pointing from the left border into Δ. ∎

The reader should check that R never contains an edge and its reverse. In fact, the digraph R is rather complicated, for instance Figure 10.3.i shows that R is not planar for $n \geq 2$.

We have a well-defined notion of flows on R as R is a digraph with two distinguished vertices s and t. We assign now to a flow f on R a flow class $\tilde{f}$ on G by defining the corresponding throughput map $E(\Delta) \to \mathbb{R}, k \mapsto \delta(k, \tilde{f})$ as follows. An edge k of Δ lies in exactly one upright hive triangle $\triangle$. Let $\triangle$ and $\triangle$ denote

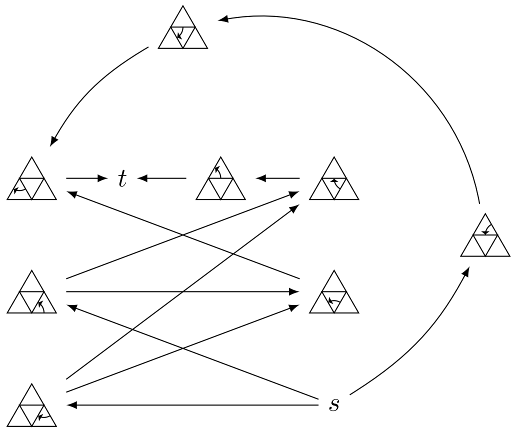


Figure 10.3.i: A subgraph of R for $n = 2$ that is a subdivision of the graph $K_{3,3}$. According to Kuratowski's theorem, this shows nonplanarity of R for $n = 2$. This easily generalizes to $n \geq 2$.

the two turns in $\triangle$ pointing towards the white vertex on k. Further, let $\triangle$ and $\triangle$ denote the turns obtained when reversing $\triangle$ and $\triangle$. We define

$$\delta(k, \tilde{f}) \;=\; \mathrel{\diamondsuit}(f) \;:=\; \mathsf{inflow}(\triangle, f) + \mathsf{inflow}(\triangle, f) - \mathsf{outflow}(\triangle, f) - \mathsf{outflow}(\triangle, f). \tag{10.3.3}$$

More explicitly,

$$\mathrel{\diamondsuit}(f) = f(\lozenge) + f(\lozenge) + f(\lozenge) + f(\lozenge) - f(\lozenge) - f(\lozenge) - f(\lozenge) - f(\lozenge).$$

From (10.3.3) it is straightforward to check that the closedness condition (10.1.4) is satisfied in the hive triangle $\triangle$.

Therefore, the flow class $\tilde{f} \in \overline{F}(G)$ is well defined by (10.3.3). So we have defined the linear map

$$\pi \colon F(R) \to \overline{F}(G), \;\; f \mapsto \tilde{f}$$

which moreover maps integral flows to integral flows.

Let us stress that we are only interested in the *cone* $K(R)$ *of nonnegative flows* on R. We define the *slack* $\sigma(\varrho, f)$ of rhombus ϱ with respect to a flow $f \in K(R)$ by $\sigma(\varrho, f) := \sigma(\varrho, \pi(f))$. Similarly, we define the *throughput* $\mathrel{\diamondsuit}(f) := \mathrel{\diamondsuit}(\pi(f))$ of f through an edge k, and we call $\delta(f) := \delta(\pi(f))$ the *overall throughput* $\delta(f)$ of $f \in K(R)$.

For the sake of clarity, paths and cycles in R shall be called *turnpaths* and *turncycles*. Correspondingly, we have the notions of s-t-turnpaths and t-s-turnpaths. By a *complete turnpath* we understand an s-t-turnpaths, a t-s-turnpaths, or a turncycle (which may pass through s or t). A complete turnpath p defines a flow on R, again denoted by p, by putting the flow value of 1 on each turnedge used. A turnpath or turncycle is called *proper*, if it neither uses s nor t, cp. Subsection 10.1 (A).

If a turnpath or turncycle uses only at most one turnvertex in each hive triangle, then it is is called a *planar turnpath* or *planar turncycle*, respectively. There is a canonical bijection between the set of paths on G and the set of planar turnpaths, which is defined via identifying each turn with its corresponding turnvertex. Proper paths on G correspond to proper planar turnpaths under this bijection.

10.3.4 Example. 1. The flow $\pi(p)$ induced by a complete turnpath p in R is not necessarily given by a complete path on G. E.g., it is possible that p uses both turnedges $\lozenge$ and $\lozenge$ (which do not share a turnvertex). Then $\mathrel{\diamondsuit}(\pi(p)) = 2$, while for a complete path q of G we always have $\mathrel{\diamondsuit}(q) \in \{-1, 0, 1\}$. ■

If p is a complete turnpath in R and x is a turnvertex or turnedge, then it will be convenient to write $\mathbb{1}_p(x) = 1$ if x occurs p, and $\mathbb{1}_p(x) = 0$ otherwise.

We reconsider now Definition 10.2.2 and interpret the sets $\Psi_+(\Diamond)$, $\Psi_-(\Diamond)$, and $\Psi_0(\Diamond)$ of slack contributions of a rhombus $\Diamond$ —instead of subsets of $E(G)$— as subsets of $V(R) \cup E(R)$. Note that these sets are pairwise disjoint.

10.3.5 Lemma. *Let p be a complete turnpath in R. Then we have for any rhombus ϱ,*

$$\sigma(\varrho, p) = \sum_{x \in \Psi_+(\varrho)} \mathbb{1}_p(x) - \sum_{x \in \Psi_-(\varrho)} \mathbb{1}_p(x).$$

Moreover, $\sigma(\varrho, p) \in \{-4, -3, \dots, 3, 4\}$. □

Proof. For each rhombus $\Diamond$ we have in pictorial notation $\mathbb{1}_p(\Diamond) + \mathbb{1}_p(\Diamond) = \mathbb{1}_p(\Diamond) + \mathbb{1}_p(\Diamond)$. Moreover,

$$\begin{aligned}
\sigma(\Diamond, p) &= \Diamond(p) + \Diamond(p) \\
&= \big(\mathbb{1}_p(\Diamond) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \big(\mathbb{1}_p(\Diamond) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) \\
&= \big(\mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) + \big(\mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) \\
&= \big(\mathbb{1}_p(\Diamond) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) \\
&\qquad + \big(\mathbb{1}_p(\Diamond) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) \\
&= \big(\mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) + \big(\mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond)\big) + \mathbb{1}_p(\Diamond) - \mathbb{1}_p(\Diamond) \\
&= \sum_{x \in \Psi_+(\Diamond)} \mathbb{1}_p(x) - \sum_{x \in \Psi_-(\Diamond)} \mathbb{1}_p(x).
\end{aligned}$$

The assertion on the possible values of $\sigma(\varrho, p)$ follows immediately. □

10.3.6 Example. There are complete turnpaths p and q, that use all the turnvertices in $\Psi_+(\varrho)$ and $\Psi_-(\varrho)$, respectively, resulting in the slacks $\sigma(\rho, p) = 4$ and $\sigma(\rho, q) = -4$. ■

We construct now the digraph R_f from R by deleting the negative slack contributions in f-flat rhombi, and removing all edges of R crossing capacity achieving edges of Δ at the border of Δ. Recall from Definition 10.1.10 the definition of the throughput capacities $b(k)$ of the border edges of Δ, given by λ, μ, ν.

10.3.7 Definition. Let $f \in B(\lambda, \mu, \nu)$. The *residual digraph* $R_f := R_f(\lambda, \mu, \nu)$ is obtained from R by deleting the turnvertices and turnedges in $\Psi_-(\varrho)$ in f-flat rhombi ϱ. Moreover, for all edges k on the right and bottom border of Δ satisfying $\delta(k, f) = b(k)$, we delete all four edges of R crossing k. Similarly, for all edges k' on the left border of Δ satisfying $-\delta(k', f) = b(k')$, we delete all four edges of R crossing k'. Let $\mathcal{P}_f$ denote the set of *complete turnpaths in R_f*. ■

The following is an immediate consequence of the construction of R_f and Lemma 10.3.5.

10.3.8 Lemma. *We have $\sigma(\varrho, p) \geq 0$ for any $p \in \mathcal{P}_f$ and any f-flat rhombus ϱ.* □

We denote by $K(R_f)$ the cone of nonnegative flows on R_f. As R_f is a subgraph of R, a nonnegative flow $f \in K(R_f)$ can be interpreted as a flow $f \in K(R)$ with value zero on the turnedges not present in R_f, making $K(R_f)$ a subcone of $K(R)$.

Let $f \in B_{\mathbb{Z}}$ and p be an s-t-turnpath in R_f. Do we have $f + \pi(p) \in B_{\mathbb{Z}}$?

By construction of R_f, if p crosses the border edge k, then $\delta(k, f) < b(k)$ if k is on the right or bottom border of Δ. Similarly, $-\delta(k', f) < b(k')$ if k' is on the left border of Δ. Thus the flow $f + \pi(p)$ does not violate the border capacity constraints.

In order to see whether $f + \pi(p)$ is a hive flow, we note that if ϱ is an f-flat rhombus, then $\sigma(\varrho, f + \pi(p)) = \sigma(\varrho, f) + \sigma(\varrho, \pi(p)) = \sigma(\varrho, p) \geq 0$ by Lemma 10.3.8. However, for rhombi ϱ that are not f-flat, it may be that $\sigma(\varrho, f) + \sigma(\varrho, \pi(p)) < 0$. Fortunately, it turns out that if p is an s-t-turnpath of minimal length, then this cannot happen!

The proof of the following result is astonishingly delicate and postponed to Section 12.2.

10.3.9 Theorem (Shortest Turnpath Theorem)**.** *Let $f \in B_{\mathbb{Z}}$ and let p be a shortest s-t-turnpath in R_f. Then $f + \pi(p) \in B_{\mathbb{Z}}$.*

To investigate in a more general context to what extent the hive conditions are preserved when adding a flow $d \in \overline{F}(G)$ to $f \in B$, we make the following definition, extending Definition 10.2.8.

10.3.10 Definition. For a hive flow $f \in B$, a flow $d \in \overline{F}(G)$ is called *f-hive preserving* if $f + \varepsilon d \in B$ for sufficiently small $\varepsilon > 0$. ■

We note that the set of f-hive preserving flows forms a cone, which was called "cone of feasible directions" in [BI09].

10.3.11 Lemma. *Let $f \in B$ and $d' \in K(R_f)$. Then $\pi(d')$ is f-hive preserving.*

Proof. According to Lemma 10.1.1, there are complete turnpaths $p_1, \ldots, p_m \in \mathcal{P}_f$ and $\alpha_1, \ldots, \alpha_m \geq 0$ such that $d' = \sum_{i=1}^{m} \alpha_i p_i$. Lemma 10.3.8 tells us that $\sigma(\varrho, p_i) \geq 0$ if ϱ is f-flat.

By construction of R_f, if p_i crosses a border edge k on the right or bottom side of Δ, then $\delta(k, f) < b(k)$. This implies that $\delta(k, f + \varepsilon d') = \delta(k, f) + \varepsilon \sum_i \alpha_i \delta(k, p_i) < b(k)$ for sufficiently small $\varepsilon > 0$. The argument is analogous for the left border edges.

We show now that $f + \varepsilon\pi(d')$ is a hive flow for sufficiently small $\varepsilon > 0$. By the linearity of the slack, this means to show that for all rhombi ϱ, we have $\sigma(\varrho, f) + \varepsilon \sum_i \alpha_i \sigma(\varrho, p_i) \geq 0$ for sufficiently small $\varepsilon > 0$. In the case $\sigma(\varrho, f) > 0$, this is obvious. On the other hand, if $\sigma(\varrho, f) = 0$, this follows from $\sigma(\varrho, p_i) \geq 0$. □

10.3 (B) Flatspaces

Our goal here is to get a detailed understanding of how turnpaths in R_f behave. For this, we first have to recall the concept of flatspaces from [KT99, Buc00]. In the following we fix $f \in B$.

By a *convex set L in the triangular graph* Δ we shall understand a union of hive triangles which is convex. It is obvious that the angles at the corners of a convex set L are either acute of 60° or obtuse of 120°.

We call two hive triangles *adjacent* if they share a side and form an f-flat rhombus. This defines a graph whose vertices are the hive triangles. An *f-flatspace* is defined to be a connected component of this graph, see Figure 10.1.ii. We simply write *flatspace* if it is clear, which flow f is meant. Also, we will identify flatspaces with the union of their hive triangles.

The following was observed in [Buc00].

10.3.12 Lemma. *(1) f-flatspaces are convex sets.*

(2) A side of an f-flatspace is either on the border of Δ, or it is also a side of a neighboring flatspace.

(3) There are exactly five types of convex sets: triangles, parallelograms, trapezoids, pentagons and hexagons.

Proof. The first and second claim follow from Corollary 10.2.11. The third claim is just the enumeration of convex shapes on the triangular grid. □

We show next that turnpaths in R_f can move in f-flatspaces only in a very limited way: namely along the border in counterclockwise direction, and they can enter and leave the flatspace only through a few distinguished edges that we define next (cf. Figure 10.3.ii).

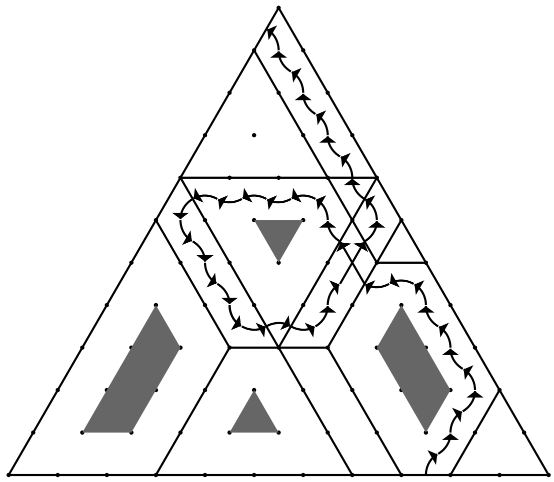


Figure 10.3.ii: The same situation as in Figure 10.1.ii. The inner triangles are shaded while the border triangles are white. An exemplary turnpath in R_f is also depicted, where the two straight arrows represent crossing turnedges.

Let L be a convex set and a be one of its sides. For $k \in E(\Delta)$ we write $k \subseteq a$ to express that k is contained in a. Let $k_1, \dots, k_r$ denote the edges contained in a in clockwise order. We call $a_{\to L} := k_1$ the *L-entrance edge* of a and $a_{L\to} := k_r$ the *L-exit edge* of a. We may have $r = 1$ in which case the entrance and exit edges coincide. Note that if M is a convex set adjacent to L, sharing with it the joint side a, then the M-entrance edge of a is at the same time the L-exit edge of L, that is, $a_{\to M} = a_{L\to}$.

The hive triangles in a convex set L either touch the border of L or lie inside L. Correspondingly, we will speak about *border triangles* and *inner triangles* of L. Recall from Definition 10.3.7 the set $\mathcal{P}_f$ of complete turnpaths in R_f.

10.3.13 Proposition. *Let $p \in \mathcal{P}_f$ and L be an f-flatspace. Then:*

(1) p can enter L only by crossing entrance edges of L. Similarly, p can leave L only by crossing exit edges of L.

(2) p uses only turnvertices in border triangles of L and traverses the border of L in counterclockwise direction.

Proof. We call turnvertices, which lie in L and start at entrance edges of L, *entrance turnvertices*. Diagonals of non-f-flat rhombi shall be called *dividers*.

(a) If p enters L with a counterclockwise turn , then must be a divider. Hence is an entrance turnvertex.

(b) If p enters L with a clockwise and a counterclockwise turn , then is a divider and hence is an entrance turnvertex.

(c) If p enters L with two clockwise turns , then is a divider and hence is an entrance turnvertex.

Analogous arguments hold for exits with the situations , and . This proves the first assertion.

We now show the second assertion. Consider an inner triangle △. All rhombi in the shaded area △ are f-flat. By the definition of R_f, the counterclockwise turnvertices △, △, and △ are not vertices of R_f. For the same reason, the clockwise turnvertices △, △, and △ have no incident turnedge in R_f. This shows that turnpaths and turncycles in R_f can only use turnvertices in border triangles.

Finally, the fact that a counterclockwise turn in p implies that is a divider, shows that p traverses the border triangles of f-flatspaces in counterclockwise direction. □

Let $d \in \overline{F}(G)$ be a flow and $k \in E(G)$ be an edge lying at the border of a convex set L. If the hive triangle in L having the side k is upright, we define $\delta(k, \to L, d) := \delta(k, f)$, otherwise we set $\delta(k, \to L, d) := -\delta(k, f)$. We call $\delta(k, \to L, d)$ the *throughput of d into L through k*. It will be convenient to call $\delta(k, L\to, d) := -\delta(k, \to L, d)$ the *throughput of d out of L through k*.

Note that if the convex sets L and M are adjacent, sharing an edge k, then $\delta(k, \to M, d) = \delta(k, L\to, d)$.

For some of the following properties of throughputs compare Figure 10.1.ii.

10.3.14 Lemma. *Let L be a convex set contained in an f-flatspace and a be a side of L. Further, let $k_1, \dots, k_r \in E(\Delta)$ be the edges contained in a in clockwise order. Then $\delta(k_1, \to L, f) = \ldots = \delta(k_r, \to L, f)$. Moreover, if $d \in \overline{F}(G)$ is f-hive preserving, then $\delta(k_1, \to L, d) \geq \ldots \geq \delta(k_r, \to L, d)$.*

Proof. It is sufficient to show this for adjacent edges $k_1 =$ ◇ and $k_2 =$ ◇, where the rhombi ◇ and ◇ are f-flat. Since $0 = \sigma(◇, f) =$ ◇$(f) +$ ◇(f) and $0 = \sigma(◇, f) =$ ◇$(f) +$ ◇(f), it follows ◇$(f) =$ ◇(f).

We have $\sigma(◇, d) \geq 0$ and $\sigma(◇, d) \geq 0$ as d is f-hive preserving and ◇ and ◇ are f-flat. The second statement follows now similarly as before. □

10.3.15 Observation. Let L be an f-flatspace with a side a lying on the left border of Δ. Then the maximum of the capacities $b(k)$ (cf. Definition 10.1.10) over all edges $k \subseteq a$ is attained at the exit edge of a. An analogous statement holds for the right and bottom border and entrance edges.

Proof. This follows directly from the fact that $\nu_1 \geq \ldots \geq \nu_n$ and the definition of the throughput capacities $b(k)$ of the border edges k of Δ, cf. Figure 10.1.i(c). Similarly for λ and μ. □

It will be important to decompose the throughput $\delta(k, \to L, d)$ into its positive and negative part. Recall that $\delta(k, \to L, d) = -\delta(k, L\to, d)$.

10.3.16 Definition. Let $d \in \overline{F}(G)$, L be an f-flatspace, and $k \in E(\Delta)$ be an edge at the border of L. The *L-inflow of d through k* and *L-outflow of d through k* are defined as

$$\omega(k, \to L, d) := \max\{\delta(k, \to L, d), 0\}, \quad \omega(k, L\to, d) := \max\{\delta(k, L\to, d), 0\}.$$

Further, for a side a of L, we define the *L-inflow of d through a* and the *L-outflow of d through a* by

$$\omega(a, \to L, d) := \sum_{k \subseteq a} \omega(k, \to L, d), \quad \omega(a, L\to, d) := \sum_{k \subseteq a} \omega(k, L\to, d).$$

We write $\omega(a, \to\Delta, d) := \omega(a, \to L, d)$ and $\omega(a, \Delta\to, d) := \omega(a, L\to, d)$ if the side a is on the border of Δ. ■

Note that $\delta(k, \to L, d) = \omega(k, \to L, d) - \omega(k, L\to, d)$. Further, if L and M are adjacent convex sets sharing a side a, then $\omega(k, \to L, d) = \omega(k, M\to, d)$ for $k \subseteq a$ and hence

$$\omega(a, \to L, d) = \omega(a, M\to, d). \tag{10.3.17}$$

The partition of Δ into f-flatspaces leads to a partition of the border of Δ. Let $\mathcal{S}_f$ denote the set of sides of f-flatspaces that lie on the right or bottom border of Δ.

10.3.18 Lemma. *For $f \in B$ and $d \in \overline{F}(G)$ we have*

$$\delta(d) = \sum_{a \in \mathcal{S}_f} \big(\omega(a, \to\Delta, d) - \omega(a, \Delta\to, d)\big).$$

Proof. By the definition of the overall throughput, and since s is connected in G only to the vertices on the right or bottom border of Δ, we have

$$\delta(d) \;=\; \sum_{e_{\mathsf{start}}=s} d(e) - \sum_{e_{\mathsf{end}}=s} d(e) \;=\; \sum_{k} \delta(k, \to\Delta, d),$$

where the right-hand sum is over all edges $k \in E(\Delta)$ on the right or bottom border of Δ. Recall that $\delta(k, \to\Delta, d) = \omega(k, \to\Delta, d) - \omega(k, \Delta\to, d)$. By Definition 10.3.16,

$$\sum_{k} \omega(k, \to\Delta, d) \;=\; \sum_{a \in \mathcal{S}_f} \sum_{k \subseteq a} \omega(k, \to\Delta, d) \;=\; \sum_{a \in \mathcal{S}_f} \omega(a, \to\Delta, d).$$

Similarly, $\sum_k \omega(k, \Delta\to, d) = \sum_{a \in \mathcal{S}_f} \omega(a, \Delta\to, d)$ and the assertion follows. □

10.3 (C) The Rerouting Theorem

We fix $f \in B$. Recall the set $\mathcal{P}_f$ of complete turnpaths in R_f from Definition 10.3.7. Let $\mathcal{P}_{st}$, $\mathcal{P}_{ts}$, and $\mathcal{P}_c$ denote the sets of s-t-turnpaths, t-s-turnpaths, and turncycles in R_f, respectively. Then we have the disjoint decomposition $\mathcal{P}_f = \mathcal{P}_{st} \cup \mathcal{P}_{ts} \cup \mathcal{P}_c$. Note that every $p \in \mathcal{P}_{st}$ enters Δ through exactly one edge on the right or bottom side of Δ, and leaves Δ through exactly one edge on the left side of Δ (otherwise s or t would be used more than once). Similarly, every $p \in \mathcal{P}_{ts}$ enters Δ through exactly one edge on the left side of Δ and leaves Δ through the right or bottom side of Δ. The reader should also note that turncycles $p \in \mathcal{P}_c$ may pass through s or t (or both of them).

10.3.19 Definition. A *weighted family φ of complete turnpaths* in R_f is defined as a map $\varphi\colon \mathcal{P}_f \to \mathbb{R}_{\geq 0}$. If φ takes values in $\mathbb{N}$, we call φ a *multiset of complete turnpaths* in R_f. In this case, we interpret $\varphi(p)$ as the multiplicity with which p occurs in the multiset φ. ■

A weighted family φ of complete turnpaths in R_f defines the nonnegative flow $\sum_{p \in \mathcal{P}_f} \varphi(p) p$ in R_f. On the other hand, by Lemma 10.1.1, any nonnegative flow $d' \in F_+(R_f)$ can be written in this form.

The flow $d' := \sum_p \varphi(p) p$ on R_f defined by the weighted family φ satisfies

$$\delta(d') \;=\; \sum_{p \in \mathcal{P}_{st}} \varphi(p) - \sum_{p \in \mathcal{P}_{ts}} \varphi(p). \tag{10.3.20}$$

To motivate the next definition, recall from Proposition 10.3.13 that a complete turnpath $p \in \mathcal{P}_f$ can enter an f-flatspace L only through an entrance edge $a_{\to L}$ of a side a of L, and leave only through an exit edge $a_{L\to}$.

10.3.21 Definition. Let a be a side of an f-flatspace L. We denote by $\mathcal{P}_f(\to L, a)$ the set of $p \in \mathcal{P}_f$ that enter L through the edge $a_{\to L}$. The set $\mathcal{P}_f(L\to, a)$ denotes the set of $p \in \mathcal{P}_f$ that leave L through the edge $a_{L\to}$. For a weighted family φ of complete turnpaths and an f-flatspace L we define the *entrance weight* and the *exit weight* of a side a of L as follows:

$$\omega(a, \to L, \varphi) \; := \sum_{p \in \mathcal{P}_f(\to L, a)} \varphi(p), \quad \omega(a, L\to, \varphi) \; := \sum_{p \in \mathcal{P}_f(L\to, a)} \varphi(p).$$

If the side a is on the border of Δ we write $\omega(a, \to\Delta, \varphi) := \omega(a, \to L, \varphi)$, $\omega(a, \Delta\to, \varphi) := \omega(a, L\to, \varphi)$, and $\mathcal{P}_f(\to\Delta, a) := \mathcal{P}_f(\to L, a)$. ■

The following remarkable result tells us that for any f-hive preserving flow $d \in \overline{F}(G)$, there is a weighted family φ of complete turnpaths such that the inflows and outflows of d through the sides a of the f-flatspaces are given by the entrance weight and exit weight of φ through a, respectively.

10.3.22 Theorem (Rerouting Theorem)**.** *Let $f \in B$ and $d \in \overline{F}(G)$ be f-hive preserving. Then there exists a weighed family φ of complete turnpaths in R_f such that $\omega(a, \to L, d) = \omega(a, \to L, \varphi)$ and $\omega(a, L\to, d) = \omega(a, L\to, \varphi)$ for all f-flatspaces L and all sides a of L. If d is integral, then we may assume that φ is a multiset.*

Let us draw an immediate consequence.

