Lie groups of mappings on non-compact spaces and manifolds

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Abstract

We study the existence of Lie group structures on topological groups of differentiable maps $C^r(M, K)$ from a non-compact manifold M to a possibly infinite dimensional Lie group K, and on weighted function spaces $CV(X, \mathfrak{g})$ from a completely regular Hausdorff space X to a Lie algebra $\mathfrak{g}$. As a tool to deal with the groups $C^r(M, K)$, we develop a differential calculus of partially differentiable mappings on multiple products of locally convex spaces and establish exponential laws for such mappings, which also admit applications in other parts of infinite-dimensional Lie theory.

Deutsche Zusammenfassung

Die Arbeit ist dem Studium von Liegruppenstrukturen auf topologischen Gruppen der Form $C^r(M,K)$ gewidmet, wobei M eine nicht-kompakte Mannigfaltigkeit und K eine endlich- oder unendlichdimensionale Liegruppe ist. Zudem werden Liegruppen zu Funktionräumen $CV(X,\mathfrak{g})$ untersucht, wobei X ein vollständig regulärer topologischer Raum ist und $\mathfrak{g}$ eine topologische Liealgebra. Als ein Werkzeug zum Umgang mit den Gruppen $C^r(M,K)$ entwickeln wir eine Differentialrechnung partiell differenzierbarer Abbildungen auf Produkten mehrerer lokal konvexer Räume und beweisen Exponentialgesetze für solche Abbildungen, welche auch in anderen Teilen der unendlich-dimensionalen Lietheorie von Nutzen sind.

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To my parents

Contents

Chapter 1

Introduction

Infinite-dimensional Lie groups are of importance for mathematical physics. This thesis introduces two classes of infinite-dimensional Lie groups. The first class are certain mapping groups which generalize loop groups and current groups, that have attracted much interest in mathematics and also appear in quantum theory. The second class are Lie groups modelled on weighted function spaces, special cases of which have a gain been applied in various branches of mathematics and physics.

It is a well-known fact that the set of smooth maps $C^\infty(M, K)$ from a compact smooth manifold M to a Lie group K modelled on a locally convex space carries a natural Lie group structure (see [27], [23] and [25]). For non-compact manifolds M this statement is, in general; not true, in this case, the topological group $C^\infty(M, K)$ may fail to admit a manifold structure. In [31], Neeb and Wagemann developed Lie theory for this class of groups. Notably, they gave sufficient conditions for the existence of Lie group structures on such groups. In this thesis, we study the group $C^k(M, K)$ for finite k. We show that there exists a natural Lie group structure compatible with evaluations on $C^k(M, K)$ if the image of the left logarithmic derivative carries a natural manifold structure. We then obtain a manifold structure on the group $C^k_*(M, K) := \{f \in C^k(M, K) : f(m_0) = 1\}$ and hence on $C^k(M, K) \cong K \ltimes C^k_*(M, K)$ a C^s-regular Lie group structure compatible with evaluations, for $k \geq s + 1$ (Theorem 142).

Let X be a completely regular Hausdorff space, E be a topological vector space and V be a Nachbin family of weights on X (Definition 150). The weighted spaces $CV_0(X)$ and $CV(X)$ were introduced in the scalar case by Nachbin [28], and the corresponding E–valued functions weighted spaces analogues $CV_0(X, E)$ and $CV(X, E)$ were introduced and studied by Bierstedt [5] and Prolla [35]. In general these spaces need not be algebras if E is an algebra. In [33] and [34] Oubbi presented necessary and sufficient conditions for these spaces to be locally convex algebras of a certain type. In this thesis we study such weighted spaces in an infinite-dimensional Lie theory setting. More precisely, we shall consider the Lie algebra–valued weighted functions space $CV(X, \mathfrak{g})$ and we shall give conditions on the weight making this weighted space a topological Lie algebra. We shall also consider Lie group structures on such spaces if $\mathfrak{g}$ is nilpotent.

Thesis outline and statement of results

This thesis consists of two parts. The first five chapters comprising the first part are devoted to the study of the Lie group structures on mapping groups. The remaining part deals with the Lie-theoretical weighted spaces.

Chapter 2 presents some preliminaries on infinite-dimensional Lie theory. We collect a few results concerning the differential calculus in locally convex spaces which will be important later. We also briefly review some basic concepts and results concerning manifolds, infinite-dimensional Lie groups and spaces of mappings.

Chapter 3 gives a systematic treatment of the calculus of mappings on products with different degrees of differentiability in the two factors, called $C^{r,s}$-mappings. We shall develop their basic properties and some refined tools. We study such mappings in an infinite-dimensional setting, which is analogous to the approach to C^r-maps between locally convex spaces known as Keller's C^r_c-theory [24]. We first introduce the notion of a $C^{r,s}$-mapping: Let E_1, E_2 and F be locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ be open subsets and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. We say that a map $f\colon U \times V \to F$ is $C^{r,s}$ if the iterated directional derivatives

$$(D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y)$$

exist for all $i, j \in \mathbb{N}_0$ such that $i \leq r$ and $j \leq s$, and are continuous functions in $(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) \in U \times V \times E_1^i \times E_2^j$ (see Definition 25 for details). To enable choices like $U = [0, 1]$, and also with a view towards manifolds with boundary, more generally we consider $C^{r,s}$-maps if U and V are locally convex (in the sense that each point has a convex neighbourhood) and have dense interior (see Definition 26). These properties are satisfied by all open sets.

The first aim of this chapter is to develop necessary tools like a version of the Theorem of Schwarz and various versions of the Chain Rule. After that we turn to an advanced tool, the exponential law for spaces of mappings on products (Theorem 52). We endow spaces of C^r-maps with the usual compact-open C^r-topology (as recalled in Definition 20) and spaces of $C^{r,s}$-maps with the analogous compact-open $C^{r,s}$-topology (see Definitions 45 and 56). The main results of Section 3 (Theorems 49 and 52) subsume:

Theorem A. *Let E_1, E_2 and F be locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ be locally convex subsets with dense interior, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Then $\gamma^\vee : U \to C^s(V, F)$, $x \mapsto \gamma(x, \bullet)$ is C^r for each $\gamma \in C^{r,s}(U \times V, F)$, and the map*

$$\Phi \colon C^{r,s}(U \times V, F) \to C^r(U, C^s(V, F)), \quad \gamma \mapsto \gamma^\vee \tag{1.1}$$

is linear and a topological embedding. If $U \times V \times E_1 \times E_2$ is a k-space [1] *or V is locally compact, then Φ is an isomorphism of topological vector spaces.*

This is a generalisation of the classical exponential law for smooth maps. Since C^∞-maps and $C^{\infty,\infty}$-maps on products coincide (see Lemma 40, Remark 41 and Lemma 46), we obtain as a special case that

$$\Phi \colon C^\infty(U \times V, F) \to C^\infty(U, C^\infty(V, F)) \tag{1.2}$$

is an isomorphism of topological vector spaces if V is locally compact or $U \times V \times E_1 \times E_2$ is a k-space.

Naturally one would like to apply the exponential law (1.1) to a pair of smooth manifolds M_1 and M_2 modelled on locally convex spaces E_1 and E_2, respectively. In Section 3.4, we extend our results to $C^{r,s}$-maps on products of manifolds. Beyond ordinary manifolds, we can consider (with increasing generality) manifolds with smooth boundary, manifolds with corners and manifolds with rough boundary (all modelled on locally convex spaces) – see Definition 54. It turns out that if the modelling space of the manifold is well behaved, the exponential law holds in these cases (Theorem 59). The main results of Section 3.4 subsume:

[1] A topological space X is called a *k-space* if it is Hausdorff and its topology is the final topology with respect to the inclusion maps $K \to X$ of compact subsets of X. For example, all locally compact spaces and all metrizable topological spaces are k-spaces.

Theorem B. *Let M_1 and M_2 be smooth manifolds (possibly with rough boundary) modelled on locally convex spaces E_1 and E_2, respectively. Let F be a locally convex space and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Then $\gamma^\vee \in C^r(M_1, C^s(M_2, F))$ for all $\gamma \in C^{r,s}(M_1 \times M_2, F)$, and the map*

$$\Phi\colon C^{r,s}(M_1 \times M_2, F) \to C^r(M_1, C^s(M_2, F)), \quad \gamma \mapsto \gamma^\vee \tag{1.3}$$

is linear and a topological embedding. If E_1 and E_2 are metrizable, then Φ is an isomorphism of topological vector spaces.

The same conclusion holds if M_2 is finite-dimensional or $E_1 \times E_2 \times E_1 \times E_2$ is a k-space, provided that M_1 and M_2 are manifolds without boundary, manifolds with smooth boundary or manifolds with corners.

Chapter 4. In this chapter we generalize the results of the previous chapter. We introduce and study mappings on multiple products of locally convex spaces (resp. manifolds modelled on locally convex spaces) with different degrees of differentiability in the individual factors (C^α-maps). We first introduce the notion of a C^α-mapping: For all $i \in \{1, \ldots, n\}$, let E_i and F be locally convex spaces, U_i be an open subset of E_i and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ such that $\alpha := (\alpha_1, \ldots, \alpha_n)$. Suppose that $\breve{D}_i$ is the iterated directional derivatives in the i-th component. we say that a map $f\colon U_1 \times \cdots \times U_n \to F$ is C^α if the iterated directional derivatives

$$(\breve{D}_1 \cdots \breve{D}_n f)(x)$$

exist and are continuous functions on $U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$ such that $\beta_i \in \mathbb{N}_0$, $\beta_i \leq \alpha_i$ (see Definition 65 for details). More generally, we consider C^α-maps if U_i is a locally convex subset with dense interior (see Definition 66). Using this definition, most results of this chapter in the C^α setting are analogous to those of Chapter 3, also for the results concerning exponential laws (Theorems 94 and 96).

Chapter 5. In this chapter we discuss the C^k-regularity concept. After recalling some definitions and results (mainly from [32], [27], [17] and [21]), we shall introduce a version of the Fundamental Theorem for $\mathfrak{g}$-valued functions (Theorem 132). The main result in this chapter is the following:

Theorem C. *Let M be a smooth manifold (possibly with boundary and modelled on a locally convex space), $2 \leq k \in \mathbb{N}$ and G be a C^{k-2}-regular Lie group with Lie algebra $\mathfrak{g}$. If $\alpha \in \Omega^1_{C^k}(M, \mathfrak{g})$ satisfies $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$, then α is locally integrable.*

Chapter 6. In this chapter, we study Lie group structures on groups of the form $C^k(M, K)$, where M is a non-compact smooth manifold and K is a, possibly infinite-dimensional, Lie group. Using that the map

$$\delta : C^k_*(M, K) \to \Omega^1_{C^{k-1}}(M, \mathfrak{k})$$

is a topological embedding (Theorem 141), we prove the following theorems (Theorems 142 and 147) :

Theorem D. *Let $s, k \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq s+1$, M be a connected finite-dimensional smooth manifold (with boundary) and K a C^s-regular Lie group. Assume that the subset $\delta(C^k_*(M, K))$ is a smooth submanifold of $\Omega^1_{C^{k-1}}(M, \mathfrak{k})$. Endow $C^k_*(M, K)$ with the smooth manifold structure for which $\delta : C^k_*(M, K) \to \mathrm{im}(\delta)$ is a diffeomorphism and*

$$C^k(M, K) \cong K \ltimes C^k_*(M, K)$$

with the product manifold structure. Assume that L_j for $j \in J$ are compact submanifolds (with boundary) of M whose interiors L_j° cover M, and such that

$$\delta_j : C^k_*(L_j, K) \to \Omega^1_{C^{k-1}}(L_j, \mathfrak{k}),$$

is an embedding of smooth manifolds onto a submanifold of $\Omega^1_{C^{k-1}}(L_j, \mathfrak{k})$. Then the following assertions hold:

(a) *For each $r \in \mathbb{N}_0 \cup \{\infty\}$ and locally convex C^r-manifold N, a map $f : N \times M \to K$ is $C^{r,k}$ if and only if for all $n \in N$, $f_n : M \to K$, $m \mapsto f(n, m)$ are C^k and the corresponding map $f^\vee : N \to C^k(M, K)$, $n \mapsto f_n$ is C^r.*

(b) *K acts smoothly by conjugation on $C^k_*(M, K)$, and $C^k(M, K)$ carries a C^s-regular Lie group structure compatible with evaluations.*

Theorem E. *Let K be a C^{k-1}-regular Lie group and N and M finite-dimensional smooth manifolds. We assume that $G := C^k(M, K)$ carries a C^{k-1}-regular Lie group structure compatible with evaluations and the smooth compact-open topology. If $C^r(N, G)$ also carries a regular Lie group structure compatible with evaluations and the compact-open C^k-topology, then $C^{r,k}(N \times M, K)$ carries a C^k-regular Lie group structure compatible with evaluations. Moreover, the canonical map*

$$\Phi : C^{r,k}(N \times M, K) \to C^r(N, G), \quad f \mapsto f^\vee$$

is an isomorphism of Lie groups.

Theorem D ensures, in particular, that $C^k(\mathbb{R}, K)$ is a Lie group for each $k \in \mathbb{N}$ and C^{k-1}-regular Lie group K. Theorem E implies that, if $k, r, s \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq s+1$ and $r \geq s+3$, then $C^{r,k}(\mathbb{R} \times \mathbb{R}, K)$ admits a C^s-regular Lie group structure compatible with evaluation and the compact-open $C^{r,k}$-topology.

Chapter 7. In this chapter we study the weighted spaces of continuous functions

$$CV(X,E) = \{f \in C(X,E) : |f|\, v \text{ is bounded for every } v \in V\},$$
$$CV_0(X,E) = \{f \in C(X,E) : fv \text{ vanishes at infinity for every } v \in V\},$$

such that X is a completely regular Hausdorff space, E is a topological vector space and V is a Nachbin family of weights on X. In Section 7.2, we recall from [33] and [34] some facts concerning these spaces as algebras. Analogous to those facts, we describe a condition on the weights that makes $CV(X, \mathfrak{g})$ a topological Lie algebra (Corollary 172):

Theorem F. *If $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a locally convex topological Lie algebra, X a Hausdorff topological space and V any Nachbin family on X such that $V \leq VV$, then $CV(X, \mathfrak{g})$ is a locally convex topological Lie algebra with the Lie bracket*

$$[\cdot,\cdot] : CV(X,\mathfrak{g}) \times CV(X,\mathfrak{g}) \to CV(X,\mathfrak{g}),\ (\gamma,\eta) \mapsto [\gamma,\eta]$$

with $[\gamma,\eta](x) := [\gamma(x), \eta(x)]_{\mathfrak{g}}$.

Using the fact that the Baker-Campbell-Hausdorff formula defines a group structure on any nilpotent Lie algebra ([21]), we obtain an analytic Lie group structure on $CV(X, \mathfrak{g})$, if $\mathfrak{g}$ is a nilpotent topological Lie algebra.

Also, for any Banach Lie group H with Lie algebra $\mathfrak{h}$, we use $CV(X, \mathfrak{h})$ to create a Lie group structure on

$$\langle \exp_H \circ \gamma : \gamma \in CV(X, \mathfrak{h}) \rangle\,,$$

if $1 \in V$ (see Section 7.3 for details).

Remark. This text slightly deviates from the version of the thesis submitted to the Institut für Mathematik in February 2013, as it takes comments of the referees into account.

Chapter 2

Preliminaries

This chapter briefly reviews some of the basic concepts and material concerning differential calculus in locally convex spaces, infinite-dimensional Lie groups and spaces of mappings.

The letter $\mathbb{K}$ always stands for $\mathbb{R}$ or $\mathbb{C}$. All vector spaces will be $\mathbb{K}$-vector spaces and all linear maps will be $\mathbb{K}$-linear, unless the contrary is stated.

2.1 Differential calculus in locally convex spaces

In this section we recall the C^r-maps in the Michal-Bastiani sense, also known as Keller's C^r_c-map [24] (see [26], [23], [27], [15] and [21] for streamlined expositions, cf. also [4]). For C^r-maps on suitable non-open domains, see [21] and [41].

Definition 1. Let E and F be locally convex topological vector spaces, $U \subseteq E$ open and $f : U \to F$ a map. Then the *derivative of f at x in the direction of h* is defined as

$$df(x,h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever the limit exists. The function f is called differentiable at x if $df(x,h)$ exists for all $h \in E$. The function f is called *continuously differentiable or C^1* if f is continuous and differentiable at all points of U and

$$df : U \times E \to F, \quad (x,h) \mapsto df(x,h)$$

is a continuous map. The function f is called a C^r-map if f is C^1 and df is a C^{r-1}-map, and C^∞ (or *smooth*) if f is C^r for all $r \in \mathbb{N}$.

Definition 2. **(C^r-maps on non-open sets).** Let $U \subseteq E$ be a locally convex subset with dense interior. A mapping $f\colon U \to F$ is called C^r if $f|_{U^\circ}\colon U^\circ \to F$ is C^r and each of the maps $d^{(k)}(f|_{U^\circ})\colon U^\circ \times E^k \to F$ admits a (necessarily unique) continuous extension $d^{(k)}f\colon U \times E^k \to F$.

We shall use some fundamental facts of the theory of C^r-maps. For details, the reader is referred to [15, 21, 23, 26, 27] (cf. also [4]):

Lemma 3. *If $f : E \supseteq U \to F$ is C^1, then $f'(x) := df(x, \bullet) : E \to F$ is a continuous linear map, for each $x \in U$.*

Proposition 4. **(Schwarz' Theorem).** *([16, Proposition 1.13]) Let E and F be locally convex spaces, $f : U \to F$ be a C^r-map on a locally convex set $U \subseteq E$ with dense interior, where $r \in \mathbb{N}_0 \cup \{\infty\}$. Then $d^{(k)}f(x, \bullet)\colon E^k \to F$ is symmetric, $\mathbb{K}$-linear, for each $x \in U$.*

The compositions of composable C^r-maps are C^r.

Lemma 5. **(Chain Rule).** *([21]) Let E, F and G be locally convex spaces, $U \subseteq E$, $V \subseteq F$ be locally convex sets with dense interior, and $f : U \to F$, $g : V \to G$ be C^r-maps such that $f(U) \subseteq V$, where $r \in \mathbb{N}_0 \cup \{\infty\}$. Then also $g \circ f : U \to G$ is C^r.*

Proposition 6. **(Parameter-dependent integrals).** *([6, Proposition 3.5]) Let E and F be locally convex spaces, $f : [a, b] \times X \to F$ be a continuous map such that $g(x) := \int_a^b f(t, x)\, dt$ exists in F for every x in a topological space X. Then $g : X \to F$ is continuous. Suppose, in addition, that $\partial_2 f : [a, b] \times U \times E \to F$ exists and is continuous, and that $g_1(x, v) := \int_a^b f(t, x; v)\, dt$ exists in F for every x in an open $U \subseteq E$ and every $v \in E$. Then g is a C^1-map with $dg = g_1$.*

Lemma 7. *A map $f\colon E \supseteq U \to F$ is C^{r+1} if and only if f is C^1 and $df\colon U \times E \to F$ is C^r.*

We shall also use the Rule on Partial Differentials:

Lemma 8. *([21]) Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $f : U \times V \to F$ be a continuous map. Assume that there exist continuous functions*

$$d^{(1,0)}f : U \times V \times E_1 \to F$$

$$d^{(0,1)}f : U \times V \times E_2 \to F$$

such that $D_{(w,0)}f(x,y)$ exists and coincides with $d^{(1,0)}f(x,y,w)$ for all $(x,y) \in U^0 \times V^0$ and $w \in E_1$, and $D_{(0,v)}f(x,y)$ exists and coincides with $d^{(0,1)}f(x,y,v)$ for all $(x,y) \in U^0 \times V^0$ and $v \in E_2$. Then f is C^1 and

$$df((x,y),(w,v)) = d^{(1,0)}f(x,y,w) + d^{(0,1)}f(x,y,v). \tag{2.1}$$

Using the method of the proof of Lemma 8 as in [21], one obtains the following proposition.

Proposition 9. (Rule on Partial Differentials). *Let $E_1, \dots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \dots, n\}$, $U := U_1 \times \dots \times U_n$ and $f : U_1 \times \dots \times U_n \to F$ be a continuous map. Assume that there exist continuous functions $d_i f : U_1 \times \dots \times U_n \times E_i \to F$ such that $D_{(w_i)^*}f(x_1, \dots, x_n)$ exists and coincides with $d_i f|_{U^0}(x_1, \dots, x_n, w_i)$ for all $i \in \{1, \dots, n\}$ and for all $(x_1, \dots, x_n) \in U^0$, $w_i \in E_i$ and the corresponding element $(w_i)^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \dots \times E_n$. Then f is C^1 and*

$$df((x_1, \dots, x_n),(w_1, \dots, w_n)) = \sum_{i=1}^{n} d_i f(x_1, \dots, x_n, w_i). \tag{2.2}$$

Proof. Assume that $d_i f$ exists for all $i \in \{1, \dots, n\}$. If we can show that $f|_{U^0}$ is C^1 and (2.2) holds for $f|_{U^0}$, then the right hand side of (2.2) provides a continuous extension of $d(f|_{U^0})$ to $U_1 \times \dots \times U_n \times (E_1 \times \dots \times E_n)$, whence f is C^1 and (2.2) holds. We may therefore assume that $U_1 \times \dots \times U_n$ is open in $E_1 \times \dots \times E_n$. Given $(x_1, \dots, x_n) \in U_1 \times \dots \times U_n$ and $w_i \in E_i$ for all $i \in \{1, \dots, n\}$, there exists $\epsilon > 0$ such that $(x_1, \dots, x_n) + \mathbb{D}_\epsilon w_1 \times \dots \times \mathbb{D}_\epsilon w_n \subseteq U_1 \times \dots \times U_n$, where $\mathbb{D}_\epsilon := \{z \in \mathbb{K} : |z| \leq \epsilon\}$. Then $(x_1, \dots, x_n) + [0,1]tw_1 \times \dots \times [0,1]tw_n \subseteq U_1 \times \dots \times U_n$ for each $0 \neq t \in \mathbb{D}_\epsilon$. By the Mean Value Theorem (see [21]), we obtain

$$\begin{aligned}
&\frac{1}{t}(f((x_1, \dots, x_n) + t(w_1, \dots, w_n)) - f(x_1, \dots, x_n)) \\
&= \sum_{j=1}^{n} \frac{1}{t} f(x_1 + tw_1, \dots, x_j + tw_j, x_{j+1}, \dots, x_n) \\
&- \sum_{j=2}^{n} \frac{1}{t} f(x_1 + tw_1, \dots, x_{j-1} + tw_{j-1}, x_j, \dots, x_n) - \frac{1}{t} f(x_1 \dots, x_n) \\
&= \sum_{j=1}^{n} \int_0^1 d_j f(x_1 + tw_1, \dots, x_{j-1} + tw_{j-1}, x_j + \sigma t w_j, x_{j+1}, \dots, x_n, w_j)\, d\sigma.
\end{aligned} \tag{2.3}$$

Note that the integrals in (2.3) make sense also for $t = 0$ (the integrands are then constants), and hence define mappings $I_1, \dots, I_n : \mathbb{D}_\epsilon \to F$. The map $\mathbb{D}_\epsilon \times [0,1] \to$

F, $(t, \sigma) \mapsto d_i f(x_1 + tw_1, \ldots, x_i + \sigma t w_i, x_{i+1}, \ldots, x_n, w_i)$ being continuous for all $i \in \{1, \ldots, n\}$, the parameter-dependent integral I_i is continuous (see [21]). Hence the right hand side of (2.3) converges as $t \to 0$, with limit $I_1(0)+\cdots+I_2(0) = d_1 f(x_1, \ldots, x_n, w_1)+ \cdots + d_n f(x_1, \ldots, x_n, w_n)$. Hence df exists and is given by the right- hand side of (2.2) and hence continuous, whence f is C^1. □

2.2 Manifolds

Since the composition of C^r maps between locally convex spaces is a C^r map, we can define C^r-manifolds M as in the finite-dimensional case (see [26], [23], [27], [15] and [21]) .

Definition 10. (a) A *smooth manifold modelled on a locally convex topological vector space* E is a Hausdorff topological space M, together with a set $\mathcal{A}$ of homeomorphisms (charts) $\varphi : U \to V$ from open subsets of M onto open subsets of E, such that the domains cover M and the transition maps $\varphi \circ \psi^{-1}$ are smooth on their domain, for all φ, $\psi \in \mathcal{A}$.

(b) If the transition maps $\varphi \circ \psi^{-1}$ are just C^r on their domain, for all φ, $\psi \in \mathcal{A}$, then it is called a *C^r-manifold.*

(c) A manifold modelled on Banach space is called a *Banach manifold.*

Products of manifolds and smoothness of maps between manifolds are defined also as in the finite-dimensional case.

Remark 11. For $i \in \{1, \ldots, n\}$, let M_i be a smooth manifold modelled on the space E_i. Then the product set $M := M_1 \times \cdots \times M_n$ carries a natural manifold structure with model space $E = \prod_{i=1}^{n} E_i$.

Definition 12. A mapping $f : M \to N$ between manifolds is said to be C^k if for each $x \in M$ and each chart (V, ψ) on N with $f(x) \in V$ there is a chart (U, ϕ) on M with $x \in U, f(U) \subseteq V$, and $\psi \circ f \circ \phi^{-1}$ is C^k. We will denote by $C^k(M, N)$ the space of all C^k-mappings from M to N. A C^k-mapping $f : M \to N$ is called a *C^k-diffeomorphism* if $f^{-1} : N \to M$ exists and is also C^k. Two manifolds are called *diffeomorphic* if there exists a diffeomorphism between them.

Definition 13. Let M be a manifold modelled on the space E, and $N \subseteq M$ a subset.

(a) N is called a *submanifold* of M if there exists a closed vector subspace $F \subseteq E$ and for each $x \in N$ there exists an E-chart (U, φ) of M with $x \in U$ and $\varphi(U \cap N) = \varphi(U) \cap F$.

(b) N is called a split submanifold of M if, in addition, there exists a vector subspace $G \subseteq E$ for which the addition map $F \times G \to E$, $(f, g) \mapsto f + g$ is a topological isomorphism.

2.3 Infinite-dimensional Lie groups and their Lie algebras

Definition 14. A *Lie group* G is a group, equipped with a smooth manifold structure modelled on a locally convex space E such that the group operations are smooth maps. Similarly, an analytic Lie group is a group G equipped with an analytic manifold structure turning the group operations into analytic maps.

We write $1 \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G.

Remark 15. It is easy to see that the group operations are smooth if the map $G \times G \to G$, $(x, y) \mapsto xy^{-1}$ is smooth.

Smooth and analytic Lie groups can be described locally:

Proposition 16. (Local description of Lie groups). *Suppose that a subset U of a group G is equipped with a smooth (resp., $\mathbb{K}$-analytic) manifold structure modelled on a locally convex space E. Furthermore, assume that there exists $V \subseteq U$ open such that $1 \in V$, $VV \subseteq U, V = V^{-1}$ and*

(a) *$V \times V \to U$, $(g, h) \mapsto gh$ is smooth (resp., $\mathbb{K}$-analytic),*
(b) *$V \to V$, $g \mapsto g^{-1}$ is smooth (resp., $\mathbb{K}$-analytic),*
(c) *For all $g \in G$, there exists an open unit neighbourhood $W \subseteq U$ such that $g^{-1}Wg \subseteq U$ and the map $W \to U$, $h \mapsto g^{-1}hg$ is smooth (resp., $\mathbb{K}$-analytic).*

Then there is a unique smooth (resp., $\mathbb{K}$-analytic) Lie group structure on G which makes V, equipped with the above manifold structure, an open submanifold of G.

Proof. The proof of [10], Proposition III.1.9.18 carries over without changes. □

Remark 17. If V generates the group G (i.e., if G is the smallest subgroup of G containing V), then Condition (c) can be omitted in Proposition (16) (as it follows from (a) and (b)).

The Lie algebra of a locally convex Lie group. As in finite dimensions, the tangent space [1] $\mathrm{L}(G) := T_1(G) \cong E$ at the identity element of a Lie group G can be made a topological Lie algebra via the identification with the Lie algebra of left invariant vector

[1]For definitions and details of tangent spaces and tangent bundles, we refer to [21] and [30].

fields on G. We recall that a *vector field* X on a locally convex Lie group G is called left invariant if

$$X \circ \lambda_g = T\lambda_g \circ X$$

as mappings $G \to TG$. Then each $x \in T_1(G)$ corresponds to a unique left invariant vector field x_l with $x_l(g) := d\lambda_g(1).x$, $g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_1(G)$ a continuous Lie bracket (see [21]) which is uniquely determined by $[x, y]_l = [x_l, y_l]$ for $x, y \in \mathfrak{g}$.

Definition 18. (Exponential function). Let G be a locally convex Lie group. The group G is said to have an exponential function if for each $x \in \mathfrak{g}$ the initial value problem

$$\gamma(0) = 1,\ \gamma'(t) = T_1\lambda_{\gamma(t)}.x$$

has a solution $\gamma_x \in C^\infty(\mathbb{R}, G)$ and the function

$$\exp_G : \mathfrak{g} \to G, x \mapsto \gamma_x(1)$$

is smooth.

Definition 19. (The Lie functor). For a Lie group G, the locally convex Lie algebra $\mathrm{L}(G) := (T_1(G), [.,.])$ is called the Lie algebra of G.

To each morphism $\varphi : G \to H$ of Lie groups we further associate its tangent map $\mathrm{L}(\varphi) := T_1(\varphi) : \mathrm{L}(G) \to \mathrm{L}(H)$, and the usual argument with related vector fields implies that $\mathrm{L}(\varphi)$ is a homomorphism of Lie algebras.

Adjoint Representation. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For each $g \in G$, we define the conjugation or the inner automorphism by the map $c_g : G \to G$, $x \mapsto gxg^{-1}$.

This defines a smooth action of G on itself by automorphisms, hence induces continuous linear automorphisms

$$\mathrm{Ad}(g) := \mathrm{L}(c_g) : \mathfrak{g} \to \mathfrak{g}.$$

Thus the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

is given by $\mathrm{Ad}(g) = T_1(c_g) : \mathfrak{g} \to \mathfrak{g}$ for $g \in G$. By Definition 19, $\mathrm{Ad}(g)$ is a Lie algebra homomorphism. We also define for $x \in \mathfrak{g}$ a linear map

$$\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g},\ \mathrm{ad}\, x(y) := T\,\mathrm{Ad}(x, 0_y)$$

2.4 Spaces of mappings

Definition 20. Let X and Y be Hausdorff topological spaces.

(a) Given a compact subset $K \subseteq X$ and open subset $U \subseteq Y$, we define

$$\lfloor K, U \rfloor := \{\gamma \in C(X,Y) : \gamma(K) \subseteq U\}.$$

Then the sets

$$\lfloor K_1, U_1 \rfloor \cap \cdots \cap \lfloor K_n, U_n \rfloor$$

where $n \in \mathbb{N}$, $K_1, \ldots, K_n$ are compact subsets of X and $U_1, \ldots, U_n$ open subsets of Y, form a basis for a topology on $C(X,Y)$, called the *compact-open topology*. We write $C(X,Y)_{c.o.}$ for $C(X,Y)$, equipped with the compact-open topology.

(b) If G is a topological group, then $C(X,G)$ is a group with respect to the pointwise product. Then the compact-open topology on $C(X,G)$ coincides with the topology of uniform convergence on compact subsets of X, for which the sets $\lfloor K, U \rfloor$, $K \subseteq X$ compact and $U \subseteq G$ a 1-neighbourhood, form a basis of 1-neighbourhoods. In particular, $C(X,G)_{c.o.}$ is a topological group.

(c) We topologize for two smooth manifolds M (possibly with boundary) and N, the space $C^k(M,N)$ by the embedding

$$C^k(M,N) \hookrightarrow \prod_{n=0}^{k} C(T^n(M), T^n(N))_{c.o.}, \quad f \mapsto (T^n(f))_{\substack{n \in \mathbb{N}_0, \\ n \leq k}}, \tag{2.4}$$

where the spaces $C(T^n(M), T^n(N))_{c.o.}$ carry the compact-open topology. The so obtained topology on $C^k(M,N)$ is called the *compact open C^k-topology.*

Remark 21. ([31]) Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. The tangent map $T(m_G)$ of the multiplication map $m_G : G \times G \to G$ defines a Lie group structure on the tangent bundle TG (cf.[21]). Iterating this procedure, we obtain a Lie group structure on all higher tangent bundles T^nG. For each $n \in \mathbb{N}_0$, we thus obtain topological groups $C(T^nM, T^nG)_{c.o.}$. We also observe that for two smooth maps $f_1, f_2 : M \to G$, the functoriality of T yields

$$T(f_1 \cdot f_2) = T(m_G \circ (f_1 \times f_2)) = T(m_G) \circ (Tf_1 \times Tf_2) = Tf_1 \cdot Tf_2.$$

Therefore the inclusion map $C^k(M,G) \hookrightarrow \prod_{n=0}^{k} C(T^nM, T^nG)_{c.o.}$ from (2.4) is a group homomorphism, so that the inverse image of the product topology from the right hand side is a group topology on $C^k(M,G)$ and thus turns $C^k(M,G)$ into a topological group, even if M and G are infinite-dimensional.

The following assertions readily follow from the definitions:

Remark 22. ([19])

(a) For every $r \geq s$, the inclusion map $C^r(U,E) \to C^s(U,E)$ is a continuous linear map. The topology on $C^\infty(U,E)$ is initial with respect to the family of inclusion maps $C^\infty(U,E) \to C^k(U,E)$, where $k \in \mathbb{N}_0$. Furthermore, $C^\infty(U,E) = \varprojlim C^k(U,E)$. Accordingly, $C^k(U, C^\infty(V,E)) = \varprojlim C^k(U, C^r(V,E))$.

(b) For every $k \in \mathbb{N}_0$, the topology on $C^{k+1}(U,E)$ is initial with respect to the inclusion map $C^{k+1}(U,E) \to C(U,E)$ together with the mapping $C^{k+1}(U,E) \to C^k(U \times E, E)$, $\gamma \mapsto \mathrm{d}\gamma$.

We recall from ([21]) the following proposition and lemma for later use.

Proposition 23. *Let X_1, X_2 and Y be Hausdorff topological spaces. Then the following holds:*

(a) *If $f : X_1 \to X_2$ a continuous map, then also the pullback*

$$C(f,Y) : C(X_2,Y)_{c.o.} \to C(X_1,Y)_{c.o.},\ \gamma \mapsto \gamma \circ f$$

is continuous.

(b) *If $g : X_1 \times X_2 \to Y$ is continuous, then the map*

$$g^\vee : X_1 \to C(X_2,Y)_{c.o.},\ g^\vee(x) := g(x, \bullet)$$

is continuous.

(c) *If X_2 is locally compact and $h : X_1 \to C(X_2,Y)_{c.o.}$ is continuous, then the map*

$$h^\wedge : X_1 \times X_2 \to Y,\ h^\wedge(x_1,x_2) := h(x_1)(x_2)$$

is continuous.

Lemma 24. *Suppose that the topology on E is initial with respect to a family $(\lambda_i)_{i\in I}$ of $\mathbb{K}$-linear maps $\lambda_i : E \to E_i$ into topological $\mathbb{K}$-vector spaces E_i. Then the topology on $C^r(M,E)$ is initial with respect to the family $(C^r(M,\lambda_i))_{i\in I}$ of the linear mappings $C^r(M,\lambda_i) : C^r(M,E) \to C^r(M,E_i)$.*

Chapter 3

$C^{r,s}$-Mappings

This chapter gives a systematic treatment of the calculus of mappings on products with different degrees of differentiability in the two factors, called $C^{r,s}$-mappings [1]. We shall develop their basic properties and some refined tools. We study such mappings in an infinite-dimensional setting, which is analogous to the approach to C^r-maps between locally convex spaces. We first introduce the notion of a $C^{r,s}$-mapping:

Definition 25. Let E_1, E_2 and F be locally convex spaces, U and V open subsets of E_1 and E_2 respectively and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. A mapping $f\colon U \times V \to F$ is called a $C^{r,s}$-map, if for all $i, j \in \mathbb{N}_0$ such that $i \leq r, j \leq s$ the iterated directional derivative

$$d^{(i,j)} f(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) := (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y)$$

exists for all $x \in U, y \in V, w_1, \ldots, w_i \in E_1, v_1, \ldots, v_j \in E_2$ and

$$d^{(i,j)} f\colon U \times V \times E_1^i \times E_2^j \to F,$$

$$(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) \mapsto (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y)$$

is continuous.

More generally, it is useful to have a definition of $C^{r,s}$-maps on not necessarily open domains available:

Definition 26. Let E_1, E_2 and F be locally convex spaces, U and V are locally convex subsets with dense interior of E_1 and E_2, respectively, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$, then we say

[1]For examples of projects which benefit from the results developed in this chapter see [1]

that a continuous map $f\colon U \times V \to F$ is $C^{r,s}$, if $f|_{U^0\times V^0}\colon U^0 \times V^0 \to F$ is $C^{r,s}$-map and for all $i, j \in \mathbb{N}_0$ such that $i \le r, j \le s$, the map

$$d^{(i,j)}(f|_{U^0\times V^0}) : U^0 \times V^0 \times E_1^i \times E_2^j \to F$$

admits a continuous extension

$$d^{(i,j)}f\colon U \times V \times E_1^i \times E_2^j \to F.$$

Remark 27. Variants and special cases of $C^{r,s}$-mappings are encountered in many parts of analysis. For example [2] considers analogues of $C^{0,r}$-maps on Banach spaces based on continuous Fréchet differentiability; [15, 1.4] for $C^{0,r}$-maps; [14] for $C^{r,s}$-maps on finite-dimensional domains; and [13, p. 135] for certain $\mathrm{Lip}^{r,s}$-maps in the convenient setting of analysis. Cf. also [29], [18] for ultrametric analogues in finite dimensions. Furthermore, a key result concerning $C^{r,s}$-maps was conjectured in [19, p.10].

Definitions 25 and 26 can be rephrased as follows:

Lemma 28. *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Then $f\colon U \times V \to F$ is $C^{r,s}$-map if and only if all of the following conditions are satisfied:*

(a) *For each $x \in U$, the map $f_x := f(x, \bullet)\colon V \to F$, $y \mapsto f_x(y) := f(x, y)$ is C^s.*

(b) *For all $y \in V$ and $j \in \mathbb{N}_0$ such that $j \le s$ and $v := (v_1, \ldots, v_j) \in E_2^j$, the map $d^{(j)}f_\bullet(y, v)\colon U \to F$, $x \mapsto (d^{(j)}f_x)(y, v)$ is C^r.*

(c) *$d^{(i,j)}f\colon U \times V \times E_1^i \times E_2^j \to F$, $(x, y, w, v) \mapsto d^{(i)}(d^{(j)}f_\bullet(y, v))(x, w)$ is continuous, for all j as in* (b)*, $i \in \mathbb{N}_0$ such that $i \le r$ and $w := (w_1, \ldots, w_i) \in E_1^i$.*

Proof. Step 1. If U, V are open subsets, then the equivalence is clear.
Now the general case: Assume that f is a $C^{r,s}$-map.
Step 2. If $x \in U^0$, then for $j \in \mathbb{N}_0$, $j \le s$

$$D_{(0,v_j)} \cdots D_{(0,v_1)} f(x, y) = D_{v_j} \cdots D_{v_1} f_x(y)$$

exists for all $y \in V^0$ and $v_1, \ldots, v_j \in E_2$, with continuous extension

$$(y, v_1, \ldots, v_j) \mapsto d^{(0,j)}f(x, y, v_1, \ldots, v_j)$$

to $V \times E_2^j \to F$. Hence $f_x\colon V \to F$ is C^s.
If $x \in U$ is arbitrary, $y \in V^0$ and $v_1 \in E_2$, we show that $D_{v_1} f_x(y)$ exists and equals $d^{(0,1)}f(x, y, v_1)$. There exists $R > 0$ such that $y + tv_1 \in V$ for all $t \in \mathbb{R}$, $|t| \le R$ and there exists a relatively open convex neighbourhood $W \subseteq U$ of x in U. Because U^0 is

dense, there exists $z \in U^0 \cap W$. Since W is convex, we have $x + \tau(z-x) \in W$ for all $\tau \in [0,1]$. Moreover, since $z \in W^0$, we have $x + \tau(z-x) \in W^0 \subseteq U^0$ for all $\tau \in (0,1]$. Hence, for $\tau \in (0,1]$, $f(x+\tau(z-x),y)$ is C^s in y, and thus for $t \neq 0$

$$\frac{1}{t}(f(x+\tau(z-x),y+tv_1)-f(x+\tau(z-x),y)) = \int_0^1 d^{(0,1)}f(x+\tau(z-x),y+\sigma t v_1,v_1)\,d\sigma$$

by the Mean Value Theorem. Now let $\tilde{F}$ be a completion of F. Because

$$h\colon\ [0,1]\times[-R,R]\times[0,1] \to \tilde{F},\ (\tau,t,\sigma) \mapsto d^{(0,1)}f(x+\tau(z-x),y+\sigma t v_1,v_1)$$

is continuous, also the parameter-dependent integral

$$g\colon\ [0,1]\times[-R,R] \to \tilde{F},\ g(\tau,t) := \int_0^1 h(\tau,t,\sigma)\,d\sigma$$

is continuous. Fix $t \neq 0$ in $[-R,R]$. Then

$$g(\tau,t) = \frac{1}{t}(f(x+\tau(z-x),y+tv_1)-f(x+\tau(z-x),y)) \tag{3.1}$$

for all $\tau \in (0,1]$. By continuity of both sides in τ, (3.1) also holds for $\tau = 0$. Hence

$$\frac{1}{t}(f(x,y+tv_1)-f(x,y)) = g(0,t) \to g(0,0)$$

as $t \to 0$. Thus $D_{v_1}f_x(y)$ exists and is given by

$$g(0,0) = \int_0^1 d^{(0,1)}f(x,y,v_1)\,d\sigma = d^{(0,1)}f(x,y,v_1).$$

Holding $(v_1,\ldots,v_{j-1})$ fixed, we can repeat the argument to see that $D_{v_j}\cdots D_{v_1}f_x(y)$ exists for all $y \in V^0$ and $j \in \mathbb{N}_0$ such that $j \leq s$ and all $v_1,\ldots,v_j \in E_2$, and is given by

$$D_{v_j}\cdots D_{v_1}f_x(y) = d^{(0,j)}f(x,y,v_1,\ldots,v_j).$$

Since the right-hand side makes sense for $(y,v_1,\ldots,v_j) \in V \times E_2^j$ and is continuous there, f_x is C^s.

Step 3 Holding $v_1,\ldots,v_j \in E_2^j$ fixed, the function $(x,y) \mapsto d^{(0,j)}f(x,y,v_1,\ldots,v_j)$ is $C^{r,0}$. By Step 2 (applied to the $C^{0,r}$ function $(y,x) \mapsto d^{(0,j)}f(x,y,v_1,\ldots,v_j)$) we see that for each $y \in V$, the function $U \to F,\ x \mapsto d^{(0,j)}f(x,y,v_1,\ldots,v_j)$ is C^r and $d^{(i)}(d^{(j)}f_\bullet(y,v))(x,w) = d^{(i,j)}f(x,y,w,v)$, which is continuous in $(x,y,w,v) \in U \times V \times E_1^i \times E_2^j$. Hence if f is $C^{r,s}$, then (a),(b) and (c) hold.

Step 4. Conversely. Assume that (a),(b) and (c) hold. By Step 1, $f|_{U^0\times V^0}$ is $C^{r,s}$ and

$$d^{(i,j)}f|_{U^0\times V^0}(x,y,w,v) = d^{(i)}(d^{(j)}f_\bullet(y,v))(x,w) \tag{3.2}$$

for $(x,y) \in U^0 \times V^0$, $w \in E_1^i$, $v \in E_2^j$. By (c), the right-hand side of (3.2) extends to a continuous function $d^{(i,j)}f\colon U \times V \times E_1^i \times E_2^j \to F$. Hence f is a $C^{r,s}$-map. $\square$

3.1 Elementary properties

The following lemma will enable us to prove a version of the Theorem of Schwarz for $C^{r,s}$-maps.