10.3.23 Corollary. *If an f-hive preserving flow d with zero throughput on the border of Δ has nonzero throughput through an edge of side a of an f-flatspace, then there exists a turncycle $\bar{c}$ in R_f that crosses a.*

Another corollary is the following.

10.3.24 Corollary. *Under the assumptions of Theorem 10.3.22, the nonnegative flow $d' := \sum_{p \in \mathcal{P}_f} \varphi(p)p$ on R_f satisfies $\delta(d') = \delta(d)$.*

Proof. Recall that $\mathcal{S}_f$ denotes the set of sides of f-flatspaces that lie on the right or bottom border of Δ. We have

$$\begin{aligned}\sum_{p \in \mathcal{P}_{st}} \varphi(p) + \sum_{\substack{p \in \mathcal{P}_c \\ s \in p}} \varphi(p) &= \sum_{a \in \mathcal{S}_f} \sum_{p \in \mathcal{P}_f(\to\Delta, a)} \varphi(p) \\ &= \sum_{a \in \mathcal{S}_f} \omega(a, \to\Delta, \varphi) = \sum_{a \in \mathcal{S}_f} \omega(a, \to\Delta, d),\end{aligned}$$

where we have used Theorem 10.3.22 for the last equality. Similarly,

$$\sum_{p \in \mathcal{P}_{ts}} \varphi(p) + \sum_{\substack{p \in \mathcal{P}_c \\ s \in p}} \varphi(p) \;=\; \sum_{a \in \mathcal{S}_f} \sum_{p \in \mathcal{P}_f(\Delta\to, a)} \varphi(p) \;=\; \sum_{a \in \mathcal{S}_f} \omega(a, \Delta\to, d).$$

Subtracting and using (10.3.20) we get

$$\delta(d') \;=\; \sum_{p \in \mathcal{P}_{st}} \varphi(p) - \sum_{p \in \mathcal{P}_{ts}} \varphi(p) \;=\; \sum_{a \in \mathcal{S}_f} \omega(a, \to\Delta, d) - \sum_{a \in \mathcal{S}_f} \omega(a, \Delta\to, d) \;=\; \delta(d),$$

where we have used Lemma 10.3.18 for the last equality. □

The proof of the Rerouting Theorem 10.3.22 is postponed to Section 12.1. The rough idea of the proof is to define a notion of canonical turnpaths within a convex set L, that specializes to the complete turnpaths in R_f restricted to L, in case L is an f-flatspace. We shall successively cut L into convex subsets by straight lines and recursively build up the required canonical turnpaths by operations of concatenation and straightening.

10.3.25 Remark. For given $f \in B$, let C_f denote the cone of f-hive preserving flows on G. Lemma 10.3.11 states that $\pi(K(R_f)) \subseteq C_f$. Using Proposition 10.3.13, it is easy to see that this inclusion may be strict for some $f \in B$. On the other hand, one can show that $\pi(K(R_f)) = C_f$ if the hive triangles and rhombi are the only f-flatspaces. Hive flows f satisfying the latter property were called *shattered* in [Ike08, BI09]. We note that if f is shattered, then the Rerouting Theorem is not needed, and hence the optimality criterion stated in Proposition 11.1.1 below is much easier to prove. ∎

Chapter 11

Algorithms

In this chapter we develop the algorithms for (1) deciding $c^{\nu}_{\lambda\mu} > 0$ in time $\mathcal{O}\big(\ell(\nu)^3 \log \nu_1\big)$, see Theorem 11.2.4 and (2) deciding $c^{\nu}_{\lambda\mu} \geq t$ in time $\mathcal{O}\big(t^2 \cdot \mathsf{poly}(n)\big)$, see Theorem 11.3.2. Moreover, the second algorithm can be used for computing $c^{\nu}_{\lambda\mu}$ in time $\mathcal{O}\big((c^{\nu}_{\lambda\mu})^2 \cdot \mathsf{poly}(n)\big)$, see Theorem 11.3.3.

All algorithms that we present only use additions, multiplications, and comparisons and the running time is defined to be the number of these operations. Moreover, the occuring numbers are all polynomially bounded in bitsize, so our algorithms are efficient in the bit model as well.

11.1 A First Max-flow Algorithm

Fix partitions λ, μ, ν such that $|\nu| = |\lambda| + |\mu|$. Recall from Definition 10.1.10 the polytope $B = B(\lambda, \mu, \nu)$ of bounded hive flows and the linear overall throughput function $\delta \colon B \to \mathbb{R}$. Moreover, recall that $\delta(f) \leq |\nu|$ for all $f \in B$ and that a hive flow $f \in B$ is capacity achieving iff $\delta(f) = |\nu|$. The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ counts the number of integral capacity achieving hive flows. Therefore, in order to decide positivity of $c^{\nu}_{\lambda\mu}$ we shall maximize the overall throughput function δ on the polytope B. (We will see in Corollary 11.1.3 that we can always find an integral optimum.) In order to implement this idea, we need a criterion that tells us when $f \in B$ is optimal with respect to δ.

11.1.1 Proposition (Optimality Criterion)**.** *Let $f \in B$. Then $\delta(f) = \max_{g \in B} \delta(g)$ iff there exists no s-t-turnpath in R_f.*

Proof. We call $f \in B$ *optimal* iff $\delta(f) = \max_{g \in B} \delta(g)$.

If p is an s-t-turnpath in R_f, then by Lemma 10.3.11, we have $f + \varepsilon\pi(p) \in B$ for some $\varepsilon > 0$. Since $\delta(f + \varepsilon p) = \delta(f) + \varepsilon > \delta(f)$, the flow f is not optimal.

Now suppose that f is not optimal and let $g \in B$ such that $\delta(g) > \delta(f)$. Clearly, $d \coloneqq g - f$ is f-hive preserving and satisfies $\delta(d) > 0$. Let φ be the weighted family of complete turnpaths corresponding to d as provided by the Rerouting Theorem 10.3.22, and put $d' \coloneqq \sum_p \varphi(p)p$. Corollary 10.3.24 shows that $\delta(d') = \delta(d) > 0$ and (10.3.20) implies that there exists an s-t-turnpath in R_f. □

Consider the following Algorithm 1 for deciding positivity of Littlewood-Richardson coefficients.

Algorithm 1 LR-Positivity (slow version)

Input: partitions λ, μ, ν with $|\nu| = |\lambda| + |\mu|$.
Output: TRUE, if $c^{\nu}_{\lambda\mu} > 0$. **FALSE** otherwise.

1: $f \leftarrow 0$.
2: **while** there is a shortest s-t-turnpath p in R_f **do**
3: $\quad f \leftarrow f + \pi(p)$.
4: **end while**
5: **return** whether $\delta(f) = |\nu|$.

11.1.2 Theorem. *Algorithm 1 returns whether* $c^{\nu}_{\lambda\mu} > 0$.

Proof. Clearly f stays integral during the run of Algorithm 1. The Shortest Turnpath Theorem 10.3.9 ensures that during the run of Algorithm 1 we always have $f \in B_{\mathbb{Z}}$. If Algorithm 1 returns **TRUE**, then we know that the final value of f is an integral and capacity achieving hive flow in B. Hence Proposition 10.1.11 implies $c^{\nu}_{\lambda\mu} > 0$.

On the other hand, if Algorithm 1 returns **FALSE**, we have $\delta(f) < |\nu|$ and according to Proposition 11.1.1, the flow f has the maximum value of δ among all flows in B. Hence there is no capacity achieving flow in B and Proposition 10.1.11 implies that $c^{\nu}_{\lambda\mu} = 0$. □

We note the following important integrality property.

11.1.3 Corollary. *For all* λ, μ, ν*, the overall throughput function* δ *attains the maximal value on* $B(\lambda, \mu, \nu)$ *at an integer flow.*

Proof. In the last line executed by Algorithm 1, there exists no s-t-turnpath in R_f. Hence, by Proposition 11.1.1, the integral flow f has the maximal value on B. □

As an application of the foregoing, we deduce here the saturation property of the Littlewood-Richardson coefficients, which was first shown in [KT99].

11.1.4 Corollary. $c^{N\nu}_{N\lambda N\mu} > 0$ *for some* $N \geq 1$ *implies* $c^{\nu}_{\lambda\mu} > 0$.

Proof. If $c^{N\nu}_{N\lambda N\mu} > 0$, then there exists an integral capacity achieving hive flow $f \in B(N\lambda, N\mu, N\nu)$, by Proposition 10.1.11 . Hence $\frac{f}{N} \in B(\lambda, \mu, \nu)$ satisfies $\delta(\frac{f}{N}) = |\nu|$ and maximizes δ on $B(\lambda, \mu, \nu)$. Even though $\frac{f}{N}$ may not be integral, Corollary 11.1.3 implies that there exists an integral optimal hive flow $g \in B(\lambda, \mu, \nu)$ such that $\delta(g) = |\nu|$. Hence $c^{\nu}_{\lambda\mu} > 0$ by Proposition 10.1.11. □

11.2 A Polynomial Time Decision Algorithm

In this section we use the capacity scaling approach (see, e.g., [AMO93, ch. 7.3]) to turn Algorithm 1 into a polynomial-time algorithm. During this method, $f \in B$ stays 2^{ℓ}*-integral*, for $\ell \in \mathbb{N}$, which means that all flow values are an integral multiple of 2^{ℓ}. The incrementation step in Algorithm 1, line 3, is replaced by adding $2^{\ell}\pi(p)$. Further, ℓ is decreased in the course of the algorithm. So our algorithm at first makes big increments which over time decrease.

To implement this idea, we will search for a shortest s-t-turnpath in the subgraph R^{ℓ}_f of R_f defined next. By construction we will have $R^0_f = R_f$. Recall that the polytope $B = B(\lambda, \mu, \nu)$ has the border capacity constraints as in Definition 10.1.10.

11.2.1 Definition. Let $\ell \in \mathbb{N}$ and let $f \in B$ be 2^{ℓ}-integral. The digraph R^{ℓ}_f is obtained from R_f by deleting all four edges crossing an edge k on the right or bottom border of Δ satisfying $\delta(k, f) + 2^{\ell} > b(k)$, and by deleting all four edges crossing an edge k' on the left border of Δ satisfying $-\delta(k', f) + 2^{\ell} > b(k')$. ■

Algorithm 2 stated below is now fully specified. Note that w.l.o.g. $\nu_1 \geq \max\{\lambda_1, \mu_1\}$, because otherwise $c^{\nu}_{\lambda\mu} = 0$, see Corollary 4.5.6.

Algorithm 2 LR-Positivity

Input: partitions λ, μ, ν with $|\nu| = |\lambda| + |\mu|$ and $\nu_1 \geq \max\{\lambda_1, \mu_1\}$.
Output: **TRUE**, if $c^{\nu}_{\lambda\mu} > 0$. **FALSE** otherwise.

```
f ← 0.
for ℓ from ⌈log ν_1⌉ down to 0 do
    while there is a shortest s-t-turnpath p in R_f^ℓ do
        f ← f + 2^ℓ π(p).
    end while
end for
return whether δ(f) = |ν|.
```

It is clear that f stays 2^ℓ-integral during the run of Algorithm 2.

11.2.2 Claim. *During the run of Algorithm 2, the flow f always is in B.*

Proof. Given a 2^ℓ-integral hive flow $f \in B = B(\lambda, \mu, \nu)$. First we note that the set of s-t-turnpaths on R^ℓ_f equals the set of s-t-turnpaths on $R_{\tilde{f}}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$, where $\tilde{f} := f/2^\ell$, $\tilde{\lambda} := \lfloor \frac{\lambda}{2^\ell} \rfloor$, $\tilde{\mu} := \lfloor \frac{\mu}{2^\ell} \rfloor$, $\tilde{\nu} := \lfloor \frac{\nu}{2^\ell} \rfloor$, and division and rounding of partitions is defined componentwise. Let p be a shortest s-t-turnpath on R^ℓ_f and hence also a shortest s-t-turnpath on $R_{\tilde{f}}(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$. According to the Shortest Turnpath Theorem 10.3.9 we have $\frac{f}{2^\ell} + \pi(p) \in B(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$. Therefore, we obtain $f + 2^\ell \pi(p) \in B(2^\ell \tilde{\lambda}, 2^\ell \tilde{\mu}, 2^\ell \tilde{\nu}) \subseteq B(\lambda, \mu, \nu) = B$. □

The last iteration of the for-loop of Algorithm 2 (where $\ell = 0$) operates like Algorithm 1 and hence Theorem 11.1.2 implies that Algorithm 2 works according to its specification.

For the analysis of the running time we need the following auxiliary result, relying on the Rerouting Theorem 10.3.22.

11.2.3 Lemma. *Let $f \in B_{\mathbb{Z}}$ be 2^ℓ-integral and $\ell \in \mathbb{N}$ be such that R^ℓ_f has no s-t-turnpath. Then $\delta_{max} - \delta(f) < 3n2^\ell$, where $\delta_{max} := \max_{g \in B} \delta(g)$.*

Proof. Let $g \in B$ with $\delta(g) = \delta_{\max}$ and put $d := g - f \in \overline{F}(G)$. Hence $\delta_{\max} - \delta(f) = \delta(d)$. Let φ be the family of complete weighted turnpaths corresponding to d as provided by the Rerouting Theorem 10.3.22. We decompose the set $\mathcal{P}_f = \mathcal{P}_{st} \cup \mathcal{P}_{ts} \cup \mathcal{P}_c$ of complete turnpaths in R_f into the sets $\mathcal{P}_{st}$, $\mathcal{P}_{ts}$, and $\mathcal{P}_c$ of s-t-turnpaths, t-s-turnpaths, and turncycles, respectively. Then the flow $d' := \sum_p \varphi(p)p$ on R_f defined by φ satisfies by (10.3.20) and Corollary 10.3.24,

$$\delta(d) \;=\; \delta(d') \;=\; \sum_{p \in \mathcal{P}_{st}} \varphi(p) - \sum_{p \in \mathcal{P}_{ts}} \varphi(p) \;\leq\; \sum_{p \in \mathcal{P}_{st}} \varphi(p). \qquad \text{(i)}$$

A turnpath $p \in \mathcal{P}_{st}$ enters Δ exactly once (through the right or bottom border) and leaves Δ exactly once (through the left border). For an edge k on the right or bottom border of Δ, let $\mathcal{P}_{st}(k)$ denote the set of $p \in \mathcal{P}_{st}$ that enter Δ through k. Further, for an edge k' on the left border of Δ, let $\mathcal{P}_{st}(k')$ denote the set of $p \in \mathcal{P}_{st}$ that leave Δ through k'.

We call an edge k on the right or bottom border of Δ *small*, if $\delta(k, f) + 2^\ell > b(k)$. Let $\mathscr{E}$ denote the set of these edges. Note that for $k \in \mathscr{E}$ we have

$$\delta(k, d) \;=\; \delta(k, g) - \delta(k, f) \;\leq\; b(k) - \delta(k, f) \;<\; 2^\ell. \qquad \text{(ii)}$$

Similarly, we call an edge $k' \in E(\Delta)$ on the left border of Δ *small*, if $-\delta(k', f)+2^\ell > b(k')$ and denote the set of these edges by $\mathscr{E}'$. Border edges that are not small are called *big*.

The point is that an s-t-turnpath $p \in \mathcal{P}_{st}$ in R_f that crosses two big edges is also an s-t-turnpath in R_f^ℓ. Hence, by our assumption, there are no s-t-turnpaths in R_f that cross two big edges. We conclude that for all $p \in \mathcal{P}_{st}$, there exists $k \in \mathscr{E} \cup \mathscr{E}'$ such that $p \in \mathcal{P}_{st}(k)$. Therefore,

$$\sum_{p \in \mathcal{P}_{st}} \varphi(p) \;\leq\; \sum_{k \in \mathscr{E}} \sum_{p \in \mathcal{P}_{st}(k)} \varphi(p) + \sum_{k' \in \mathscr{E}'} \sum_{p \in \mathcal{P}_{st}(k')} \varphi(p). \tag{iii}$$

To bound the right-hand sums, suppose first that $k \in \mathscr{E}$. By Proposition 10.3.13, $p \in \mathcal{P}_{st}(k)$ implies that k is the entrance edge of the side a of an f-flatspace L, in which case $k = a_{\to L}$. We have $\mathcal{P}_{st}(k) = \mathcal{P}_f(\to L, a) \cap \mathcal{P}_{st}$ and hence, by Definition 10.3.21,

$$\sum_{p \in \mathcal{P}_{st}(k)} \varphi(p) \;\leq\; \omega(a, \to L, \varphi) = \omega(a, \to L, d),$$

where the last equality is guaranteed by the Rerouting Theorem 10.3.22.

Lemma 10.3.14 and Observation 10.3.15 imply that, since k is a small edge, all the other edges contained in a are small as well. Note that $\delta(\tilde{k}, \to L, d) < 2^\ell$ implies $\omega(\tilde{k}, \to L, d) < 2^\ell$ for all edges $\tilde{k} \subseteq a$ by Definition 10.3.16. Therefore we can use (ii) to deduce

$$\omega(a, \to L, d) \;=\; \sum_{\tilde{k} \subseteq a} \omega(\tilde{k}, \to L, d) \;<\; |a|\, 2^\ell,$$

where $|a|$ denotes the number of edges of Δ contained in a. Summarizing, we conclude for $k \in \mathscr{E}$,

$$\sum_{p \in \mathcal{P}_f(st)} \varphi(p) \;<\; |a|\, 2^\ell,$$

where a is the side of the f-flatspace, in which k lies.

The same bound holds for $k' \in \mathscr{E}'$ by an analogous argument. Combining these bounds with (i) and (iii) we obtain

$$\delta(d) = \delta(d') \;\leq\; \sum_{p \in \mathcal{P}_{st}} \varphi(p) \;<\; 3n2^\ell$$

since there are $3n$ edges on the border of Δ. □

11.2.4 Theorem. *Algorithm 2 decides the positivity of the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ in time $\mathcal{O}\big(\ell(\nu)^3 \log \nu_1\big)$.*

Proof. Again let $\delta_{\max} := \max_{g \in B} \delta(g)$. After ending the while-loop for the value ℓ, there is no s-t-turnpath in R_f^ℓ and hence $\delta_{\max} - \delta(f) < 3n2^\ell = 6n\, 2^{\ell-1}$. Hence in the next iteration of the while-loop, for the value $\ell - 1$, at most $6n$ s-t-turnpaths can be found. Moreover, note that the initial value of ℓ is so large, that in the first iteration of the while-loop at most one s-t-turnpath can be found.

So Algorithm 2 finds at most $6n\lceil \log \nu_1 \rceil$ many s-t-turnpaths and searches at most $\log \nu_1$ many times for an s-t-turnpath without finding one. Note that searching for a shortest s-t-turnpath requires at most time $\mathcal{O}(n^2)$ using breadth-first-search, since there are $\mathcal{O}(n^2)$ turnvertices and turnedges. Hence we get a total running time of $\mathcal{O}\big(n^3 \log \nu_1\big)$. □

11.3 Enumerating Hive Flows

In this section we give a nontechnical overview of the enumeration algorithms used to compute $c^{\nu}_{\lambda\mu}$.

Recall Definition 10.2.7 of the neighborhood $\Gamma(f)$. The following theorem states that (a superset of) $\Gamma(f)$ can be efficiently enumerated. We postpone the proof of Theorem 11.3.1 to Section 11.4.

11.3.1 Theorem (Neighbourhood generator)**.** *There is an algorithm* NeighGen *which on input $f \in P_{\mathbb{Z}}$ outputs the elements of a set $\tilde{\Gamma}(f) \subseteq P_{\mathbb{Z}}$ one by one such that $\Gamma(f) \subseteq \tilde{\Gamma}(f)$. The computation of the first k elements takes time $\mathcal{O}\big(k \cdot \mathsf{poly}(n)\big)$.*

We define the directed graph $\tilde{P}_{\mathbb{Z}}$ to be the graph with vertex set $V(P_{\mathbb{Z}})$ and (possibly asymmetric) neighborhood function $\tilde{\Gamma}$ as given by Theorem 11.3.1. Since $P_{\mathbb{Z}}$ is connected by Theorem 10.2.14, $\tilde{P}_{\mathbb{Z}}$ is strongly connected. Therefore, breadth-first-search on $\tilde{P}_{\mathbb{Z}}$ started at any hive flow in $P_{\mathbb{Z}}$ visits all flows in $P_{\mathbb{Z}}$. A variant of this breadth-first-search is realized in the following Algorithm 3, which gets an additional threshold parameter t, so that Algorithm 3 visits at most t flows.

Algorithm 3 LR-Threshold

Input: partitions λ, μ, ν with $|\lambda| + |\mu| = |\nu|$; $t \in \mathbb{N}_{>0}$
Output: **TRUE**, if $c^{\nu}_{\lambda\mu} \geq t$. **FALSE** otherwise.

```
1: If P(λ, μ, ν) = ∅, return FALSE.
2: Compute an integral hive flow f ∈ P(λ, μ, ν)_Z.
3: Initially, set S ← {f}, T ← ∅.
4: while T ⊊ S do
5:    Choose an f ∈ S \ T.
6:    for each g ∈ Γ̃(f) generated one by one by NeighGen via Thm. 11.3.1 do
7:       S ← S ∪ {g}.
8:       If |S| ≥ t, then return TRUE.
9:    end for
10:   T ← T ∪ {f}.
11: end while
12: return FALSE.
```

The first two lines of Algorithm 3 deal with computing a hive flow $f \in P_{\mathbb{Z}}$ if there exists one. This can be done in time strongly polynomial in n using Tardos' algorithm [Tar86, GLS93] as stated in [MS05] and [DLM06]. We can also use Algorithm 2 for this purpose, which is especially designed for this problem, but note that although it has a much smaller exponent in the running time, its running time depends on the bitsize of the input partitions. To achieve so-called strongly polynomial running time, we focus on algorithms whose number of arithmetic operations and comparisons does not depend on the input bitsize. If Tardos' algorithm is used as a subalgorithm in Algorithm 3, then this is the case and hence we choose this option. In the bit-model however, Algorithm 2 is much faster than Tardos' algorithm.

11.3.2 Theorem. *Given partitions λ, μ, ν with $|\lambda|+|\mu| = |\nu|$ and a natural number $t \geq 1$, then Algorithm 3 decides $c^{\nu}_{\lambda\mu} \geq t$ in time $\mathcal{O}\big(t^2 \cdot \mathsf{poly}(n)\big)$.*

Proof. Recall that according to Proposition 10.1.11 we have $|P_{\mathbb{Z}}| = c^{\nu}_{\lambda\mu}$. Now observe that, starting after line 3, Algorithm 3 preserves the three invariants $T \subseteq S \subseteq P_{\mathbb{Z}}$, $|S| \leq t$ and $\forall f \in T : \tilde{\Gamma}(f) \subseteq S$.

If the algorithm returns **TRUE**, then $|S| \geq t$. As $S \subseteq P_{\mathbb{Z}}$ and $|P_{\mathbb{Z}}| = c^{\nu}_{\lambda\mu}$, we have $c^{\nu}_{\lambda\mu} \geq t$.

If the algorithm returns **FALSE**, then $|S| < t$ and $S = T$. Moreover, $\tilde{\Gamma}(f) \subseteq S$ for all $f \in S$. Since the digraph $\tilde{P}_{\mathbb{Z}}$ is strongly connected, it follows that $P_{\mathbb{Z}} = S$. Therefore we have $c^{\nu}_{\lambda\mu} = |P_{\mathbb{Z}}| = |S| < t$.

We have shown that the algorithm works correctly.

Now we analyze its running time. Recall that the first two lines of Algorithm 3 run in time $\mathsf{poly}(n)$, because of Tardos' algorithm. The outer loop runs at most t times, because in each iteration, $|T|$ increases and $|T| \leq |S| \leq t$. If in the inner loop we have $|\tilde{\Gamma}(f)| < t$, then the inner loop runs for at most $t-1$ iterations and hence $\tilde{\Gamma}(f)$ can be generated in time $\mathcal{O}\big(t \cdot \mathsf{poly}(n)\big)$ via Theorem 11.3.1. If in the inner loop we have $|\tilde{\Gamma}(f)| \geq t$, then after t iterations we have $|S| \geq t$ and the algorithm returns immediately. The first t elements of $\tilde{\Gamma}(f)$ can be generated via Theorem 11.3.1 in time $\mathcal{O}\big(t \cdot \mathsf{poly}(n)\big)$. Therefore we get an overall running time of $\mathcal{O}\big(t^2 \cdot \mathsf{poly}(n)\big)$. □

11.3.3 Theorem. *Given partitions $|\lambda| + |\mu| = |\nu|$. Then $c^{\nu}_{\lambda\mu}$ can be computed in time $\mathcal{O}\big((c^{\nu}_{\lambda\mu})^2 \cdot \mathsf{poly}(n)\big)$ by a variant of Algorithm 3.*

Proof. Use Algorithm 3 with the input $t = \infty$ as a formal symbol, but instead of returning **FALSE** in line 1, return 0, and instead of returning **FALSE** in line 12, return $|S|$. Note that the algorithm never returns **TRUE**, because "$|S| > \infty$" in line 8 is always false. If the algorithm terminates, then $P_{\mathbb{Z}} = S$ and thus the algorithm works correctly. Note that if started with $t = \infty$ the algorithm behaves exactly as if started with $t = c^{\nu}_{\lambda\mu} + 1$. Thus it runs in time $\mathcal{O}\big((c^{\nu}_{\lambda\mu})^2 \cdot \mathsf{poly}(n)\big)$. □

11.4 The Neighbourhood Generator

This section is devoted to the proof of Theorem 11.3.1 by describing and analyzing the algorithm NeighGen. This algorithm is inspired by the binary partitioning method used in [FM94].