Lemma 29. *Let E_1, E_2 and F be locally convex spaces, $f\colon U \times V \to F$ be a $C^{1,1}$-map on open subsets $U \subseteq E_1$, $V \subseteq E_2$. Let $w \in E_1$ and $v \in E_2$. Then $D_{(0,v)}D_{(w,0)}f$ exists and coincides with $D_{(w,0)}D_{(0,v)}f$.*

Proof. After replacing F with a completion, we may assume that F is complete. Fix $x \in U$, $y \in V$. There is $\varepsilon > 0$ such that $x + sw \in U$ and $y + tv \in V$ for all $s, t \in B^{\mathbb{R}}_{\varepsilon}(0)$. For $t \neq 0$ as before, we have

$$\frac{1}{t}(f(x + sw, y + tv) - f(x + sw, y)) = \int_0^1 D_{(0,v)}f(x + sw, y + rtv)dr. \tag{3.3}$$

For fixed t, consider the map

$$g\colon B^{\mathbb{R}}_{\varepsilon}(0) \to F,\ g(s) := \int_0^1 D_{(0,v)}f(x + sw, y + rtv)dr.$$

The map $[0, 1] \times B^{\mathbb{R}}_{\varepsilon}(0) \to F$, $(r, s) \mapsto D_{(0,v)}f(x + sw, y + rt)$ is differentiable in s, with partial derivative $D_{(w,0)}D_{(0,v)}f(x + sw, y + rtv)$ which is continuous in (r, s). Hence, by [6, Proposition 3.5], g is C^1 and

$$g'(0) = \int_0^1 D_{(w,0)}D_{(0,v)}f(x, y + rtv)dr.$$

Hence (3.3) can be differentiated with respect to s, and

$$\frac{1}{t}(D_{(w,0)}f(x, y + tv) - D_{(w,0)}f(x, y)) = \int_0^1 D_{(w,0)}D_{(0,v)}f(x, y + rtv)dr. \tag{3.4}$$

Note that, for fixed x, v and w, the integrand in (3.4) also makes sense for $t = 0$, and defines a continuous function $h\colon [0, 1] \times B^{\mathbb{R}}_{\varepsilon}(0) \to F$ of (r, t). By [6, Proposition 3.5], the function

$$H\colon B^{\mathbb{R}}_{\varepsilon}(0) \to F,\ H(t) := \int_0^1 h(r, t)dr$$

is continuous. If $t \neq 0$ this function coincides with $\frac{1}{t}(D_{(w,0)}f(x, y+tv) - D_{(w,0)}f(x, y))$, by (3.4). Hence

$$\begin{aligned} D_{(0,v)}D_{(w,0)}f(x, y) &= \lim_{t\to 0} \frac{1}{t}(D_{(w,0)}f(x, y + tv) - D_{(w,0)}f(x, y)) \\ &= \lim_{t\to 0} H(t) = H(0) = \int_0^1 h(r, 0)dr = h(r, 0) = D_{(w,0)}D_{(0,v)}f(x, y) \end{aligned}$$

exists and has the asserted form. $\square$

Lemma 30. *Let E_1, E_2 and F be locally convex spaces, U and V be open subsets of E_1 and E_2, respectively, and $r \in \mathbb{N}_0 \cup \{\infty\}$. If $f\colon U \times V \to F$ is a $C^{r,1}$-map, then*

$$D_{(0,v)} D_{(w_i,0)} \cdots D_{(w_1,0)} f(x,y)$$

exists for all $i \in \mathbb{N}$ such that $i \leq r$, $(x,y) \in U \times V$, $v \in E_2$ and $w_1, \ldots, w_i \in E_1$, and it coincides with $d^{(i,1)} f(x,y,w_1,\ldots,w_i,v)$.

Proof. The proof is by induction on i. *The case $i = 1$.* This is covered by Lemma 29. *Induction step.* Assume that $i > 1$. By induction, we know that

$$D_{(0,v)} D_{(w_{i-1},0)} \cdots D_{(w_1,0)} f(x,y)$$

exists and coincides with

$$d^{(i-1,1)} f(x,y,w_1,\ldots,w_{i-1},v). \tag{3.5}$$

Define $g\colon U \times V \to F$ via

$$g(x,y) = D_{(w_{i-1},0)} \cdots D_{(w_1,0)} f(x,y) = d^{(i-1,0)} f(x,y,w_1,\ldots,w_{i-1}).$$

Then g is $C^{1,0}$ (f is $C^{r,1}$ and $r \geq i$, hence we can differentiate once more in the first variable). By induction, g is differentiable in the second variable with

$$\begin{aligned} D_{(0,v)} g(x,y) &= d^{(i-1,1)} f(x,y,w_1,\ldots,w_{i-1},v) \qquad (3.6) \\ &= D_{(w_{i-1},0)} \cdots D_{(w_1,0)} D_{(0,v)} f(x,y), \qquad (3.7) \end{aligned}$$

which is continuous in (v,x,y). Hence g is $C^{0,1}$ and $d^{(0,1)} g(x,y,v)$ is given by (3.5). Because f is $C^{r,1}$ and $r \geq i$, the right-hand side of (3.6) can be differentiated once more in the first variable, hence also $D_{(0,v)} g(x,y)$, with

$$\begin{aligned} d^{(1,1)} g(x,y,w_i,v) = D_{(w_i,0)} D_{(0,v)} g(x,y) &= D_{(w_i,0)} D_{(w_{i-1},0)} \cdots D_{(w_1,0)} D_{(0,v)} f(x,y) \\ &= d^{(i,1)} f(x,y,w_1,\ldots,w_i,v). \end{aligned}$$

As this map is continuous, g is $C^{1,1}$. By Lemma 29, also $D_{(0,v)} D_{(w_i,0)} g(x,y)$ exists and is given by $D_{(w_i,0)} D_{(0,v)} g(x,y) = d^{(i,1)} f(x,y,w_1,\ldots,w_i,v)$ (where we used (3.7)). But, by definition of g,

$$D_{(0,v)} D_{(w_i,0)} g(x,y) = D_{(0,v)} D_{(w_i,0)} D_{(w_{i-1},0)} \cdots D_{(w_1,0)} f(x,y).$$

Hence $D_{(0,v)} D_{(w_i,0)} \cdots D_{(w_1,0)} f(x,y) = d^{(i,1)} f(x,y,w_1,\ldots,w_i,v)$. □

Proposition 31. (Schwarz' Theorem). *Let E_1, E_2 and F be locally convex spaces and $f\colon U \times V \to F$ be a $C^{r,s}$-map on open subsets $U \subseteq E_1$, $V \subseteq E_2$. Let $i, j \in \mathbb{N}_0$ with $i \leq r, j \leq s$ and $\sigma \in S_{i+j}$ be a permutation of $\{1, \ldots, i+j\}$. Let $x \in U$, $y \in V$, $w_1, \ldots, w_i \in E_1$ and $w_{i+1}, \ldots, w_{i+j} \in E_2$. Define $w_k^* := (w_k, 0)$ if $k \in \{1, \ldots, i\}$ and $w_k^* := (0, w_k)$ if $k \in \{i+1, \ldots, i+j\}$. Then the iterated directional derivative*

$$(D_{w^*_{\sigma(1)}} \cdots D_{w^*_{\sigma(i+j)}} f)(x, y)$$

exists and coincides with

$$d^{(i,j)} f(x, y, w_1, \ldots, w_i, w_{i+1}, \ldots, w_{i+j}).$$

Proof. The proof is by induction on $i + j$. *The case* $i + j = 0$ is trivial.
The case $i = 0$ or $j = 0$. If $i = 0$, then the assertion follows from Schwarz' Theorem for the C^s-function $f(x, \bullet)\colon V \to F$. Likewise, if $j = 0$, then the assertion follows from Schwarz' Theorem for the C^r-function $f(\bullet, y)\colon U \to F$ (see [21]).
The case $i, j \geq 1$. If $\sigma(1) \in \{1, \ldots, i\}$, then by induction,

$$\begin{aligned} &D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(i+j)}} f(x, y) \\ &= d^{(i-1,j)} f(x, y, w_1, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_i, w_{i+1}, \ldots, w_{i+j}). \end{aligned}$$

Because f is $C^{i,j}$, we can differentiate once more in the first variable:

$$\begin{aligned} &D_{w^*_{\sigma(1)}} \cdots D_{w^*_{\sigma(i+j)}} f(x, y) \\ &= d^{(i,j)} f(x, y, w_1, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_i, w_{\sigma(1)}; w_{i+1}, \ldots, w_{i+j}) \\ &= d^{(i,j)} f(x, y, w_1, \ldots, w_i; w_{i+1}, \ldots, w_{i+j}). \end{aligned}$$

For the final equality we used that

$$d^{(i,j)} f(x, y, z_1, \ldots, z_i, v_1, \ldots, v_j) = d^{(i)}(d^{(j)} f_\bullet(y, v_1, \ldots, v_j))(x, z_1, \ldots, z_j)$$

is symmetric in $z_1, \ldots, z_j$, as $g(x) := d^{(j)} f_x(y, v_1, \ldots, v_j)$ is C^r in x (see Lemma 28).

If $\sigma(1) \in \{i+1, \ldots, i+j\}$, then by induction,

$$D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(i+j)}} f(x, y) = d^{(i,j-1)} f(x, y, w_1, \ldots, w_i, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_{i+j}).$$

For fixed $w_{i+1}, \ldots, w_{i+j}$, consider the function $h\colon U \times V \to F$,

$$h(x, y) := d^{(0,j-1)} f(x, y, w_{i+1}, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_{i+j}),$$

which is $C^{r,s-(j-1)}$.
By Lemma 30,

$$D_{w^*_{\sigma(1)}} D_{w^*_i} \cdots D_{w^*_1} h(x,y)$$

exists and coincides with

$$D_{w^*_i} \cdots D_{w^*_1} D_{w^*_{\sigma(1)}} h(x,y).$$

Now

$$\begin{aligned} D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(i+j)}} f(x,y) &= d^{(i,j-1)} f(x,y,w_1,\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{i+j}) \\ &= D_{w^*_i} \cdots D_{w^*_1} h(x,y). \end{aligned}$$

By the preceding, we can apply $D_{w^*_{\sigma(1)}}$, i.e., $D_{w^*_{\sigma(1)}} \cdots D_{w^*_{\sigma(i+j)}} f(x,y)$ exists and coincides with

$$\begin{aligned} & D_{w^*_i} \cdots D_{w^*_1} D_{w^*_{\sigma(1)}} h(x,y) \\ & = d^{(i,j)} f(x,y,w_1,\ldots,w_i,w_{i+1},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{i+j},w_{\sigma(1)}) \\ & = d^{(i)}(d^{(j)} f_\bullet(y,w_{i+1},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{i+j},w_{\sigma(1)}))(x,w_1,\ldots,w_i) \end{aligned}$$

where $d^{(j)} f_x(y,v_1,\ldots,v_j)$ is symmetric in $v_1,\ldots,v_j$ by the Schwarz Theorem for the C^s-function f_x. Hence

$$d^{(j)} f_x(y,w_{i+1},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{i+j},w_{\sigma(1)}) = d^{(j)} f_x(y,w_{i+1},\ldots,w_{i+j})$$

for all x. Hence also after differentiations in x:

$$d^{(i)}(d^{(j)} f_\bullet(y,w_{i+1},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{i+j},w_{\sigma(1)}))(x,w_1,\ldots,w_i)$$

coincides with

$$d^{(i,j)} f(x,y,w_1,\ldots,w_{i+j}) = d^{(i)}(d^{(j)} f_\bullet(y,w_{i+1},\ldots,w_{i+j}))(x,w_1,\ldots,w_i).$$

□

Remark 32. If U and V are merely locally convex subsets with dense interior in the situation of Proposition 31, then

$$(D_{w^*_{\sigma(1)}} \cdots D_{w^*_{\sigma(i+j)}} f)(x,y) \tag{3.8}$$

exists for all $x \in U^0$, $y \in V^0$, and the map $d^{(i,j)} f(x,y,w_1,\ldots,w_{i+j})$ provides a continuous extension of (3.8) to all of $U \times V \times E_1^i \times E_2^j$.

Corollary 33. *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively. If $f\colon U \times V \to F$ is $C^{r,s}$, then*

$$g\colon V \times U \to F,\ (y,x) \mapsto f(x,y)$$

is a $C^{s,r}$-map, and

$$d^{(j,i)}g(y,x,v_1,\ldots,v_j,w_1,\ldots,w_i) = d^{(i,j)}f(x,y,w_1,\ldots,w_i,v_1,\ldots,v_j)$$

for all $i,j \in \mathbb{N}_0$ with $i \leq r$, $j \leq s$, $x \in U, y \in V$, $w_1,\ldots,w_i \in E_1$ and $v_1,\ldots,v_j \in E_2$.

Lemma 34. *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively. If $f\colon U \times V \to F$ is $C^{r,s}$ and $\lambda\colon F \to H$ is a continuous linear map to a locally convex space H, then $\lambda \circ f$ is $C^{r,s}$ and $d^{(i,j)}(\lambda \circ f) = \lambda \circ d^{(i,j)}f$.*

Proof. Follows from the fact that directional derivatives and continuous linear maps can be interchanged. □

Lemma 35. (Mappings to products). *Let E_1, E_2 be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $(F_\alpha)_{\alpha\in A}$ be a family of locally convex spaces with direct product $F := \prod_{\alpha\in A} F_\alpha$ and the projections $\pi_\alpha\colon F \to F_\alpha$ onto the components. Let $r,s \in \mathbb{N}_0 \cup \{\infty\}$ and $f\colon U \times V \to F$ be a map. Then f is $C^{r,s}$ if and only if all of its components $f_\alpha := \pi_\alpha \circ f$ are $C^{r,s}$. In this case*

$$d^{(i,j)}f = (d^{(i,j)}f_\alpha)_{\alpha\in A}, \tag{3.9}$$

for all $i,j \in \mathbb{N}_0$ such that $i \leq r$ and $j \leq s$.

Proof. π_α is continuous linear. Hence if f is $C^{r,s}$, then $f_\alpha = \pi_\alpha \circ f$ is $C^{r,s}$, by Lemma 34, with $d^{(i,j)}f_\alpha = \pi_\alpha \circ d^{(i,j)}f$. Hence (3.9) holds.

Conversely, assume that each f_α is $C^{r,s}$. Because the limits in products can be formed component-wise, we see that

$$d^{(i,j)}f(x,y,w_1,\ldots,w_i,v_1,\ldots,v_j) = D_{(w_i,0)}\cdots D_{(w_1,0)}D_{(0,v_j)}\cdots D_{(0,v_1)}f(x,y)$$

exists for all $(x,y) \in U^0 \times V^0$ and $w_1,\ldots,w_i \in E_1,\quad v_1,\ldots,v_j \in E_2$, and is given by

$$(d^{(i,j)}f_\alpha(x,y,w_1,\ldots,w_i,v_1,\ldots,v_j))_{\alpha\in A}. \tag{3.10}$$

Now (3.10) defines a continuous function $U \times V \times E_1^i \times E_2^j \to F$ for all $i,j \in \mathbb{N}_0$ such that $i \leq r$ and $j \leq s$. Hence f is $C^{r,s}$. □

Lemma 36. *Let $r, s \in \mathbb{N}_0 \cup \{\infty\}$, $s \geq 1$, E_1, E_2, F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively. Let $f\colon U \times V \to F$ be a map. Then f is $C^{r,s}$ if and only if f is $C^{r,0}$, f is $C^{0,1}$ and $d^{(0,1)}f\colon U \times (V \times E_2) \to F$ is $C^{r,s-1}$.*

Proof. The implication " $\Rightarrow$ " will be established after Lemma 38, and shall not be used before. To prove " $\Leftarrow$ ", let $i, j \in \mathbb{N}_0$ such that $i \leq r$ and $j \leq s$, and $(x, y) \in U^0 \times V^0$ and $w_1, \ldots, w_i \in E_1$ and $v_1, \ldots, v_j \in E_2$.
If $j = 0$, then $D_{(w_i,0)} \cdots D_{(w_1,0)} f(x, y)$ exists as f is $C^{r,0}$, and is given by

$$d^{(i,0)} f(x, y, w_1, \ldots, w_i)$$

which extends continuously to $U \times V \times E_1^i$.
If $j > 0$, then $D_{(0,v_1)} f(x, y) = d^{(0,1)} f(x, y, v_1)$ exists because f is $C^{0,1}$ and since $d^{(0,1)}f$ is $C^{r,s-1}$, also the directional derivatives

$$\begin{aligned} &D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f(x, y) \\ &= D_{(w_i,(0,0))} \cdots D_{(w_1,(0,0))} D_{(0,(v_j,0))} \cdots D_{(0,(v_2,0))} (d^{(0,1)} f)(x, y, v_1) \end{aligned}$$

exist and the right-hand side extends continuously to $(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) \in U \times V \times E_1^i \times E_2^j$. Hence f is $C^{r,s}$. ☐

Lemma 37. *Let $r, s \in \mathbb{N}_0 \cup \{\infty\}$, E_1, E_2, H_1, H_2, F be locally convex spaces, U, V, P and Q be locally convex subsets with dense interior of E_1, E_2, H_1 and H_2, respectively. If $f\colon U \times V \to F$ is a $C^{r,s}$-map and $\lambda_1\colon H_1 \to E_1$ as well as $\lambda_2\colon H_2 \to E_2$ are continuous linear maps such that $\lambda_1(P) \subseteq U$ and $\lambda_2(Q) \subseteq V$, then $f \circ (\lambda_1 \times \lambda_2)|_{P \times Q}\colon P \times Q \to F$ is $C^{r,s}$.*

Proof. Let $(p, q) \in P^0 \times Q^0$ and $w_1, \ldots, w_i \in H_1$, $v_1, \ldots, v_j \in H_2$. Let $X \subseteq U$ be a convex neighbourhood of $\lambda_1(p)$ and $Y \subseteq V$ be a convex neighbourhood of $\lambda_2(q)$. For $t \in \mathbb{R}$ so small that $\lambda_2(q) + t\lambda_2(v_1) \in Y$, we have

$$\begin{aligned} &\frac{1}{t}(f(\lambda_1(p), \lambda_2(q) + t\lambda_2(v_1)) - f(\lambda_1(p), \lambda_2(q))) \\ &= \int_0^1 d^{(0,1)} f(\lambda_1(p), \lambda_2(q) + st\lambda_2(v_1), v_1)\, ds \end{aligned}$$

by the Mean value Theorem for the C^1-map $f(\lambda_1(p), \bullet)$. Hence

$$\begin{aligned} &D_{(0,v_1)}(f \circ (\lambda_1 \times \lambda_2))(p, q) \\ &= \lim_{t \to 0} \int_0^1 d^{(0,1)} f(\lambda_1(p), \lambda_2(q) + st\lambda_2(v_1), v_1)\, ds \end{aligned}$$

exists and is given by

$$\begin{aligned}&\int_0^1 d^{(0,1)} f(\lambda_1(p), \lambda_2(q), v_1)\, ds\\ &= d^{(0,1)} f(\lambda_1(p), \lambda_2(q), v_1)\end{aligned}$$

and recursively,we obtain

$$\begin{aligned}&D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} (f \circ (\lambda_1 \times \lambda_2))(p, q)\\ &= d^{(i,j)} f(\lambda_1(p), \lambda_2(q), \lambda_1(w_1), \ldots, \lambda_1(w_i), \lambda_2(v_1), \ldots, \lambda_2(v_j)).\end{aligned}$$

The right-hand side defines a continuous function of $(p, q, w_1, \ldots, w_i, v_1, \ldots, v_j) \in P \times Q \times H_1^i \times H_2^j$. Hence the assertion follows. $\square$

Lemma 38. *Let $r, s \in \mathbb{N}_0 \cup \{\infty\}$, $E_1, E_2, H_1, \ldots, H_n, F$ be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2, respectively, and*

$$f \colon U \times V \times H_1 \times \cdots \times H_n \to F$$

be a continuous map with the following properties:

(a) *$f(x, y, \bullet) \colon H_1 \times \cdots \times H_n \to F$ is n-linear for all $x \in U$, $y \in V$;*

(b) *The directional derivatives $D_{(w_i,0,0)} \cdots D_{(w_1,0,0)} D_{(0,v_j,0)} \cdots D_{(0,v_1,0)} f(x, y, h)$ exist for all $i, j \in \mathbb{N}_0$ such that $i \le r, j \le s$, $(x, y) \in U^0 \times V^0$, $h \in H_1 \times \cdots \times H_n$ and $w_1, \ldots, w_i \in E_1$, $v_1, \ldots, v_j \in E_2$, and extend continuously to functions*

$$U \times V \times H_1 \times \cdots \times H_n \times E_1^i \times E_2^j \to F.$$

Then $f \colon U \times (V \times H_1 \times \cdots \times H_n) \to F$ is $C^{r,s}$. Also $g \colon (U \times H_1 \times \cdots \times H_n) \times V \to F$, $((x, h), y) \mapsto f(x, y, h)$ is $C^{r,s}$.

Proof. Holding $h \in H := H_1 \times \cdots \times H_n$ fixed, the map $f(\bullet, h)$ is $C^{r,s}$ and hence

$$\varphi \colon V \times U \to F,\ (x, y) \mapsto f(y, x, h)$$

is $C^{s,r}$, by Corollary 33, with

$$\begin{aligned}&D_{(v_j,0)} \cdots D_{(v_1,0)} D_{(0,w_i)} \cdots D_{(0,w_1)} \varphi(x, y)\\ &= D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f(y, x, h).\end{aligned}$$

Hence $f_1 \colon V \times (U \times H) \to F$, $f_1(y, x, h) := f(x, y, h)$ satisfies hypotheses analogous to those for f (with r and s interchanged) and will be $C^{s,r}$ if the first assertion holds.

Using Corollary 33, this implies that g is $C^{r,s}$. Hence we only need to prove the first assertion.

We may assume that $r, s < \infty$; the proof is by induction on s.
The case $s = 0$. Then f is $C^{r,0}$ by the hypotheses.
Induction step. Let $v \in E_2$, $z = (z_1, \ldots, z_n) \in H$. By hypothesis, $D_{(0,v,0)}f(x,y,h)$ exists for $(x,y,h) \in U^0 \times V^0 \times H$ and extends to a continuous map $U \times V \times H \times E_2 \to F$ in (x,y,h,v). Because $f(x,y,\bullet)\colon H \to F$ is continuous and linear, it is C^1 with

$$D_{(0,0,z)}f(x,y,h) = \sum_{k=1}^{n} f(x,y,h_1,\ldots,h_{k-1},z_k,h_{k+1},\ldots,h_n).$$

This formula defines a continuous function $U \times V \times H \times H \to F$. Holding $x \in U$ fixed, we deduce with the Rule on Partial Differentials (Lemma 8) that the map

$$V \times H \to F, (y,h) \mapsto f(x,y,h)$$

is C^1, with

$$D_{(0,v,z)}f(x,y,h) = D_{(0,v,0)}f(x,y,h) + \sum_{k=1}^{n} f(x,y,h_1,\ldots,h_{k-1},z_k,h_{k+1},\ldots,h_n). \quad (3.11)$$

Now $f\colon U \times (V \times H) \to F$ is $C^{r,0}$ (see the case $s=0$). Also, $f\colon U \times (V \times H) \to F$ is $C^{0,1}$, because we have just seen that $d^{(0,1)}f(x,(y,h),(v,z))$ exists and is given by (3.11), which extends continuously to $U \times (V \times H) \times (E_2 \times H)$.

We claim that $d^{(0,1)}f\colon U \times ((V \times H) \times (E_2 \times H))$ is $C^{r,s-1}$. If this is true, then f is $C^{r,s}$, by Lemma 36. To prove the claim, for fixed $k \in \{1,\ldots,n\}$, consider

$$\phi\colon U \times (V \times H \times E_2 \times H) \to F,\ (x,y,h,v,z) \mapsto f(x,y,h_1,\ldots,h_{k-1},z_k,h_{k+1},\ldots,h_n).$$

The map

$$\psi\colon U \times V \times H_1 \times \cdots \times H_{n-1} \times (H_n \times E_2 \times H) \to F,$$
$$(x,y,h_1,\ldots,h_{n-1},(h_n,v,z)) \mapsto f(x,y,h_1,\ldots,h_n)$$

is n-linear in $(h_1,\ldots,h_{n-1},(h_n,v,z))$. By induction, ψ is $C^{r,s-1}$ as a map on $U \times (V \times H_1 \times \cdots \times H_{n-1} \times H_n \times E_2 \times H)$. By Lemma 37, also ϕ is $C^{r,s-1}$. Hence each of the final k summands in (3.11) is $C^{r,s-1}$ in $(x,(y,h,v,z))$. It remains to observe that $\theta\colon U \times V \times (H \times (E_2 \times H)) \to F$, $(x,y,h,v,z) \mapsto D_{(0,v,0)}f(x,y,h)$ is $(n+1)$-linear in the final argument and satisfies hypotheses analogous to those of f, with r,s replaced by $r, s-1$. Hence $\theta\colon U \times (V \times H \times E_2 \times H) \to F$ is $C^{r,s-1}$, by induction. As a consequence, $d^{(0,1)}f$ is $C^{r,s-1}$ (like each of the summands in (3.11)).

□

Taking $E_2 = \{0\}$, Lemma 38 readily entails:

Lemma 39. *Let $r \in \mathbb{N}_0 \cup \{\infty\}$, $E, H_1, \ldots, H_n, F$ locally convex spaces, U be a locally convex subset with dense interior of E and $f\colon U \times (H_1 \times \cdots \times H_n) \to F$ be a $C^{r,0}$-map which is n-linear for fixed first argument. Then f is $C^{r,\infty}$.*

Proof of Lemma 36, completed. If f is $C^{r,s}$, then f is $C^{0,1}$ and f is $C^{r,0}$. Moreover $d^{(0,1)}f\colon U \times V \times E_2 \to F$ is linear in the E_2-variable and

$$D_{(w_i,0,0)} \cdots D_{(w_1,0,0)} D_{(0,v_j,0)} \cdots D_{(0,v_1,0)} (d^{(0,1)}f)(x,y,z)$$
$$= d^{(i,j+1)} f(x,y,w_1,\ldots,w_i,z,v_1,\ldots,v_j)$$

exists for all $i,j \in \mathbb{N}_0$ such that $i \le r$, $j \le s-1$, if $(x,y) \in U^0 \times V^0$, and extends to a continuous function in $(x,y,z,w_1,\ldots,w_i,v_1,\ldots,v_j) \in U \times V \times E_2 \times E_1^i \times E_2^j$. Hence by Lemma 38, $d^{(0,1)}f$ is $C^{r,s-1}$.

Lemma 40. *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $r \in \mathbb{N}_0 \cup \{\infty\}$. If $f\colon U \times V \to F$ is $C^{r,r}$, then f is C^r.*

Proof. We may assume that $r < \infty$, the proof is by the induction on $r \in \mathbb{N}_0$, *The case* $r = 0$. If f is $C^{0,0}$, then f is continuous and hence C^0. *The case* $r \ge 1$. Assume U, V are open subsets. Then $D_{(w,0)}f(x,y)$ exists and is continuous in (x,y,w), and $D_{(0,v)}f(x,y)$ exists and is continuous in (x,y,v). Hence by (2.1) f is C^1 and

$$df((x,y),(w,v)) = D_{(w,0)}f(x,y) + D_{(0,v)}f(x,y), \tag{3.12}$$

which is continuous in (x,y,w,v). Thus f is C^1. In the general case, the right hand side of (3.12) is continuous for $(x,y,w,v) \in U \times V \times E_1 \times E_2$ and extends $d(f|_{U^0 \times V^0})$. Hence f is C^1. Next, note that $D_{(w,0)}f(x,y)$ and $D_{(0,v)}f(x,y)$ are $C^{r-1,r-1}$-mappings in $((x,w),y)$ and $(x,(y,v))$, respectively, by Lemma 36 and Corollary 33. Hence df is C^{r-1}, by induction. Since f is a C^1 and df is C^{r-1}, the map f is C^r. □

Remark 41. If $r = \infty$, then a map $f\colon U \times V \to F$ is C^∞ if and only if it is $C^{\infty,\infty}$ (as an immediate consequence of Lemma 40).

Proposition 42. *Let E be a finite-dimensional vector space, F a locally convex space, U be a locally convex and locally compact subset with dense interior of E and $s \in \mathbb{N}_0 \cup \{\infty\}$. Then the evaluation map*

$$\varepsilon : C^s(U,F) \times U \to F, \ \varepsilon(\gamma,x) := \gamma(x)$$

of $C^s(U,F)$ is $C^{\infty,s}$.

Proof. Without loss of generality, we may assume that $s < \infty$. The proof is by induction on s.
If $s = 0$, then ε is continuous because U is locally compact [11, Theorem 3.4.3]. Also, ε is linear in the first argument. Hence ε is $C^{\infty,0}$, by Lemma 39 and Corollary 33.
Let $s \geq 1$. For $x \in U^0$, $w \in E$, $\gamma \in C^s(U,F)$ and small $t \in \mathbb{R} \setminus \{0\}$,

$$\frac{1}{t}(\varepsilon(\gamma, x+tw) - \varepsilon(\gamma, x)) = \frac{1}{t}(\gamma(x+tw) - \gamma(x)) \to d\gamma(x,w) \text{ as } t \to 0.$$

Hence $d^{(0,1)}\varepsilon(\gamma,x,w)$ exists and is given by

$$d^{(0,1)}\varepsilon(\gamma,x,w) = d\gamma(x,w) = \varepsilon_1(d\gamma,(x,w)), \tag{3.13}$$

where $\varepsilon_1 \colon C^{s-1}(U \times E, F) \times (U \times E) \to F$, $(\zeta, z) \mapsto \zeta(z)$ is $C^{\infty,s-1}$, by induction. The right-hand side of (3.13) defines a continuous map (indeed a $C^{\infty,s-1}$-map)

$$C^s(U,F) \times (U \times E) \to F$$

by induction and Lemma 37, using that

$$C^s(U,F) \to C^{s-1}(U \times E, F),\ \gamma \mapsto d\gamma$$

is continuous linear. Thus, by Lemma 36, ε is $C^{\infty,s}$. □

3.2 Chain Rules for $C^{r,s}$-mappings

Lemma 43. (Chain Rule 1). *Let X_1, X_2, E_1, E_2 and F be locally convex spaces, P, Q, U and V be locally convex subsets with dense interior of X_1, X_2, E_1 and E_2 respectively, $r,s \in \mathbb{N}_0 \cup \{\infty\}$, $f \colon U \times V \to F$ a $C^{r,s}$-map, $g_1 \colon P \to U$ a C^r-map and $g_2 \colon Q \to V$ a C^s-map. Then*

$$f \circ (g_1 \times g_2) \colon P \times Q \to F,\ (p,q) \mapsto f(g_1(p), g_2(q))$$

is a $C^{r,s}$-map.

Proof. Without loss of generality, we may assume that $r,s < \infty$. The proof is by induction on r.
The case $r = 0$. If $s = 0$, $f \circ (g_1 \times g_2)$ is just a composition of continuous maps, which is continuous.
Now let $s > 0$. For fixed $x \in U$, $f_x \colon V \to F$ is C^s. Hence, for fixed $p \in P$, $f_{g_1(p)} \colon V \to F$

is C^s and $f_{g_1(p)} \circ g_2 \colon Q \to F$ is C^s by the Chain Rule for C^s-maps (see [21]). In particular, the latter is C^1, whence

$$D_{(0,z)}(f \circ (g_1 \times g_2))(p,q) = d(f_{g_1(p)} \circ g_2)(q,z) = df_{g_1(p)}(g_2(q), dg_2(q,z))$$

exists for $z \in X_2$ and $q \in Q^0$. Hence,

$$d^{(0,1)}(f \circ (g_1 \times g_2))(p,q,z) = \underbrace{d^{(0,1)}f}_{C^{0,s-1}}(\underbrace{g_1(p)}_{C^0 \text{ in } p}, \underbrace{g_2(q), dg_2(q,z)}_{C^{s-1} \text{ in } (q,z)})$$

exists. By induction on s, the map $d^{(0,1)}(f \circ (g_1 \times g_2))$ is $C^{0,s-1}$. Hence, by Lemma 36, $f \circ (g_1 \times g_2)$ is $C^{0,s}$.
Induction step ($r > 0$). If $s = 0$, we see as in the first part of the proof that $h := f \circ (g_1, g_2)$ is $C^{r,0}$.
If $s > 0$, we know that

$$d^{(0,1)}h(p,q,z) = \underbrace{d^{(0,1)}f}_{C^{r,s-1}}(\underbrace{g_1(p)}_{C^r}, \underbrace{g_2(q), dg_2(q,z)}_{C^{s-1}}).$$

By induction on s, this is $C^{r,s-1}$. Hence, by Lemma 36, h is $C^{r,s}$. □

Lemma 44. (Chain Rule 2). *Let E_1, E_2, F and Y be locally convex spaces, U, V and W be locally convex subsets with dense interior of E_1, E_2 and F respectively, $r, s \in \mathbb{N}_0 \cup \{\infty\}$, $f \colon U \times V \to F$ a $C^{r,s}$-map with $f(U \times V) \subseteq W$ and $g \colon W \to Y$ be a C^{r+s}-map. Then*

$$g \circ f \colon U \times V \to Y$$

is a $C^{r,s}$-map.

Proof. Without loss of generality, we may assume that $r, s < \infty$. The proof is by induction on r.
The case $r = 0$. If $s = 0$, $g \circ f$ is just a composition of continuous maps, which is continuous.
Now let $s > 0$. For fixed $x \in U$, $f_x \colon V \to F$ is C^s and $g \colon W \to Y$ is C^s. Hence $g \circ f_x \colon V \to Y$ is C^s by the Chain Rule for C^s-maps (see [21]). In particular, the latter is C^1, whence

$$D_{(0,v)}(g \circ f)(x,y) = d(g \circ f_x)(y,v) = dg(f_x(y), df_x(y,v)) = dg(f(x,y), d^{(0,1)}f(x,y,v))$$

exists for $v \in E_2$, if $x \in U^0$, $y \in V^0$. Now

$$d^{(0,1)}(g \circ f) \colon U \times (V \times E_2) \to Y$$

$$(x,y,v) \mapsto \underbrace{dg}_{C^{r+s-1}} \underbrace{(f(x,y), d^{(0,1)}f(x,y,v))}_{C^{0,s-1}}$$

is a $C^{0,s-1}$-map, by Lemma 36, Lemma 37 and induction on s. Hence, by Lemma 36, $g \circ f$ is $C^{0,s}$.

Induction step ($r > 0$). If $s = 0$, we see as in the first part of the proof that $h := g \circ f$ is $C^{r,0}$.

If $s > 0$, we know that h is $C^{r,0}$ by the preceding. Moreover,

$$d^{(0,1)}h(x,y,v) = \underbrace{dg}_{C^{r+s-1}} \underbrace{(f(x,y), d^{(0,1)}f(x,y,v))}_{C^{r,s-1}}.$$

Hence, by induction on s the map $d^{(0,1)}h$ is $C^{r,s-1}$. Hence by Lemma 36, h is $C^{r,s}$. $\square$

3.3 The Exponential Law for $C^{r,s}$-mappings

Definition 45. Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$.
Give $C^{r,s}(U \times V, F)$ the initial topology with respect to the mappings

$$d^{(i,j)} \colon C^{r,s}(U \times V, F) \to C(U \times V \times E_1^i \times E_2^j, F),\ \gamma \mapsto d^{(i,j)}\gamma$$

for $i, j \in \mathbb{N}_0$ such that $i \leq r, j \leq s$, where the right-hand side is equipped with the compact-open topology.

Lemma 46. *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, then*

$$C^{\infty,\infty}(U \times V, F) = C^{\infty}(U \times V, F)$$

as topological vector spaces.

Proof. By Lemma 40 and Remark 41 both spaces coincide as sets. Thus it suffices to show that the $C^{\infty,\infty}$- topology coincides with the C^{∞}-topology. As both topologies are initial topologies, we only have to prove that the families of maps inducing the topologies are continuous with respect to the other topology. For $x \in U$, $y \in V$, $w := (w_1, \ldots, w_i) \in E_1^i$ and $v := (v_1, \ldots, v_j) \in E_2^j$, we have

$$d^{(i,j)}f(x,y,w,v) = d^{(i+j)}f(x,y,(w_1,0),\ldots,(w_i,0),(0,v_1),\ldots,(0,v_j)).$$

Let $g \colon U \times V \times E_1^i \times E_2^j \to U \times V \times (E_1 \times E_2)^{i+j}$, $(x,y,w_1,\ldots,w_i,v_1,\ldots,v_j) \mapsto (x,y,(w_1,0),\ldots,(w_i,0),(0,v_1),\ldots,(0,v_j))$. As g is continuous linear, by [19, Proposition 4.4], the pullback g^* is continuous. Hence by continuity of $d^{(i+j)}$, $d^{(i,j)} = g^* \circ d^{(i+j)}$

is continuous with respect to the C^∞-topology. This proves that the $C^{\infty,\infty}$-topology is coarser than the C^∞-topology. To show the converse we recall that $d^{(k)}f(x,y,\bullet)$ is multilinear. Writing $(w_i,v_i)=(w_i,0)+(0,v_i)$ we obtain

$$d^{(k)}f=\sum_{I\subseteq\{1,\ldots,k\}} g_I^*(d^{(|I|,k-|I|)}f),$$

where we defined $g_I(x,y,(w_1,v_1),\ldots,(w_k,v_k)):=(x,y,w_{i_1},\ldots,w_{i_{|I|}},v_{j_1},\ldots,v_{j_{k-|I|}})$ for $I=\{i_1,\ldots,i_{|I|}\}$ and $\{1,\ldots,k\}\setminus I=\{j_1,\ldots,j_{k-|I|}\}$. Clearly each g_I is continuous linear, hence smooth and we deduce from [19, Proposition 4.4] that $d^{(k)}$ is continuous with respect to the $C^{\infty,\infty}$-topology. Hence the assertion follows. □

Lemma 47. *Let E and F be locally convex spaces, U be a locally convex subset with dense interior of E and $r\in\mathbb{N}_0\cup\{\infty\}$. Then sets of the form*

$$\bigcap_{i=0}^{k}\{\gamma\in C^r(U,F): d^{(i)}\gamma(K_i)\subseteq Q_i\}$$

form a basis of 0-neighbourhoods in $C^r(U,F)$, for $k\in\mathbb{N}_0$ such that $k\leq r$, compact sets $K_i\subseteq U\times E^i$ and 0-neighbourhoods $Q_i\subseteq F$.

Proof. The topology on $C^r(U,F)$ is initial with respect to the maps

$$d^{(i)}\colon C^r(U,F)\to C(U\times E^i,F)_{c.o},\ \gamma\mapsto d^{(i)}\gamma.$$

Therefore the map

$$\Psi\colon C^r(U,F)\to\prod_{\mathbb{N}_0\ni i\leq r} C(U\times E^i,F),\ \gamma\mapsto(d^{(i)}\gamma)_{\mathbb{N}_0\ni i\leq r}$$

is a topological embedding. Sets of the form

$$W:=\{(\eta_i)_{\mathbb{N}_0\ni i\leq r}\in\prod_{\mathbb{N}_0\ni i\leq r} C(U\times E^i,F):\ \eta_i(K_i)\subseteq Q_i \text{ for } i=0,\ldots,k\}$$

(with $k\in\mathbb{N}_0$ such that $k\leq r$, compact sets $K_i\subseteq U\times E^i$ and 0-neighbourhoods $Q_i\subseteq F$), form a basis of 0-neighbourhoods in $\prod_{\mathbb{N}_0\ni i\leq r}C(U\times E^i,F)$. Hence the sets $\Phi^{-1}(W)$ form a basis of 0-neighbourhoods in $C^r(U,F)$. □

Similarly:

Lemma 48. *Let E_1,E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $r,s\in\mathbb{N}_0\cup\{\infty\}$. The sets*

$$W=\{\gamma\in C^{r,s}(U\times V,F)\colon d^{(i,j)}\gamma(K_{i,j})\subseteq P_{i,j} \text{ for } i=0,\ldots,k \text{ and } j=0,\ldots,l\}$$

(where $k \in \mathbb{N}_0$ such that $k \leq r$, $l \in \mathbb{N}_0$ such that $l \leq s$, $P_{i,j} \subseteq F$ are 0-neighbourhoods and $K_{i,j} \subseteq U \times V \times E_1^i \times E_2^j$ is compact) form a basis of 0-neighbourhoods for $C^{r,s}(U \times V, F)$.

Theorem 49. *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Then*

(a) *If $\gamma\colon U \times V \to F$ is $C^{r,s}$, then $\gamma_x\colon V \to F$ is C^s for all $x \in U$ and*

$$\gamma^\vee\colon U \to C^s(V, F),\ x \mapsto \gamma_x$$

is C^r.

(b) *The map*

$$\Phi\colon C^{r,s}(U \times V, F) \to C^r(U, C^s(V, F)),\ \gamma \mapsto \gamma^\vee$$

is linear and a topological embedding.

Proof.

(a) $\gamma_x\colon V \to F$ is C^s for all $x \in U$ by Lemma 28.
By Remark 22,

$$C^r(U, C^\infty(V, F)) = \varprojlim_{s \in \mathbb{N}_0} C^r(U, C^s(V, F)).$$

It therefore suffices to prove the assertion when $s \in \mathbb{N}_0$ (cf. [4, Lemma 10.3]). We may assume that r is finite. The proof is by induction on r.
The case $r = 0$. If $s = 0$ then the assertion follows from [11, Theorem 3.4.1].
If $s \geq 1$, the topology on $C^s(V, F)$ is initial with respect to the maps

$$d^{(j)}\colon C^s(V, F) \to C(V \times E_2^j, F)_{c.o},\ \gamma \mapsto d^{(j)}\gamma, \text{ for } j \in \mathbb{N}_0 \text{ such that } j \leq s.$$

Hence, we only need that $d^{(j)} \circ f^\vee\colon U \to C(V \times E_2^j, F)_{c.o}$ is continuous for $j \in \{0, 1, \ldots, s\}$. Now

$$d^{(j)}(f^\vee(x)) = d^{(j)}(f(x, \bullet)) = d^{(0,j)} f(x, \bullet) = (d^{(0,j)} f)^\vee(x).$$

Thus $d^{(j)} \circ f^\vee = (d^{(0,j)} f)^\vee\colon U \to C(V \times E_2^j, F)_{c.o}$, which is continuous by induction. As a consequence, $\gamma^\vee\colon U \to C^s(V, F)$ is continuous.
The case $r \geq 1$. If $s = 0$, then $f^\vee\colon U \to C(V, F)$. Let $x \in U^0$, $z \in E_1$. Then $x + tz \in U^0$, for small $t \in \mathbb{R} \setminus \{\infty\}$; we show that

$$\frac{1}{t}(f^\vee(x + tz) - f^\vee(x)) \to d^{(1,0)} f(x, \bullet, z)$$

in $C(V,F)$ as $t \to 0$. For this, let $K \subseteq V$ be compact. We have to show that

$$(\frac{1}{t}(f^\vee(x+tz) - f^\vee(x)))|_K \to (d^{(1,0)}f(x,\bullet,z))|_K$$

uniformly as $t \to 0$. Let $W \subseteq F$ be a 0-neighbourhood. Without loss of generality, W is closed and absolutely convex. There is $\varepsilon \geq 0$ such that $x + B_\varepsilon^{\mathbb{R}}(0)z \subseteq U^0$. For $y \in K$ and $t \in \mathbb{R} \setminus \{0\}$ such that $|t| < \varepsilon$, we have

$$\begin{aligned}
\Delta(t,y) :&= \frac{1}{t}(f^\vee(x+tz) - f^\vee(x))(y) - d^{(1,0)}f(x,y,z) \\
&= \frac{1}{t}(f(x+tz,y) - f(x,y)) - d^{(1,0)}f(x,y,z) \\
&= \int_0^1 d^{(1,0)}f(x+\sigma tz,y,z)d\sigma - d^{(1,0)}f(x,y,z) \\
&= \int_0^1 (d^{(1,0)}f(x+\sigma tz,y,z) - d^{(1,0)}f(x,y,z))\, d\sigma.
\end{aligned}$$

The function

$$g\colon B_\varepsilon^{\mathbb{R}}(0) \times K \times [0,1] \to F, (t,y,\sigma) \longmapsto d^{(1,0)}f(x+\sigma tz,y,z) - d^{(1,0)}f(x,y,z)$$

is continuous and $g(0,y,\sigma) = 0$ for all $(y,\sigma) \in K \times [0,1]$. Because $K \times [0,1]$ is compact, by the Wallace Lemma (see [11, 3.2.10]), there exists $\delta \in (0,\varepsilon]$ such that $g(B_\delta^{\mathbb{R}}(0) \times K \times [0,1]) \subseteq W$. Hence $\Delta(t,y) = \int_0^1 g(t,y,\sigma)d\sigma \in W$ for all $y \in K$ and all $t \in B_\delta^{\mathbb{R}}(0) \setminus \{0\}$. Because this holds for all $y \in K$, we see that $\Delta(t,\bullet) \to 0$ uniformly, as required. Thus $df^\vee(x,z)$ exists for all $x \in U^0$, $z \in E_1$ and is given by $df^\vee(x,z) = d^{(1,0)}f(x,\bullet,z)$. Now

$$U \to C(V,F), x \mapsto d^{(1,0)}f(x,\bullet,z)$$

is a continuous function in all of U (by $r = 0$); so $f^\vee$ is C^1 on U, and $df^\vee(x,z) = d^{(1,0)}f(x,\bullet,z)$. Because

$$h\colon (U \times E_1) \times V \to F,\ ((x,z),y) \mapsto d^{(1,0)}f(x,y,z)$$

is $C^{r-1,0}$ (see Lemma 36 and Corollary 33), by induction $d(f^\vee) = h^\vee\colon U \times E_1 \to C(V,F)$ is C^{r-1}. Hence f is C^r.