The following definition of secure turnpaths is related to the Definition 10.2.8 of secure cycles.

11.4.1 Definition (Secure turnpaths)**.** A proper turnpath p on R_f is called *f-secure*, if p is planar and if additionally p does not use both counterclockwise turn-vertices ♢ and ♢ at the acute angles of any nearly f-flat rhombus ♢. We define f-secure turncycles analogously. ■

Since f-secure turncycles are planar, the bijection between the set of proper turncycles and the set of proper cycles on G induces a bijection between the set of f-secure cycles and the set of f-secure turncycles. To prove Theorem 11.3.1, we want to list all f-secure cycles (cf. Prop. 10.2.9). Hence we may as well list the f-secure turncycles.

Given $f \in P_{\mathbb{Z}}$, NeighGen prints out the elements of a set $\tilde{\Gamma}(f)$ with $\Gamma(f) \subseteq \tilde{\Gamma}(f) \subseteq P_{\mathbb{Z}}$. Note that we would like to have a direct algorithm that prints the elements of $\Gamma(f)$, but we do not know how to do this efficiently.

Although we can treat f-hive preserving turncycles algorithmically, there are problems when it comes to f-secure turncycles. In fact we do not know how to solve the following crucial Secure Extension Problem 11.4.2.

11.4.2 Problem (Secure extension problem)**.** *Given $f \in P_{\mathbb{Z}}$ and an f-secure turnpath p, decide in time $\mathsf{poly}(n)$ whether there exists an f-secure turncycle c containing p or there exists no such c.*

If in Problem 11.4.2 a turncycle c exists for a given p, then we call p *f-securely extendable*.

The usefulness of having a solution to Problem 11.4.2 will be made clear in the next subsection, where we introduce an algorithm NeighGen' that proves Theorem 11.3.1 under the assumption that Problem 11.4.2 is has a positive solution.

11.4 (A) A First Approach

Assume that $\mathscr{A}$ is an algorithm that on input (f, p) with $f \in P_{\mathbb{Z}}$ and p an f-secure turnpath in R_f returns whether p is f-securely extendable or not. Notationally,

$$\mathscr{A}(f,p) = \begin{cases} \textbf{TRUE}, \text{ if } p \text{ is } f\text{-securely extendable} \\ \textbf{FALSE}, \text{ otherwise.} \end{cases} \tag{†}$$

We denote by $T(\mathscr{A}, n)$ the worst case running time of $\mathscr{A}(f, p)$ over all partitions λ, μ, ν into n parts, all $f \in P(\lambda, \mu, \nu)_{\mathbb{Z}}$ and all f-secure turnpaths p in R_f. If $\mathscr{A}$ solves Problem 11.4.2, then $T(\mathscr{A}, n)$ is polynomially bounded in n — but remember that we do not know of such $\mathscr{A}$.

The algorithms presented in this subsection use $\mathscr{A}$ as a subroutine and hence they are only polynomial time algorithms if $T(\mathscr{A}, n)$ is polynomially bounded. In fact this subsection is only meant to prepare the reader for the more complicated approach used in the Subsection 11.4 (B), where $\mathscr{A}$ is modified in a way such that polynomial running time is achieved.

The main subalgorithm of this subsection is Algorithm 4 below.

Algorithm 4 FindNeighWithBlackBox

Input: $f \in P_{\mathbb{Z}}$; p an f-securely extendable turnpath on R_f; $\mathscr{A}$ as in (†)
Output: (1) Prints all integral flows $f + c \in P_{\mathbb{Z}}$, where $c \in C(G)$ is a cycle that contains p. (2) Prints at least one element.

1: **if** p is not just a turnpath, but a turncycle **then**
2:     `print`$(f + \pi(p))$ and **return**.
3: **end if**
4: **for both** turnedges $e := (\mathsf{end}(p), z) \in E(R_f)$ **do**
5:     Concatenate $p' \leftarrow pe$.
6:     If p' is not f-secure, **continue** with the next e.
7:     **if** $\mathscr{A}(f, p')$ **then**
8:         Recursively call FindNeighWithBlackBox$(f, p', \mathscr{A})$.
9:     **end if**
10: **end for**

Note that the statement **for both** in line 4 means *for all*, as there are at most two turnedges e in R_f starting at $\mathsf{end}(p)$. Here $\mathsf{end}(p)$ denotes the last vertex of the turnpath p.

11.4.3 Lemma. *Algorithm 4 works according to its output specification.*

Proof. Since the input p is f-securely extendable, there exists at least one turnedge $e = (\mathsf{end}(p), z)$ such that $p' = pe$ is f-secure in line 6 and f-securely extendable in line 7. Hence Algorithm 4 prints at least one element or calls itself recursively. The lemma follows now easily by induction on the number of turns in G not used by p. □

We will see that it is crucial for the running time that for each call of Algorithm 4 we can ensure that an element is printed.

11.4.4 Lemma. *Let $f \in P_{\mathbb{Z}}$. On input $f \in P_{\mathbb{Z}}$, Algorithm 4 prints out distinct elements. The first k elements are printed in time $\mathcal{O}(kn^2 T(\mathscr{A}, n))$.*

Proof. Algorithm 4 traverses a binary recursion tree of depth at most $|E(R_f)| = \mathcal{O}(n^2)$ with depth-first-search. The time needed at each recursion tree node is $\mathcal{O}(T(\mathscr{A}))$, if the implementation is done in a reasonable manner: Checking if a turnpath is a turncycle can be done in time $\mathcal{O}(1)$ and testing p' for f-security under the assumption that p was f-secure can also be in time $\mathcal{O}(1)$. Thus the time the algorithm spends between two leafs is at most $\mathcal{O}(n^2 T(\mathscr{A}))$. The lemma is proved by the fact that at each leaf a distinct element is printed. □

We can define an algorithm NeighGen' as required for Theorem 11.3.1 (besides polynomial running time) as follows:

Let $f \in P_{\mathbb{Z}}$ be an input.
for all turnedges $p \in E(R_f)$ **do**
  Compute $\mathscr{A}(f,p)$.
  if $\mathscr{A}(f,p) =$ **TRUE then**
    Call Algorithm 4 on $(f,p,\mathscr{A})$.
  end if
end for

Lemma 11.4.3 ensures that $\Gamma(f)$ is printed by NeighGen'. We now analyze the running time of NeighGen'.

Since each element in $\Gamma(f)$ is printed at most once during each of the $\mathcal{O}(n^2)$ calls of Algorithm 4 and each call of Algorithm 4 prints pairwise distinct elements, it follows that each element is printed by NeighGen' at most $\mathcal{O}(n^2)$ times. Hence, according to Lemma 11.4.4, the first k elements are printed in time $\mathcal{O}(k{\cdot}n^4{\cdot}T(\mathscr{A}, n))$.

Since the existence of $\mathscr{A}$ was hypothetical, we have to bypass Problem 11.4.2. This is achieved in the next subsection.

11.4 (B) Bypassing the Secure Extension Problem

We need a polynomial time algorithm that solves a problem similar to Problem 11.4.2. A first approach for this is the following (which will fail for several reasons explained below): Instead of extending p to an f-secure cycle, we compute a *trivial extension q of p*, which is a shortest turnpath q in R_f starting at $\mathsf{end}(p)$ and ending at $\mathsf{start}(p)$. A turncycle c containing p can then be obtained as the concatenation $c = pq$. But c might not be secure and might not even be planar and in the worst case we could have $f + \pi(pq) \notin P$. It will be crucial in the following to find q such that $f + \pi(pq) \in P$, so in the upcoming examples we have a look at the difficulties that may arise for a trivial extension q and how we can fix them.

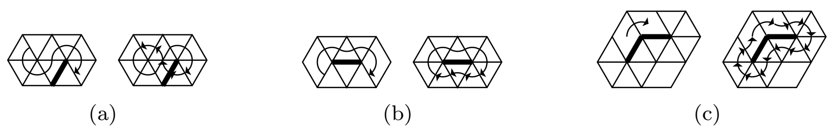


Figure 11.4.i: Illustration of the examples in Subsection 11.4 (B). Rhombi where the diagonal is not drawn are f-flat. All other rhombi are not f-flat. The secure turnpath p is drawn as a single long arrow, while the shortest turnpath q from $\mathsf{end}(p)$ to $\mathsf{start}(p)$ is drawn as several single turns. For better visibility, in the left versions only p is drawn. Fat lines indicate diagonals of nearly f-flat rhombi ϱ which satisfy $\sigma(\varrho, pq) = -2$. Hence $f + \pi(pq) \notin P$.

Example (a): Figure 11.4.i(a) shows a secure turnpath p and a trivial extension q that uses no turnvertices of p. Nevertheless, $f + \pi(pq) \notin P$. In the light of

Example (a) we want to consider only those q where q uses no turnvertices in those hive triangles where p uses turnvertices.

Example (b): Figure 11.4.i(b) shows a secure turnpath p and a trivial extension q that uses no turnvertices of p and uses no turnvertices in hive triangles where p uses turnvertices. But still we have $f + \pi(pq) \notin P$. Example (b) gives the idea to consider only those q that use no turnvertices in nearly f-flat rhombi where p uses a negative slack contribution.

Example (c): Figure 11.4.i(c) shows a secure turnpath p and a trivial extension q that uses no turnvertices of p and uses no turnvertices in hive triangles where p uses turnvertices. Moreover, q uses no turnvertices in nearly f-rhombi in which p uses a negative slack contribution. Nevertheless, $f + \pi(pq) \notin P$. A situation as in Example (c) is possible, because p consists of only 2 turns. We will see that 3 turns are enough to avoid these problems.

Considering more and more special subclasses of turnpaths q in R_f leads to the forthcoming definition of the digraph R_f^p. The turnpaths q will be shortest turnpaths on R_f^p.

11.4.5 Definition (the digraph R_f^p)**.** Let $f \in P_{\mathbb{Z}}$ and p be an f-secure turnpath on R_f. We define the digraph R_f^p by further deleting from R_f a subset of its turnvertices: Delete from R_f each turnvertex lying in a hive triangle in which p uses turnvertices. Moreover, for all nearly f-flat rhombi $\Diamond$ in which p uses $\hat{\Diamond}$ or $\check{\Diamond}$, delete all turnvertices of $\Diamond$. If we deleted $\mathsf{start}(p)$ or $\mathsf{end}(p)$ in the process, we add them back. We denote with $C'(R_f^p)$ the set of proper turnpaths in R_f^p from $\mathsf{end}(p)$ to $\mathsf{start}(p)$. ∎

Since $f \in P_{\mathbb{Z}}$ is capacity achieving, the source and sink vertices are isolated in R_f^p and hence all turnpaths in R_f^p are proper. Therefore formally there was no need to explicitly require the turnpaths in $C'(R_f^p)$ to be proper in Definition 11.4.5.

For each $q \in C'(R_f^p)$ we have that $pq \in C(R_f)$ and pq is a proper turncycle. If for an f-secure turnpath p in R_f we have that $C'(R_f^p) \neq \emptyset$, then we call p *f-extendable*. Note the following crucial fact: All f-securely extendable turnpaths are also f-extendable. Unlike f-secure extendability, f-extendability can be handled efficiently: We can check in time $\mathcal{O}(n^2)$ whether an f-secure turnpath p is f-extendable or not by breadth-first-search on R_f^p.

Definition 11.4.5 is precisely what we want, which can be seen in the following result, which is a variant of Theorem 10.3.9.

11.4.6 Theorem (Shortest Turncycle Theorem)**.** *Let p be an f-secure turnpath in R_f, consisting of at least 3 turns. Let q be a shortest turnpath in $C'(R_f^p)$. Then $f + \pi(pq) \in P$.*

The proof is postponed to Section 12.2.

Consider now Algorithm 5, which is a refinement of Algorithm 4.

We will see in Remark 11.4.8 in which situations the condition of line 12 is satisfied.

11.4.7 Lemma. *Algorithm 5 works according to its output specification.*

Proof. Recall that by definition, all f-extendable turnpaths are f-secure. Hence in line 2, only elements of $P_{\mathbb{Z}}$ are printed. The flows printed in line 13 are elements of $P_{\mathbb{Z}}$ because of the Shortest Turncycle Theorem 11.4.6.

If p is f-securely extendable, then the binary recursion tree of Algorithm 4 is a subtree of the binary recursion tree of Algorithm 5, because f-secure extendability implies f-extendability. Hence in this case, Algorithm 5 meets the output specification requirement (1).

Algorithm 5 FindNeigh

Input: $f \in P_{\mathbb{Z}}$; an f-extendable turnpath p in R_f consisting of at least 3 turns
Output: (1) Prints at least all integral flows $f + c \in P_{\mathbb{Z}}$, where $c \in C(G)$ is a cycle that contains p, but may print other elements of $P_{\mathbb{Z}}$ as well. (2) Prints at least one element.

```
1: if start(p) = end(p) then
2:   print(f + π(p)) and return.
3: end if
4: local foundpath ← FALSE.
5: for both turnedges e := (end(p), z) ∈ E(R_f^p) do
6:   Concatenate p' ← pe.
7:   if p' is f-extendable then
8:     Recursively call FindNeigh(f, p').
9:     foundpath ← TRUE.
10:  end if
11: end for
12: if not foundpath then
13:   print(f + π(pq)) with a shortest q ∈ C'(R_f^p) and return.
14: end if
```

If p is not f-securely extendable, then there exists no f-secure turncycle c containing p such that $f + \pi(c) \in P_{\mathbb{Z}}$, see Proposition 10.2.9. Thus in this case, Algorithm 5 trivially meets the output specification requirement (1).

The output specification requirement (2) is satisfied because exactly one of the following three cases occurs: (a) Algorithm 5 prints an element in line 2 and returns or (b) Algorithm 5 calls itself recursively or (c) Algorithm 5 prints an element in line 13 and returns. □

11.4.8 Remark. If during a run of Algorithm 5 we have `foundpath`=**FALSE** in line 12, then we have an f-extendable turnpath p such that its concatenation $p' \leftarrow pe$ with any further turnedge e results in p' being not f-extendable. For example, this can happen if p ends with and $q \in C'(R_f^p)$ continues with , but q also uses . Then q uses two turnvertices in the hive triangle . If q is the only element in $C'(R_f^p)$, then adding a turnedge to p destroys f-extendability: A turnpath $q' \in C'(R_f^{p'})$ would mean that there exists a concatenated turncycle $p'q'$ that uses only a single turnvertex in the hive triangle , a contradiction to the uniqueness of q. In this situation, $f + \pi(pq)$ is printed in line 13 and since pq is not planar, this means that an element of $P_{\mathbb{Z}}$ is printed that does lie in $\Gamma(f)$. ■

11.4.9 Lemma. *Let $f \in P_{\mathbb{Z}}$. On input $f \in P_{\mathbb{Z}}$, Algorithm 5 prints out distinct elements. The first k elements are printed in time $\mathcal{O}(k \cdot n^4)$.*

Proof. We use the fact that f-extendability can be decided in time $\mathcal{O}(n^2)$. The rest of the proof is analogous to the proof of Lemma 11.4.4. Note that it is crucial in this proof that for each recursive algorithm call we can ensure that at least one element of $P_{\mathbb{Z}}$ is printed. This is guaranteed by line 13. □

11.4.10 Remark. If we delete line 13 from Algorithm 5, then its execution gives a recursion tree where at some leafs no elements are printed. Then it is not clear how much time is spent visiting elementless leafs and we cannot prove Lemma 11.4.9. ■

Analogously to the definition of NeighGen', we can define the algorithm NeighGen as required for Theorem 11.3.1 as follows: Call Algorithm 5 several times with fixed $f \in P_{\mathbb{Z}}$, but each time with a different secure turnpath $p \in E(R_f)$

consisting of 3 turns such that p is f-extendable. We now prove the correctness and running time of NeighGen.

Let $\tilde{\Gamma}(f)$ be the set of flows printed by NeighGen. Since f-secure extendability implies f-extendability, each flow printed by NeighGen' is also printed by NeighGen, so $\Gamma(f) \subseteq \tilde{\Gamma}(f)$. Lemma 11.4.7 implies $\tilde{\Gamma}(f) \subseteq P_{\mathbb{Z}}$.

Since we have $\mathcal{O}(n^2)$ many calls of Algorithm 5, we get a total running time of $\mathcal{O}(kn^6)$.

This proves Theorem 11.3.1 with one hole remaining: The proof of the Shortest Turncycle Theorem 11.4.6.

Chapter 12

Proofs

This chapter covers the proof of the Rerouting Theorem 10.3.22 in Section 12.1, the proofs of the Shortest Turnpath and Turncycle Theorems 10.3.9 and 11.4.6 in Section 12.2, and the proof of the Connectedness Theorem 10.2.14 in Section 12.3. Additionally we prove the King-Tollu-Toumazet conjecture (Thm. 12.4.1) in Section 12.4. While the Sections 12.1, 12.2, and 12.3 are logically independent, Section 12.4 requires results from all three previous sections.

12.1 Proof of the Rerouting Theorem

We will easily derive the Rerouting Theorem 10.3.22 by a glueing argument from the Canonical Turnpath Theorem 12.1.6 below. The latter is a general result for hive flows on convex sets in the triangular graph. In the next subsection we introduce the necessary terminology to state the Canonical Turnpath Theorem, which is then proved by induction in the remainder of this section, considering separately the different possible shapes of convex sets.

12.1 (A) Canonical Turnpaths in Convex Sets

Let L be a convex set in the triangular graph Δ, cf. Figure 10.1.i. We define the graph G_L as the induced subgraph of the honeycomb graph G obtained by restricting to the set of vertices lying in L (including the vertices on the border of L, but omitting s and t). A *flow on* G_L is defined as a map $E(G_L) \to \mathbb{R}$ satisfying the flow conservation laws at all vertices of G_L that do not lie at the border of L. The vector space $\overline{F}(G_L)$ of *flow classes on* G_L is defined as in (10.1.2) by factoring out the null flows. As in Definition 10.1.9, we define a *hive flow* f *on* L as a flow class in $\overline{F}(G_L)$ satisfying $\sigma(\varrho, f) \geq 0$ for all rhombi ϱ lying in L. Similarly, we define the notion of a *flow on* R *(or* R_f*) restricted to* L, by restricting to the subgraph induced by the turnvertices lying in L.

For a fixed convex set L we are going to define a set $\mathcal{P}_L$ of distinguished turnpaths p in L starting at some entrance edge $a_{\to L}$ and ending at some exit edge $b_{L\to}$. The goal is to achieve that p is a turnpath in R_f whenever L is an f-flatspace. Proposition 10.3.13 will guide us how to make the right definition.

Let $r \geq 3$ and $a_1, \ldots, a_r$ be a sequence of successive sides of L in counterclockwise order, where $a_2, \ldots, a_{r-1}$ are different. Further, we assume that the angles between a_{i-1} and a_i are obtuse for $i = 3, \ldots, r-1$. We then form a unique turnpath p moving within the border triangles of L from $(a_1)_{\to L}$ to $(a_r)_{L\to}$ in counterclockwise direction, cf. Figure 12.1.i. The turnpath p alternatively takes clockwise and

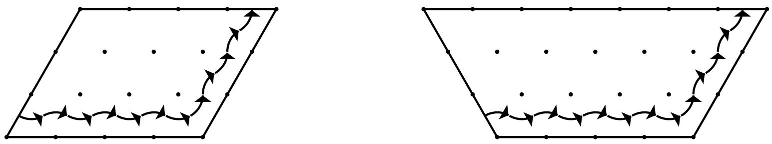


Figure 12.1.i: Canonical turnpaths in a parallelogram and in a trapezoid. The left hand turnpath starts with a counterclockwise turn (acute angle) and the right-hand turnpath starts with a clockwise turn (obtuse angle).

counterclockwise turns, except at the (obtuse) angles of L between a_{i-1} and a_i (for $i = 3, \dots, i-1$), where p takes two consecutive counterclockwise turns to go around. If a_1, a_2 form an acute angle, then p starts with a counterclockwise turn, otherwise p starts with a clockwise turn. Moreover, if a_{r-1}, a_r form an acute angle, then p ends with a counterclockwise turn, otherwise p ends with a clockwise turn. We call the resulting turnpath a *canonical turnpath* of L. We shall also consider the turnpaths consisting of a single clockwise turnvertex at an acute angle as a canonical turnpath of L.

12.1.1 Definition. The symbol $\mathcal{P}_L$ denotes the set of all canonical turnpaths of the convex set L. For $p \in \mathcal{P}_L$, we denote by $\mathsf{start}(p) = (a_1)_{\to L}$ the edge of Δ from which p starts and by $\mathsf{end}(p) = (a_r)_{L\to}$ the edge of Δ where p ends. ■

12.1.2 Example. A triangle has exactly six canonical turnpaths, cf. Figure 12.1.ii. A parallelogram has exactly eight canonical turnpaths, cf. Figure 12.1.iii. In particular, this holds true for rhombi. A trapezoid has exactly nine canonical turnpaths, cf. Figure 12.1.iv. A pentagon has exactly 16 canonical turnpaths, cf. Figure 12.1.v. A hexagon has six canonical turnpaths up to rotations, which makes a total of 36 turnpaths, see Figure 12.1.vi. ■

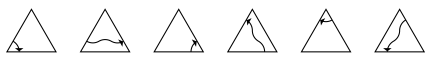


Figure 12.1.ii: The six canonical turnpaths in a triangle.

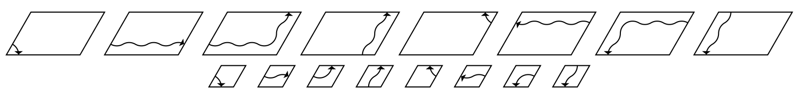


Figure 12.1.iii: Top row: The eight canonical turnpaths in a parallelogram.
Bottom row: The same list in the special case of a rhombus.

12.1.3 Lemma. *Let f be a hive flow and L be one of its f-flatspaces. Then any canonical turnpath $p \in \mathcal{P}_L$ of L is a turnpath in R_f.*

Proof. We need to ensure that p uses no negative contributions in f-flat rhombi. But p does not use any pair of successive clockwise turnvertices at all. And whenever p uses a counterclockwise turnvertex , then  is at the border of an f-flatspace. □

We have to extend some of the notions introduced in Section 10.3 (C).

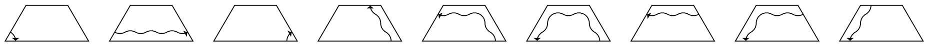


Figure 12.1.iv: The nine canonical turnpaths in a trapezoid.

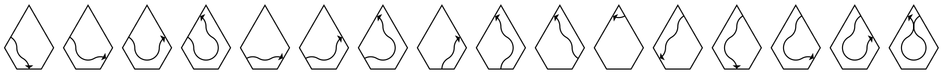


Figure 12.1.v: The 16 canonical turnpaths in a pentagon.

12.1.4 Definition. A *multiset* φ *of canonical turnpaths* in a convex set L is defined as a map $\varphi\colon \mathcal{P}_L \to \mathbb{N}_{\geq 0}$. Let a be a side of L. The number of turnpaths of φ starting from $a_{\to L}$ or ending at $a_{L\to}$, respectively, is denoted by

$$\omega(a, \to L, \varphi) \;:=\; \sum_{\mathsf{start}(p)=a_{\to L}} \varphi(p), \quad \omega(a, L\to, \varphi) \;:=\; \sum_{\mathsf{end}(p)=a_{L\to}} \varphi(p).$$

■

Note that the weighted sum $\sum_{p\in\mathcal{P}_L} \varphi(p)p$ defines a nonnegative flow d'_L on R restricted to L. Moreover, if L is an f-flatspace, then d'_L is a flow on R_f restricted to L, as a consequence of Lemma 12.1.3.

12.1.5 Definition. Let d be a hive flow on a convex set L. A multiset φ of canonical turnpaths on L is called *compatible with* d, if $\omega(a, \to L, \varphi) = \omega(a, \to L, d)$ and $\omega(a, L\to, \varphi) = \omega(a, L\to, d)$ for all edges a of L. ■

The key result, amenable to an inductive proof along L, is the following.

12.1.6 Theorem (Canonical Turnpath Theorem)**.** *Let L be a convex set and d be an integral hive flow on L. Then there exists a multiset φ of canonical turnpaths on L which is* compatible with d.

12.1.7 Lemma. *The Canonical Turnpath Theorem 12.1.6 implies the Rerouting Theorem 10.3.22.*

Proof. We first note that it suffices to prove the Rerouting Theorem 10.3.22 for an integral flow $d \in \overline{F}(G)$. Indeed, then it trivially must hold for a rational d. A standard continuity argument then shows the assertion for a real d.

So let $f \in B$ and $d \in \overline{F}(G)$ be integral and f-hive preserving. Theorem 12.1.6 applied to every f-flatspace L and the hive flow d restricted to L yields a multiset φ_L of canonical turnpaths on L compatible with d restricted to L. By Lemma 12.1.3, canonical turnpaths of L are in R_f.

Suppose that L and M are adjacent f-flatspaces sharing the side a. Then we have, using Theorem 12.1.6 and (10.3.17),

$$\omega(a, \to L, \varphi_L) = \omega(a, \to L, d) = \omega(a, M\to, d) = \omega(a, M\to, \varphi_M). \tag{12.1.8}$$

We set up an arbitrary bijection between the turnpaths p_M in φ_M ending at $a_{M\to}$ and the turnpaths p_L in φ_L starting from $a_{\to L} = a_{M\to}$ and concatenate these turnpaths correspondingly. It is essential to note that the additional turnedges used for joining p_M and p_L lie in R_f, since the rhombus with diagonal $a_{\to L}$ is not f-flat!