Let $s \geq 1$. Because

$$C^s(V,F) \to C(V,F) \times C^{s-1}(V \times E_2, F),\ \gamma \mapsto (\gamma, d\gamma)$$

is a linear topological embedding with closed image, $f^\vee\colon U \to C^s(V,F)$ will be C^r if $f^\vee\colon U \to C(V,F)$ is C^r (which holds by induction) and the map

$$h\colon U \to C^{s-1}(V \times E_2, F),\ x \mapsto d(f^\vee(x))$$

is C^r (see [21]; cf. [4, Lemma 10.1]). For $x \in U$, $y \in V$ and $z \in E_2$, we have

$$h(x)(y,z) = d(f^\vee(x))(y,z) = d(f(x,\bullet))(y,z) = d^{(0,1)}f(x,y,z),$$

thus $h = (d^{(0,1)}f)^\vee$ for $d^{(0,1)}f\colon U \times (V \times E_2) \to F$. This function is $C^{r,s-1}$ by Lemma 36. Hence h is C^r by induction.

(b) The linearity of Φ is clear. For $y \in V$, the point evaluation $\lambda\colon C^s(V,F) \to F$, $\eta \mapsto \eta(y)$ is continuous linear. Hence, for $i \leq r$,

$$\begin{aligned}(d^{(i)}f^\vee)(x,w_1,\ldots,w_i)(y) &= \lambda((d^{(i)}f^\vee)(x,w_1,\ldots,w_i))\\ &= d^{(i)}(\lambda \circ f^\vee)(x,w_1,\ldots,w_i)\\ &= d^{(i)}(f(\bullet,y))(x,w_1,\ldots,w_i)\\ &= d^{(i,0)}f(x,y,w_1,\ldots,w_i),\end{aligned}$$

using that $(\lambda \circ f^\vee)(x) = \lambda(f^\vee(x)) = f^\vee(x)(y) = f(x,y)$. Hence

$$(d^{(i)}f^\vee)(x,w_1,\ldots,w_i) = (d^{(i,0)}f)(x,\bullet,w_1,\ldots,w_i).$$

Hence by Schwarz' Theorem (Proposition 31)

$$d^{(j)}((d^{(i)}f^\vee)(x,w_1,\ldots,w_i))(y,v_1,\ldots,v_j) = d^{(i,j)}f(x,y,w_1,\ldots,w_i,v_1,\ldots,v_j).$$

Φ is continuous at 0. Let $W \subseteq C^r(U,C^s(V,F))$ be a 0-neighbourhood. After shrinking W, without loss of generality

$$W = \bigcap_{i=0}^{k}\{\gamma \in C^r(U,C^s(V,F))\colon d^{(i)}\gamma(K_i) \subseteq Q_i\}$$

where $k \in \mathbb{N}_0$ with $k \leq r$, $K_i \subseteq U \times E_1^i$ is compact and $Q_i \subseteq C^s(V,F)$ is a 0-neighbourhood (see Lemma 47). Using Lemma 47 again, after shrinking Q_i we may assume that

$$Q_i = \bigcap_{j=0}^{l_i}\{\eta \in C^s(V,F)\colon d^{(j)}\eta(L_{i,j}) \subseteq P_{i,j}\}$$

with $l_i \in \mathbb{N}_0$ such that $l_i \leq s$, compact sets $L_{i,j} \subseteq V \times E_2^j$ and 0-neighbourhoods $P_{i,j} \subseteq F$. Shrinking Q_i further, we may assume that $l_i = l$ is independent of i. Then W is the set of all $\gamma \in C^r(U,C^s(V,F))$ such that $d^{(j)}(d^{(i)}\gamma(x,w))(y,v) \in P_{i,j}$ for all $i = 0,\ldots,k$ and $j = 0,\ldots,l$, $(x,w) \in K_i \subseteq U \times E_1^i$ and $(y,v) \in L_{i,j} \subseteq V \times E_2^j$. The projections of $U \times E_1^i$ onto the factors U and E_1^i are continuous, hence the images K_i^1 and

K_i^2 of K_i under these projections are compact. After replacing K_i by $K_i^1 \times K_i^2$, without loss of generality $K_i = K_i^1 \times K_i^2$. Likewise, without loss of generality $L_{i,j} = L_{i,j}^1 \times L_{i,j}^2$ with compact sets $L_{i,j}^1 \subseteq V$ and $L_{i,j}^2 \subseteq E_2^j$.
Now if $\gamma \in C^{r,s}(U \times V, F)$ then $d^{(j)}(d^{(i)}\gamma^\vee(x,w))(y,v) = d^{(i,j)}\gamma(x,y,w,v)$. Hence $\gamma^\vee \in W$ if and only if $d^{(i,j)}\gamma(K_i^1 \times L_{i,j}^1 \times K_i^2 \times L_{i,j}^2) \subseteq P_{i,j}$ for all $i = 0,\dots,k$ and $j = 0,\dots,l$. This is a basic neighbourhood in $C^{r,s}(U \times V, F)$ (see Lemma 48). Thus $\Phi^{-1}(W)$ is a 0-neighbourhood, whence Φ is continuous at 0, and hence Φ is continuous.
It is clear that Φ is injective. To see that Φ is an embedding, it remains to show that $\Phi(W)$ is a 0-neighbourhood in $\mathrm{im}(\Phi)$ for each W in a basis of 0-neighbourhoods in $C^{r,s}(U \times V, F)$.
Take W as in Lemma 48; without loss of generality, after increasing $K_{i,j}$, we may assume $K_{i,j} = K_{i,j}^1 \times L_{i,j}^1 \times K_{i,j}^2 \times L_{i,j}^2$ with compact sets $K_{i,j}^1 \subseteq U$, $L_{i,j}^1 \subseteq V$, $K_{i,j}^2 \subseteq E_1^i$ and $L_{i,j}^2 \subseteq E_2^j$. Then $\Phi(W) = \{\eta \in \mathrm{im}(\Phi) \colon d^{(j)}(d^{(i)}\eta(x,w))(y,v) \in P_{i,j}$ for all $i = 0,\dots,k$, $j = 0,\dots,l$, $x \in K_{i,j}^1$, $y \in L_{i,j}^1$, $w \in K_{i,j}^2$ and $v \in L_{i,j}^2\}$, which is a 0-neighbourhood in $\mathrm{im}(\Phi)$, by Lemma 47. □

Lemma 50. *Let X be a topological space, E and F be locally convex spaces, $k \in \mathbb{N}$, and $f \colon X \times E^k \to F$ be a map such that $f(x, \bullet) \colon E^k \to F$ is symmetric k-linear for each $x \in X$. Then f is continuous if and only if $g \colon X \times E \to F$, $(x,w) \mapsto f(x,w,\dots,w)$ is continuous.*

Proof. The continuity of g follows directly from the continuity of f. If, conversely, g is continuous, then by the Polarization Identity [8, Theorem A]

$$f(x, w_1, \dots, w_k) = \frac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k = 0}^{1} (-1)^{k-(\varepsilon_1+\cdots+\varepsilon_k)} g(x, \varepsilon_1 w_1 + \cdots + \varepsilon_k w_k),$$

which is continuous.

□

Lemma 51. *Let X be a topological space, E_1, E_2 and F be locally convex spaces, $k, l \in \mathbb{N}$, and $f \colon X \times E_1^k \times E_2^l \to F$ be a map such that $f(x, \bullet, w_1, \dots, w_l) \colon E_1^k \to F$ is symmetric k-linear for all $x \in X$ and $w_1,\dots,w_l \in E_2$, and $f(x, v_1, \dots, v_k, \bullet) \colon E_2^l \to F$ is symmetric l-linear for all $x \in X$ and $v_1,\dots,v_k \in E_1$. Then f is continuous if and only if $g \colon X \times E_1 \times E_2 \to F$, $g(x,v,w) := f(x,v,\dots,v,w,\dots,w)$ is continuous.*

Proof. The continuity of g follows directly from the continuity of f. If, conversely, g is continuous, then two applications of the Polarization Identity show that

$$
\begin{aligned}
& f(x, v_1, \ldots, v_k, w_1, \ldots, w_l) \\
&= \frac{1}{l!} \sum_{\varepsilon_1,\ldots,\varepsilon_l=0}^{1} (-1)^{l-(\varepsilon_1+\cdots+\varepsilon_l)} f(x, v_1, \ldots, v_k, \sum_{j=1}^{l} \varepsilon_j w_j, \ldots, \sum_{j=1}^{l} \varepsilon_j w_j) \\
&= \frac{1}{k!\, l!} \sum_{\varepsilon_1,\ldots,\varepsilon_l,\delta_1,\ldots,\delta_k=0}^{1} (-1)^{l-(\varepsilon_1+\cdots+\varepsilon_l)} (-1)^{k-(\delta_1+\cdots+\delta_k)} g(x, \sum_{i=1}^{k} \delta_i v_i, \sum_{j=1}^{l} \varepsilon_j w_j),
\end{aligned}
$$

whence f is continuous.

□

Theorem 52. (Exponential Law). *Let E_1, E_2 and F be locally convex spaces, U and V be locally convex subsets with dense interior of E_1 and E_2 respectively, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Assume that at least one of the following conditions is satisfied:*

(a) *V is locally compact.*
(b) *$r = s = 0$ and $U \times V$ is a k-space.*
(c) *$r \geq 1,\ s = 0$ and $U \times V \times E_1$ is a k-space.*
(d) *$r = 0,\ s \geq 1$ and $U \times V \times E_2$ is a k-space.*
(e) *$r \geq 1,\ s \geq 1$ and $U \times V \times E_1 \times E_2$ is a k-space.*

Then

$$\Phi\colon C^{r,s}(U \times V, F) \to C^r(U, C^s(V, F)),\ f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Moreover, if $g\colon U \to C^s(V, F)$ is C^r, then

$$g^\wedge\colon U \times V \to F,\ g^\wedge(x, y) := g(x)(y)$$

is $C^{r,s}$.

Proof. We only need to show the final assertion. Indeed, given $g \in C^r(U, C^s(V, F))$, the map $g^\wedge$ will be $C^{r,s}$ and hence $g = (g^\wedge)^\vee = \Phi(g^\wedge)$. Thus Φ will be surjective. Hence by Theorem 49, Φ will be an isomorphism of topological vector spaces.

(a) $g^\wedge(x, y) = g(x)(y) = \varepsilon(g(x), y)$ where $\varepsilon\colon C^s(V, F) \times V \to F,\ (\gamma, y) \mapsto \gamma(y)$ is $C^{\infty,s}$ (Proposition 42). Hence $g^\wedge$ is $C^{r,s}$ by Chain Rule 1 (Lemma 43).

(b), (c), (d) and (e) If $g\colon U \to C^s(V, F)$ is C^r, define $g^\wedge\colon U \times V \to F,\ g^\wedge(x, y) = g(x)(y)$. For fixed $x \in U$, we have $g^\wedge(x, \bullet) = g(x)$ which is C^s, hence

$$
\begin{aligned}
(D_{(0,v_j)} \cdots D_{(0,v_1)} g^\wedge)(x, y) &= d^{(j)}(g(x))(y, v_1, \ldots, v_j) \\
&= (d^{(j)} \circ g)(x)(y, v_1, \ldots, v_j)
\end{aligned}
$$

exists for $j \in \mathbb{N}_0$ such that $j \leq s$, $y \in V^0$ and $v_1, \ldots, v_j \in E_2$. Also,

$$(D_{(0,v_j)} \cdots D_{(0,v_1)} g^\wedge)(x,y) = (\varepsilon_{(y,v_1,\ldots,v_j)} \circ d^{(j)} \circ g)(x),$$

where $\varepsilon_{(y,v_1,\ldots,v_j)} \colon C^{s-j}(V \times E_2^j, F) \to F$, $f \mapsto f(y, v_1, \ldots, v_j)$. For fixed $(y, v_1, \ldots, v_j)$, this is the function $\varepsilon_{(y,v_1,\ldots,v_j)} \circ d^{(j)} \circ g$ of x, which is C^r. Since $\varepsilon_{(y,v_1,\ldots,v_j)}$ and $d^{(j)} \colon C^s(V,F) \to C^{s-j}(V \times E_2^j, F)$ are continuous linear, we obtain the directional derivatives

$$\begin{aligned}
&(D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} g)(x,y) \\
&= \varepsilon_{(y,v_1,\ldots,v_j)}(d^{(j)}(d^{(i)} g(x, w_1, \ldots, w_i))) \\
&= d^{(j)}(d^{(i)} g(x, w_1, \ldots, w_i))(y, v_1, \ldots, v_j) \\
&= (d^{(j)} \circ (d^{(i)} g))(x, w_1, \ldots, w_i)(y, v_1, \ldots, v_j) \\
&= (d^{(j)} \circ (d^{(i)} g))^\wedge((x, w_1, \ldots, w_i), (y, v_1, \ldots, v_j))
\end{aligned}$$

for $x \in U^0$, $w_1, \ldots, w_i \in E_1$, and $i \in \mathbb{N}_0$ such that $i \leq r$. To see that $g^\wedge$ is $C^{r,s}$, it therefore suffices to show that

$$h := (d^{(j)} \circ (d^{(i)} g))^\wedge \colon U \times E_1^i \times V \times E_2^j \to F$$

is continuous for all $i, j \in \mathbb{N}_0$ such that $i \leq r, j \leq s$.
The case $i = 0$, $j = 0$. Then $h = g^\wedge$, which is continuous by the case of topological spaces with $U \times V$ a k-space (see [18, Proposition B.15]).
The case $i = 0$, $j \geq 1$. Then

$$h \colon (U \times V) \times E_2^j \to F, \text{ where } h(x, y, \bullet) := d^{(j)}(g(x))(y, \bullet) \colon E_2^j \to F$$

is symmetric j-linear. Hence, by Lemma 50 and its proof, h is continuous if we can show that $f \colon U \times V \times E_2 \to F$, $(x, y, v) \mapsto d^{(j)}(g(x))(y, v, \ldots, v) = h(x, y, v, \ldots, v)$ is continuous.
Now

$$\begin{array}{ccc}
C^s(V,F) & \xrightarrow{d^{(j)}} & C^0(V \times E_2^j, F) \\
\big\uparrow{\scriptstyle g} & & \big\downarrow{\scriptstyle C^0(\varphi,F)} \\
U & \xrightarrow{\eta} & C^0(V \times E_2, F).
\end{array}$$

where $\varphi \colon V \times E_2 \to V \times E_2^j$, $(y, v) \mapsto (y, v, \ldots, v)$ and $C^0(\varphi, F) \colon C^0(V \times E_2^j, F) \to C^0(V \times E_2, F)$, $\gamma \mapsto \gamma \circ \varphi$ is the pullback which is continuous linear (see [21]; cf. [19, Lemma 4.4]).

Hence $\eta := C^0(\varphi, F) \circ d^{(j)} \circ g \colon U \to C^0(V \times E_2, F)$ is continuous. Because $U \times (V \times E_2)$ is a k-space by hypothesis, we know from the case of topological spaces (see [20,

Proposition B.15]) that $f = \eta^\wedge\colon U \times (V \times E_2) \to F$ is continuous.
The case $i \geq 1,\ j = 0$. Then

$$h\colon U \times E_1^i \times V \to F,\ h(x, w_1, \ldots, w_i, y) = (d^{(i)}g)(x, w_1, \ldots, w_j)(y).$$

By Lemma 50 and its proof, h is continuous if we can show that $f\colon U \times E_1 \times V \to F,\ f(x, w, y) := (d^{(i)}g)(x, w, \ldots, w)(y)$ is continuous. But $f = \psi^\wedge$ for the continuous map $\psi\colon U \times E_1 \to C^0(V, F),\ (x, w) \mapsto (d^{(i)}g)(x, w, \ldots, w)$. Hence f is continuous because $U \times E_1 \times V$ is a k-space by hypothesis.
The case $i \geq 1,\ j \geq 1$. By Lemma 51 and its proof, h will be continuous if we can show that

$$f\colon U \times E_1 \times V \times E_2 \to F,\ f(x, w, y, v) := h(x, \underbrace{w, \ldots, w}_{i-\text{times}}, y, \underbrace{v, \ldots, v}_{j-\text{times}})$$

is continuous. Now $\psi\colon U \times E_1 \to U \times E_1^i,\ (x, w) \mapsto (x, w, \ldots, w)$ is continuous and

$$\theta := C^0(\varphi, F) \circ d^{(j)} \circ d^{(i)}g \circ \psi\colon U \times E_1 \to C^0(V \times E_2, F)$$

is continuous. Since $U \times E_1 \times V \times E_2$ is a k-space by hypothesis, it follows that $\theta^\wedge\colon U \times E_1 \times V \times E_2 \to F$ is continuous (see [20, Proposition B.15]). But $\theta^\wedge = f$, and thus f is continuous. $\square$

Since C^∞-maps and $C^{\infty,\infty}$-maps coincide on products (see Lemma 40, Remark 41 and Lemma 46), we obtain as a special case that

$$\Phi\colon C^\infty(U \times V, F) \to C^\infty(U, C^\infty(V, F)) \tag{3.14}$$

is an isomorphism of topological vector spaces if V is locally compact or $U \times V \times E_1 \times E_2$ is a k-space.

Remark 53. For open sets U and V, the latter was known if E_2 is finite-dimensional or both E_1 and E_2 are metrizable (see [7] and [21]; cf. [19, Propositions 12.2 (b) and 12.6 (c)], where also manifolds are considered). In the inequivalent setting of differential calculus developed by E. G. F. Thomas,[1] an exponential law for smooth functions on open sets (analogous to (3.14)) holds without any conditions on the spaces, see [39, Theorem 5.1]. Related earlier results can be found in [37, p. 90, Lemma 17]. In the inequivalent "convenient setting" of analysis, (3.14) always is an isomorphism of *bornological* vector spaces (see [13] and [25], also for the case of manifolds) – but rarely

[1] Thomas replaces continuity of a function or its differentials with continuity on compact sets, and only considers quasi-complete locally convex spaces.

an isomorphism of topological vector spaces [7] (in this setting other topologies on the function spaces are used). Analogues of Theorems 49 and 52 for finite-dimensional vector spaces over a complete ultrametric field can be found in [18].

3.4 The Exponential Law for $C^{r,s}$-mappings on manifolds

Definition 54. We recall from [21] that a *manifold with rough boundary* modelled on a locally convex space E is a Hausdorff topological space M with an atlas of smoothly compatible homeomorphisms $\phi\colon U_\phi \to V_\phi$ from open subsets U_ϕ of M onto locally convex subsets $V_\phi \subseteq E$ with dense interior. If each V_ϕ is open, M is an ordinary manifold (without boundary). If each V_ϕ is relatively open in a closed hyperplane $\lambda^{-1}([0,\infty[)$, where $\lambda \in E'$ (the space of continuous linear functionals on E), then M is a *manifold with smooth boundary.* In the case of a *manifold with corners*, each V_ϕ is a relatively open subset of $\lambda_1^{-1}([0,\infty[) \cap \cdots \cap \lambda_n^{-1}([0,\infty[)$, for suitable $n \in \mathbb{N}$ (which may depend on ϕ) and linearly independent $\lambda_1, \ldots, \lambda_n \in E'$.

Definition 55. Let M_1 and M_2 be smooth manifolds (possibly with rough boundary) modelled on locally convex spaces, $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and F be a locally convex space. A map $f\colon M_1 \times M_2 \to F$ is called $C^{r,s}$ if $f \circ (\varphi^{-1} \times \psi^{-1})\colon V_\varphi \times V_\psi \to F$ is $C^{r,s}$ for all charts $\varphi\colon U_\varphi \to V_\varphi$ of M_1 and $\psi\colon U_\psi \to V_\psi$ of M_2. Then f is continuous in particular.

Definition 56. In the situation of Definition 55, let $C^{r,s}(M_1 \times M_2, F)$ be the space of all $C^{r,s}$-maps $f\colon M_1 \times M_2 \to F$. Endow $C^{r,s}(M_1 \times M_2, F)$ with the initial topology with respect to the maps $C^{r,s}(M_1 \times M_2, F) \to C^{r,s}(V_\varphi \times V_\psi, F)$, $f \mapsto f \circ (\varphi^{-1} \times \psi^{-1})$, for φ and ψ in the maximal smooth atlas of M_1 and M_2, respectively.

The following fact is well known (cf. [11, Proposition 2.3.2]).

Lemma 57. *Let $(\theta_j)_{j\in J}$ be a family of topological embeddings $\theta_j\colon X_j \to Y_j$ between topological spaces. Then also*

$$\theta := \prod_{j\in J} \theta_j \colon \prod_{j\in J} X_j \to \prod_{j\in J} Y_j,\ (x_j)_{j\in J} \mapsto (\theta_j(x_j))_{j\in J}$$

is a topological embedding.

Proposition 58. *Let M_1 and M_2 be smooth manifolds (possibly with rough boundary) modelled on locally convex spaces, $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and F be a locally convex space. Then*

(a) *$f^\vee \in C^r(M_1, C^s(M_2, F))$ for all $f \in C^{r,s}(M_1 \times M_2, F)$.*

(b) *The map*

$$\Phi\colon C^{r,s}(M_1 \times M_2, F) \to C^r(M_1, C^s(M_2, F)),\ f \mapsto f^\vee$$

is linear and a topological embedding.