Similarly, we have $\omega(a, \to M, \varphi_M) = \omega(a, L\to, \varphi_L)$ and we concatenate the turnpaths in φ_L ending at $a_{L\to}$ with the turnpaths in φ_M starting from $a_{\to M} = a_{L\to}$ correspondingly.

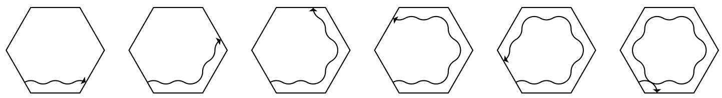


Figure 12.1.vi: The six canonical turnpaths in a hexagon starting from a fixed side.

Doing so for all sides a shared by different f-flatspaces, we obtain a multiset of turncycles in R_f and a multiset of turnpaths in R_f going from a side of Δ to a side of Δ. These turnpaths can be extended to complete turnpaths. Altogether, we obtain a multiset φ of complete turnpaths in R_f.

Then we have $\omega(a, \to L, \varphi) = \omega(a, \to L, \varphi_L)$ and $\omega(a, L\to, \varphi) = \omega(a, L\to, \varphi_L)$ for any side a of an f-flatspace L. Hence, $\omega(a, \to L, \varphi) = \omega(a, \to L, d)$ and $\omega(a, L\to, \varphi) = \omega(a, L\to, d)$ by (12.1.8). So the multiset φ is as required. □

In the subsequent sections we shall prove Theorem 12.1.6 for the five possible shapes of L. Although the arguments are quite similar for the different shapes, there are subtle differences. We begin with the case of parallelograms.

12.1 (B) Parallelograms

By the *size* of a convex set L we understand the number of hive triangles contained in L. We will prove the Canonical Turnpath Theorem 12.1.6 for parallelograms L by induction on the size of L. The induction start is provided by the following lemma. Recall the definition of the symbol $\Psi_+(\varrho)$ from Definition 10.2.2.

12.1.9 Lemma. *The assertion of the Canonical Turnpath Theorem 12.1.6 is true if $L = \varrho$ is a rhombus. More specifically, if d is an integral hive flow on ϱ, then there is a multiset φ of canonical turnpaths compatible with d, such that for all $p \in \Psi_+(\varrho)$ occuring in φ we have $p \subseteq \mathrm{supp}(d)$.*

Proof. The canonical turnpaths in a rhombus ϱ are exactly the eight contributions in $\Psi_+(\varrho) \cup \Psi_0(\varrho)$, see Figure 12.1.iii.

Given an integral hive flow d on ϱ. If $p \subseteq \mathrm{supp}(d)$ for some $p \in \Psi_-(\varrho)$, then $p' \subseteq \mathrm{supp}(d)$ by Lemma 10.2.5 on antipodal contributions. Since $\sigma(\varrho, p + p') = 0$ it follows that $d - (p + p')$ is a hive flow. So we can successively subtract flows of the form $p + p'$ from d to arrive at a flow decomposition $d = \sum_i m_i(p_i + p_i') + h$, where $m_i \in \mathbb{N}$, h is a hive flow on ϱ, and $p \not\subseteq \mathrm{supp}(h)$ for all $p \in \Psi_-(\varrho)$. Moreover, $\mathrm{supp}(h) \subseteq \mathrm{supp}(d)$ by construction.

It is straighforward to check that h must be a nonnegative integer linear combination of turnpaths $p \in \Psi_+(\varrho) \cup \Psi_0(\varrho)$ such that $p \subseteq \mathrm{supp}(h)$.

Now we replace the sums $p_i + p_i'$ by sums $n_i + n_i'$ of two neutral slack contributions as follows: we replace ♢ + ♢ by ♢ + ♢, we replace ♢ + ♢ by ♢ + ♢, and similarly in the situations rotated by 180°. Since this exchange does not alter the number of turnpaths entering and leaving a side of ϱ, this leads to a multiset of canonical turnpaths of ϱ satisfing the desired requirements. □

The induction step will be based on the following result on *straightening* canonical turnpaths.

12.1.10 Proposition. *Let L be a parallelogram cut into two parallelograms L_1 and L_2 by a straight line parallel to one the sides of L. Further let p be a turnpath going from the side a of L to the side b of L such that p is either a canonical turnpath of L_1, or p is obtained by concatenating a canonical turnpath p_1 of L_1 with a canonical turnpath p_2 of L_2. Then p can be* straightened, *that is, there exists a canonical turnpath of L going from $a_{\to L}$ to $b_{L\to}$.*

Proof. It suffices to check the various cases. Recall the possible canonical turnpaths in a parallelogram from Figure 12.1.iii. Figure 12.1.vii(a) shows how to treat the four possible canonical turnpaths of L_1 going from a side of L to a side of L. Note that only in two of these four cases, the turnpath has to be changed (by "stretching" or moving parallely). Figure 12.1.vii(b) shows how to treat the six possible cases of a concatenation p of a turnpath in L_1 with one in L_2. Only in two of the six cases, the turnpath has to be changed (by "shrinking"). □

(a) The modifications of the four canonical turnpaths of L_1 going from a side of L to a side of L.

(b) The modifications of the six possible cases of a concatenation p of a turnpath in L_1 with one in L_2.

Figure 12.1.vii: Illustration of the proof of Proposition 12.1.10.

12.1.11 Proposition. *The assertion of the Canonical Turnpath Theorem 12.1.6 is true if L is a parallelogram.*

Proof. We proceed by induction on the size of L. The induction start is provided by Lemma 12.1.9. We suppose now that L has size greater than two and we cut L into two parallelograms L_1 and L_2 by a straight line parallel to one the sides of L. The induction hypothesis yields the existence of multisets φ_i of canonical turnpaths of L_i compatible with d restricted to L_i, for $i = 1, 2$.

Let a denote the side shared by L_1 and L_2 and note that $a_{L_1\rightarrow} = a_{\rightarrow L_2}$. The reader should note that it is not possible for a canonical turnpath in L_i to start and end at a, see Figure 12.1.iii. Using Definition 12.1.4, the induction hypothesis, and (10.3.17), we get

$$\omega(a, L_1\!\rightarrow, \varphi_1) = \omega(a, L_1\!\rightarrow, d) = \omega(a, \rightarrow\! L_2, d) = \omega(a, \rightarrow\! L_2, \varphi_2)$$

This means that the number of turnpaths p_1 in φ_1 ending at $a_{L_1\rightarrow}$ equals the number of turnpaths p_2 in φ_2 starting at $a_{\rightarrow L_2}$. It is therefore possible to set up a bijection between the set of turnpaths p_1 in φ_1 ending at $a_{L_1\rightarrow}$ with the set of turnpaths p_2 in φ_2 starting at $a_{\rightarrow L_2}$, and to concatenate each p_1 with some partner p_2 to obtain a turnpath q in L starting from a side of L and ending at a side of L. However, the turnpath q may not be canonical for L. But now we use Proposition 12.1.10 to replace q by a canonical turnpath of L starting and ending at the same sides of L as q does.

Similarly, $a_{L_2\rightarrow} = a_{\rightarrow L_1}$ and we get $\omega(a, L_2\!\rightarrow, \varphi_2) = \omega(a, \rightarrow\! L_1, \varphi_1)$ As before, we can match and concatenate the turnpaths p_2 in φ_2 ending at $a_{L_2\rightarrow}$ with the turnpaths p_1 in φ_1 starting at $a_{\rightarrow L_1}$. Again, we use Proposition 12.1.10 to replace the resulting turnpaths by canonical turnpaths of L without changing the starting and ending side.

We also apply Proposition 12.1.10 to the turnpaths in φ_1 and φ_2 going from a side of L to a side of L.

After performing these procedures, we obtain a multiset φ of canonical turnpaths of L. Let b be the side of L_1 parallel to a. Then we have by construction

$$\omega(b, \to L, \varphi) = \omega(b, \to L_1, \varphi_1) = \omega(b, \to L_1, d) = \omega(b, \to L, d).$$

Similarly for b being the side of L_2 parallel to a. Now let b be a side of L cut by a into line segments b_1 and b_2. Then we have

$$\begin{aligned}\omega(b, \to L, \varphi) &= \omega(b_1, \to L_1, \varphi) + \omega(b_2, \to L_2, \varphi)\\ &= \omega(b_1, \to L_1, d) + \omega(b_2, \to L_2, d) = \omega(b, \to L, d).\end{aligned}$$

It follows that φ is compatible with d. □

12.1 (C) Trapezoids, Pentagons, and Hexagons

We first treat the case of trapezoids. Again, the strategy is to proceed by induction, cutting the trapezoid into a smaller trapezoids or parallelograms. But now, unlike the case of parallelograms before, the cutting has to be done in a certain way in order to ensure the straightening of canonical turnpaths. The following result identifies the critical cases to be avoided. The straightforward proof is similar to the one of Proposition 12.1.10 and left to the reader, who should consult Figures 12.1.iii–12.1.iv for the possible canonical turnpaths in a parallelogram or a trapezoid, respectively.

The *height* of a convex set L is defined as the number of its edges on its shortest side.

12.1.12 Proposition. *Let L be a trapezoid cut into convex sets L_1 and L_2 by a straight line a. Further let p be a turnpath going from the side b of L to the side c of L such that p is either a canonical turnpath of L_1, a canonical turnpath of L_2, or p is obtained by concatenating canonical turnpaths of L_1 with canonical turnpaths L_2 (in any order).*

(1) If a is parallel to the longest side of L so that L_1 and L_2 are trapezoids, then p can be straightened, i.e., there exists a canonical turnpath of L going from $b_{\to L}$ to $c_{L\to}$.

(2) Suppose that L has the height 1 and that L is cut by a into a trapezoid (or a triangle) and a rhombus (there are two possibilities to do so). Then p can be straightened unless in the four critical cases depicted in Figure 12.1.viii(a)-(b). □

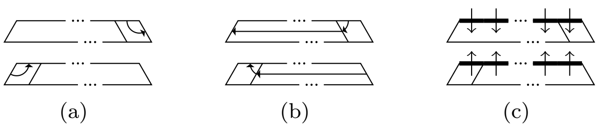


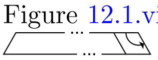
Figure 12.1.viii: A trapezoid of height 1 cut into a trapezoid and a rhombus with the four critical cases of a turnpath p that cannot be straightened.

12.1.13 Proposition. *The assertion of the Canonical Turnpath Theorem 12.1.6 is true if L is a trapezoid.*

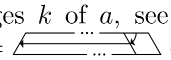
Proof. We make induction on the size of L. The case where L is a hive triangle, which we consider a degenerate trapezoid, is trivial. So suppose that L has size at least three and let d be an integral hive flow on L.

(1) If L has height greater than 1, then we cut L into two trapezoids L_1 and L_2 by a straight line parallel to the longest side of L and apply the induction hypothesis to L_1 and L_2 to obtain multisets φ_i of canonical turnpaths of L_i compatible with d restricted to L_i, for $i = 1, 2$. Proceeding as in the proof of Proposition 12.1.11 and using Proposition 12.1.12(1), we construct from φ_1, φ_2 a multiset φ of canonical turnpaths of L satisfying $\omega(a, \to L, \varphi) = \omega(a, \to L, d)$ and $\omega(a, L\to, \varphi) = \omega(a, L\to, d)$ for all sides a of L.

(2) Now suppose that L has height 1. There are two possibilities **t** and **b** of cutting L by a straight line into a trapezoid and a rhombus, as depicted in the top and bottom row of Figure 12.1.viii.

Choose the **t** version of cutting and apply the induction hypothesis and Proposition 12.1.11 to L_1 and L_2, respectively, to obtain multisets φ_i of canonical turnpaths of L_i. Then we apply the straightening of Proposition 12.1.12(2) as before, which succeeds unless we are in one of the critical cases as depicted in the top row of Figure 12.1.viii. For instance, assume that L_2 is a rhombus and the turnpath $p =$  occurs in φ_2 as in Figure 12.1.viii(a). Let k_r denote the edge of L_2 where p starts and a be the corresponding side of L. Then, using Definition 10.3.16, we have

$$\omega(k_r, \to L_2, d) = \omega(k_r, \to L_2, \varphi_2) \geq 1,$$

hence $\delta(k_r, \to L_2, d) > 0$. Lemma 10.3.14 implies that $\delta(k, \to L, d) > 0$ for all edges k of a, see Figure 12.1.viii(c). The same conclusion can be drawn when $p =$  occurs in φ_2.

The clue is now that if we cut L in the other possible way (**b** version, cf. bottom row of Figure 12.1.viii), then no critical case can occur. Indeed, otherwise, by an analogous reasoning as before, we had $\delta(k, L\to, d) > 0$ for all edges k of a, which contradicts $\delta(k, \to L, d) > 0$.

Similarly one shows that if we start with the **b** version of cutting L and a critical case occurs, then cutting with the **t** version succeeds. □

For later use, we note the following observation resulting from the above proof.

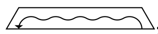
12.1.14 Observation. Let L be a trapezoid of height 1 and d be an integral hive flow on L. If the multiset φ of canonical turnpaths compatible with d resulting from the proof of Proposition 12.1.13 contains the turnpath $q =$ , then there is a rhombus ϱ and a turnedge $p \in \Psi_+(\rho)$ as in Figure 12.1.ix such that $p \subseteq \text{supp}(d)$.

Figure 12.1.ix: On Observation 12.1.14.

Proof. Tracing part (2) of the inductive proof of Proposition 12.1.13 shows that q results from a smaller turnpath $\tilde{q}$ either by stretching to the right or left, or by appending to $\tilde{q}$ a turnedge p in the right or left rhombus ϱ, see Figure 12.1.x. In

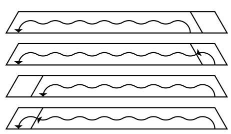


Figure 12.1.x: Inductive construction of the turnpath q along the proof of Proposition 12.1.13. In the first and third case stretching is necessary.

the case p is appended, we know that $p \subseteq \mathrm{supp}(d)$ by Lemma 12.1.9. Otherwise, we conclude by the induction hypothesis. □

We settle now the case of pentagons.

12.1.15 Proposition. *The assertion of the Canonical Turnpath Theorem 12.1.6 is true if L is a pentagon.*

Proof. We proceed by induction on the size of L, cutting the pentagon by a straight line into a pentagon and a trapezoid, or a parallelogram and a trapezoid. The two critical cases, where a straightening fails, are depicted in Figure 12.1.xi. As in the

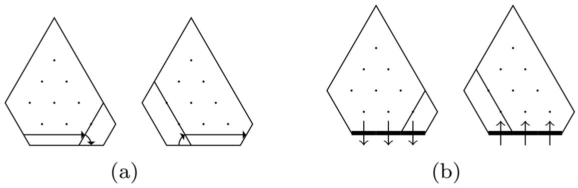


Figure 12.1.xi: Two ways of cutting a pentagon with the two critical cases of turnpaths that cannot be straightened and the resulting throughputs.

proof of Proposition 12.1.13, one can show in that one of the two possible ways of cutting L, no critical case can occur. The details are left to the reader. □

The case where L is a hexagon is the simplest one of all and does not require an inductive argument as before.

12.1.16 Proposition. *The assertion of the Canonical Turnpath Theorem 12.1.6 is true if L is a hexagon.*

Proof. Let $a_1, \dots, a_6$ denote the six sides of a hexagon L. We write $a_i^- := (a_i)_{\to L}$ and $a_i^+ := (a_i)_{L\to}$ for the entrance and exit edges of L, respectively.

The essential observation is that for any pair (i, j) there exists a canonical turnpath of L going from a_i^- to a_j^+. This is easily verified by looking at Figure 12.1.vi.

Let d be an integral flow on L and put $\mathsf{in}(i) := \omega(a_i, \to L, d)$ and $\mathsf{out}(i) := \omega(a_i, L\to, d)$ for $1 \le i \le 6$. The flow conservation laws imply that $\sum_i \mathsf{in}(i) = \sum_i \mathsf{out}(i)$.

We form a list $\mathcal{L}^-$ of entrance edges in which a_i^- occurs $\mathsf{in}(i)$ many times and we form a list $\mathcal{L}^+$of exit edges in which a_i^+ occurs $\mathsf{out}(i)$ many times. Both lists have the same length. We now connect, for all j, the jth element of $\mathcal{L}^-$ with the jth element of $\mathcal{L}^+$ by a canonical turnpath p_j of L. This is possible by the observation made at the beginning of the proof. The resulting multiset φ of canonical turnpaths of L satisfies $\omega(a_i, \to L, \varphi) = \omega(a_i, \to L, d)$ and $\omega(a_i, L\to, \varphi) = \omega(a_i, L\to, d)$ by construction. □

12.1 (D) Triangles

We need the following flow propagation lemma.

12.1.17 Lemma. *Let L be a trapezoid and d be a hive flow on L.*

1. Let p be the path in Figure 12.1.xii(a) and suppose that $p \subseteq \mathrm{supp}(d)$. Then all the edges of G belonging to the turns in Figure 12.1.xii(b) belong to $\mathrm{supp}(d)$ as well. Moreover, $\mathrm{supp}(d)$ cannot contain the paths $q_1, q_2 \in \Psi_+(\varrho)$ in the shaded rhombi ϱ depicted in Figure 12.1.xii(c).

2. If the path $\tilde{p}$ in Figure 12.1.xii(a') satisfies $\tilde{p} \subseteq \mathrm{supp}(d)$, then a similar conclusion can be drawn, see Figures 12.1.xii(b')-(c').

Proof. 1. The first assertion on $\mathrm{supp}(d)$ follows by successively applying Lemma 10.2.6 on flow propagation.

The assertion $q_i \not\subseteq \mathrm{supp}(d)$ follows by inspecting the edges of G involved in the paths appearing in the bottom row of the trapezoid in Figure 12.1.xii(c) and noting that $\mathrm{supp}(d)$ cannot contain an edge $k \in E(G)$ and its reverse.

2. The second case is treated similarly. □

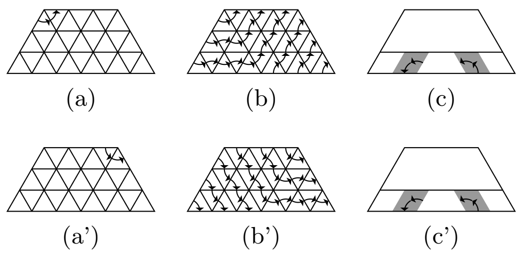


Figure 12.1.xii: If d is a hive flow and the path on the left figure is contained in $\mathrm{supp}(d)$, then the turns in the middle figure are contained in $\mathrm{supp}(d)$. The two paths on the right figure cannot be contained in $\mathrm{supp}(d)$.

The following proposition completes the proof of the Canonical Turnpath Theorem 12.1.6 for any shapes of L.

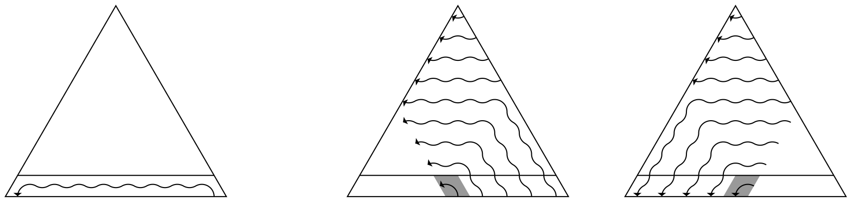
12.1.18 Proposition. *The assertion of the Canonical Turnpath Theorem 12.1.6 is true if L is a triangle.*

Proof. Again we proceed by induction on the size of L, the start of a hive triangle being trivial. For the induction step, suppose that d is an integral hive flow on L, and note that there are three ways of cutting L into a trapezoid L_1 of height 1 and a triangle L_2. We choose one as in Figure 12.1.xiii(a). The induction hypothesis and Proposition 12.1.13 yield multisets φ_i compatible with d restricted to L_1 and L_2, respectively. Using Figure 12.1.ii and Figure 12.1.iv showing the possible canonical turnpaths in triangles and trapezoids, the reader should verify that the procedure of concatenation and straightening, as explained in the proof of Proposition 12.1.11, can only fail in the critical case where φ_1 contains a turnpath q as depicted in Figure 12.1.xiii(a).

By Observation 12.1.14 applied to the trapezoid L_1 we may assume that there is rhombus ϱ (shaded in Figure 12.1.xiii(b)) and a path $p \in \Psi_+(\rho)$ such that $p \subseteq \mathrm{supp}(d)$. Now we can apply Lemma 12.1.17 as depicted in Figure 12.1.xiii(b) and conclude that all the turns depicted in this figure are contained in $\mathrm{supp}(d)$. Suppose we are in the left-hand situation of Figure 12.1.xiii(b). Then we can decompose L into a trapezoid of height 1 and a triangle by cutting along the right-hand side of L. Lemma 12.1.17 implies that no critical case can arise, so that in this situation, the procedure of concatenation and straightening works. If we are in the right-hand situation of Figure 12.1.xiii(b), we can decompose L into a trapezoid of height 1 and a triangle by cutting along the left-hand side of L and argue analogously. □

12.2 Proof of the Shortest Turncycle Theorem

In this section we prove Theorem 11.4.6, the proof of Theorem 10.3.9 being highly similar. Recall from Subsection 11.4 (B) the definition of $C'(R_f^p)$ as the set of proper turnpaths in R_f^p from $\mathsf{end}(p)$ to $\mathsf{start}(p)$. We slightly generalize the term "turnpath" in the next definition. We will need this generalization in Section 12.4.



(a) Cutting a triangle into a trapezoid and a triangle with the only critical case of a turnpath q that cannot be straightened.

(b) The turnpath in the shaded rhombus is contained in $\mathrm{supp}(d)$, implying flow propagation according to Lemma 12.1.17.

Figure 12.1.xiii: On the proof of Proposition 12.1.18.

12.2.1 Definition. Let $k \in E(\Delta)$ and let α be one of the two directions in which k can be crossed by turnedges. Then $p := (k, \alpha)$ is called a *turnpath of length 0*. For turnpaths of length 0, the symbol $C'(R_f^p)$ is defined as the set of all proper turnpaths in R_f that cross k in direction α. By a *generalized turnpath* we understand a turnpath or a turnpath of length 0. Turnpaths of length 0 are defined to be secure. ∎

We first prove the following preliminary Proposition 12.2.2, which rules out complex behaviour of shortest turnpaths and turncycles.

Each turnvertex in R has a so-called *reverse turnvertex* in R that points in the other direction, e.g., the reverse turnvertex of ◊ is ◊.

Recall that f-secure generalized turnpaths are planar by definition.

12.2.2 Proposition. *Let $f \in B_{\mathbb{Z}}$. Let p be an f-secure generalized turnpath. For a shortest turnpath $q \in C'(R_f^p)$ we have:*

(1) q does not use a turnvertex and its reverse.

(2) If q crosses the diagonal of a rhombus ◊ twice, then q uses one of the following four patterns of turnedges in ◊ and no other turnvertex: ◊ , ◊ , ◊ , or ◊ .

(3) The rhombi ◊ described in (2) do not overlap.

The proof of this result is provided in the next subsection.

12.2 (A) Special Rhombi

For the whole subsection we fix $f \in B_{\mathbb{Z}}$ and a secure generalized turnpath p. Additionally, we fix a *shortest* $q \in C'(R_f^p)$. Flatness shall always refer to f.

We will show in several steps that the minimal length of q poses severe restrictions on the way q may pass a rhombus. Here and in the following, statements involving the pictorial description ◊ include the possibility of a rotation by 180°.

Proof of Proposition 12.2.2(1). We first treat the case of p having nonzero length. By way of contradiction, assume that q uses a pair of reverse turnvertices. Let w be the *first* turnvertex in q whose reverse turnvertex is also used by q. Then q does not use the reverse turnvertex of the predecessor v of w. Let ◊ stand for the rhombus that contains both w and v. Note that there are two cases: $v \in p$ or $v \in q$.

Suppose first that v is a clockwise turn: $v =$ ◊. Since the reverse of w is in q, the turnpath q must use ◊ or ◊. But ◊ is excluded because it is the reverse turnvertex of v. Hence q uses ◊. We see that $v \in q$, because $v \in p$ would prohibit q using turnvertices in the hive triangle ◊ . If ◊ is not flat, then it is easy to check that q can be rerouted via ◊. This contradicts the minimal length of q. Note here that this rerouting is possible: By construction of R_f^p, the rerouting would only be prohibited by p using ◊ (and ◊ being nearly f-flat), but this would be a contradiction to ◊ $\in q$.

Therefore ◊ is flat. Then ◊ $\notin E(R_f)$ by construction of R_f and thus ◊ $\in q$. Hence $w =$ ◊, which implies ◊ $\in q$. But since ◊ $\notin E(R_f)$, it follows that ◊ is used by q, which is the reverse turnvertex of v: Contradiction!

It remains to analyze the case where v is a counterclockwise turn: $v =$ ◊. As before we must have ◊ $\in q$ and $v \in q$. The existence of counterclockwise turns at acute angles implies that ◊ and ◊ are not flat. Hence q can be rerouted via ◊, in contradiction with the minimal length of q.