Proof. (a) It is clear that $f^\vee(x) = f(x, \bullet)$ is a C^s-map $M_2 \to F$. It suffices to show that $f \circ \varphi^{-1}\colon U_\varphi \to C^s(M_2, F)$ is C^r for each chart $\varphi\colon U_\varphi \to V_\varphi$ of M_1. For $i = 1, 2$, let $\mathcal{A}_i$ be the maximal smooth atlas for M_i. Because the map

$$\Psi\colon C^s(M_2, F) \to \prod_{\psi \in \mathcal{A}_2} C^s(V_\psi, F),\ h \mapsto (h \circ \psi^{-1})_{\psi \in \mathcal{A}_2}$$

is a linear topological embedding with closed image (see [21]; cf. [19, 4.7 and Proposition 4.19(d)]), $f^\vee \circ \varphi^{-1}$ is C^r if and only if $\Psi \circ f \circ \varphi^{-1}$ is C^r (see [21]; cf. [4, Lemma 10.2]), which holds if all components are C^r. Hence we only need that

$$\theta\colon V_\varphi \to C^s(V_\psi, F),\ x \mapsto f^\vee(\varphi^{-1}(x)) \circ \psi^{-1} = (f \circ (\varphi^{-1} \times \psi^{-1}))^\vee(x)$$

is C^r. But $\theta = (f \circ (\varphi^{-1} \times \psi^{-1}))^\vee$ where $f \circ (\varphi^{-1} \times \psi^{-1})\colon V_\varphi \times V_\psi \to F$ is $C^{r,s}$, hence θ is C^r by Theorem 49.
(b) It is clear that Φ is linear and injective. Because Ψ is linear and a topological embedding, also

$$C^r(M_1, \Psi)\colon C^r(M_1, C^s(M_2, F)) \to C^r(M_1, \prod_{\psi \in \mathcal{A}_2} C^s(V_\psi, F)),\ f \mapsto \Psi \circ f$$

is a topological embedding (see [21]).
Let $P := \prod_{\psi \in \mathcal{A}_2} C^s(V_\psi, F)$. The map

$$\Xi\colon C^r(M_1, P) \to \prod_{\varphi \in \mathcal{A}_1} C^r(V_\varphi, P),\ f \mapsto (f \circ \varphi^{-1})_{\varphi \in \mathcal{A}_1}$$

is a linear topological embedding. Using the isomorphism

$$\prod_{\varphi \in \mathcal{A}_1} C^r(V_\varphi, P) \cong \prod_{\varphi \in \mathcal{A}_1} \prod_{\psi \in \mathcal{A}_2} C^r(V_\varphi, C^s(V_\psi, F))$$

we obtain a linear topological embedding

$$\Gamma := \Xi \circ C^r(M_1, \Psi)\colon C^r(M_1, C^s(M_2, F)) \to \prod_{\varphi \in \mathcal{A}_1} \prod_{\psi \in \mathcal{A}_2} C^r(V_\varphi, C^s(V_\psi, F)),$$
$$f \mapsto (C^s(\psi^{-1}, F) \circ f \circ \varphi^{-1})_{\substack{\varphi \in \mathcal{A}_1, \\ \psi \in \mathcal{A}_2}}$$

where $C^s(\psi^{-1}, F)\colon C^s(M_2, F) \to C^s(V_\psi, F),\ f \mapsto f \circ \psi^{-1}$. Also the map

$$\omega\colon C^{r,s}(M_1 \times M_2, F) \to \prod_{\substack{\varphi\in\mathcal{A}_1,\\ \psi\in\mathcal{A}_2}} C^{r,s}(V_\varphi \times V_\psi, F),\ f \mapsto (f \circ (\varphi^{-1} \times \psi^{-1}))_{\substack{\varphi\in\mathcal{A}_1,\\ \psi\in\mathcal{A}_2}}$$

is a topological embedding, by Definition 56. Now we have the commutative diagram.

$$\begin{array}{ccc}
C^{r,s}(M_1 \times M_2, F) & \xrightarrow{\Phi} & C^r(M_1, C^s(M_2, F)) \\
\big\downarrow{\scriptstyle\omega} & & \big\downarrow{\scriptstyle\Gamma} \\
\prod\limits_{\substack{\varphi\in\mathcal{A}_1,\\ \psi\in\mathcal{A}_2}} C^{r,s}(V_\varphi \times V_\psi, F) & \xrightarrow{\eta} & \prod\limits_{\substack{\varphi\in\mathcal{A}_1,\\ \psi\in\mathcal{A}_2}} C^r(V_\varphi, C^s(V_\psi, F))
\end{array}$$

where η is the map $(f_{\varphi,\psi})_{\varphi\in\mathcal{A}_1,\psi\in\mathcal{A}_2} \mapsto (f^\vee_{\varphi,\psi})_{\varphi\in\mathcal{A}_1,\psi\in\mathcal{A}_2}$. Because the vertical arrows are topological embeddings and also the horizontal arrow at the bottom (by Lemma 57 and Theorem 49) is a topological embbeding, we deduce that the map Φ at the top has to be a topological embedding as well. $\square$

Theorem 59. *Let M_1 and M_2 be smooth manifolds (possibly with rough boundary) modelled on locally convex spaces E_1 and E_2 respectively, F be a locally convex space and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Assume that M_2 is locally compact or that one of the following conditions is satisfied:*

(a) *$r = s = 0$ and $M_1 \times M_2$ is a k-space.*
(b) *$r \geq 1,\ s = 0$ and $M_1 \times M_2 \times E_1$ is a k-space.*
(c) *$r = 0,\ s \geq 1$ and $M_1 \times M_2 \times E_2$ is a k-space.*
(d) *$r \geq 1,\ s \geq 1$ and $M_1 \times M_2 \times E_1 \times E_2$ is a k-space.*

Then

$$\Phi\colon C^{r,s}(M_1 \times M_2, F) \to C^r(M_1, C^s(M_2, F)),\ f \mapsto f^\vee \tag{3.15}$$

is an isomorphism of topological vector spaces. Moreover, a map $g\colon M_1 \to C^s(M_2, F)$ is C^r if and only if

$$g^\wedge\colon M_1 \times M_2 \to F,\ g^\wedge(x, y) := g(x)(y)$$

is $C^{r,s}$.

Proof. By Proposition 58, we only need to show that Φ is surjective. To this end, let $g \in C^r(M_1, C^s(M_2, F))$ and define

$$f := g^\wedge\colon M_1 \times M_2 \to F,\ f(x, y) := g(x)(y).$$

Let $\varphi : U_\varphi \to V_\varphi$ and $\psi : U_\psi \to V_\psi$ be charts for M_1 and M_2, respectively. Then

$$f \circ (\varphi^{-1} \times \psi^{-1}) \colon V_\varphi \times V_\psi \to F,\ (x, y) \mapsto (C^s(\psi^{-1}, F) \circ g \circ \varphi^{-1})^\wedge(x, y)$$

with $C^s(\psi^{-1}, F) \colon C^s(M_2, F) \to C^s(V_\psi, F), h \mapsto h \circ \psi^{-1}$ continuous linear. Hence $C^s(\psi^{-1}, F) \circ g \circ \varphi^{-1} \colon V_\varphi \to C^s(V_\psi, F)$ is C^r. Hence $f \circ (\varphi^{-1} \times \psi^{-1})$ is $C^{r,s}$ by the exponential law (Theorem 52).

Note. In (d) $V_\varphi \times V_\psi \times E_1 \times E_2$ is homeomorphic to the open subset $U_\varphi \times U_\psi \times E_1 \times E_2$ of the k-space $M_1 \times M_2 \times E_1 \times E_2$ and hence a k-space. Similarly in (a), (b) and (c). Hence the Exponential Law (Theorem 52) applies. If M_2 is locally compact, then the open subsets U_ψ are locally compact and hence also the V_ψ. Again, the Exponential Law (Theorem 52) applies. □

Remark 60. The same conclusion holds if M_2 is finite-dimensional or $E_1 \times E_2 \times E_1 \times E_2$ is a k-space, provided that M_1 and M_2 are manifolds without boundary, manifolds with smooth boundary or manifolds with corners. Recall that direct products of k-spaces need not be k-spaces. However, the direct product of two metrizable spaces is metrizable (and hence a k-space). Likewise, the product of two hemicompact k-spaces[1] (also known as k_ω-spaces) is a hemicompact k-space and hence a k-space (see [12] for further information and [22], including analogues for spaces which are only locally k_ω). Thus $E_1 \times E_2 \times E_1 \times E_2$ is a k-space whenever both E_1 and E_2 are k_ω. For example, the dual E' of a metrizable locally convex space E always is k_ω when equipped with the compact-open topology (cf. [3, Corollary 4.7]). Consequently (3.15) is an isomorphism in the case of manifolds with corners if M_2 is finite-dimensional or both E_1 and E_2 are metrizable, respectively, both are hemicompact k-spaces (Corollary 62).

To deduce a corollary, we use the following lemma.

Lemma 61. *Let X be a Hausdorff topological space. If $X = \bigcup_{j \in J} V_j$ with open subsets $V_j \subseteq X$ which are k-spaces, then X is a k-space.*

Proof. Let $W \subseteq X$ be a subset such that $W \cap K$ is relatively open in K for each compact subset $K \subseteq X$. We show that W is open in X. Since $W = \bigcup_{j \in J}(V_j \cap W)$, it suffices to show that each $V_j \cap W$ is open. For each compact subset $K \subseteq V_j$, $K \cap (V_j \cap W) = K \cap W$ is relatively open in K by hypothesis, thus $V_j \cap W$ is open in V_j, hence open in X. □

Corollary 62. *Let M_1 and M_2 be smooth manifolds (possibly with rough boundary) modelled on locally convex spaces E_1 and E_2 respectively, F be a locally convex space and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Assume that* (a) *or* (b) *is satisfied:*

[1] A topological space X is called hemicompact if it is the union of an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact sets and each compact subset of X is contained in some K_n.

(a) *E_1 and E_2 are metrizable.*

(b) *M_1 and M_2 are manifolds with corners. Moreover, E_2 is finite-dimensional or both of E_1 and E_2 are hemicompact k-spaces.*

Then

$$\Phi\colon C^{r,s}(M_1 \times M_2, F) \to C^r(M_1, C^s(M_2, F)),\ f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Moreover, a map $g\colon M_1 \to C^s(M_2, F)$ is C^r if and only if

$$g^\wedge\colon M_1 \times M_2 \to F,\ g^\wedge(x, y) := g(x)(y)$$

is $C^{r,s}$.

Proof. Case M_2 a finite-dimensional manifold with corners. Let M_2 be of dimension n. Then each point of M_2 has an open neighbourhood homeomorphic to an open subset V of $[0, \infty[^n$. Hence V is locally compact, thus M_2 is locally compact. Thus Theorem 59 applies.

Case E_1, E_2 metrizable. Then all points $x \in M_1, y \in M_2$ have open neighbourhoods $U_1 \subseteq M_1$, $U_2 \subseteq M_2$ homeomorphic to subsets $V_1 \subseteq E_1$ and $V_2 \subseteq E_2$, respectively. Since $V_1 \times V_2$ is metrizable, it follows that $U_1 \times U_2 \times E_1 \times E_2$ is metrizable and hence a k-space. Hence, by Lemma 61, $M_1 \times M_2 \times E_1 \times E_2$ is a k-space and Theorem 59 applies.

Case E_1 and E_2 are k_ω-spaces and M_1 and M_2 are manifolds with corners. For all $x \in M_1$ and $y \in M_2$, there are open neighbourhoods $U_1 \subseteq M_1$, $U_2 \subseteq M_2$ homeomorphic to open subsets V_1 and V_2, respectively, of finite intersections of closed half-spaces in E_1 and E_2, respectively. Hence $V_1 \times V_2 \times E_1 \times E_2$ is a (relatively) open subset of a closed subset of $E_1 \times E_2 \times E_1 \times E_2$. The latter product is k_ω since E_1 and E_2 are k_ω-spaces (see [22, Proposition 4.2(i)]), and hence a k-space.

Since open subsets (and also closed subsets) of k-spaces are k-spaces, it follows that $V_1 \times V_2 \times E_1 \times E_2$ is a k-space. Now Lemma 61 shows that $M_1 \times M_2 \times E_1 \times E_2$ is a k-space, and thus Theorem 59 applies.

□

Proof for the Remark 60. All assertions are covered by Corollary 62, except for the case when M_1, M_2 are manifolds with corners and $E_1 \times E_2 \times E_1 \times E_2$ is a k-space. But this case can be proved like the result for k_ω-spaces in Corollary 62.

Remark 63. If $s = 0$, then $C^{r,s}$-maps $f\colon U \times V \to F$ can be defined just as well if V is any Hausdorff topological space (and $U \subseteq E_1$ as before).

If $r = 0$, then $C^{r,s}$-maps $f\colon U \times V \to F$ make sense if U is a Hausdorff topological space. All results carry over to this situation (with obvious modifications).

Remark 64. If F is a complex locally convex space, we obtain analogous results if E_1 is a locally convex space over $\mathbb{K}_1 \in \{\mathbb{R}, \mathbb{C}\}$, E_2 is a locally convex space over $\mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$, and all directional derivatives in the first and second variable are considered as derivatives over the ground field $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively. The corresponding maps could be called $C^{r,s}_{\mathbb{K}_1,\mathbb{K}_2}$-maps.

Chapter 4

C^α-Mappings

In this chapter we develop the calculus of mappings on products with different degrees of differentiability, called C^α-mappings, which generalize the concept of $C^{r,s}$-mappings in Chapter 3. We study their basic properties and some refined tools in an infinite-dimensional setting. In section 4.4, we introduce the exponential laws for such mappings on products of manifolds modelled on locally convex spaces.

Definition 65. Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be an open subset of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$.
A continuous mapping $f : U_1 \times \cdots \times U_n \to F$ is called a C^α-map, if for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$ the iterated directional derivative

$$d^\beta f(x, w_1, \ldots, w_n) := (\breve{D}_1 \cdots \breve{D}_n f)(x)$$

where $(\breve{D}_i f)(x) := (D_{(w_i)^*_{\beta_i}} \cdots D_{(w_i)^*_1} f)(x)$, exists for all $x := (x_1, \ldots, x_n)$ where $x_i \in U_i$, $w_i := ((w_i)_1, \ldots, (w_i)_{\beta_i})$ such that $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$, $(w_i)^*_1, \ldots, (w_i)^*_{\beta_i} \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$ and the maps

$$d^\beta f : U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F,$$

so obtained are continuous.

More generally, the following definition allows us to speak about C^α-maps on non-open sets, like products of compact intervals.

Definition 66. Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. We say that a continuous map $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map, if

$f|_{U_1^0\times\cdots\times U_n^0}\colon U_1^0\times\cdots\times U_n^0\to F$ is a C^α-map and for all $\beta_i\in\mathbb{N}_0$ such that $\beta_i\leq\alpha_i$ and $\beta:=(\beta_1,\ldots,\beta_n)$, and the map

$$d^\beta(f|_{U_1^0\times\cdots\times U_n^0}):U_1^0\times\cdots\times U_n^0\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}\longrightarrow F$$

admits a continuous extension

$$d^\beta f:U_1\times\cdots\times U_n\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}\longrightarrow F.$$

Definitions 65 and 66 can be rephrased as follows:

Lemma 67. *For all $i\in\{1,\ldots,n\}$, let E_i and F be locally convex spaces, $U_i\subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$. For $j\in\mathbb{N}$, $2\leq j\leq n$, let $U:=U_1\times\cdots\times U_{j-1}$, $V:=U_j\times\cdots\times U_n$, $\gamma:=(\alpha_1,\ldots,\alpha_{j-1})$ and $\eta:=(\alpha_j,\ldots,\alpha_n)$. Then $f:U_1\times\cdots\times U_n\to F$ is a $C^{(\gamma,\eta)}$-map if and only if all of the following conditions are satisfied:*

(a) *For all $x:=(x_1,\ldots,x_{j-1})\in U$, the map $f_x:=f(x,\bullet):V\to F$ taking $y:=(x_j,\ldots,x_n)\in V$ to $f_x(y):=f(x_1,\ldots,x_n)$ is C^η.*

(b) *For all $y\in V$ and $w_i:=((w_i)_1,\ldots,(w_i)_{\beta_i})\in E_i^{\beta_i}$, the map $U\to F, x\mapsto d^{(\beta_j,\ldots,\beta_n)}f_x(y,w_j,\ldots,w_n)$ is C^γ, where $\beta_i\in\mathbb{N}_0$, $\beta_i\leq\alpha_i$.*

(c) *For $\beta:=(\beta_1,\ldots,\beta_n)$, the map $d^\beta f:U_1\times\cdots\times U_n\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}\to F$, $(x,y,w_1,\ldots,w_n)\mapsto d^{(\beta_1,\ldots,\beta_{j-1})}(d^{(\beta_j,\ldots,\beta_n)}f_\bullet(y,w_j,\ldots,w_n))(x,w_1,\ldots,w_{j-1})$, is continuous.*

Proof. Step 1. If $U_i\subseteq E_i$ is an open subset for all $i\in\{1,\ldots,n\}$, then the equivalence follows by the definition of a $C^{(\gamma,\eta)}$-map.
Now the general case. Assume that f is a $C^{(\gamma,\eta)}$-map.
Step 2. For $x\in U^0:=U_1^0\times\cdots\times U_{j-1}^0$ and $v_k:=((v_k)_1,\ldots,(v_k)_{\beta_k})\in E_k^{\beta_k}$ for $k\in\{j,\ldots,n\}$ with corresponding elements $(v_k)_1^*,\ldots,(v_k)_{\beta_k}^*\in(\{0\})^{k-j}\times E_k\times(\{0\})^{n-k}\subseteq E_{j+1}\times\cdots\times E_n$, the iterated directional derivative

$$(\breve{D}_n\cdots\breve{D}_j)f(x,y)=D_{(v_n)_{\beta_n}^*}\cdots D_{(v_j)_1^*}f_x(y)$$

exists for all $y\in V^0:=U_j^0\times\cdots\times U_n^0$, with continuous extension

$$(y,v_j,v_{j+1},\ldots,v_n)\mapsto d^{(0,\ldots,0,\beta_j,\ldots,\beta_n)}f(x,y,v_j,v_{j+1},\ldots,v_n)$$

to $V\times E_j^{\beta_j}\times\cdots\times E_n^{\beta_n}\to F$. Hence $f_x\colon V\to F$ is C^η. If $x\in U$ is arbitrary, $y\in V^0$, we show that $D_{(v_n)_1^*}f_x(y)$ exists and equals $d^{(0,0,\ldots,0,1)}f(x,y,(v_j)_1)$ with j-th entry 1. There exists $R>0$ such that $y+t(v_j)_1^*\in V$ for all $t\in\mathbb{R}$, $|t|\leq R$ and there exists a relatively open convex neighbourhood $W \subseteq U$ of x in U. Because U^0 is dense, there exists $z \in U^0 \cap W$. Since W is convex, we have $x + \tau(z-x) \in W$ for all $\tau \in [0,1]$. Moreover, since $z \in W^0$, we have $x + \tau(z-x) \in W^0$ for all $\tau \in (0,1]$. Hence, for $\tau \in (0,1]$, $f(x+\tau(z-x), y)$ is C^η in y, and thus for $t \neq 0$

$$\begin{aligned}&\frac{1}{t}(f(x+\tau(z-x), y+t(v_n)_1^*) - f(x+\tau(z-x), y))\\&= \int_0^1 d^{(0,\ldots,0,1,0\ldots,0)} f(x+\tau(z-x), y+\sigma t(v_n)_1^*, (v_n)_1^*)\, d\sigma\end{aligned}$$

by the Mean Value Theorem. Now let $\tilde{F}$ be a completion of F. Because

$$h\colon [0,1]\times[-R,R]\times[0,1] \to \tilde{F},\ (\tau,t,\sigma) \mapsto d^{(0,\ldots,0,1,0\ldots,0)} f(x+\tau(z-x), y+\sigma t(v_n)_1^*, (v_n)_1^*)$$

is continuous, also the parameter-dependent integral

$$g\colon [0,1]\times[-R,R] \to \tilde{F},\ g(\tau,t) := \int_0^1 h(\tau,t,\sigma)\, d\sigma$$

is continuous. Fix $t \neq 0$ in $[-R,R]$. Then

$$g(\tau,t) = \frac{1}{t}(f(x+\tau(z-x), y+t(v_j)_1^*) - f(x+\tau(z-x), y)) \tag{4.1}$$

for all $\tau \in (0,1]$. By continuity of both sides in τ, (4.1) also holds for $\tau = 0$. Hence

$$\frac{1}{t}(f(x, y+t(v_j)_1^*) - f(x,y)) = g(0,t) \to g(0,0)$$

as $t \to 0$. Thus $D_{(v_n)_1^*} f_x(y)$ exists and is given by

$$g(0,0) = \int_0^1 d^{(0,\ldots,0,1,0\ldots,0)} f(x,y,(v_n)_1)\, d\sigma = d^{(0,\ldots,0,1,0\ldots,0)} f(x,y,(v_n)_1).$$

Holding $(v_n)_1$ fixed, we can repeat the argument to see that $D_{(v_n)_{\beta_n}^*} \cdots D_{(v_n)_1^*} f_x(y)$ exists for all $y \in V^0$ and is given by $D_{(v_n)_{\beta_n}^*} \cdots D_{(v_n)_1^*} f_x(y) = d^{(0,\ldots,0,\beta_n)} f(x,y,v_n)$. Again we can repeat the argument to see that $D_{(v_j)_{\beta_j}^*} \cdots D_{(v_n)_1^*} f_x(y)$ exists for all $y \in V^0$, $v := (v_j, v_{j+1}, \ldots, v_n) \in E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$ and is given by

$$D_{(v_j)_{\beta_j}^*} \cdots D_{(v_n)_1^*} f_x(y) = d^{(0,\ldots,0,\beta_j,\beta_{j+1},\ldots,\beta_n)} f(x,y,v).$$

Since the right-hand side makes sense for $(y,v) \in V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$ and is continuous there, f_x is C^η.

Step 3 Holding $v \in E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$ fixed, the function

$$(x,y) \mapsto d^{(0,\ldots,0,\beta_j,\beta_{j+1},\ldots,\beta_n)} f(x,y,v)$$

is $C^{(\gamma,0)}$. By Step 2 (applied to the $C^{(0,\gamma)}$ function $(y,x)\mapsto d^{(0,\ldots,0,\beta_j,\beta_{j+1},\ldots,\beta_n)}f(x,y,v)$) we see that for each $y\in V$, the function

$$U\to F,\ x\mapsto d^{(0,\ldots,0,\beta_j,\beta j+1,\ldots,\beta_n)}f(x,y,v)$$

is C^γ and for $w\in E_1^{\beta_1}\times\cdots\times E_{j-1}^{\beta_{j-1}}$, we get

$$d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}(d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}f_\bullet(y,v))(x,w)=d^\beta f(x,y,w,v),$$

which is continuous in $(x,y,w,v)\in U\times V\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}$. Hence if f is $C^{(\gamma,\eta)}$, then (a),(b) and (c) hold.