For the proof of the conjecture by King, Tollu, and Toumazet in Section 12.4 we need to consider the degenerate case of p having length zero. The whole proof works in this case as well, but the following detail must be considered: If w is the first turnvertex of q, then v is the last turnvertex of q and it could be the case that q uses v and also the reverse turnvertex of v. But in our proof we require that q does not use the reverse turnvertex of v. Hence, if p has length zero, then we choose a different w as follows: In this case, the vertex w is defined to be the first turnvertex in q whose reverse turnvertex is also used by q and where q does not use the reverse turnvertex of the predecessor v of w. Such w exists, because q cannot consist of pairs of reverse turnvertices only. □

We continue now by analyzing the possible ways q may pass through a rhombus. Note that, due to Proposition 12.2.2(1), the turnpath q can cross the diagonal ◊ of a rhombus at most twice.

12.2.3 Lemma. *If ◊ is not flat, then its diagonal ◊ is crossed by q at most once.*

Proof. Assume by way of contradiction that both ◊ and ◊ occur in q. Since q cannot use a turnvertex twice, there are only two possibilites:

either ◊ and ◊ are edges of q or ◊ and ◊ are edges of q. (※)

In both cases, q can be rerouted, contradicting the minimal length of q. Note that the rerouting in the second case is possible since ◊ is assumed to be not flat. □

We now focus on the rhombi in which q crosses the diagonal twice.

12.2.4 Definition. A rhombus ϱ is called *special* if the turnpath q crosses its diagonal twice. If the crossing is in the same direction, then ϱ is called *confluent*, otherwise, if the crossing is in opposite directions, ϱ is called *contrafluent*. ■

By Lemma 12.2.3, special rhombi are necessarily flat. Recall the slack contributions of a rhombus ϱ introduced in Definition 10.2.2.

12.2.5 Lemma. *In a confluent rhombus ◊ the turnpath q uses at least the two contributions ◊ and ◊.*

Proof. Suppose that both ◊ and ◊ occur in q. Then, as before, there are only the two possibilites of (※) in the proof of Claim 12.2.3. Since ◊ is flat, the first case is impossible. □

We can now completely determine how q passes through contrafluent rhombi.

12.2.6 Lemma. *In a contrafluent rhombus ◊, the turnpath q uses the contributions ◊ and ◊ and no other contributions in this rhombus.*

Proof. Assume first that ◊ and ◊ are in q. Then ◊ and ◊ are both not flat. Hence q can be rerouted via ◊, contradicting the minimal length of q.

We are therefore left with the case where ◊ and ◊ are in q. By construction, ◊ and ◊ are not edges of R_f and thus ◊ and ◊ are both turnedges of q. It remains to show that q uses no other contribution in ◊.

Proposition 12.2.2(1) combined with the fact that ◇ is flat easily implies that ◇ and ◇ are the only contributions that q may possibly use. We exclude now these two cases.

Suppose that ◇ occurs in q. Then, as ◇ $\in q$, Lemma 12.2.3 implies that ◇ is flat. However, this contradicts ◇ $\in q$.

We are left with the case that ◇ occurs in q. Then ◇ and ◇ are both contrafluent. Applying what we have learned so far about contrafluent rhombi, we get the situation depicted in Figure 12.2.i. Note that the depicted triangle of side

Figure 12.2.i: The situation in Lemma 12.2.6. Left: The arrows represent q. All three overlapping rhombi are contrafluent and thus flat. On the right: The turnvertices that can be used for rerouting.

length 2 is not only contained in a flatspace, but it is a flatspace itself: The reason is that q traverses flatspaces in counterclockwise direction at their border, see Proposition 10.3.13. This implies that q can be rerouted as seen in Figure 12.2.i, which is a contradiction to the minimal length of q. □

The following lemma proves Proposition 12.2.2(3).

12.2.7 Lemma. *1. Contrafluent rhombi cannot overlap with contrafluent or confluent rhombi.*

2. Confluent rhombi cannot overlap.

Proof. 1. Assume that a contrafluent rhombus ◇ overlaps with a shaded confluent or contrafluent rhombus as in Figure 12.2.ii (the other cases are similar). The

Figure 12.2.ii: The situation in the proof of Lemma 12.2.7.

turnpath q uses at least the turnedges drawn in the left figure, where the directions are irrelevant and hence omitted. Hence the turnvertex in the right figure is used by q. But then, Lemma 12.2.6 implies that ◇ cannot be contrafluent, contradiction!

2. Let ◇ be confluent and assume that ◇ and ◇ occur in q. Assume that ◇ is confluent and hence q contains ◇ and ◇. Then ◇ is contrafluent and overlapping with ◇, which is contradicting part one of this lemma. The same argument works for the other three overlapping cases. □

Proof of Proposition 12.2.2(2). It suffices to show that in a special rhombus the turnpath q uses exactly two neutral slack contributions and no other turnvertex. By Lemma 12.2.6 it remains to consider the case of a confluent rhombus. We improve on Lemma 12.2.5. If q would use any additional contribution, then ◇ would overlap with a confluent or contrafluent rhombus, which is impossible due to Lemma 12.2.7. □

Proposition 12.2.2 is now fully proved.

12.2 (B) Rigid and Critical Rhombi

The rest of this section is devoted to the proof of the Shortest Turncycle Theorem 11.4.6.

Recall the polyhedron B of bounded hive flows associated with chosen partitions λ, μ, ν. Again we fix $f \in B_{\mathbb{Z}}$. Additionally, for the rest of this section we fix a secure turnpath p consisting of at least 3 turns and a *shortest* turnpath $q \in R_f^p$. We set

$$\varepsilon := \max\{t \in \mathbb{R} \mid f + t\pi(pq) \in B\}, \quad g := f + \varepsilon\pi(pq).$$

Then we have $\varepsilon > 0$ by Lemma 10.3.11 and $g \in B$. For the proof of the Shortest Turncycle Theorem 11.4.6 it suffices to show that $\varepsilon \geq 1$, since then $f + \pi(pq) \in B_{\mathbb{Z}}$.

If all rhombi are f-flat, then there exists no proper turncycle in R_f and we are done.

In the following we suppose that not all rhombi are f-flat. We shall argue indirectly and assume that $\varepsilon < 1$ for the rest of this subsection. After going through numerous detailed case distinctions, describing the possible local situations, we will finally end up with a contradiction, which then finishes the proof of the Shortest Turncycle Theorem 11.4.6. Our main tools will be Proposition 12.2.2 on special rhombi and the hexagon equality (Claim 10.2.10). Unfortunately, we see no way of considerably simplifying the tedious arguments.

12.2.8 Definition. A rhombus is called *critical* if it is not f-flat, but g-flat. Moreover, we call a rhombus *rigid* if it is both f-flat and g-flat. ∎

12.2.9 Claim. *There exists a critical rhombus.*

Proof. Let $S \neq \emptyset$ denote the set of rhombi which are not f-flat and consider the continuous function of $t \in \mathbb{R}$

$$F(t) := \min_{\varrho \in S} \sigma(\varrho, f + t\pi(pq)).$$

It is sufficient to show that $F(\varepsilon) = 0$.

By the definition of ε we have $F(\varepsilon) \geq 0$. Further, for $\varepsilon < t < 1$ we have $f + t\pi(pq) \notin B$. Since the flow $f + t\pi(pq)$ satisfies the border capacity constraints, there is a rhombus ϱ with $\sigma(\varrho, f + t\pi(pq)) < 0$. We must have $\varrho \in S$, since otherwise $\sigma(\varrho, pq) \geq 0$ (cf. Lemma 10.3.8), which would lead to the contradiction $\sigma(\varrho, f + t\pi(pq)) \geq 0$. We have thus shown that $F(t) < 0$. Since t can be arbitrarily close to ε, we get $F(\varepsilon) \leq 0$. Altogether, we conclude $F(\varepsilon) = 0$. □

12.2.10 Claim. *Each critical rhombus ϱ satisfies $\sigma(\varrho, pq) \leq -2$.*

Proof. We have $\sigma(\varrho, f) \geq 1$ and $\sigma(\varrho, f + \varepsilon\pi(pq)) = 0$, hence $\sigma(\varrho, pq) = -\frac{1}{\varepsilon}\,\sigma(\varrho, f)$. Using $0 < \varepsilon < 1$ we conclude $\sigma(\varrho, pq) < -1$. □

12.2.11 Lemma. *A rhombus ϱ is rigid iff it is f-flat and pq uses in it only neutral contributions. All special rhombi are rigid.*

Proof. The first assertion follows immediately from Lemma 10.3.5. The second assertion is a consequence of the first and Proposition 12.2.2(2). □

For the rest of this subsection, we denote by $\Diamond$ be a *first critical rhombus* visited by pq (a priori it might not be unique, because critical rhombi could overlap). By Claim 12.2.10, pq uses at least two negative slack contributions in $\Diamond$, cf. Lemma 10.3.5. In particular, pq uses at least one turnvertex among $\Diamond, \Diamond, \Diamond, \Diamond$: let $\Diamond \in \{\Diamond, \Diamond\}$ denote the *first one* used by pq in this set. (Rotating with 180° we may assume so without loss of generality.) Further, let $\Diamond$ denote the predecessor of $\Diamond$ in pq.

Our goal is to analyze the route of pq through ◊ and nearby rhombi. Narrowing down the possibilities will finally lead to a contradiction.

The next claim shows that ◊ is the first critical rhombus in which q uses turnvertices.

12.2.12 Claim. *The turnpath p does not use turnvertices in critical rhombi.*

Proof. Define $c := pq$. Assume that p uses turnvertices in a critical rhombus ◊. W.l.o.g. let p use a turnvertex in the hive triangle ◊ . If p uses ◊, then ◊$(c) = 1$. Since ◊$(c) \in \{-2, -1, 0, 1, 2\}$ we have $\sigma($◊$, c) \geq -1$. This implies that ◊ is not critical according to Claim 12.2.10. Analogous arguments show that p does not use ◊ or ◊.

If p uses ◊ and $\sigma($◊$, f) = 1$, then by construction of R_f^p, c uses no further turnvertex in ◊ and hence ◊ is not critical, in contradiction to the definition of ◊. If p uses ◊ and $\sigma($◊$, f) > 1$, then $\sigma($◊$, c) \leq -3$. This implies ◊$(c) = 2$ and ◊$(c) = 2$, which results in overlapping special rhombi ◊ and ◊. This is a contradiction to Proposition 12.2.2(3).

Assume that p uses ◊. If ◊$(c) = 2$, then q uses ◊ and ◊, in contradiction to p using ◊. Hence ◊$(c) \leq 1$. Since ◊ is critical, it follows ◊$(c) = 1$. But since p uses ◊, q uses both ◊ and ◊. Therefore, ◊ is f-flat. The hexagon equality (Claim 10.2.10) implies that either we have $\sigma($◊$, f) = 0$ and $\sigma($◊$, f) = 1$ or we have $\sigma($◊$, f) = 1$ and $\sigma($◊$, f) = 0$. In the former case, p continues with ◊. This implies that all turnvertices in the hive triangle ◊ are deleted in R_f^p, in contradiction to q using ◊. Therefore $\sigma($◊$, f) = 1$ and $\sigma($◊$, f) = 0$. But by construction of R_f, c uses ◊ . The turnpath p can continue as ◊ or as ◊, but both cases will yield a contradiction. Assume p continues as ◊. Then the hive triangle ◊ has no turnvertices in R_f^p, in contradiction to c using ◊ . So now assume that p continues as ◊. Since ◊ is f-flat, p continues as ◊, also in contradiction to c using ◊ .

The proof that p does not use ◊ is analogous to above argument.

Note that we used the fact that p consists of at least 3 turns. □

12.2.13 Claim. ◊ *is not g-flat.*

Proof. By way of contradiction, assume that ◊ is g-flat. Then ◊ and ◊ are both g-flat by Corollary 10.2.11 applied to g. So they are either rigid or critical. Since ◊ is not f-flat, it follows from Corollary 10.2.11 applied to f that not both ◊ and ◊ are rigid, so at least one of them is critical. It remains to exclude the following two cases:

If ◊ is critical, then the critical rhombus ◊ is passed by q before ◊, contradicting the minimal choice of ◊.

If ◊ is rigid, then ◊ $\notin V(R_f)$ by the definition of R_f and hence ◊ = ◊ and q passes the critical rhombus ◊ before ◊, because p cannot just consist of the turnvertex ◊. This again contradicts the minimal choice of ◊. □

12.2.14 Claim. *The turncycle pq goes directly from ◊ to ◊, which is the only time that pq leaves ◊ over ◊. Additionally, pq leaves ◊ exactly once over ◊. In particular, $\sigma($◊$, pq) = -2$, $\sigma($◊$, f) = 1$ and $\varepsilon = \frac{1}{2}$.*

Proof. We prove the following claims, tacitly using Proposition 12.2.2 on special rhombi and Claim 12.2.12:

- If pq leaves ◊ at ◊, then pq goes directly from ◊ to ◊. Proof: Otherwise pq could be rerouted via ◊, a contradiction.
- pq does not leave ◊ twice at ◊. Proof: Otherwise ◊ would be special and hence pq would use both ◊ and ◊. Then pq could be rerouted via ◊, a contradiction.

- pq does not leave $\Diamond$ twice at $\Diamond$. Proof: Otherwise $\Diamond$ would be special and hence rigid by Lemma 12.2.11. However, this is prohibited by Claim 12.2.13.

The fact that pq leaves $\Diamond$ over $\Diamond$ at most once and over $\Diamond$ at most once implies $\sigma(\Diamond, pq) = \Diamond(pq) + \Diamond(pq) \geq -2$. On the other hand, since $\Diamond$ is critical, we have $\sigma(\Diamond, pq) \leq -2$ by Claim 12.2.10. Therefore, $\sigma(\Diamond, pq) = -2$. Hence $\sigma(\Diamond, f) = -\varepsilon\sigma(\Diamond, pq) = 2\varepsilon$, so $\varepsilon = \frac{1}{2}\sigma(\Diamond, f)$ and since $0 < \varepsilon < 1$ we obtain $\sigma(\Diamond, f) = 1$ and $\varepsilon = \frac{1}{2}$. □

Despite the fact that, due to Claim 12.2.14, q enters $\Diamond$ at $\Diamond$ and leaves $\Diamond$ at $\Diamond$, pq cannot be rerouted via $\Diamond$ to a shorter turncycle, because q has minimal length. This can have two reasons, which leads to the following namings (compare the proof of Theorem 10.2.13)

> If $\Diamond$ is f-flat and pq uses $\Diamond$, then we say that pq *enters nonreroutably*, otherwise pq *enters reroutably*. If $\Diamond$ is f-flat and pq uses $\Diamond$, then we say that pq *leaves nonreroutably*, otherwise pq *leaves reroutably*.

An explanation of these namings can be found in the proof of the following claim.

12.2.15 Claim. *The turncycle pq enters nonreroutably or leaves nonreroutably.*

Proof. Assume the contrary, i.e., pq enters reroutably and leaves reroutably. Recall that $\Diamond$ is critical and hence not f-flat. We make a distinction of four cases, tacitly using Claim 12.2.12 for rerouting purposes.

1. Let $\Diamond$ be not f-flat and $\Diamond$ not be f-flat. Then pq can be rerouted using $\Diamond$.
2. Let $\Diamond$ be f-flat and $\Diamond$ not be f-flat. Then pq uses $\Diamond$ by our assumption on pq at the beginning of the proof. Hence pq can be rerouted with $\Diamond$.
3. Let $\Diamond$ not be f-flat and $\Diamond$ be f-flat. Assume that pq uses $\Diamond$. Then pq uses $\Diamond$, which is a contradiction to the fact that pq leaves reroutably. Hence pq uses $\Diamond$ and pq can be rerouted using $\Diamond$.
4. Let $\Diamond$ and $\Diamond$ be both f-flat. Then pq uses $\Diamond$ and $\Diamond$ by our assumption on p at the beginning of the proof. Hence pq can be rerouted with $\Diamond$. □

The possible reroutability of pq upon entering or leaving the critical rhombus $\Diamond$ gives rise to a distinction of four cases, one of which is dealt with in Claim 12.2.15. In the rest of this subsection we deal with the other three cases. But first we prove the following auxiliary Claim 12.2.16.

12.2.16 Claim. $\Diamond$ *is not rigid.*

Proof. Assume the contrary. Then, according to Lemma 12.2.11 and Claim 12.2.14, pq uses $\Diamond$. Hence pq enters nonreroutably. Corollary 10.2.11 implies that $\Diamond$ and $\Diamond$ are both g-flat. The hexagon equality (Claim 10.2.10) and $\sigma(\Diamond, f) = 1$ imply that $\sigma(\Diamond, f) + \sigma(\Diamond, f) = 1$. Integrality of f implies that one of the two shaded rhombi of $\Diamond$ and $\Diamond$ is critical and the other one is rigid.

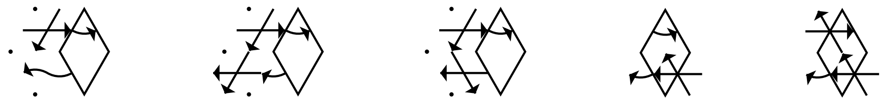
The rhombus $\Diamond$ cannot be critical since otherwise q would pass the critical $\Diamond$ before $\Diamond$, which contradicts the choice of $\Diamond$ as the first critical rhombus in which p uses turnvertices. (Note that we used here the lower bound on the number of turns of p.) Hence $\Diamond$ is rigid, which implies that $\Diamond$ is not a turnvertex of R_f. Thus pq uses $\Diamond$ and hence q passes the critical rhombus $\Diamond$ before $\Diamond$. Again, this is a contradiction. Here we used that p consists of at least 3 turns. □

12.2.17 Claim. *pq enters reroutably.*

Proof. We suppose the contrary, so assume that pq uses $\Diamond$ and $\Diamond$ is f-flat. Since $\Diamond$ is not rigid by Claim 12.2.16, pq must use a positive slack contribution in $\Diamond$. We claim that only $\Diamond$ is possible. This is shown by the following case distinction, leading to contradictions in all three cases.

1. ◊ uses a turnvertex already used by pq.
2. ◊ $\in pq$ implies that pq leaves ◊ over ◊ more than once, which is impossible due to Claim 12.2.14.
3. ◊ $\in pq$ contradicts Proposition 12.2.2(1).

The fact ◊ $\in pq$ implies that ◊ is special and hence we have ◊ $\in q$. Corollary 10.2.11 implies that both ◊ and ◊ are f-flat. Moreover, $\sigma($◊$, f) = 1$ (see Claim 12.2.14) and the hexagon equality (Claim 10.2.10) implies $\sigma($◊$, f) = 1$. Since ◊ $\notin V(R_f)$ and ◊ $\notin E(R_f)$, pq must leave ◊ at ◊ via ◊, see Figure 12.2.iii(a).



(a) (b) (c) (d) (e)

Figure 12.2.iii: The situations in the proof of Claims 12.2.17–12.2.18. We use straight arrows to depict parts of p which are contained in special rhombi.

- If pq continues from ◊ to ◊, then ◊ is special, see Figure 12.2.iii(b). Slack computation shows that $\sigma($◊$, pq) = -2$, which implies with $\varepsilon = \frac{1}{2}$ (Claim 12.2.14) that $\sigma($◊$, g) = 0$ and hence ◊ is critical. But q passes ◊ before ◊, in contradiction with the choice of ◊.
- If pq continues from ◊ to ◊, then ◊ is special (see Figure 12.2.iii(c)). Note that ◊ $\in pq$ implies that the rhombi ◊ and ◊ are not f-flat. Therefore we can reroute pq via ◊, which is in contradiction to the minimal length of q. □

Claim 12.2.15 and Claim 12.2.17 imply that pq enters reroutably and leaves nonreroutably. It remains to show that this leads to a contradiction. Since ◊ is f-flat and Claim 12.2.13 ensures that ◊ is not g-flat, pq must use a positive slack contribution in ◊.

12.2.18 Claim. *From the positive slack contributions of pq in ◊, only ◊ is possible.*

Proof. The turnedge ◊ uses a turnvertex already used by pq. The turnedge ◊ $\subseteq pq$ contradicts Proposition 12.2.2(1). Now assume that ◊ $\subseteq pq$. Then ◊ is special and in particular f-flat. Corollary 10.2.11 implies that ◊ and ◊ are f-flat. Since p enters reroutably, pq uses ◊, which is a contradiction to ◊ $\notin V(R_f)$. □

It follows that ◊ is special. The situation is depicted in Figure 12.2.iii(d). Now ◊ cannot continue to ◊, because, according to Claim 12.2.14, pq leaves ◊ over ◊ only once. Hence pq continues to ◊ and ◊ is special, see Figure 12.2.iii(e). But then, pq enters nonreroutably, which is a contradiction to Claim 12.2.17.

This shows that the assumption $\varepsilon < 1$ is absurd. Hence the Shortest Turncycle Theorem 11.4.6 is finally proved.

Remarks on the Shortest Turnpath Theorem 10.3.9. The Shortest Turnpath Theorem 10.3.9 is proved analogously to the Shortest Turncycle Theorem 11.4.6. The turnpath q is chosen from the set of shortest s-t-turnpaths instead of from the set $C'(R_f^p)$. The subtleties when using the length requirement of p get replaced by straightforward arguments about the behaviour of q at the border of Δ. We refer the reader to [BI12] for a detailed treatment of the proof of Theorem 10.3.9.

12.3 Proof of the Connectedness Theorem

Recall from (10.1.5) the linear isomorphism $\overline{F}(G) \simeq Z$ mapping flow classes f to their throughput function $E(\Delta) \to \mathbb{R}, k \mapsto \delta(k, f)$. The L_1-norm on Z induces the following *norm* on $\overline{F}(G)$: for $f \in \overline{F}(G)$, we set

$$\|f\| := \textstyle\sum_{k \in E(\Delta)} |\delta(k, f)|.$$

Correspondingly, we define the *distance* between $f, g \in \overline{F}(G)$ as $\mathrm{dist}(f, g) = \|g - f\|$,

In order to prove the Connectedness Theorem 10.2.14 we have to show that for all $f, g \in P(\lambda, \mu, \nu)_{\mathbb{Z}}$ there exists a finite sequence $f = f_0, f_1, f_2, \dots, f_\ell = g$ such that each f_{i+1} equals $f_i + c_i$ for some f_i-secure cycle c_i (cf. Proposition 10.2.9). We will construct this sequence with the additional property that $\mathrm{dist}(f_{i+1}, g) < \mathrm{dist}(f_i, g)$ for all i. To achieve this, it suffices to show the following Proposition 12.3.1.

12.3.1 Proposition. *For all distinct $f, g \in P(\lambda, \mu, \nu)_{\mathbb{Z}}$ there exists an f-secure cycle c such that $\mathrm{dist}(f + c, g) < \mathrm{dist}(f, g)$.*

In the rest of this section we prove Proposition 12.3.1 by explicitly constructing c. We fix $f, g \in P(\lambda, \mu, \nu)_{\mathbb{Z}}$ and set $d := g - f$. We can ensure the distance property $\mathrm{dist}(f + c, g) < \mathrm{dist}(f, g)$ for a proper cycle c by using the following result.

12.3.2 Lemma. *If more than half of the edges of a proper cycle c on G are contained in $\mathrm{supp}(d)$, then we have $\mathrm{dist}(f + c, g) < \mathrm{dist}(f, g)$.*

Proof. Let $K \subseteq E(\Delta)$ be the set of edges of Δ crossed by c. Then

$$\mathrm{dist}(f, g) - \mathrm{dist}(f + c, g) = \sum_{k \in E(\Delta)} \big(|\delta(k, d)| - |\delta(k, d - c)|\big) = \sum_{k \in K} \big(|\delta(k, d)| - |\delta(k, d - c)|\big).$$

But for edges $\diamond \in K$ we easily calculate

$$|\delta(\diamond, d)| - |\delta(\diamond, d - c)| = \begin{cases} 1 & \text{if } \mathrm{sgn}\big(\diamond(c)\big) = \mathrm{sgn}\big(\diamond(d)\big) \\ -1 & \text{if } \mathrm{sgn}\big(\diamond(c)\big) \neq \mathrm{sgn}\big(\diamond(d)\big) \end{cases}.$$

If more than half the edges of c are contained in $\mathrm{supp}(d)$, then also more than half of the summands are 1. The claim follows. □

In the light of Lemma 12.3.2, we will try to make c use as much edges contained in $\mathrm{supp}(d)$ as possible. Note that since f and g are both capacity achieving, we have that

$$\delta(k, d) = 0 \text{ for all edges } k \text{ at the border of } \Delta. \qquad (\dagger)$$

For the construction of c we distinguish two situations. The first one turns out to be considerably easier.

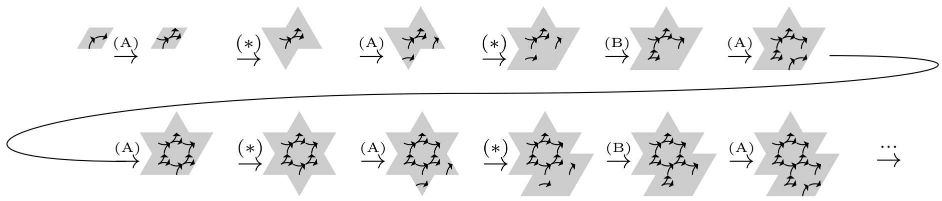
Situation 1. We assume that d does not cross the sides of f-flatspaces, that is,

$$\delta(k, d) = 0 \text{ for all diagonals } k \text{ of non-}f\text{-flat rhombi} \qquad (*)$$

12.3.3 Claim. *In the situation $(*)$, $\mathrm{supp}(d)$ does not contain a path in G consisting of two consecutive clockwise turns.*

Proof. Assume the contrary. We create a contradiction by using three type of arguments: (A) Lemma 10.2.5 on antipodal pairs, (B) the flow conservation laws, and the fact $(*)$.

The following sequence of pictures shows paths contained in $\mathrm{supp}(d)$ and how rules (A), (B) and $(*)$ imply that additional paths are contained in $\mathrm{supp}(d)$. All rhombi that are known to be f-flat are drawn shaded:

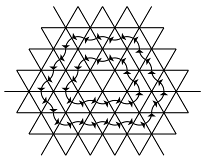


The process of repeatedly applying $\big[(\mathrm{A}), (*), (\mathrm{A}), (*), (\mathrm{B}), (\mathrm{A})\big]$ can be continued infinitely while extending the f-flat region to the lower right. This is a contradiction to the finite size of Δ. □

By Lemma 10.1.1 we have a decomposition $d = \sum_j \alpha_j c_j$ with $\alpha_j > 0$ and cycles c_j in G that are contained in $\mathrm{supp}(d)$. According to $(*)$, each c_j runs in a single f-flatspace and does not cross any f-flatspace border. Let c be any of the cycles c_j and suppose that c runs inside the f-flatspace L. Claim 12.3.3 implies that c runs in counterclockwise direction. We will next show that L is a hexagon.

Let γ denote the polygon (without self-intersections) obtained from c by linearly interpolating between the successive white vertices of c. Following the white vertices of c (in counterclockwise order) reveals that two consecutive counterclockwise turns lead to an angle of 120° in γ. Further, an alternating sequence of clockwise and counterclockwise turns in c is represented by a line segment in γ. By an elementary geometric argument we see that γ must be a hexagon.

Let $\tilde{c}$ be the counterclockwise cycle *surrounding* c: more specifically, $\tilde{c}$ consists of the clockwise antipodal contributions of all counterclockwise turns in c and, additionally, of the necessary counterclockwise turns in between, as illustrated below:



The Flow Propagation Lemma 10.2.6 implies that all turns of $\tilde{c}$ lying inside L are contained in $\mathrm{supp}(d)$. Hence, by $(*)$, $\tilde{c}$ cannot pass the border of L. Therefore, $\tilde{c}$ either lies completely inside L or completely outside L. If $\tilde{c}$ lies completely inside L, we can form the cycle surrounding $\tilde{c}$ and continue inductively, until we find a cycle $c' \subseteq \mathrm{supp}(d)$ which lies inside L and such that $\tilde{c}'$ lies outside L. Since the polygon γ' corresponding to c' is a hexagon, it follows that L must be a hexagon. Summarizing, we see that the cycle c' runs counterclockwise through the border triangles of a hexagon L. Such c' is clearly f-secure. Moreover, since $c' \subseteq \mathrm{supp}(d)$, we have $\mathrm{dist}(f + c', g) < \mathrm{dist}(f, g)$ by to Lemma 12.3.2. This proves Proposition 12.3.1 in Situation 1.

Situation 2. We now treat the case where d has nonzero throughput through some edge k of an f-flatspace L. By (†), k is not at the border of Δ. By Lemma 10.3.14, we can assume w.l.o.g. that k is an L-entrance edge and $\delta(k, \to L, d) > 0$. Let $p \subseteq \mathrm{supp}(d)$ be a turn in L starting at k.

We will show that the following Algorithm 6 constructs a desired c.

Algorithm 6 FINDSECURECYCLE

Input: $f, g \in P(\lambda, \mu, \nu)_{\mathbb{Z}}$, an L-entrance edge k such that $\delta(k, \to L, d) > 0$, where $d := g - f$, and a turn p in $\mathrm{supp}(d)$ starting at k.
Output: An f-secure cycle c such that $\mathrm{dist}(f + c, g) < \mathrm{dist}(f, g)$.

1: **while** p does not contain a vertex more than once **do**
2: **if** one can append to p a turn ϑ contained in $\mathrm{supp}(d)$ such that, after appending, p does not use a negative contribution in any f-flat rhombus **then**
3: Append ϑ to p.
4: **else**
5: Append a clockwise turn followed by a counterclockwise turn to p.
6: **end if**
7: **end while**
8: Generate a cycle c from the edges of p by “truncation and concatenation”.
9: **return** c.

We postpone the definition of the procedure used in line 8 for the construction of c from p. Later on, we will give a precedence rule to determine what should happen in line 2 when both turns, clockwise and counterclockwise, are possible to append.

What strikes about Algorithm 6 is that it is a priori unclear that line 5 can be executed (without p leaving Δ). We next explain why this is the case.

To ease notation we index the intermediate results that occur during the construction of p by $p_0, p_1, \ldots$, where p_i either has one or two more turns than p_{i-1}, depending on whether in the while loop there has been appended only one turn or (in case of line 5) two turns to p_{i-1}. The paths q_i are defined such that each p_{i+1} is the result of the concatenation of p_i and q_i. We denote by the term *swerve* each q_i that is not a single turn, i.e., those q_i that consist of a clockwise turn followed by a counterclockwise turn. For a swerve q_i we denote by $\varrho(q_i)$ the rhombus in which both turns of q_i lie.

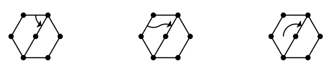
12.3.4 Claim. *For all i we have the following properties:*

(1) Let ◊ $\in \{$◊, ◊$\}$ denote the last turn of p_i and suppose that line 5 is about to be executed. Then $\lozenge \in E(\Delta)$ is not at the border of Δ, which means that $q_i =$ ◊ can be appended to p_i in line 5 without leaving Δ.
(2) The first and last edge of each q_i are contained in $\mathrm{supp}(d)$.
(3) Each p_i does not use negative contributions in f-flat rhombi.
(4) The rhombus $\varrho(q_i)$ is f-flat for each swerve q_i.

Before proving Claim 12.3.4 we start out with a fairly easy lemma that will prove useful.

12.3.5 Lemma. *Given a walk p in G starting with a turn at a side of an f-flatspace, for some fixed $f \in B$. Further assume that p does not use negative contributions in f-flat rhombi. If the trapezoid consists of two overlapping f-flat rhombi, then p does not end with one of the two turns .*

Proof. According to the hexagon inequality (Claim 10.2.10), both trapezoids are f-flat. Note that the following three possible cases, which could precede , all use negative contributions in f-flat rhombi, which contradicts our assumption:



□

Proof of Claim 12.3.4. We prove all claims simultaneously by induction on i. If q_{i-1} is a single turn, then $q_{i-1} \subseteq \mathrm{supp}(d)$ and (by definition of the if-clause in Algorithm 6) p_i does not use any negative contributions in f-flat rhombi, which proves (2) and (3) in this case.

It remains to consider the case where q_{i-1} is a swerve, that is, line 5 is about to execute. Let $\lozenge \in \{\lozenge, \lozenge\}$ be the last turn of p_{i-1}. The induction hypothesis (2) ensures $\lozenge(d) > 0$.

We first show (1). For the sake of contradiction, we assume that the edge $\lozenge \in E(\Delta)$ is at the border of Δ. Then, considering (†), it follows that $\lozenge \not\subseteq \mathrm{supp}(d)$, but $\lozenge \subseteq \mathrm{supp}(d)$. Thus Algorithm 6 uses line 3 and $q_{i-1} = \lozenge$. This is a contradiction to the assumption that line 5 is about to be executed. Hence $\lozenge \in E(\Delta)$ is not at the border of Δ. This proves (1).

It remains to show (2), (3) and (4). The fact that line 5 is about to execute can have the following two reasons (a) and (b):

(a) $\lozenge \subseteq \mathrm{supp}(d)$, but $\lozenge$ cannot be appended to p_{i-1} in line 3.
Then $\lozenge$ is f-flat and $\lozenge = \lozenge \subseteq \mathrm{supp}(d)$ as this turn was appended in line 3. Lemma 12.3.5 applied to $\lozenge$ yields that $\lozenge$ is not f-flat. Since $\lozenge \subseteq \mathrm{supp}(d)$ we have $\lozenge \subseteq \mathrm{supp}(d)$ by Lemma 10.2.5. Therefore, p_{i-1} can be continued via $q_{i-1} = \lozenge$ in line 3, in contradiction to the fact that line 5 is about to execute.

(b) $\lozenge \subseteq \mathrm{supp}(d)$, but $\lozenge$ cannot be appended to p_{i-1} in line 3.
Then $\lozenge$ is f-flat, which shows (4). Lemma 10.2.5 implies that $\lozenge \subseteq \mathrm{supp}(d)$. In line 5, the turns $q_{i-1} = \lozenge$ are appended to p_{i-1}, which shows (2). It remains to show that appending q_{i-1} does not result in negative contributions in f-flat rhombi. But if $\lozenge$ leads to a negative contribution in an f-flat rhombus, then $\lozenge$ is f-flat and if $\lozenge$ leads to a negative contribution in an f-flat rhombus, then $\lozenge$ is f-flat. In both cases, this contradicts Lemma 12.3.5, for the f-flat trapezoid $\lozenge$ and $\lozenge$, respectively. This shows (3). □

We specify now the precedence rule (‡) for breaking ties in line 2 of Algorithm 6.

> If p_{i-1} ends at the diagonal of an f-flat rhombus, then Algorithm 6 appends *counterclockwise* turns; if p_{i-1} ends at the diagonal of a non-f-flat rhombus, then Algorithm 6 appends *clockwise* turns. (‡)

Finally, to fully specify Algorithm 6, we now define how the cycle c is generated from p in line 8: When line 8 is about to execute, then p has used a vertex more than once. Let v denote the first vertex of p that is used more than once. Note that v is a black vertex. Let q denote the q_i that was appended last. We note that q consists of either two edges or four edges. Now we truncate everything of p previous to the first occurence of v and everything after the last occurence of v, thus generating the cycle c. We denote by ϑ the first turn of p that uses v and by ϑ' the turn of c that uses v.

For example, suppose p uses the swerve $\lozenge$ and $q = \lozenge$. Then $\vartheta = \lozenge$ and $\vartheta' = \lozenge$. Note that, in this case, the turn ϑ' is contained in c but not contained in p.

Since p uses no negative contributions in f-flat rhombi, the first assertion of Claim 12.3.6 below is plausible, but needs proof as c may contain turns that are not contained in p. In fact, we must ensure that no negative contributions exist in c "near v".

12.3.6 Claim. *(1) The cycle c is f-hive preserving.*
(2) If ϑ' is not used by p, then $\vartheta' \subseteq \mathrm{supp}(d)$.

Let us first show that once Claim 12.3.6 is shown, we are done.

Proof of Proposition 12.3.1. We first show that Algorithm 6 produces an f-secure cycle c. Claim 12.3.6 already tells us that c is f-hive preserving. Assume that c uses both $\lozenge$ in a rhombus $\lozenge$. Claim 12.3.4(2) and Claim 12.3.6(2) imply that at least the second edge of every counterclockwise turn in c is contained in $\mathrm{supp}(d)$. Hence $\lozenge(d) > 0$ and $\lozenge(d) > 0$, which implies $\sigma(\lozenge, d) \leq -2$. The fact $\sigma(\lozenge, f + d) \geq 0$ implies $\sigma(\lozenge, f) \geq 2$ and hence $\lozenge$ is not nearly f-flat. It follows that c is f-secure.

Claim 12.3.6(2) combined with Claim 12.3.4(2) also ensures that the only turns in c, that are not contained in $\mathrm{supp}(d)$, are turns of swerves. Hence at least half of the edges of c are contained in $\mathrm{supp}(d)$. This inequality is strict, because c cannot consist of swerves only. Lemma 12.3.2 implies $\mathrm{dist}(f + c, g) < \mathrm{dist}(f, g)$. □

From now on, swerves (e.g.) will be drawn as straight arrows with a filled triangular head, e.g. . (They are not to be confused with throughput arrows like , which have a different head and are always drawn crossing fat edges.)

Proof of Claim 12.3.6. Since p uses no negative contributions in f-rhombi by construction, the proof that the cycle c is f-hive preserving breaks down into the following parts:

(neg1) The turn ϑ' is not counterclockwise at the acute angle of an f-flat rhombus.

(neg2) The turn ϑ' is not both clockwise and preceded in c by another clockwise turn such that both turns lie in the same f-flat rhombus.

(neg3) The turn ϑ' is not both clockwise and succeded in c by another clockwise turn such that both turns lie in the same f-flat rhombus.

We also need to prove the following property:

(su) If ϑ' is not used by p, then $\vartheta' \subseteq \mathrm{supp}(d)$.

Recall that q denotes the q_i which was appended last. Three cases can appear: (a) q is a counterclockwise turn, (b) q is a clockwise turn, (c) q is a swerve. All three cases are significantly different and require careful attention to detail. We start with the simplest one, which does not require the precedence rule (‡):

(a) Assume that q is a counterclockwise turn, pictorially $q =$. Considering Algorithm 6, we see that $q \subseteq \mathrm{supp}(d)$. There are two possibilities for ϑ: $\vartheta =$ or $\vartheta =$, because the other four turns lead to a contradiction to the fact that Algorithm 6 stops as soon as p contains a vertex twice.

(a1) Suppose first $\vartheta =$. In this case, we have $\vartheta' = q$. The statement (neg1) holds, because ϑ' is used by p and p uses no negative contributions in f-flat rhombi. We also see that (su) holds in this case, because ϑ' is used by p.

(a2) Suppose now $\vartheta =$. Since $(d) > 0$, it follows that is part of a swerve: The situation of p can be depicted as . By construction, $\vartheta' =$, so ϑ' consists of an edge of q and the last edge of a swerve. Hence $q \subseteq \mathrm{supp}(d)$ and (su) follows in this case. It remains to verify (neg2) and (neg3). Note that Algorithm 6 ensures that the counterclockwise turn q is not a negative contribution in f-flat rhombi and hence the shaded rhombus is not f-flat. This proves (neg3). If we assume the contrary of (neg2), then the path is a negative contribution in an f-flat rhombus. But since swerves lie in f-flat rhombi according to Claim 12.3.4(4), it follows that the trapezoid is f-flat, which is a contradiction to Lemma 12.3.5, applied to . This proves (neg2).

(b) Assume that q is a clockwise turn, pictorially $q =$. As in case (a), we have two possibilities: $\vartheta =$ or $\vartheta =$. Since $q \subseteq \mathrm{supp}(d)$, we have $(d) > 0$. If we had $\vartheta =$, then $(d) > 0$, which is a contradiction. Hence $\vartheta =$ and thus $\vartheta' = q$. This proves (su) in this case. The fact that $\vartheta' = q$ is a part of p shows (neg2). It remains to show (neg3). Assume the contrary. Then the rhombus is f-flat. Hence $(d) > 0$ implies $(d) > 0$. The rhombus is not f-flat by Lemma 12.3.5 applied to . But the precedence rule (‡) of Algorithm 6 implies that p continues from with the counterclockwise turn . This is a contradiction, proving (neg3) in this case.

(c) Assume that q is a swerve, pictorially $q =$. The rhombus which contains the swerve is f-flat by Claim 12.3.4(4). Since p does not use negative contributions in f-flat rhombi, we get that is not f-flat. The possibilities for ϑ here are , , (note that is ruled out, because is f-flat). We distinguish the following three cases:

(c1) Suppose $\vartheta =$. Here $\vartheta' =$, which is part of q. This proves (su) in this case. The fact that is not f-flat implies (neg1) in this case.

(c2) Suppose $\vartheta =$. Here $\vartheta' =$, which is part of q, which again proves (su) in this case. The fact (neg2) follows because p uses no negative contributions in f-flat rhombi. It remains to show (neg3). Note that $(d) > 0$, because the first edge of is contained in $\mathrm{supp}(d)$. Further, $(d) > 0$, because the second edge of the counterclockwise turn is contained in $\mathrm{supp}(d)$ (this is always the case for counterclockwise turns by construction of p). Hence $\subseteq \mathrm{supp}(d)$. If were not f-flat, then Algorithm 6 would have appended the clockwise turn over appending the swerve . Hence is f-flat. The hexagon equality (Claim 10.2.10) implies that the trapezoid if f-flat. The fact (neg3) follows from Lemma 12.3.5 applied to .

(c3) Suppose $\vartheta =$. Here $\vartheta' =$, which is a negative contribution in the f-flat rhombus . Hence we need to show that this case leads to a contradiction. Recall that is not f-flat. Clearly, $(d) > 0$ and $(d) > 0$, which implies $(d) > 0$. This means that ϑ is preceded in p by the counterclockwise turn . This is a contradiction to the precedence rule (‡), because Algorithm 6 would have chosen instead of . □

12.4 Proof of the King-Tollu-Toumazet Conjecture

In this section we prove the following Theorem 12.4.1, conjectured by King, Tollu, and Toumazet in [KTT04].

12.4.1 Theorem ([Ike12])**.** *Given partitions λ, μ and ν such that $|\nu| = |\lambda| + |\mu|$. Then $c^{\nu}_{\lambda\mu} = 2$ implies $c^{M\nu}_{M\lambda M\mu} = M + 1$ for all $M \in \mathbb{N}$.*

More precisely, we prove the following equivalent geometric formulation.

12.4.2 Theorem. *Let $c^{\nu}_{\lambda\mu} = 2$ and let f_1 and f_2 be the two integral points of $P(\lambda, \mu, \nu)$. Then $P(\lambda, \mu, \nu)$ is exactly the line segment between f_1 and f_2.*

12.4.3 Claim. *Theorem 12.4.2 is equivalent to Theorem 12.4.1.*

Proof. Let $c^{\nu}_{\lambda\mu} = 2$ and let $P = P(\lambda, \mu, \nu)$. For every natural number M, the stretched polytope MP contains at least the $M + 1$ integral points

$$\kappa_M := \{Mf_1,\ (M-1)f_1 + f_2,\ (M-2)f_1 + 2f_2,\ \ldots,\ f_1 + (M-1)f_2,\ Mf_2\}$$

and hence $c^{M\nu}_{M\lambda M\mu} \geq M + 1$.

If P contains a point besides the line segment between f_1 and f_2, then P contains a rational point x besides this line segment. But then there exists M such that Mx is integral and hence $c^{M\nu}_{M\lambda M\mu} > M + 1$.

On the other hand, if P contains no other point besides the line segment, then it is easy to see that these $M + 1$ points are the only integral points in MP and hence $c^{M\nu}_{M\lambda M\mu} = M + 1$. □

From now on, fix partitions λ, μ and ν such that $c^{\nu}_{\lambda\mu} = 2$. Let f_1, f_2 be the two flows in $P(\lambda, \mu, \nu)_{\mathbb{Z}}$. Let $c_1 := f_2 - f_1$ denote the f_1-secure cycle that connects the integral points in P, and analogously define $c_2 := -c_1 = f_1 - f_2$ the f_2-secure cycle running in the other direction. W.l.o.g. c_1 runs in counterclockwise direction, otherwise we switch f_1 and f_2.

For $M \in \mathbb{N}$, the stretched polytope MP contains at least the set of integral points

$$\kappa_M := \{Mf_1 + mc_1 \mid m \in \mathbb{N},\ 0 \le m \le M\}.$$

To prove Theorem 12.4.2 it remains to show that for all $M \in \mathbb{N}_{\ge 2}$ these are the only integral points in MP. According to the Connectedness Theorem 10.2.14, this is equivalent to the statement that for each integral flow $\xi \in \kappa_M$ the neighborhood $\Gamma(\xi)$ is contained in κ_M. To prove this, we fix some $M \in \mathbb{N}_{\ge 2}$. Now we choose an arbitrary $\xi \in \kappa_M$, which means choosing a natural number $0 \le m \le M$ such that $\xi = Mf_1 + mc_1$. If $\xi \in \{Mf_1, Mf_2\}$, we say that ξ is *extremal*, otherwise ξ is called *inner*. We are interested in the neighborhood $\Gamma(\xi)$.

If $\xi \ne f_2$, then $\xi + c_1 = Mf_1 + (m+1)c_1 \in MP$, which implies that the cycle c_1 is ξ-secure by Proposition 10.2.9. Analogously, c_2 is ξ-secure for $\xi \ne f_1$. It follows that both c_1 and c_2 are ξ-secure for an inner ξ. It remains to show that for an inner ξ the only ξ-secure proper cycles are c_1 and c_2, and for both $\iota \in \{1, 2\}$ the only f_ι-secure proper cycle is c_ι. In fact, we are going to show the following slightly stronger statement.

> For an inner ξ, the proper cycles c_1 and c_2 are the only ξ-hive preserving proper cycles. For both $\iota \in \{1, 2\}$ the only f_ι-hive preserving proper cycle is c_ι. (※)

Note that a proper cycle c is ξ-hive preserving iff c is $\frac{\xi}{M}$-hive preserving. Hence for proving (※) we can assume w.l.o.g. that $\xi = xf_1 + (1-x)f_2$ with $0 \le x \le 1$ is a convex combination of f_1 and f_2. To classify all ξ-hive preserving proper cycles we use the following important proposition which we prove in the next subsection.

12.4.4 Proposition. *Let $c^\nu_{\lambda\mu} = 2$ and let f_1 and f_2 be the two integral hive flows in $P(\lambda, \mu, \nu)$. For each convex combination $\xi = xf_1 + (1-x)f_2$, $0 \le x \le 1$ we have that each ξ-hive preserving proper cycle is* either *f_1-hive preserving* or *f_2-hive preserving.*

According to Proposition 12.4.4, for proving (※) it suffices to show that c_1 is the only f_1-hive preserving proper cycle and c_2 is the only f_2-hive preserving proper cycle. But this can be seen with the following key lemma, whose proof we postpone to Subsection 12.4 (B).

12.4.5 Key Lemma. *Let $c^\nu_{\lambda\mu} = 2$ and let f_1 and f_2 be the two integral points of $P(\lambda, \mu, \nu)$. For each $\iota \in \{1, 2\}$ each planar turncycle in R_{f_ι} is f_ι-secure.*

We apply Key Lemma 12.4.5 as follows. Let $\iota \in \{1, 2\}$ and let $c' \ne c_\iota$ be an f_ι-hive preserving proper cycle. Then Key Lemma 12.4.5 implies that c' is f_ι-secure. The fact that $c^\nu_{\lambda\mu} = 2$ implies $c' = c_\iota$. Therefore we have shown that c_ι is the only f_ι-hive preserving proper cycle and are done proving Theorem 12.4.2.

It remains to prove Key Lemma 12.4.5 and Proposition 12.4.4.

12.4 (A) Proof of Proposition 12.4.4

In this subsection we already use Key Lemma 12.4.5, which is proved later in Subsection 12.4 (B).

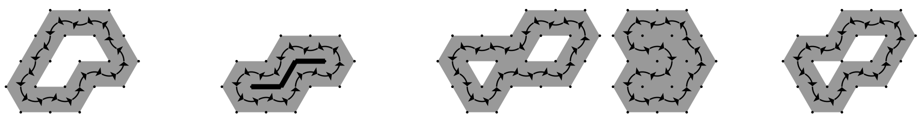
We first prove Proposition 12.4.4 for extremal $\xi \in \{f_1, f_2\}$. In this case it remains to show that there is no proper cycle that is both f_1-hive preserving and f_2-hive preserving. Indeed, for the sake of contradiction, assume there is such a cycle c. Key Lemma 12.4.5 implies that c is both f_1-secure and f_2-secure, in contradiction to $c^\nu_{\lambda\mu} = 2$.

From now on, we assume that ξ is inner. Recall that in this case both c_1 and c_2 are ξ-hive preserving.

First of all, we note that if a rhombus $\Diamond$ is f_1-flat and f_2-flat, then $\Diamond$ is also ξ-flat, because ξ lies between f_1 and f_2. The converse is also true: if a rhombus $\Diamond$ is ξ-flat, then $\Diamond$ is both f_1-flat and f_2-flat, because $\sigma(\Diamond, \xi) = 0$ and $\sigma(\Diamond, f_1) > 0$ would imply $\sigma(\Diamond, f_2) < 0$, in contradiction to $f_2 \in P$.

From this consideration, it follows that if a cycle c is f_1-hive preserving or f_2-hive preserving, then it is ξ-hive preserving as well.

In the following Claim 12.4.7 we begin to rule out the existence of ξ-hive preserving cycles whose curves have no intersection with the curve of c_1. We first introduce some terminology (see Figure 12.4.i): The set of hive triangles from which c_1 uses



(a) A pipe enclosing a single region, for example for $c^{(4,3,3,2,2,1)}_{(3,2,2,2)(2,2,1,1)}$. The pipe is thin.

(b) A pipe enclosing no region, for example for $c^{(14,14,8,8,5,4)}_{(6,6,6,3)(11,8,8,2,2,1)}$. The pipe is thick and the path ζ is drawn fat.

(c) The left pipe encloses two regions. The right one encloses no region. Both pipes are neither thin nor thick.

(d) A pipe enclosing two regions. The pipe is neither thin nor thick.

Figure 12.4.i: Pipes and their enclosed regions. The terms "thin" and "thick" refer to Definition 12.4.8.

turns is called the *pipe*. Note that this coincides with the set of hive triangles from which c_2 uses turns. A hive triangle of the pipe is called a *pipe triangle*. The pipe partitions the plane into several connected components: The pipe itself, the *outer region* and the *inner regions* enclosed by the pipe. The *pipe border* is defined to be the set of edges between the regions and can be divided into the *inner pipe border* and the *outer pipe border*.