Conversely, assume that (a),(b) and (c) hold. By step 1, $f|_{U^0\times V^0}$ is $C^{(\gamma,\eta)}$ and

$$d^\beta f|_{U^0\times V^0}(x,y,w,v)=d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}(d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}f_\bullet(y,v))(x,w) \tag{4.2}$$

for $(x,y)\in U^0\times V^0$. By (c), the right-hand side of (4.2) extends to a continuous function $d^\beta f\colon U\times V\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}\to F$. Hence f is a $C^{(\gamma,\eta)}$-map. □

4.1 Elementary properties

The following lemma will enable us to prove a version of the Theorem of Schwarz for C^α-maps.

Lemma 68. *Let $E_1,\ldots,E_n$ and F be locally convex spaces, U_i be an open subset of E_i, $x_i\in U_i$ for $i\in\{1,\ldots,n\}$, $x:=(x_1,\ldots,x_n)$ and $\alpha:=(\alpha_1,\ldots,\alpha_{n-1},1)$ such that $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$. If $f\colon U_1\times\cdots\times U_n\to F$ is a C^α-map, then*

$$D_{(w_n)_1^*}D_{(w_1)_{\beta_1}^*}\cdots D_{(w_{n-1})_1^*}f(x) \tag{4.3}$$

exists for all $\beta_i\in\mathbb{N}_0$ with $\beta_i\le\alpha_i$, for all $(w_i)_1,\ldots,(w_i)_{\beta_i}\in E_i$ and corresponding elements $(w_i)_1^*,\ldots,(w_i)_{\beta_i}^*\in(\{0\})^{i-1}\times E_i\times(\{0\})^{n-i}\subseteq E_1\times\cdots\times E_n$, and it coincides with

$$d^{(\beta_1,\ldots,\beta_{n-1},1)}f(x,(w_1)_1,\ldots,(w_{n-1})_{\beta_{n-1}},(w_n)_1). \tag{4.4}$$

Proof. The proof is by induction on n. If $n=1$, there is nothing to show. Let $n\ge 2$. Now the proof is by induction on β_1. If $\beta_1=0$, holding the first variable fixed, we see that (4.3) exists and coincides with (4.4), by the case $n-1$. Now assume that $\beta_1\ge 1$. If $\beta_i=0$ for all $i=2,\ldots,n-1$, the assertion follows from Lemma 30. Now assume that $\beta_i\ge 1$ for some $i=2,\ldots,n-1$. By induction on β_1, we know that

$$D_{(w_n)_1^*}D_{(w_1)_{\beta_1-1}^*}D_{(w_1)_{\beta_1-2}^*}\cdots D_{(w_{n-1})_1^*}f(x)$$

exists and coincides with

$$d^{(\beta_1-1,\beta_2,\ldots,\beta_{n-1},1)}f(x,(w_1)_1,\ldots,(w_1)_{\beta_1-1},(w_2)_1,\ldots,(w_n)_1). \tag{4.5}$$

Define $g\colon U_1\times\cdots\times U_n\to F$ via

$$\begin{aligned} g(x) &:= D_{(w_1)^*_{\beta_1-1}}D_{(w_1)^*_{\beta_1-2}}\cdots D_{(w_{n-1})^*_1}f(x) \\ &= d^{(\beta_1-1,\beta_2,\ldots,\beta_{n-1},0)}f(x,(w_1)_1,\ldots,(w_1)_{\beta_1-1},(w_2)_1,(w_2)_2,\ldots,(w_{n-1})_{\beta_{n-1}}). \end{aligned}$$

By the preceding, g is differentiable in the n-th variable and

$$D_{(w_n)^*_1}g(x) \tag{4.6}$$
$$= d^{(\beta_1-1,\beta_2,\ldots,\beta_{n-1},1)}f(x,(w_1)_1,\ldots,(w_1)_{\beta_1-1},\ldots,(w_{n-1})_{\beta_{n-1}},(w_n)_1) \tag{4.7}$$
$$= D_{(w_1)^*_{\beta_1-1}}D_{(w_1)^*_{\beta_1-2}}\cdots D_{(w_{n-1})^*_1}D_{(w_n)^*_1}f(x), \tag{4.8}$$

which is continuous in $((w_n)_1,x)$. Hence g is $C^{(0,\ldots,0,1)}$ and $d^{(0,\ldots,0,1)}g(x,(w_n)_1)$ is given by (4.5). Because f is C^{α} and $\alpha_1\geq\beta_1$, (4.7) can be differentiated once more in the first variable, hence also $D_{(w_n)^*_1}g(x)$, with

$$\begin{aligned} &d^{(1,0,0,\ldots,0,1)}g(x,(w_n)_1,(w_1)_{\beta_1}) \\ &= D_{(w_1)^*_{\beta_1}}D_{(w_n)^*_1}g(x) \\ &= D_{(w_1)^*_{\beta_1}}\cdots D_{(w_{n-1})^*_{\beta_{n-1}}}D_{(w_n)^*_1}f(x) \\ &= d^{(\beta_1,\ldots,\beta_{n-1},1)}f(x,(w_1)_1,\ldots,(w_{n-1})_{\beta_{n-1}},(w_n)_1). \end{aligned}$$

As this map is continuous, g is $C^{(1,0,\ldots,0,1)}$. By Lemma 29, also $D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}g(x)$ exists and is given by $D_{(w_1)^*_{\beta_1}}D_{(w_n)^*_1}g(x) = d^{(\beta_1,\ldots,\beta_{n-1},1)}f(x,(w_1)_1,\ldots,(w_{n-1})_{\beta_{n-1}},(w_n)_1)$. But, by definition of g, $D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}g(x) = D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}\cdots D_{(w_{n-1})^*_{n-1}}f(x)$. Hence $D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}\cdots D_{(w_{n-1})^*_{n-1}}f(x) = d^{(\beta_1,\ldots,\beta_{n-1},1)}f(x,(w_1)_1,\ldots,(w_{n-1})_{\beta_{n-1}},(w_n)_1)$.

□

Proposition 69. (Schwarz' Theorem for C^{α}-mappings). *For $i\in\{1,\ldots,n\}$, let E_i and F be locally convex spaces, $U_i\subseteq E_i$ an open subset, $x_i\in U_i$ and $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$ with $\alpha:=(\alpha_1,\ldots,\alpha_n)$. For $\beta_i\in\mathbb{N}_0$ such that $\beta_i\leq\alpha_i$, we define $\beta:=(\beta_1,\ldots,\beta_n)$, $\xi_i:=\sum_{m=1}^{i-1}\beta_m+1$, $\rho_i:=\sum_{m=1}^{i}\beta_m$, $w^*_{\xi_i},\ldots,w^*_{\rho_i}\in(\{0\})^{i-1}\times E_i\times(\{0\})^{n-i}\subseteq E_1\times\cdots\times E_n$ with entries $w_{\xi_i},\ldots,w_{\rho_i}$ in the E_i-coordinate. If $\sigma\in S_{\rho_n}$ is a permutation of $\{1,\ldots,\rho_n\}$ and $f:U_1\times\cdots\times U_n\to F$ is a C^{α}-map, then the iterated directional derivative*

$$(D_{w^*_{\sigma(1)}}\cdots D_{w^*_{\sigma(\rho_n)}}f)(x_1,\ldots,x_n)$$

exists and coincides with $d^{\beta}f(x_1,\ldots,x_n,w_1,\ldots,w_{\rho_n})$.

Proof. The case $n = 2$ having been settled in Proposition 31, we may assume that $n \geq 3$ and assume that the assertion holds when n is replaced with $n-1$. We prove the n-th case by induction on ρ_n. *The case $\rho_n = 0$ is trivial.* *If at least one of the $\beta_i = 0$ for $i = 1, \ldots, n$,* then the assertion follows from the assumption that n has been replaced with $n-1$. *The case $\beta_i \geq 1$ for all $i = 1, \ldots, n$.* If $\sigma(1) \in \{1, \ldots, \beta_1\}$, then by induction,

$$\begin{aligned}
&D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(\rho_n)}} f(x_1, \ldots, x_n) \\
&= d^{(\beta_1 - 1, \beta_2, \ldots, \beta_n)} f(x_1, \ldots, x_n, w_1, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_{\beta_1}, \ldots, w_{\rho_n}).
\end{aligned}$$

Because f is C^α, we can differentiate once more in the first variable:

$$\begin{aligned}
&D_{w^*_{\sigma(1)}} \cdots D_{w^*_{\sigma(\rho_n)}} f(x_1, \ldots, x_n) \\
&= d^\beta f(x_1, \ldots, x_n, w_1, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_{\rho_1}, w_{\sigma(1)}; w_{\xi_2}, w_{\xi_2+1}, \ldots, w_{\rho_n}) \\
&= d^\beta f(x_1, \ldots, x_n, w_1, w_2, \ldots, w_{\rho_n}).
\end{aligned}$$

For the final equality we used that, for $v_{\xi_i}, \ldots, v_{\rho_i} \in E_i$,

$$\begin{aligned}
&d^\beta f(x_1, \ldots, x_n, v_1, v_2, \ldots, v_{\rho_n}) \\
&= d^{\beta_1}(d^{(\beta_2, \ldots, \beta_n)} f_\bullet(x_2, \ldots, x_n, v_{\xi_2}, v_{\xi_2+1}, \ldots, v_{\rho_n}))(x_1, v_1, \ldots, v_{\beta_1})
\end{aligned}$$

is symmetric in $v_1, \ldots, v_{\beta_1} \in E_1$, as

$$g(x_1) := d^{(\beta_2, \ldots, \beta_n)} f_{x_1}(x_2, \ldots, x_n, v_{\xi_2}, v_{\xi_2+1}, \ldots, v_{\rho_n})$$

is C^{α_1} in x_1 (see Lemma 67).
If $\sigma(1) \in \{\xi_i, \ldots, \rho_i\}$ for some $i \in \{2, \ldots, n\}$, then

$$\begin{aligned}
&D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(\rho_n)}} f(x_1, \ldots, x_n) \\
&= d^{(\beta_1, \ldots, \beta_i - 1, \beta_{i+1}, \ldots, \beta_n)} f(x_1, \ldots, x_n, w_1, \ldots, w_{\xi_i}, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_{\rho_i}, \ldots, w_{\rho_n}).
\end{aligned}$$

For fixed $w_{\xi_i}, \ldots, w_{\rho_n}$, consider the function $h\colon U_1 \times \cdots \times U_n \to F$, $h(x_1, \ldots, x_n) := d^{(0, \ldots, 0, \beta_i - 1, \beta_{i+1}, \ldots, \beta_n)} f(x_1, \ldots, x_n, w_{\xi_i}, \ldots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \ldots, w_{\rho_i}, \ldots, w_{\rho_n})$.
Holding $x_{i+1}, \ldots, x_n$ fixed, we can apply Lemma 68 and find that

$$D_{w^*_{\sigma(1)}} D_{w^*_{\rho_1}} \cdots D_{w^*_{\xi_{i-1}}} h(x_1, \ldots, x_n)$$

exists and coincides with

$$D_{w^*_{\rho_1}} \cdots D_{w^*_{\xi_{i-1}}} D_{w^*_{\sigma(1)}} h(x_1, \ldots, x_n).$$

Now, by induction,

$$
\begin{aligned}
&D_{w^*_{\sigma(2)}}\cdots D_{w^*_{\sigma(\rho_n)}} f(x_1,\ldots,x_n)\\
&= d^{(\beta_1,\ldots,\beta_i-1,\beta_{i+1},\ldots,\beta_n)} f(x_1,\ldots,x_n,w_1,\ldots,w_{\xi_i},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{\rho_i},\ldots,w_{\rho_n})\\
&= D_{w^*_{\rho_1}}\cdots D_{w^*_{\xi_{i-1}}} h(x_1,\ldots,x_n).
\end{aligned}
$$

Let ψ denote

$$d^{(\beta_2,\ldots,\beta_n)} f_\bullet(x_2,\ldots,x_n,w_{\xi_2},\ldots,w_{\xi_i},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{\rho_i},w_{\sigma(1)},w_{\xi_{i+1}},\ldots,w_{\rho_n}).$$

By the preceding, we can apply, $D_{w^*_{\sigma(1)}}$, i.e., $D_{w^*_{\sigma(1)}}\cdots D_{w^*_{\sigma(\rho_n)}} f(x_1,\ldots,x_n)$ exists and coincides with

$$
\begin{aligned}
&D_{w^*_{\rho_1}}\cdots D_{w^*_{\xi_{i-1}}} D_{w^*_{\sigma(1)}} h(x_1,\ldots,x_n)\\
&= d^{\beta} f(x_1,\ldots,x_n,w_1,\ldots,w_{\xi_i},\ldots,w_{\sigma(1)-1},w_{\sigma(1)+1},\ldots,w_{\rho_i},w_{\sigma(1)},w_{\xi_{i+1}},\ldots,w_{\rho_n})\\
&= d^{\beta_1}\psi(x_1,w_1,\ldots,w_{\rho_1})
\end{aligned}
$$

where $d^{(\beta_2,\ldots,\beta_n)} f_{x_1}(x_2,\ldots,x_n,v_{\xi_2},v_{\xi_2+1},\ldots,v_{\rho_n})$ is symmetric in $v_i,\ldots,v_{\rho_i}\in E_i$ by induction on n for the $C^{\alpha_2,\ldots,\alpha_n}$-function f_{x_1}. Hence also after differentiations in x_1, $d^{\beta_1}\psi(x_1,w_1,\ldots,w_{\rho_1})$ coincides with $d^\beta f(x_1,\ldots,x_n,w_1,\ldots,w_{\rho_n})$. □

Corollary 70. *Let $E_1,\ldots,E_n$ and F be locally convex spaces, $U_i\subseteq E_i$ be locally convex subset with dense interior for $i\in\{1,\ldots,n\}$, $\alpha:=(\alpha_1,\ldots,\alpha_{n-1},1)$ such that $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$ and $\sigma\in S_n$. Define $\alpha_\sigma:=(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)})$. If $f\colon U_1\times\cdots\times U_n\to F$ is a C^α-map, then*

$$g_\sigma : U_{\sigma(1)}\times\cdots\times U_{\sigma(n)}\to F,\ x\mapsto f(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n)}) \tag{4.9}$$

is C^{α_σ} and

$$d^{\beta_\sigma} g(x_{\sigma(1)},\ldots,x_{\sigma(n)},w_{\sigma(1)},\ldots,w_{\sigma(n)}) = d^\beta f(x_1,\ldots,x_n,w_1,\ldots,w_n), \tag{4.10}$$

for all $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{N}_0^n$ such that $\beta\le\alpha$ and $(x_1,\ldots,x_n,w_1,\ldots,w_n)\in U_1\times\cdots\times U_n\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}$ where $\beta_\sigma:=(\beta_{\sigma_{(1)}},\ldots,\beta_{\sigma_{(n)}})$.

Lemma 71. *Let $E_1,\ldots,$ E_n and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i\in\{1,\ldots,n\}$ and $\alpha:=(\alpha_1,\ldots,\alpha_n)$ such that $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$. If $f:U_1\times\cdots\times U_n\to F$ is a C^α-map and $\lambda:F\to H$ is a continuous linear map to a locally convex space H, then $\lambda\circ f$ is C^α and $d^\beta(\lambda\circ f)=\lambda\circ d^\beta f$ for all $\beta_i\in\mathbb{N}_0$ such that $\beta_i\le\alpha_i$ and $\beta:=(\beta_1,\ldots,\beta_n)$.*

Proof. Follows from the fact that directional derivatives and continuous linear maps can be interchanged. □

Lemma 72. (Mappings to products for C^α-mappings). *Let $E_1, \ldots, E_n$ be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1, \ldots, n\}$, and $(F_j)_{j\in J}$ be a family of locally convex spaces with direct product $F := \prod_{j\in J} F_j$ and the projections $\pi_j : F \to F_j$ onto the components. Let $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and $f : U_1 \times \cdots \times U_n \to F$ be a map. Then f is C^α if and only if all of its components $f_j := \pi_j \circ f$ are C^α. In this case*

$$d^\beta f = (d^\beta f_j)_{j\in J}, \tag{4.11}$$

for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$.

Proof. π_j is continuous linear. Hence if f is C^α, then $f_j = \pi_j \circ f$ is C^α, by Lemma 71, with $d^\beta f_j = \pi_j \circ d^\beta f$. Hence (4.11) holds. Conversely, assume that each f_j is C^α. Because the limits in products can be formed component-wise, we see that for all $(x_1, \ldots, x_n) \in U_1^0 \times \cdots \times U_n^0$, $w_i := ((w_i)_1, \ldots, (w_i)_{\beta_i})$ such that $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$,

$$d^\beta f(x_1, \ldots, x_n, w_1, \ldots, w_n) = (\breve{D}_1 \cdots \breve{D}_n f)(x_1, \ldots, x_n)$$

exists and is given by

$$(d^\beta f_j(x_1, \ldots, x_n, w_1, \ldots, w_n))_{j\in J}. \tag{4.12}$$

Now (4.12) defines a continuous function

$$U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F.$$

Hence f is C^α. □

Lemma 73. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, $\alpha_n \geq 1$. If $f : U_1 \times \cdots \times U_n \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$, f is $C^{(0,\ldots,0,1)}$ and $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F$ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$, then f is C^α.*

Proof. Let $\beta_i \in \mathbb{N}_0$ with $\beta_i \leq \alpha_i$, $x := (x_1, \ldots, x_n) \in U_1^0 \times \cdots \times U_n^0$, $w_i := ((w_i)_1, \ldots, (w_i)_{\beta_i})$ where $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$. Consider also the corresponding elements $(w_i)_1^*, \ldots, (w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$.
If $\beta_n = 0$, then $(\breve{D}_1 \cdots \breve{D}_{n-1} f)(x)$ exists as f is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$, and is given by

$$d^{(\beta_1,\ldots,\beta_{n-1},0)} f(x, w_1, \ldots, w_{n-1})$$

which extends continuously to $U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}}$.
If $\beta_n > 0$, then $D_{(w_n)_1^*} f(x) = d^{(0,\dots,0,1)} f(x,(w_n)_1)$ exists because f is $C^{(0,\dots,0,1)}$ and because this function is $C^{(\alpha_1,\dots,\alpha_{n-1},\alpha_n-1)}$, also the directional derivatives

$$\begin{aligned}&(\breve{D}_1 \cdots \breve{D}_n f)(x)\\ &= (D_{((w_1)_{\beta_1}^*,0)} \cdots D_{((w_{n-1})_1^*,0)} D_{((w_n)_{\beta_n}^*,0)} \cdots D_{((w_n)_2^*,0)} d^{(0,\dots,0,1)} f)(x,(w_n)_1)\end{aligned}$$

exist and the right-hand side extends continuously to $(x,(w_1)_1,\dots,(w_n)_{\beta_n}) \in U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$. Hence f is C^α. □

Lemma 74. *Let $E_1,\dots,E_n$, $H_1,\dots,H_n$ and F be locally convex spaces, $U_i \subseteq E_i$, $P_i \subseteq H_i$ be locally convex subsets with dense interior for $i \in \{1,\dots,n\}$ and $\alpha := (\alpha_1,\dots,\alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, if $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map and $\lambda_i : H_i \to E_i$ is a continuous linear map such that $\lambda_i(P_i) \subseteq U_i$, then $f \circ (\lambda_1 \times \cdots \times \lambda_n)|_{P_1 \times \cdots \times P_n}$: $P_1 \times \cdots \times P_n \to F$ is C^α.*

Proof. Let $\beta_i \in \mathbb{N}_0$ such that $\beta_i \le \alpha_i$ and $\beta := (\beta_1,\dots,\beta_n)$. For $(p_1,\dots,p_n) \in P_1^0 \times \cdots \times P_n^0$, $(w_i)_1,\dots,(w_i)_{\beta_i} \in H_i$ and corresponding elements $(w_i)_1^*,\dots,(w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times H_i \times (\{0\})^{n-i} \subseteq H_1 \times \cdots \times H_n$, we have

$$\begin{aligned}&D_{(w_n)_1^*}(f \circ (\lambda_1 \times \cdots \times \lambda_n))(p_1,\dots,p_n)\\ &= \lim_{t\to 0} \frac{1}{t}(f(\lambda_1(p_1),\dots,\lambda_{n-1}(p_{n-1}),\lambda_n(p_n + t(w_n)_1)) - f(\lambda_1(p_1),\dots,\lambda_n(p_n)))\\ &= \lim_{t\to 0} \frac{1}{t}(f(\lambda_1(p_1),\dots,\lambda_{n-1}(p_{n-1}),\lambda_n(p_n) + t\lambda_n((w_n)_1)) - f(\lambda_1(p_1),\dots,\lambda_n(p_n)))\\ &= D_{(0,\dots,0,\lambda_n((w_n)_1))} f)(\lambda_1(p_1),\dots,\lambda_n(p_n)),\end{aligned}$$

arguing as in the proof of Lemma 37. Recursively,

$$\begin{aligned}&\breve{D}_1 \cdots \breve{D}_n (f \circ (\lambda_1 \times \cdots \times \lambda_n))(p_1,\dots,p_n)\\ &= d^\beta f(\lambda_1(p_1),\dots,\lambda_n(p_n),\lambda_1((w_1)_1),\dots,\lambda_n((w_n)_{\beta_n})).\end{aligned}$$

The right-hand side defines a continuous function of $(p_1,\dots,p_n,(w_1)_1,\dots,(w_n)_{\beta_n}) \in P_1 \times \cdots \times P_n \times H_1^{\beta_1} \times \cdots \times H_n^{\beta_n}$. Hence the assertion follows. □

Lemma 75. *Let $E_1,\dots,E_n$, $H_1,\dots,H_m$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1,\dots,n\}$, $\alpha := (\alpha_1,\dots,\alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, $H := H_1 \times \cdots \times H_m$ and $f : U_1 \times \cdots \times U_n \times H \to F$ be a map with the following properties:*

(a) *For all $x := (x_1,\dots,x_n)$, $x_i \in U_i$, the map $f(x,\bullet) : H \to F$ is m-linear;*

(b) *The map* $f : U_1 \times \cdots \times U_n \times H \to F$ *is* $C^{(\alpha,0)}$.

Then $f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \to F$ *is* C^α. *Also* $g : U_1 \times \cdots \times U_{i-1} \times (U_i \times H) \times U_{i+1} \times \cdots \times U_n \to F$, $(x_1, \ldots, x_{i-1}, (x_i, h), x_{i+1}, \ldots, x_n) \mapsto f(x,h)$ *is* C^α.