12.4.6 Claim. *Rhombi whose diagonal lies on the outer pipe border are not f_1-flat. Likewise, rhombi whose diagonal lies on the inner pipe border are not f_2-flat.*

Proof. This follows directly from c_1 traversing the pipe in counterclockwise direction. □

12.4.7 Claim. *Each ξ-hive preserving cycle uses a pipe triangle.*

Proof. First we show that each ξ-hive preserving cycle that runs only in the outer region is also f_1-hive preserving and each ξ-hive preserving cycle that runs only in an inner region is also f_2-hive preserving: Recall that the flows ξ, f_1, and f_2 only differ by multiples of c_1. So for a rhombus in which both hive triangles are not pipe triangles, ξ-flatness, f_1-flatness, and f_2-flatness coincide. The first claim follows with Claim 12.4.6.

For the sake of contradiction, assume now the existence of a ξ-hive preserving cycle c that uses no pipe triangle. Then c runs only in the outer region and is thus f_1-hive preserving, or c runs only in the inner region and is thus f_2-hive preserving. Key Lemma 12.4.5 ensures that c is f_1-secure or f_2-secure, which is a contradiction to $c^{\nu}_{\lambda\mu} = 2$. □

We show now that there are severe restrictions on the possible shape of the pipe, forcing it to have at most one inner region. The upcoming Claim 12.4.9 shows that the following two fundamentally different types, introduced in the next definition, are the only types of pipes that can appear.

12.4.8 Definition. We call the pipe *thin* if it has the following property, see Figure 12.4.i(a): Two pipe triangles share a side iff they are direct predecessors or direct successors when traversing c_1. Additionally, we require that the pipe encloses a single inner region, see Figure 12.4.i(c) and Figure 12.4.i(d) for counterexamples.

We call the pipe *thick* if it has the following property, see Figure 12.4.i(b): There exists a path ζ in Δ, called the *center curve*, such that ζ has only obtuse angles of 120° and the path c_1 runs around ζ as indicated in Figure 12.4.i(b). Additionally, we require that two pipe triangles Δ_1 and Δ_2 share a side $k \in E(\Delta)$ iff either k is an edge of ζ or Δ_1 and Δ_2 are direct predecessors or direct successors when traversing c_ι. The center curve may consist of a single vertex only. ■

12.4.9 Claim. *The pipe is either thin or thick.*

Proof. First of all, assume that c_1 runs as depicted in . Then c_1 can be rerouted in R_{f_1} to $\bar{c}$ via . However, $\bar{c}$ is f_1-hive preserving and Key Lemma 12.4.5 implies that $\bar{c}$ is f_1-secure, in contradiction to $c_{\lambda\mu}^{\nu} = 2$. This excludes the cases in Figure 12.4.i(d).

Now suppose that the pipe is not thin, i.e., two adjacent pipe triangles are not direct successors when traversing c_1. It is a simple topological fact (cp. Remark 10.2.4) that then there is a rhombus where c_1 or c_2 runs like . We choose $\iota \in \{1, 2\}$ such that c_ι runs like . So is not f_ι-flat. It suffices to construct the center curve ζ, which contains the diagonal . To achieve this, we show that "c_ι cannot diverge", which precisely means the following: If c_ι uses , then c_ι uses and ζ continues with ; if c_ι uses , then c_ι uses and ζ continues with ; and also both situations rotated by 180°.

We treat only one case, the others being similar. So let c_ι use . If is not f_ι-flat, then c_ι can be rerouted via to give an f_ι-hive preserving cycle $\bar{c}$. Key Lemma 12.4.5 ensures that $\bar{c}$ is f_ι-secure. This is a contradiction to $c_{\lambda\mu}^{\nu} = 2$. Therefore, is f_ι-flat. If c_ι uses , then c_ι can be rerouted via , again in contradiction to $c_{\lambda\mu}^{\nu} = 2$. Therefore c_ι uses . Since c_ι uses , it follows that is not f_ι-flat. The hexagon equality (Claim 10.2.10) implies that is not f_ι-flat. If were not f_ι-flat, then c_ι could be rerouted via , which would again result in a contradition to $c_{\lambda\mu}^{\nu} = 2$. Therefore is f_ι-flat. But this implies that c_ι uses . □

If the pipe is thin, then we have exactly one inner region by definition. If the pipe is thick, then we have no inner region. The center curve of the thick pipe is *not* considered part of the pipe border. In fact, the thick pipe is defined to have an empty inner pipe border.

An *f-flatspace side* is defined to be a side of an f-flatspace, defined in Subsection 10.3 (B). The pipe also has *sides*, which are defined to be maximal line segments of the pipe border. *Inner pipe sides* are contained in the inner pipe border, while *outer pipe sides* are contained in the outer pipe border.

12.4.10 Claim. *All outer pipe sides can be partitioned into f_1-flatspace sides and all inner pipe sides can be partitioned into f_2-flatspace sides.*

Proof. Claim 12.4.6 implies that all edges of outer pipe sides belong to f_1-flatspace sides. We now prove that the f_1-flatspace sides do not exceed the pipe sides. This is easy to see by looking at the following example, which represents the general case. Consider c_1 and the pipe side a: . Then the following edges are diagonals of non-f_1-flat rhombi: . This proves that a can be partitioned into f_1-flatspace sides. □

Proof of Proposition 12.4.4. **Case 1:** We first analyze ξ-hive preserving cycles that use pipe triangles only. If the pipe is *thin*, we note that a cycle that uses only pipe triangles necessarily equals c_1 or c_2. Since c_ι is f_ι-hive preserving, the assertion follows.

Now assume that the pipe is *thick* and assume by way of contradiction that there is a ξ-hive preserving cycle c that uses only pipe triangles but is neither f_1-hive preserving nor f_2-hive preserving. Hence there is a rhombus ϱ_1 such that $\sigma(\varrho_1, f_1) = 0$ and $\sigma(\varrho_1, c) < 0$. If we had $\sigma(\varrho_1, f_2) = 0$, then $\sigma(\varrho_1, \xi) = 0$, which contradicts $\sigma(\varrho_1, c) < 0$ as c is ξ-hive preserving. Therefore $\sigma(\varrho_1, f_2) > 0$. Similarly, by our assumption, there is also a rhombus ϱ_2 with $\sigma(\varrho_2, f_2) = 0$ and $\sigma(\varrho_2, c) < 0$. We know that c must use a negative contribution in ϱ_1 and in ϱ_2. We can now analyze all positions in which ϱ_1 and ϱ_2 can lie in the thick pipe and after a detailed but rather straightforward case distinction, which we omit here, we end up with a contradiction to $c^\nu_{\lambda\mu} = 2$.

Case 2: For the sake of contradiction, we now assume the existence of a ξ-hive preserving cycle c that does not use pipe triangles only. Claim 12.4.7 implies that c uses at least one pipe triangle. Since c does not use pipe triangles only, it follows that c crosses the pipe border. Use Claim 12.4.10 and choose $\iota \in \{1, 2\}$ such that c crosses the pipe border through an f_ι-flatspace side a. Choose $\tilde{x} \in \{x, 1-x\}$ such that $\xi = f_\iota + \tilde{x} c_\iota$. We have $f_\iota + \tilde{x} c_\iota + \varepsilon c \in P$ for a small $\varepsilon > 0$ and hence $d := \tilde{x} c_\iota + \varepsilon c$ is f_ι-hive preserving. Corollary 10.3.23 applied to f_ι and d ensures the existence of a turncycle in R_{f_ι} that crosses the side a. According to Claim 12.4.10, a is contained in the pipe border. Let $\bar{c}$ be a shortest turncycle in R_{f_ι} that crosses the pipe border. According to Proposition 12.2.2 (for turnpaths of length 0), $\bar{c}$ uses no reverse turnvertices and all self-intersections of the curve of $\bar{c}$ can only happen in $(\bar{c}, f_\iota)$-special rhombi, defined as follows:

For a turncycle c and a flow f we call a rhombus $\Diamond$ (c, f)-*special*, if (1) $\Diamond$ is f-flat and (2) the diagonal of $\Diamond$ is crossed by c via $\Diamond$, $\Diamond$, $\Diamond$, or $\Diamond$ and (3) c does not use any additional turnvertex in $\Diamond$.

First assume that $\bar{c}$ is planar. But in this case, Key Lemma 12.4.5 implies that $\bar{c}$ is f_ι-secure, in contradiction to $c^\nu_{\lambda\mu} = 2$.

Now assume that $\bar{c}$ is not planar, i.e., that the curve of $\bar{c}$ has self-intersections. We will refine Key Lemma 12.4.5 to suit our needs (see Lemma 12.4.11 below). To achieve this, we now precisely analyze the situation at the self-intersections. We choose a $(\bar{c}, f_\iota)$-special rhombus $\Diamond$ and reroute $\bar{c}$ in $\Diamond$ to obtain a shorter turncycle c' in R_{f_ι} as follows:


(‡)

Note that this rerouting is the unique way to reroute in an f_ι-flat rhombus $\Diamond$ such that the resulting turncycle uses no negative slack contribution in $\Diamond$. Because of the minimal length of $\bar{c}$, the turncycle c' does not cross the pipe border. According to Claim 12.4.7, c' uses pipe triangles only. We now show that c' is planar.

If the pipe is thin, this is obvious. Now assume that the pipe is thick. For the sake of contradiction, assume that c' is not planar. Since c' was obtained by rerouting from $\bar{c}$, self-intersections of the curve of c' can only appear in (c', f_ι)-special rhombi. As c' uses pipe triangles only, the diagonals of (c', f_ι)-special rhombi are contained in the center curve, see Figure 12.4.ii. But rerouting iteratively at these rhombi finally results in an planar f_ι-hive preserving turncycle, which is shorter than c_ι. According to Key Lemma 12.4.5, this turncycle is f_ι-secure, in contradiction to $c^\nu_{\lambda\mu} = 2$.

Hence c' is planar in both pipe cases. Since c' is f_ι-hive preserving, it is f_ι-secure (Key Lemma 12.4.5). The fact $c^\nu_{\lambda\mu} = 2$ implies that c' coincides with c_ι. Thus $\bar{c}$ reroutes to c_ι, no matter in which $(\bar{c}, f_\iota)$-special rhombus we reroute.

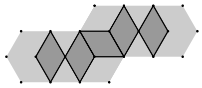


Figure 12.4.ii: An example of a thick pipe with all possible special rhombi highlighted by a darker shading.

We can see that $\bar{c}$ is f_ι-secure, in contradiction to $c^\nu_{\lambda\mu} = 2$, by using the following Lemma 12.4.11.

12.4.11 Lemma. *Let $\iota \in \{1, 2\}$. If an f_ι-hive preserving turncycle c has no reverse turnvertices and all self-intersections of the curve of c occur in (c, f_ι)-special rhombi in which c reroutes to c_ι when applying (‡), then c is f_ι-secure.*

This finishes the proof of Proposition 12.4.4. □

Lemma 12.4.11 is a refined version of Key Lemma 12.4.5. We postpone its proof to Subsection 12.4 (C), because it is based on ideas of the following subsection.

12.4 (B) Proof of Key Lemma 12.4.5

Proof. We will show that

for each planar turncycle in R_{f_ι} that is not f_ι-secure there exist two distinct shorter planar turncycles in R_{f_ι}. (∗)

If there exists an planar turncycle in R_{f_ι} that is not f_ι-secure, then take one of minimal length. It follows from (∗) that there exist two distinct planar f_ι-secure turncycles. This is a contradiction to $c^\nu_{\lambda\mu} = 2$.

It remains to prove (∗). Recall that proper planar turncycles in R are in bijection to proper cycles in G. Let c be an planar turncycle on R_{f_ι} that is not f_ι-secure. By Definition 10.3.10 and Proposition 10.2.9 we have $f_\iota + c \notin P$ and hence there exists $0 < \varepsilon < 1$ such that $f_\iota + \varepsilon c \in P$. According to Observation 10.2.3 we have $\sigma(\varrho, c) \in \{-2, -1, 0, 1, 2\}$ for all rhombi ϱ. Hence there exists a rhombus ϱ with $\sigma(\varrho, f_\iota) = 1$ and $\sigma(\varrho, c) = -2$. We call such rhombi *bad*. Let $\lozenge$ be a bad rhombus. Then c uses $\lozenge$ by Remark 10.2.4.

We begin by analyzing a very special case: If all four rhombi $\lozenge$, $\lozenge$, $\lozenge$, and $\lozenge$ are not f_ι-flat, then c can be rerouted twice: Once via $\lozenge$ and once via $\lozenge$, which results in two planar turncycles in R_{f_ι}. This proves (∗) in this special case. In the more general case, we prove the following:

In each bad rhombus $\lozenge$ the planar turncycle c in R_{f_ι} can be rerouted via $\lozenge$ or one of the three rhombi $\lozenge$, $\lozenge$ or $\lozenge$ is bad such that c uses $\lozenge$, $\lozenge$, or $\lozenge$, respectively. Additionally, in each bad rhombus $\lozenge$ the planar turncycle c in R_{f_ι} can be rerouted via $\lozenge$ or one of the three rhombi $\lozenge$, $\lozenge$ or $\lozenge$ is bad such that c uses $\lozenge$, $\lozenge$, or $\lozenge$, respectively. (∗∗)

We first show that (∗∗) implies (∗). According to (∗∗), each bad rhombus that cannot be rerouted at the left has another bad rhombus located at its left, and analogously for its right side. We continue finding bad rhombi in this manner and obtain a set of adjacent bad rhombi, which we call the *chain*. The chain has two endings at which c can be rerouted to shorter planar turncycles. Hence (∗) follows.

We now show that (∗∗) holds. First, we precisely characterize the situations in which c can be rerouted in R_{f_ι} via $\lozenge$: This is exactly the case when

both ($\lozenge$ is not f_ι-flat or c uses $\lozenge$) and ($\lozenge$ is not f_ι-flat or c uses $\lozenge$).

Now assume that c cannot be rerouted via $\Diamond$, i.e.,

$$\Big(\ \Diamond \text{ is } f_\iota\text{-flat and } c \text{ uses } \Diamond\ \Big) \quad \text{or} \quad \Big(\ \Diamond \text{ is } f_\iota\text{-flat and } c \text{ uses } \Diamond\ \Big).$$

We demonstrate how to prove (∗∗) in the following exemplary case, all others being similar: Let $\Diamond$ be f_ι-flat with c using $\Diamond$ and let $\Diamond$ be f_ι-flat with c using $\Diamond$. The hexagon equality (Claim 10.2.10) applied twice implies that we have either $\sigma(\Diamond, f_\iota) = 0$, $\sigma(\Diamond, f_\iota) = 1$, $\sigma(\Diamond, f_\iota) = 1$ or $\sigma(\Diamond, f_\iota) = 1$, $\sigma(\Diamond, f_\iota) = 0$, $\sigma(\Diamond, f_\iota) = 0$. The latter is impossible, because c uses $\Diamond$. The fact that c uses no negative contributions in f_ι-flat rhombi and that c is planar leads to c running as desired: $\Diamond$ with $\Diamond$ being bad. All other cases are similar. □

12.4 (C) Proof of Lemma 12.4.11

We now complete the proof of Proposition 12.4.4 by proving Lemma 12.4.11.

Proof. The proof is completely analogous to the proof of Key Lemma 12.4.5. We only highlight the technical differences here. Since, in contrast to Key Lemma 12.4.5, we are not dealing with planar turncycles only, we make the following definition.

12.4.12 Definition. Let $\iota \in \{1, 2\}$. If an f_ι-hive preserving proper turncycle c has no reverse turnvertices and all self-intersections of the curve of c occur in (c, f_ι)-special rhombi in which c reroutes to c_ι when applying (‡), then c is called *almost planar.* ■

Note that this notion depends on ι, which we think of being fixed in the following. In analogy to Key Lemma 12.4.5, Lemma 12.4.11 now reads as follows:

Every almost planar turncycle is f_ι-secure.

We will show the following statement:

For each almost planar turncycle in R_{f_ι} that is not f_ι-secure there exist two distinct shorter almost planar turncycles in R_{f_ι}. (∗')

As in the proof of Key Lemma 12.4.5, in order to prove Lemma 12.4.11 it suffices to show (∗'). So fix an almost planar turncycle c. First of all, since c is not necessarily planar, we have $\sigma(\Diamond, c) \in \{-4, -3, \dots, 3, 4\}$ for all rhombi $\Diamond$. But we show now that $\sigma(\Diamond, c) \geq -2$ for all rhombi $\Diamond$.

We begin by showing that there is no rhombus $\Diamond$ with $\sigma(\Diamond, c) = -4$. Recall that $\sigma(\Diamond, c) = \Diamond(c) + \Diamond(c)$. If $\sigma(\Diamond, c) = -4$, then both $\Diamond$ and $\Diamond$ are (c, f_ι)-special. But since $\sigma(\Diamond, c) = \Diamond(c) + \Diamond(c)$, it follows that $\Diamond$ and $\Diamond$ are (c, f_ι)-special as well. This is a contradiction to the fact that (c, f_ι)-special rhombi do not overlap (true by definition, cp. Proposition 12.2.2(3)). Hence $\sigma(\Diamond, c) \geq -3$ for all rhombi $\Diamond$.

We show next that $\sigma(\Diamond, c) \neq -3$. Assume the contrary. W.l.o.g. let $\Diamond(c) = -2$ and $\Diamond(c) = -1$, the other case being the same, just rotated by 180°. Then c is bound to use $\Diamond$. But, according to our assumption that in (c, f_ι)-special rhombi c reroutes to c_ι, this means that c_ι uses $\Diamond$, which is a contradiction to Claim 12.4.9.

So it follows that $\sigma(\Diamond, c) \geq -2$ for all rhombi. This is exactly the same situation as in Key Lemma 12.4.5. As in the proof of Key Lemma 12.4.5, we call rhombi ϱ *bad* if $\sigma(\varrho, c) = -2$ and $\sigma(\varrho, f_\iota) = 1$ There exists a bad rhombus $\Diamond$, since c is assumed to be not f_ι-secure. But here is a technical difference: Unlike in Key Lemma 12.4.5, c does not necessarily use exactly the turnvertices $\Diamond$ in $\Diamond$, but c uses exactly one of the following sets of turnvertices in $\Diamond$: $\Diamond$, $\Diamond$, $\Diamond$, $\Diamond$, $\Diamond$, $\Diamond$, $\Diamond$, $\Diamond$, or $\Diamond$.

We now show that only the first 5 cases can appear, because the last 4 cases cases are in contradiction to the fact that c reroutes to c_ι in (c, f_ι)-special rhombi: In the case where c uses $\Diamond$, then the cycle c_ι on G uses $\Diamond$, which is impossible. The other three cases are treated similarly.

We want to prove that there exists a chain of adjacent bad rhombi as we did in Key Lemma 12.4.5. Analogously to Key Lemma 12.4.5, we can prove the following statement, which is more technical than (∗∗):

> **Case :** For each bad rhombus where the turncycle c in R_{f_ι} uses only the turnvertices , one of the following holds: (1) c can be rerouted via , , , or , or (2) one of the three rhombi , or is bad such that c uses , , or , respectively. Additionally, as in (∗∗), this holds for the situation rotated by 180°. (∗∗')
> **Case :** For each bad rhombus where c uses exactly the turnvertices in , the turncycle c can be rerouted via and or is bad.
> **Remaining cases:** Results that are analogous to the second case hold for c using , , or , respectively.

We remark that the strange reroutings , , and occur in the cases where c uses , , or , respectively.

As in the proof of Key Lemma 12.4.5, (∗∗') can be seen to imply (∗') by constructing a chain of bad rhombi. □

Theorem 12.4.1 is completely proved.

Appendix

Appendix A

Calculations

A.1 Obstruction Candidates

Recall that in Subsection 8.1 (A) we argued that if a partition $\lambda \vdash_{n^2} dn$ satisfies the inequality (8.1.1), i.e.,

$$p_\lambda(d[n]) > \mathsf{sk}\big(\lambda; \big(n \times d\big)^2\big),$$

then the vanishing ideal $I(\mathsf{GL}_{n^2}\det_n)$ of the orbit of the determinant $\det_n$ contains a highest weight vector of weight λ^*. For ($n = 2$ and $d \leq 20$) and for ($n = 4$ and $d \leq 10$) we proved by computer calculation that no $\lambda \vdash_{n^2} dn$ satisfying (8.1.1) exists. More specifically, we showed the following.

A.1.1 Proposition.
For $(n, d) \in \{(2, 1), (2, 2), \ldots, (2, 20)\} \cup \{(4, 1), (4, 2), \ldots, (4, 10)\}$ *we have*

$$p_\lambda(d[n]) \leq \mathsf{sk}\big(\lambda; \big(n \times d\big)^2\big)$$

for all $\lambda \vdash_{n^2} dn$.

But the case $n = 3$ yielded interesting results. In the following listing, we use the following short syntax: We write

$$(\lambda_1, \ldots, \lambda_l)[p{>}s]l;$$

for $\ell(\lambda) = l$, $p_\lambda(d[3]) = p$, and $\mathsf{sk}\big(\lambda; \big(3 \times d\big)^2\big) = s$.

We remark that in the following list the shortest partitions are of length 7. There are two of them: $(13, 13, 2, 2, 2, 2, 2) \vdash_7 36$ in degree $\frac{36}{3} = 12$ and $(15, 5, 5, 5, 5, 5, 5) \vdash_7 45$ in degree $\frac{45}{3} = 15$.

$d = 10$:

(9,9,2,2,2,2,2,2)[1>0]8; (12,5,2,2,2,2,2,2,1)[1>0]9; (13,3,2,2,2,2,2,2,2)[1>0]9;

$d = 11$:

(9,9,4,2,2,2,2,2,1)[2>0]9; (10,9,2,2,2,2,2,2,2)[1>0]9; (10,9,3,2,2,2,2,2,1)[2>1]9; (11,7,3,2,2,2,2,2,2)[1>0]9;
(11,7,4,2,2,2,2,2,1)[5>4]9; (11,8,2,2,2,2,2,2,2)[2>1]9; (11,8,3,2,2,2,2,2,1)[3>2]9; (11,9,2,2,2,2,2,2,1)[1>0]9;
(11,9,3,2,2,2,2,2)[4>3]8; (11,10,2,2,2,2,2,2)[2>1]8; (11,11,2,2,2,2,2,1)[1>0]8; (12,5,4,2,2,2,2,2,2)[2>1]9;
(12,5,5,2,2,2,2,2,1)[1>0]9; (12,6,3,2,2,2,2,2,2)[2>1]9; (12,7,2,2,2,2,2,2,2)[2>1]9; (12,7,3,2,2,2,2,2,1)[4>2]9;
(12,8,2,2,2,2,2,2,1)[2>1]9; (12,9,2,2,2,2,2,2)[4>3]8; (13,5,3,2,2,2,2,2,2)[1>0]9; (13,5,4,2,2,2,2,2,1)[3>2]9;
(13,6,2,2,2,2,2,2,2)[3>1]9; (13,6,3,2,2,2,2,2,1)[3>2]9; (13,7,2,2,2,2,2,2,1)[2>1]9; (13,8,2,2,2,2,2,2)[5>4]8;
(14,4,3,2,2,2,2,2,2)[1>0]9; (14,5,2,2,2,2,2,2,2)[2>1]9; (14,5,3,2,2,2,2,2,1)[2>1]9; (14,6,2,2,2,2,2,2,1)[2>1]9;
(14,7,2,2,2,2,2,2)[4>3]8; (15,4,2,2,2,2,2,2,2)[2>1]9; (15,5,2,2,2,2,2,2,1)[1>0]9; (16,3,2,2,2,2,2,2,2)[1>0]9;
(16,5,2,2,2,2,2,2)[3>2]8;

$d = 12$:

(9,7,7,3,2,2,2,2,2)[1>0]9; (9,8,7,2,2,2,2,2,2)[1>0]9; (9,9,4,4,2,2,2,2,2)[2>0]9; (9,9,5,3,2,2,2,2,2)[2>1]9;
(9,9,6,2,2,2,2,2,2)[2>0]9; (9,9,7,2,2,2,2,2,1)[3>0]9; (9,9,8,2,2,2,2,2)[2>1]8; (10,8,5,3,2,2,2,2,2)[4>3]9;
(10,8,6,2,2,2,2,2,2)[5>4]9; (10,9,4,3,2,2,2,2,2)[4>2]9; (10,9,5,2,2,2,2,2,2)[3>0]9; (10,9,6,2,2,2,2,2,1)[8>5]9;
(10,10,4,2,2,2,2,2,2)[3>2]9; (11,7,4,4,2,2,2,2,2)[7>6]9; (11,7,5,3,2,2,2,2,2)[5>3]9; (11,7,6,2,2,2,2,2,2)[4>1]9;
(11,7,7,2,2,2,2,2,1)[4>1]9; (11,8,4,3,2,2,2,2,2)[6>3]9; (11,8,5,2,2,2,2,2,2)[6>3]9; (11,8,6,2,2,2,2,2,1)[12>10]9;
(11,9,3,3,2,2,2,2,2)[1>0]9; (11,9,4,2,2,2,2,2,2)[6>1]9; (11,9,4,3,2,2,2,2,1)[14>11]9; (11,9,5,2,2,2,2,2,1)[11>4]9;
(11,10,3,2,2,2,2,2,2)[2>0]9; (11,10,3,3,2,2,2,2,1)[3>2]9; (11,10,4,2,2,2,2,2,1)[7>3]9; (11,11,2,2,2,2,2,2,2)[1>0]9;
(11,11,3,2,2,2,2,2,1)[2>0]9; (11,11,4,2,2,2,2,2)[5>1]8; (12,5,5,4,2,2,2,2,2)[1>0]9; (12,6,5,3,2,2,2,2,2)[4>2]9;
(12,6,6,2,2,2,2,2,2)[5>4]9; (12,7,4,3,2,2,2,2,2)[7>3]9; (12,7,5,2,2,2,2,2,2)[5>1]9; (12,7,6,2,2,2,2,2,1)[10>7]9;
(12,8,3,3,2,2,2,2,2)[1>0]9; (12,8,4,2,2,2,2,2,2)[9>3]9; (12,8,5,2,2,2,2,2,1)[13>8]9; (12,9,3,2,2,2,2,2,2)[4>1]9;
(12,9,3,3,2,2,2,2,1)[5>3]9; (12,9,4,2,2,2,2,2,1)[12>5]9; (12,10,2,2,2,2,2,2,2)[3>1]9; (12,10,3,2,2,2,2,2,1)[5>2]9;
(12,11,2,2,2,2,2,2,1)[1>0]9; (12,11,3,2,2,2,2,2)[5>3]8; (13,5,4,4,2,2,2,2,2)[4>3]9; (13,5,5,3,2,2,2,2,2)[1>0]9;
(13,6,4,3,2,2,2,2,2)[6>3]9; (13,6,5,2,2,2,2,2,2)[5>1]9; (13,7,3,3,2,2,2,2,2)[1>0]9; (13,7,4,2,2,2,2,2,2)[8>3]9;
(13,7,4,3,2,2,2,2,1)[14>13]9; (13,7,5,2,2,2,2,2,1)[11>5]9; (13,8,3,2,2,2,2,2,2)[5>1]9; (13,8,3,3,2,2,2,2,1)[5>4]9;
(13,8,4,2,2,2,2,2,1)[13>7]9; (13,9,2,2,2,2,2,2,2)[4>0]9; (13,9,3,2,2,2,2,2,1)[7>2]9; (13,9,4,2,2,2,2,2)[18>17]8;
(13,10,2,2,2,2,2,2,1)[3>1]9; (13,10,3,2,2,2,2,2)[8>6]8; (13,11,2,2,2,2,2,2)[4>1]8; (13,13,2,2,2,2,2,2)[1>0]7;
(14,5,4,3,2,2,2,2,2)[3>1]9; (14,5,5,2,2,2,2,2,2)[1>0]9; (14,6,3,3,2,2,2,2,2)[1>0]9; (14,6,4,2,2,2,2,2,2)[8>3]9;
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$d = 13$:

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(11,10,8,7,2,2,2,2,1)[193>85]9; (11,10,8,7,3,2,2,1,1)[540>505]9; (11,10,8,8,2,2,2,1,1)[79>70]9; (11,10,9,3,3,3,2,2,2)[38>9]9;
(11,10,9,3,3,3,3,2,1)[42>33]9; (11,10,9,4,3,2,2,2,2)[154>28]9; (11,10,9,4,3,3,2,2,1)[290>158]9; (11,10,9,4,3,3,3,1,1)[150>142]9;
(11,10,9,4,4,2,2,2,1)[291>133]9; (11,10,9,4,4,3,2,1,1)[407>383]9; (11,10,9,5,2,2,2,2,2)[102>13]9; (11,10,9,5,3,2,2,2,1)[463>216]9;
(11,10,9,5,3,3,2,1,1)[541>501]9; (11,10,9,5,4,2,2,1,1)[632>586]9; (11,10,9,6,2,2,2,2,1)[207>86]9; (11,10,9,6,3,2,2,1,1)[553>512]9;
(11,10,9,7,2,2,2,1,1)[167>155]9; (11,10,10,3,3,2,2,2,2)[20>2]9; (11,10,10,3,3,3,2,2,1)[36>17]9; (11,10,10,4,2,2,2,2,2)[43>10]9;
(11,10,10,4,3,2,2,2,1)[131>57]9; (11,10,10,4,3,3,2,1,1)[138>135]9; (11,10,10,4,4,2,2,1,1)[119>109]9; (11,10,10,5,2,2,2,2,1)[91>39]9;
(11,10,10,5,3,2,2,1,1)[216>194]9; (11,10,10,6,2,2,2,1,1)[86>74]9; (11,11,4,4,4,4,3,2,2)[10>2]9; (11,11,4,4,4,4,4,2,1)[8>2]9;
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(11,11,5,4,4,4,2,2,2)[41>9]9; (11,11,5,4,4,4,3,2,1)[66>42]9; (11,11,5,4,4,4,4,1,1)[22>18]9; (11,11,5,5,3,3,3,2,2)[34>12]9;
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(11,11,5,5,4,4,3,1,1)[94>91]9; (11,11,5,5,5,2,2,2,2)[31>14]9; (11,11,5,5,5,3,2,2,1)[114>79]9; (11,11,5,5,5,3,3,1,1)[85>66]9;
(11,11,5,5,5,4,2,1,1)[106>100]9; (11,11,5,5,5,5,1,1,1)[22>21]9; (11,11,6,3,3,3,3,3,2)[5>1]9; (11,11,6,4,3,3,3,2,2)[47>20]9;
(11,11,6,4,3,3,3,3,1)[29>21]9; (11,11,6,4,4,3,2,2,2)[137>32]9; (11,11,6,4,4,3,3,2,1)[163>114]9; (11,11,6,4,4,4,2,2,1)[158>79]9;
(11,11,6,5,3,3,2,2,2)[136>41]9; (11,11,6,5,3,3,3,2,1)[172>111]9; (11,11,6,5,4,2,2,2,2)[163>36]9; (11,11,6,5,4,3,2,2,1)[488>288]9;
(11,11,6,5,4,3,3,1,1)[299>281]9; (11,11,6,5,4,4,2,1,1)[324>314]9; (11,11,6,5,5,2,2,2,1)[194>121]9; (11,11,6,5,5,3,2,1,1)[372>351]9;
(11,11,6,6,3,2,2,2,2)[104>21]9; (11,11,6,6,3,3,2,2,1)[225>133]9; (11,11,6,6,3,3,3,1,1)[125>119]9; (11,11,6,6,4,2,2,2,1)[276>135]9;
(11,11,6,6,4,3,2,1,1)[438>429]9; (11,11,6,6,5,2,2,1,1)[241>234]9; (11,11,7,3,3,3,3,2,2)[14>6]9; (11,11,7,3,3,3,3,3,1)[13>4]9;
(11,11,7,4,3,3,2,2,2)[130>37]9; (11,11,7,4,3,3,3,2,1)[142>97]9; (11,11,7,4,4,2,2,2,2)[135>20]9; (11,11,7,4,4,3,2,2,1)[363>192]9;
(11,11,7,4,4,4,2,1,1)[210>207]9; (11,11,7,5,3,2,2,2,2)[199>42]9; (11,11,7,5,3,3,2,2,1)[423>252]9; (11,11,7,5,3,3,3,1,1)[237>209]9;
(11,11,7,5,4,2,2,2,1)[491>255]9; (11,11,7,5,4,3,2,1,1)[775>747]9; (11,11,7,5,5,2,2,1,1)[346>340]9; (11,11,7,6,2,2,2,2,2)[78>8]9;
(11,11,7,6,3,2,2,2,1)[385>191]9; (11,11,7,6,3,3,2,1,1)[459>442]9; (11,11,7,6,4,2,2,1,1)[576>560]9; (11,11,7,7,2,2,2,2,1)[83>37]9;
(11,11,7,7,3,2,2,1,1)[233>232]9; (11,11,8,3,3,3,2,2,2)[35>15]9; (11,11,8,3,3,3,3,2,1)[38>24]9; (11,11,8,4,3,2,2,2,2)[147>22]9;
(11,11,8,4,3,3,2,2,1)[269>155]9; (11,11,8,4,3,3,3,1,1)[137>130]9; (11,11,8,4,4,2,2,2,1)[273>114]9; (11,11,8,4,4,3,2,1,1)[368>360]9;
(11,11,8,5,2,2,2,2,2)[99>12]9; (11,11,8,5,3,2,2,2,1)[425>206]9; (11,11,8,5,3,3,2,1,1)[485>459]9; (11,11,8,5,4,2,2,1,1)[568>545]9;
(11,11,8,6,2,2,2,2,1)[183>68]9; (11,11,8,6,3,2,2,1,1)[481>462]9; (11,11,8,7,2,2,2,1,1)[133>131]9; (11,11,9,3,3,2,2,2,2)[43>10]9;
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(11,11,9,4,3,3,2,1,1)[219>208]9; (11,11,9,4,4,2,2,1,1)[218>208]9; (11,11,9,5,2,2,2,2,1)[146>52]9; (11,11,9,5,3,2,2,1,1)[355>343]9;
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(11,11,10,4,2,2,2,2,1)[62>17]9; (11,11,10,4,3,2,2,1,1)[125>123]9; (11,11,11,2,2,2,2,2,2)[1>0]9; (11,11,11,3,2,2,2,2,1)[10>0]9;
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(12,6,6,6,6,3,2,2,2)[22>12]9; (12,6,6,6,6,3,3,2,1)[34>25]9; (12,6,6,6,6,4,2,2,1)[50>42]9; (12,6,6,6,6,4,3,1,1)[38>35]9;
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(12,7,5,5,5,5,4,1,1)[28>21]9; (12,7,6,4,4,4,3,3,2)[23>20]9; (12,7,6,4,4,4,4,2,2)[28>20]9; (12,7,6,5,3,3,3,3,3)[9>4]9;
(12,7,6,5,4,3,3,3,2)[75>43]9; (12,7,6,5,4,4,3,2,2)[140>78]9; (12,7,6,5,4,4,3,3,1)[114>107]9; (12,7,6,5,4,4,4,2,1)[111>108]9;
(12,7,6,5,5,3,3,2,2)[136>65]9; (12,7,6,5,5,3,3,3,1)[115>77]9; (12,7,6,5,5,4,2,2,2)[118>57]9; (12,7,6,5,5,4,3,2,1)[315>246]9;
(12,7,6,5,5,5,2,2,1)[122>92]9; (12,7,6,5,5,5,3,1,1)[140>116]9; (12,7,6,6,3,3,3,3,2)[34>17]9; (12,7,6,6,4,3,3,2,2)[129>59]9;
(12,7,6,6,4,3,3,3,1)[108>86]9; (12,7,6,6,4,4,2,2,2)[130>60]9; (12,7,6,6,4,4,3,2,1)[258>208]9; (12,7,6,6,5,3,2,2,2)[146>60]9;
(12,7,6,6,5,3,3,2,1)[287>215]9; (12,7,6,6,5,4,2,2,1)[310>230]9; (12,7,6,6,5,4,3,1,1)[300>291]9; (12,7,6,6,5,5,2,1,1)[162>159]9;
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(12,7,7,5,4,3,3,3,1)[117>92]9; (12,7,7,5,4,4,2,2,2)[127>47]9; (12,7,7,5,4,4,3,2,1)[295>238]9; (12,7,7,5,5,3,2,2,2)[158>70]9;
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(12,7,7,6,4,4,2,2,1)[321>222]9; (12,7,7,6,5,2,2,2,2)[105>35]9; (12,7,7,6,5,3,2,2,1)[452>328]9; (12,7,7,6,5,3,3,1,1)[320>312]9;
(12,7,7,6,6,2,2,2,1)[109>67]9; (12,7,7,7,3,3,2,2,2)[66>29]9; (12,7,7,7,3,3,3,2,1)[81>57]9; (12,7,7,7,4,2,2,2,2)[51>13]9;
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(12,8,5,5,3,3,3,3,3)[2>1]9; (12,8,5,5,4,3,3,3,2)[50>31]9; (12,8,5,5,4,4,3,2,2)[98>49]9; (12,8,5,5,4,4,3,3,1)[82>80]9;
(12,8,5,5,4,4,4,2,1)[78>68]9; (12,8,5,5,5,3,3,2,2)[100>49]9; (12,8,5,5,5,3,3,3,1)[66>44]9; (12,8,5,5,5,4,2,2,2)[79>33]9;
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(12,8,6,5,4,3,3,2,2)[301>132]9; (12,8,6,5,4,3,3,3,1)[221>170]9; (12,8,6,5,4,4,2,2,2)[266>110]9; (12,8,6,5,4,4,3,2,1)[536>417]9;
(12,8,6,5,5,3,2,2,2)[259>100]9; (12,8,6,5,5,3,3,2,1)[497>355]9; (12,8,6,5,5,4,2,2,1)[505>363]9; (12,8,6,5,5,4,3,1,1)[495>468]9;
(12,8,6,5,5,5,2,1,1)[251>230]9; (12,8,6,6,3,3,3,2,2)[107>40]9; (12,8,6,6,3,3,3,3,1)[93>74]9; (12,8,6,6,4,3,2,2,2)[334>122]9;
(12,8,6,6,4,3,3,2,1)[525>377]9; (12,8,6,6,4,4,2,2,1)[500>364]9; (12,8,6,6,4,4,3,1,1)[400>388]9; (12,8,6,6,5,2,2,2,2)[173>61]9;
(12,8,6,6,5,3,2,2,1)[641>437]9; (12,8,6,6,5,3,3,1,1)[490>476]9; (12,8,6,6,5,4,2,1,1)[630>624]9; (12,8,6,6,6,2,2,2,1)[165>109]9;
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(12,8,7,4,4,4,2,2,2)[167>57]9; (12,8,7,4,4,4,3,2,1)[293>220]9; (12,8,7,5,3,3,3,2,2)[199>86]9; (12,8,7,5,3,3,3,3,1)[126>93]9;
(12,8,7,5,4,3,2,2,2)[517>178]9; (12,8,7,5,4,3,3,2,1)[792>572]9; (12,8,7,5,4,4,2,2,1)[717>490]9; (12,8,7,5,5,2,2,2,2)[188>58]9;
(12,8,7,5,5,3,2,2,1)[793>549]9; (12,8,7,5,5,3,3,1,1)[562>530]9; (12,8,7,5,5,4,2,1,1)[760>757]9; (12,8,7,6,3,3,2,2,2)[283>92]9;
(12,8,7,6,3,3,3,2,1)[397>287]9; (12,8,7,6,4,2,2,2,2)[355>103]9; (12,8,7,6,4,3,2,2,1)[1174>777]9; (12,8,7,6,4,3,3,1,1)[774>770]9;
(12,8,7,6,5,2,2,2,1)[619>395]9; (12,8,7,7,3,2,2,2,2)[121>30]9; (12,8,7,7,3,3,2,2,1)[330>224]9; (12,8,7,7,3,3,3,1,1)[181>175]9;
(12,8,7,7,4,2,2,2,1)[392>235]9; (12,8,8,3,3,3,3,3,2)[9>7]9; (12,8,8,4,3,3,3,2,2)[68>25]9; (12,8,8,4,3,3,3,3,1)[47>46]9;
(12,8,8,4,4,3,2,2,2)[189>60]9; (12,8,8,4,4,3,3,2,1)[239>165]9; (12,8,8,4,4,4,2,2,1)[232>160]9; (12,8,8,4,4,4,3,1,1)[162>151]9;
(12,8,8,5,3,3,2,2,2)[196>54]9; (12,8,8,5,3,3,3,2,1)[274>198]9; (12,8,8,5,4,2,2,2,2)[259>73]9; (12,8,8,5,4,3,2,2,1)[781>498]9;
(12,8,8,5,4,3,3,1,1)[505>498]9; (12,8,8,5,4,4,2,1,1)[529>528]9; (12,8,8,5,5,2,2,2,1)[320>186]9; (12,8,8,6,3,2,2,2,2)[211>56]9;
(12,8,8,6,3,3,2,2,1)[464>280]9; (12,8,8,6,4,2,2,2,1)[601>360]9; (12,8,8,7,2,2,2,2,2)[64>17]9; (12,8,8,7,3,2,2,2,1)[289>156]9;
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(12,9,5,5,4,4,2,2,2)[125>42]9; (12,9,5,5,4,4,3,2,1)[270>202]9; (12,9,5,5,5,3,2,2,2)[134>55]9; (12,9,5,5,5,3,3,2,1)[232>156]9;
(12,9,5,5,5,4,2,2,1)[237>172]9; (12,9,5,5,5,4,3,1,1)[225>203]9; (12,9,5,5,5,5,2,1,1)[105>90]9; (12,9,6,3,3,3,3,3,3)[3>1]9;
(12,9,6,4,3,3,3,3,2)[39>25]9; (12,9,6,4,4,3,3,2,2)[189>75]9; (12,9,6,4,4,3,3,3,1)[115>105]9; (12,9,6,4,4,4,2,2,2)[205>66]9;
(12,9,6,4,4,4,3,2,1)[327>231]9; (12,9,6,4,4,4,4,1,1)[92>88]9; (12,9,6,5,3,3,3,2,2)[200>83]9; (12,9,6,5,3,3,3,3,1)[134>96]9;
(12,9,6,5,4,3,2,2,2)[550>178]9; (12,9,6,5,4,3,3,2,1)[818>573]9; (12,9,6,5,4,4,2,2,1)[749>493]9; (12,9,6,5,4,4,3,1,1)[611>599]9;
(12,9,6,5,5,2,2,2,2)[198>61]9; (12,9,6,5,5,3,2,2,1)[793>533]9; (12,9,6,5,5,3,3,1,1)[573>516]9; (12,9,6,5,5,4,2,1,1)[759>729]9;
(12,9,6,6,3,3,2,2,2)[231>70]9; (12,9,6,6,3,3,3,2,1)[323>227]9; (12,9,6,6,4,2,2,2,2)[327>91]9; (12,9,6,6,4,3,2,2,1)[977>614]9;
(12,9,6,6,4,3,3,1,1)[637>624]9; (12,9,6,6,4,4,2,1,1)[673>665]9; (12,9,6,6,5,2,2,2,1)[503>299]9; (12,9,6,6,5,3,2,1,1)[978>967]9;
(12,9,6,6,6,2,2,1,1)[229>228]9; (12,9,7,3,3,3,3,3,2)[12>6]9; (12,9,7,4,3,3,3,2,2)[151>66]9; (12,9,7,4,3,3,3,3,1)[88>72]9;
(12,9,7,4,4,3,2,2,2)[400>110]9; (12,9,7,4,4,3,3,2,1)[512>363]9; (12,9,7,4,4,4,2,2,1)[481>281]9; (12,9,7,5,3,3,2,2,2)[454>145]9;
(12,9,7,5,3,3,3,2,1)[571>397]9; (12,9,7,5,4,2,2,2,2)[512>124]9; (12,9,7,5,4,3,2,2,1)[1629>1018]9; (12,9,7,5,4,3,3,1,1)[1017>981]9;
(12,9,7,5,5,2,2,2,1)[671>417]9; (12,9,7,5,5,3,2,1,1)[1302>1265]9; (12,9,7,6,3,2,2,2,2)[401>92]9; (12,9,7,6,3,3,2,2,1)[935>585]9;
(12,9,7,6,3,3,3,1,1)[534>514]9; (12,9,7,6,4,2,2,2,1)[1143>637]9; (12,9,7,7,2,2,2,2,2)[71>8]9; (12,9,7,7,3,2,2,2,1)[442>247]9;
(12,9,7,7,3,3,2,1,1)[558>555]9; (12,9,8,3,3,3,3,2,2)[33>17]9; (12,9,8,3,3,3,3,3,1)[21>17]9; (12,9,8,4,3,3,2,2,2)[245>67]9;
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(12,9,8,5,3,2,2,2,2)[399>85]9; (12,9,8,5,3,3,2,2,1)[866>523]9; (12,9,8,5,3,3,3,1,1)[487>471]9; (12,9,8,5,4,2,2,2,1)[1023>549]9;
(12,9,8,5,4,3,2,1,1)[1628>1603]9; (12,9,8,5,5,2,2,1,1)[751>742]9; (12,9,8,6,2,2,2,2,2)[188>34]9; (12,9,8,6,3,2,2,2,1)[855>440]9;
(12,9,8,6,3,3,2,1,1)[1045>1040]9; (12,9,8,6,4,2,2,1,1)[1284>1268]9; (12,9,8,7,2,2,2,2,1)[238>109]9; (12,9,8,7,3,2,2,1,1)[665>660]9;
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(12,9,9,6,2,2,2,2,1)[197>75]9; (12,9,9,6,3,2,2,1,1)[553>547]9; (12,10,4,4,4,4,3,2,2)[27>10]9; (12,10,4,4,4,4,3,3,1)[12>7]9;
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(12,10,5,4,4,4,2,2,2)[110>32]9; (12,10,5,4,4,4,3,2,1)[168>110]9; (12,10,5,4,4,4,4,1,1)[46>41]9; (12,10,5,5,3,3,3,2,2)[90>39]9;
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(12,10,6,4,3,3,3,3,1)[69>66]9; (12,10,6,4,4,3,2,2,2)[340>92]9; (12,10,6,4,4,3,3,2,1)[403>269]9; (12,10,6,4,4,4,2,2,1)[398>241]9;
(12,10,6,4,4,4,3,1,1)[270>255]9; (12,10,6,5,3,3,2,2,2)[332>92]9; (12,10,6,5,3,3,3,2,1)[421>291]9; (12,10,6,5,4,2,2,2,2)[409>100]9;
(12,10,6,5,4,3,2,2,1)[1202>728]9; (12,10,6,5,4,3,3,1,1)[746>711]9; (12,10,6,5,4,4,2,1,1)[794>778]9; (12,10,6,5,5,2,2,2,1)[472>275]9;
(12,10,6,5,5,3,2,1,1)[923>891]9; (12,10,6,6,3,2,2,2,2)[262>62]9; (12,10,6,6,3,3,2,2,1)[535>311]9; (12,10,6,6,4,2,2,2,1)[686>384]9;
(12,10,6,6,4,3,2,1,1)[1081>1070]9; (12,10,6,6,5,2,2,1,1)[581>574]9; (12,10,7,3,3,3,3,2,2)[44>22]9; (12,10,7,3,3,3,3,3,1)[22>21]9;
(12,10,7,4,3,3,2,2,2)[309>80]9; (12,10,7,4,3,3,3,2,1)[340>244]9; (12,10,7,4,4,2,2,2,2)[351>76]9; (12,10,7,4,4,3,2,2,1)[873>497]9;
(12,10,7,4,4,3,3,1,1)[493>489]9; (12,10,7,4,4,4,2,1,1)[498>490]9; (12,10,7,5,3,2,2,2,2)[472>94]9; (12,10,7,5,3,3,2,2,1)[1028>613]9;
(12,10,7,5,3,3,3,1,1)[555>538]9; (12,10,7,5,4,2,2,2,1)[1197>635]9;(12,10,7,5,4,3,2,1,1)[1883>1859]9; (12,10,7,5,5,2,2,1,1)[864>858]9;
(12,10,7,6,2,2,2,2,2)[202>36]9; (12,10,7,6,3,2,2,2,1)[931>478]9;(12,10,7,6,3,3,2,1,1)[1119>1117]9;(12,10,7,6,4,2,2,1,1)[1374>1369]9;
(12,10,7,7,2,2,2,2,1)[195>84]9; (12,10,8,3,3,3,2,2,2)[79>18]9; (12,10,8,3,3,3,3,2,1)[88>73]9; (12,10,8,4,3,2,2,2,2)[348>65]9;
(12,10,8,4,3,3,2,2,1)[622>345]9; (12,10,8,4,4,2,2,2,1)[649>327]9; (12,10,8,4,4,3,2,1,1)[869>860]9; (12,10,8,5,2,2,2,2,2)[241>42]9;
(12,10,8,5,3,2,2,2,1)[997>482]9;(12,10,8,5,3,3,2,1,1)[1148>1142]9;(12,10,8,5,4,2,2,1,1)[1325>1297]9; (12,10,8,6,2,2,2,2,1)[442>201]9;
(12,10,8,6,3,2,2,1,1)[1115>1089]9; (12,10,8,7,2,2,2,1,1)[304>293]9; (12,10,9,3,3,2,2,2,2)[83>12]9; (12,10,9,3,3,3,2,2,1)[135>80]9;
(12,10,9,4,2,2,2,2,2)[139>20]9; (12,10,9,4,3,2,2,2,1)[481>218]9; (12,10,9,4,4,2,2,1,1)[458>446]9; (12,10,9,5,2,2,2,2,1)[323>131]9;
(12,10,9,5,3,2,2,1,1)[785>766]9; (12,10,9,6,2,2,2,1,1)[328>318]9; (12,10,10,3,2,2,2,2,2)[36>8]9; (12,10,10,3,3,2,2,2,1)[77>28]9;
(12,10,10,4,2,2,2,2,1)[117>53]9; (12,10,10,4,3,2,2,1,1)[216>208]9; (12,10,10,5,2,2,2,1,1)[138>125]9; (12,11,4,4,4,3,3,2,2)[18>5]9;
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(12,11,6,5,4,2,2,2,1)[666>348]9; (12,11,6,5,4,3,2,1,1)[1002>998]9; (12,11,6,6,2,2,2,2,2)[107>19]9; (12,11,6,6,3,2,2,2,1)[414>204]9;
(12,11,7,3,3,3,2,2,2)[72>21]9; (12,11,7,3,3,3,3,2,1)[73>58]9; (12,11,7,4,3,2,2,2,2)[300>51]9; (12,11,7,4,3,3,2,2,1)[525>303]9;
(12,11,7,4,4,2,2,2,1)[539>249]9; (12,11,7,5,2,2,2,2,2)[189>23]9; (12,11,7,5,3,2,2,2,1)[807>399]9; (12,11,7,6,2,2,2,2,1)[324>137]9;
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(12,12,8,3,2,2,2,2,2)[55>8]9; (12,12,8,3,3,2,2,2,1)[117>44]9; (12,12,8,4,2,2,2,2,1)[176>74]9; (12,12,8,5,2,2,2,1,1)[203>200]9;
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