Proof. Holding $h \in H$ fixed, the map $f(\bullet, h)$ is C^α and hence, for a permutation $\sigma \in S_n$ of $\{1, \ldots, n\}$, we have $U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} \to F$, $(x_1, \ldots, x_n) \mapsto f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}, h)$ is C^{α_σ}, where $\alpha_\sigma := \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}$, by Corollary 70. Using (4.10), we see that $f_1 : U_{\sigma(1)} \times \cdots \times U_{\sigma(n-1)} \times (U_{\sigma(n)} \times H) \to F$, $f_1(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, h) := f(x,h)$ satisfies hypotheses analogous to those for f (with $\alpha_{\sigma(i)}$ interchanged) and will be C^{α_σ} if the first assertion holds. Using Corollary 70, this implies that g is C^α. Hence we only need to prove the first assertion. We may assume that $\alpha_i < \infty$; the proof is by induction on α_n.

The case $\alpha_n = 0$. Then f is $C^{\alpha_1, \ldots, \alpha_{n-1}, 0}$ by the hypotheses.

Induction step. Let $(w_n)_1 \in E_n$, $z = (z_1, \ldots, z_m) \in H$. By hypothesis, $D_{(0,\ldots,0,(w_n)_1,0)} f(x,h)$ exists and extends to a continuous map $U_1 \times \cdots \times U_n \times H \times E_n \to F$. Because $f(x, \bullet) : H \to F$ is continuous and m-linear, it is C^1 with

$$D_{(0,\ldots,0,z)} f(x,h) = \sum_{k=1}^{m} f(x, h_1, \ldots, h_{k-1}, z_k, h_{k+1}, \ldots, h_m).$$

This formula defines a continuous function $U_1 \times \cdots \times U_n \times H \times E_n \to F$. Holding $(x_1, \ldots, x_{n-1}) \in U_1 \times \cdots \times U_{n-1}$ fixed, we deduce with the Rule on Partial Differentials (Lemma 8) that the map $U_n \times H \to F, (x_n, h) \mapsto f(x,h)$ is C^1, with

$$\begin{aligned} &D_{(0,\ldots,0,(w_n)_1,z)} f(x,h) \\ &= D_{(0,\ldots,0,(w_n)_1,0)} f(x,h) + \sum_{k=1}^{m} f(x, h_1, \ldots, h_{k-1}, z_k, h_{k+1}, \ldots, h_m). \end{aligned} \tag{4.13}$$

Because we have just seen that $d^{(0,\ldots 0,1)} f(x_1, \ldots, x_{n-1}, (x_n, h), ((w_n)_1, z))$ exists and is given by (4.13), which extends continuously to $U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \times (E_n \times H)$, $f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \to F$ is $C^{0,\ldots,0,1}$. Also, $f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \to F$ is $C^{\alpha_1, \ldots, \alpha_{n-1}, 0}$ by the hypothesis.

We claim that $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H \times E_n \times H) \to F$ is $C^{\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1}$. If this is true, then f is C^α by Lemma 73. To prove the claim, for fixed $k \in \{1, \ldots, m\}$, consider

$$\phi : U_1 \times \cdots \times U_{n-1} \times (U_n \times H \times E_n \times H) \to F,$$

$$(x, h, (w_n)_1, z) \mapsto f(x, h_1, \ldots, h_{k-1}, z_k, h_{k+1}, \ldots, h_m).$$

The map

$$\psi : U_1 \times \cdots \times U_n \times H_1 \times \cdots \times H_{m-1} \times (H_m \times E_n \times H) \to F,$$

$$(x, h_1, \ldots, h_{m-1}, (h_m, (w_n)_1, z)) \mapsto f(x, h_1, \ldots, h_m)$$

is m-linear in $(h_1, \ldots, h_{n-1}, (h_n, (w_n)_1, z))$. By induction, ψ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$ as a map on $U_1 \times \cdots \times U_{n-1} \times (U_n \times H_1 \times \cdots \times H_m \times E_n \times H)$. By Lemma 74, also ϕ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$. Hence each of the final k summands in (4.13) is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$ in $(x, h_1, \ldots, h_{m-1}, (h_m, (w_n)_1, z))$. To take care of the first summands in (4.13), observe that $\theta\colon U_1 \times \cdots \times U_n \times (H \times E_n) \to F$, $(x, h, (w_n)_1) \mapsto D_{(0,\ldots,0,(w_n)_1,0)} f(x, h)$ is $(m+1)$-linear in the final argument and satisfies hypotheses analogous to those of f, with $(\alpha_1, \ldots, \alpha_n)$ replaced by $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1)$. Hence θ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$ on $U_1 \times \cdots \times U_{n-1} \times (U_n \times H \times E_n)$, and hence the first summand of (4.13) is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$ on $U_1 \times \cdots \times U_{n-1} \times (U_n \times H \times E_n \times H)$, by Lemma 74. As a consequence, $d^{(0,\ldots,0,1)} f$ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$ (like each of the summands in (4.13)). □

Lemma 76. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, $\alpha_n \geq 1$. Then $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map if and only if f is $C^{\alpha_1,\ldots,\alpha_{n-1},0}$, f is $C^{0,\ldots,0,1}$ and $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F$ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$.*

Proof. If f is C^α, then f is $C^{\alpha_1,\ldots,\alpha_{n-1},0}$ and f is $C^{0,\ldots,0,1}$. Moreover $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_n \times E_n \to F$ is linear in the E_n-variable and for all $\beta_i \in \mathbb{N}_0$, $\beta_i \leq \alpha_i$, $\beta_n \leq \alpha_n - 1$, $(x_1, \ldots, x_n) \in U_1^0 \times \cdots \times U_n^0$, $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$ and corresponding elements $(w_i)_1^*, \ldots, (w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$,

$$\begin{aligned}&D_{((w_1)_{\beta_1}^*,0)} \cdots D_{((w_n)_1^*,0)} (d^{(0,\ldots,0,1)} f)(x_1, \ldots, x_n, z) \\ &= d^{(\beta_1,\ldots,\beta_{n-1},\beta_n+1)} f(x_1, \ldots, x_n, (w_1)_1, \ldots, (w_{n-1})_{\beta_{n-1}}, z, (w_n)_1, \ldots, (w_n)_{\beta_n})\end{aligned}$$

exists and extends to a continuous function in

$$(x_1, \ldots, x_n, (w_1)_1, \ldots, (w_{n-1})_{\beta_{n-1}}, z, (w_n)_1, \ldots, (w_n)_{\beta_n}) \in U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}.$$

Hence, by Lemma 75, $d^{(0,\ldots,0,1)} f$ is $C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$. The converse has already been established in Lemma 73. □

Lemma 77. *For $i \in \{1, \ldots, n\}$, let E_i be a locally convex space, $U_i \subseteq E_i$ be a locally convex subset with dense interior and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Assume* *that $E_n = H_1 \times H_2$ with locally convex spaces H_1, H_2 and $U_n = V \times W$ with locally convex subsets $V \subseteq H_1$ and $W \subseteq H_2$ with dense interior. Let F be a locally convex space. If a map $f : U_1 \times \cdots \times U_{n-1} \times V \times W \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},k,l)}$ for all $k, l \in \mathbb{N}_0$ with $k + l \leq \alpha_n$, then $f : U_1 \times \cdots \times U_n \to F$ is C^α.*

Proof. We may assume that $\alpha_n < \infty$. The proof is by induction on α_n. For the case $\alpha_n = 0$, the assertion follows by the definition of a C^α-map. For the case $\alpha_n > 0$, let $x := (x_1, \ldots, x_n) \in U_1 \times \ldots \times U_n$ and $(h_1, h_2), (h'_1, h'_2) \in H_1 \times H_2$. By the Rule on Partial Differentials (Proposition 9),

$$d^{(0,\ldots,0,1)} f(x, (h_1, h_2)) = d^{(0,\ldots,0,1,0)} f(x, h_1) + d^{(0,\ldots,0,1)} f(x, h_2). \tag{4.14}$$

By Lemmas 76 and 74, (4.14) is $C^{(\alpha_1,\ldots,\alpha_{n-1},k,l)}$ as a map on $U_1 \times \cdots \times U_{n-1} \times (V \times H_1) \times (W \times H_2)$ for all $k + l \leq \alpha_{n-1}$, hence by induction and again by Lemma 76, (4.14) is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$ on $U_1 \times \cdots \times (U_n \times E_n)$. Thus, $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_n \times E_n \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$ and by induction $f : U_1 \times \cdots \times U_n \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$. Hence, it is C^α, by Lemma 76.

□

Lemma 78. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1, \ldots, n\}$ and $\alpha_0 \in \mathbb{N}_0$. If the map $f : U_1 \times \cdots \times U_n \to F$ is $C^{\alpha_0,\ldots,\alpha_0}$, then f is C^{α_0}.*

Proof. The proof is by induction on α_0. *The case* $\alpha_0 = 0$. If f is $C^{0,\ldots,0}$, then f is continuous and hence C^0. *The case* $\alpha_0 \geq 1$. Assume that $U_1, \ldots, U_n$ are open subsets. Then $D_{(w_i)^*} f(x_1, \ldots, x_n)$ exists and is continuous in $(x_1, \ldots, x_n, w_i)$ for all $x_i \in U_i$ and all $i \in \{1, \ldots, n\}$, where $w_i \in E_i$, $(w_i)^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$. Hence, by Proposition 9, f is C^1 and

$$df((x_1, \ldots, x_n), (w_1, \ldots, w_n)) = D_{(w_1)^*} f(x_1, \ldots, x_n) + \cdots + D_{(w_n)^*} f(x_1, \ldots, x_n). \tag{4.15}$$

Next note that $D_{(w_i)^*} f(x_1, \ldots, x_n)$ is $C^{\alpha_0-1,\ldots,\alpha_0-1}$-mappings, by Lemma 76 and Corollary 70. Hence df is C^{α_0-1}, by induction. Since f is C^1 and df is C^{α_0-1}, then f is C^{α_0}. □

As an immediate consequence of Lemma 78, we obtain:

Remark 79. The map $f : U_1 \times \cdots \times U_n \to F$ is smooth if and only if it is $C^{\infty,\ldots,\infty}$.

Proposition 80. *Let $E_1, \ldots, E_n$ be finite-dimensional vector spaces and F be a locally convex space. For $i \in \{1, \ldots, n\}$, let U_i be a locally convex and locally compact subset with dense interior of E_i and $\alpha := (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Then the evaluation map*

$$\varepsilon : C^\alpha(U_1 \times \cdots \times U_n, F) \times U_1 \times \cdots \times U_n \to F,\ \varepsilon(\gamma, x_1, \ldots, x_n) := \gamma(x_1, \ldots, x_n)$$

is $C^{\infty,\alpha}$.

Proof. Without loss of generality, we may assume up to permutation that $\alpha_i < \infty$ for all $i \in \{1, \ldots, n\}$. The proof is by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\alpha = 0$, then ε is continuous because each U_i is locally compact [11, Theorem 3.4.3]. Also, ε is linear in the first argument. Hence ε is $C^{\infty,0,\ldots,0}$, by Lemma 75 and Corollary 70. If $\alpha \neq 0$, we may assume that $\alpha_n \geq 1$, using Corollary 70. For $x_i \in U_i^0$, $w \in E_n$, $\gamma \in C^\alpha(U_1 \times \cdots \times U_n, F)$ and small $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}&\frac{1}{t}(\varepsilon(\gamma, x_1, \ldots, x_{n-1}, x_n + tw) - \varepsilon(\gamma, x_1, \ldots, x_n))\\ &= \frac{1}{t}(\gamma(x_1, \ldots, x_{n-1}, x_n + tw) - \gamma(x_1, \ldots, x_n)) \to d^{(0,\ldots,0,1)}\gamma(x_1, \ldots, x_n, w) \text{ as } t \to 0.\end{aligned}$$

Hence $d^{(0,\ldots,0,1)}\varepsilon(\gamma, x_1, \ldots, x_n, w)$ exists and is given by

$$d^{(0,\ldots,0,1)}\varepsilon(\gamma, x_1, \ldots, x_n, w) = d^{(0,\ldots,0,1)}\gamma(x_1, \ldots, x_n, w) = \varepsilon_1(d^{(0,\ldots,0,1)}\gamma, (x_1, \ldots, x_n, w)), \tag{4.16}$$

where $\varepsilon_1 : C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}(U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n), F) \times (U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n)) \to F$, $(\zeta, x_1, \ldots, x_{n-1}, z) \mapsto \zeta(x_1, \ldots, x_{n-1}, z)$ is $C^{\infty,\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$, by induction.
The right-hand side of (4.16) defines a continuous map (indeed a $C^{\infty,\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}$-map) by induction and Lemma 74, using that

$$C^\alpha(U_1 \times \cdots \times U_n, F) \to C^{\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1}(U_1 \times \cdots \times U_{n-1} \times U_n \times E, F),\ \gamma \mapsto d^{(0,\ldots,0,1)}\gamma$$

is continuous linear. Thus, by Lemma 76, ε is $C^{\infty,\alpha}$. □

4.2 Chain Rule for C^α-mappings

Lemma 81. (Chain Rule for C^α-mappings). *For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$, let E_i, $X_{i,j}$ and F be locally convex spaces, $U_i \subseteq E_i$, $P_{i,j} \subseteq X_{i,j}$ be locally convex subsets with dense interior, $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0 \cup \{\infty\})^n$, $f : U_1 \times \cdots \times U_n \to F$ be a C^α-map and $g_i : P_{i,1} \times P_{i,2} \times \cdots \times P_{i,m_i} \to U_i$ be a C^{γ_i}-map, where $\gamma_i := (\gamma_{i,1}, \ldots, \gamma_{i,m_i}) \in (\mathbb{N}_0 \cup \{\infty\})^{m_i}$, $|\gamma_i| := \gamma_{i,1} + \cdots + \gamma_{i,m_i} \leq \alpha_i$. Then*

$$f \circ (g_1 \times \cdots \times g_n) : (P_{1,1} \times \cdots \times P_{1,m_1}) \times \cdots \times (P_{n,1} \times \cdots \times P_{n,m_n}) \to F,$$

$$(p_{1,1}, \ldots, p_{n,m_n}) \mapsto f(g_1(p_{1,1}, \ldots, p_{1,m_1}), \ldots, g_n(p_{n,1}, \ldots, p_{n,m_n}))$$

is a $C^{(\gamma_1,\ldots,\gamma_n)}$*-map.*

Proof. Without loss of generality, we may assume that $\gamma_i < \infty$. The proof is by induction on $|\gamma| := |\gamma_1| + \cdots + |\gamma_n|$. If $|\gamma| = 0$, then $f \circ (g_1 \times \cdots \times g_n)$ is just a composition of continuous maps, which is continuous, hence $C^{(0,\ldots,0)}$. Now if $|\gamma| > 0$, by Corollary 70, we may assume that $|\gamma_n| > 0$. Again by Corollary 70, we may assume that $\gamma_{n,m_n} > 0$. Let $P_i := P_{i,1} \times \cdots \times P_{i,m_i}$, for $p := (p_1, \cdots, p_n) \in P_1 \times \cdots \times P_n$ and $z \in X_{n,m_n}$, the map $d^{(0,\ldots,0,1)} g_n(p_n, z)$ is $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$, by Lemma 76. Also, the function

$$P_{n,1} \times \cdots \times P_{n,m_n-1} \times (P_{n,m_n} \times X_{n,m_n}) \to U_n,$$

$$(p_{n,1}, \ldots, p_{n,m_n-1}, (p_{n,m_n}, z)) \mapsto g_n(p_{n,1}, \ldots, p_{n,m_n-1}, p_{n,m_n})$$

is C^{γ_n}, by Lemma 75. In particular, the latter is $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$. Thus both components of

$$\varphi : P_{n,1} \times \cdots \times (P_{n,m_n} \times X_{n,m_n} \to U_n) \times E_n,\ (p_n, h) \mapsto (g_n(p_n), d^{(0,\ldots,0,1)} g_n(p_n, z))$$

are $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$, so φ is $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_i-1},\gamma_{n,m}-1)}$. By Lemma 76, $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$. Thus, by the preceding, the map $d^{(0,\ldots,0,1)}(f \circ (g_1 \times \cdots \times g_{n-1} \times \varphi))(p_1, \ldots, p_{n-1}, (p_n, z))$ is $C^{(\gamma_1,\ldots,\gamma_{n-1},\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$. Hence,

$$d^{(0,\ldots,0,1)}(f \circ (g_1 \times \cdots \times g_n))(p, z) = (d^{(0,\ldots,0,1)} f)((g_1 \times \cdots \times g_n), d^{(0,\ldots,0,1)} g_n(p_n, z))$$

is $C^{(\gamma_1,\ldots,\gamma_{n-1},\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$ and by induction, $f \circ (g_1 \times \cdots \times g_n) : (P_{1,1} \times \cdots \times P_{1,m_1}) \times \cdots \times (P_{n,1} \times \cdots \times P_{n,m_n}) \to F$ is $C^{(\gamma_1,\ldots,\gamma_{n-1},\gamma_{n,1},\ldots,\gamma_{n,m_n-1},0)}$. Hence, by Lemma 76, $f \circ (g_1 \times \cdots \times g_n)$ is a $C^{(\gamma_1,\ldots,\gamma_n)}$-map. □

4.3 The Exponential Law for C^α-mappings

Definition 82. Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Give $C^\alpha(U_1 \times \cdots \times U_n, F)$ the initial topology with respect to the mappings $d^\beta : C^\alpha(U_1 \times \cdots \times U_n, F) \to C(U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)$, $\gamma \mapsto d^\beta\gamma$ for $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$, where the right-hand side is equipped with the compact-open topology.

Lemma 83. *Let F and E_i be locally convex spaces for $i \in \{1, \dots, n\}$, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and $\beta_i \in \mathbb{N}_0$ with $\beta_i \leq \alpha_i$. Define $U := U_1 \times \cdots \times U_n$ and $\beta := (\beta_1, \beta_2, \dots, \beta_n)$. Then the sets of the form*

$$W = \{f \in C^\alpha(U, F) \colon d^\beta f(K) \subseteq P\}$$

(where $P \subseteq F$ are 0-neighbourhoods and $K \subseteq U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$ is compact), form a subbasis of 0-neighbourhoods for $C^\alpha(U, F)$, i.e., finite intersections of such sets form a basis of 0-neighbourhoods.

Proof. The topology on $C^\alpha(U, F)$ is initial with respect to the maps

$$d^\beta \colon C^\alpha(U, F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)_{c.o}, \; f \mapsto d^\beta f.$$

Therefore the map

$$\Psi \colon C^\alpha(U, F) \to \prod_{\mathbb{N}_0 \ni \beta_i \leq \alpha_i} C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F), \; f \mapsto (d^\beta f)_{\mathbb{N}_0 \ni \beta_i \leq \alpha_i}$$

is a topological embedding. Sets of the form

$$B := \{(g_\beta)_{\mathbb{N}_0 \ni \beta_i \leq \alpha_i} \in \prod_{\mathbb{N}_0 \ni \beta_i \leq \alpha_i} C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F) : \; g_\beta(K_\beta) \subseteq Q_\beta\}$$

(with compact sets $K_\beta \subseteq U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$ and 0-neighbourhoods $Q_\beta \subseteq F$), form a basis of 0-neighbourhoods in $\prod_{\mathbb{N}_0 \ni \beta_i \leq \alpha_i} C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)$, where $Q_\beta = F$ for all but finitely many β. Hence the sets $\Psi^{-1}(W)$ form a basis of 0-neighbourhoods in $C^\alpha(U, F)$. These are finite intersections 0-neighbourhoods as described in the lemma, whence the latter for a subbasis. $\square$

Lemma 84. *For $i \in \{1, \dots, n\}$, let E_i and X be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $F \subseteq X$ be a (sequentially) closed vector subspace and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ with $\alpha := (\alpha_1, \dots, \alpha_n)$. Then $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map if and only if $f : U_1 \times \cdots \times U_n \to X$ is a C^α-map.*

Proof. The inclusion map $j : F \to X$ is continuous linear and hence smooth. If $f : U_1 \times \cdots \times U_n \to F$ is C^α, then also $j \circ f$ is C^α, by the Chain Rule (Lemma 81). Conversely, assume that $j \circ f : U_1 \times \cdots \times U_n \to X$ is C^α.
Step 1. Assume that $U_1 \times \cdots \times U_n$ are open sets. Because directional derivatives can be realized as limits of sequnces of directional difference quotiants, which lie in F, we obtain

$$D_{w^*} f(x) = D_{w^*}(j \circ f)(x) \in F$$

for all $x \in U_1 \times \cdots \times U_n$ and $w \in E_i$ such that $\alpha_i \geq 1$, where $w^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i}$ is as in Definition 65. Repeating this argument, we find that

$$d^\beta f(x, w_1, \dots, w_n) = d^\beta (j \circ f)(x, w_1, \dots, w_n) \in F$$

for all $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$, $x \in U_1 \times \cdots \times U_n$ and $w_i \in E_i^{\beta_i}$ for $i = 1, \dots, n$. Because $d^\beta(j \circ f)$ is continuous and $j \circ d^\beta f = d^\beta(j \circ f)$, also $d^\beta f$ and thus f is C^α.
Step 2. If $U_1, \dots, U_n$ are arbitrary, then $f|_{U_1^0 \times \cdots \times U_n^0}$ is C^α by Step 1, and

$$d^\beta (j \circ f)(x, w_1, \dots, w_n) \in F$$

for all $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$, $x \in U_1^0 \times \cdots \times U_n^0$ and $w_i \in E_i^{\beta_i}$ for $i = 1, \dots, n$. Given $x = (x_1, \dots, x_n) \in U_1^0 \times \cdots \times U_n^0$, let $V_i \subseteq U_i$ be a convex neighbourhood of x_i. Because U_i^0 is dense in U_i there exists $y_i \in V_i^0$. By convexity, $z_{i,m} := (1 - \frac{1}{m})x_i + \frac{1}{m} y_i \in V_i$ for all $m \in \mathbb{N}$, and indeed $z_{i,m} \in V_i^0$. Hence $z_m := (z_{1,m}, \dots, z_{n,m}) \in U_1^0 \times \cdots \times U_n^0$ for all m and thus

$$d^\beta (j \circ f)(z_m, w_1, \dots, w_n) \in F.$$

Since $z_m \to x$ as $m \to \infty$ and F is sequentially closed, we deduce that

$$d^\beta (j \circ f)(x, w_1, \dots, w_n) \in F$$

thus

$$d^\beta f := d^\beta (j \circ f) : U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F$$

is a continuous extension to $d^\beta(f|_{U_1 \times \cdots \times U_n})$, and thus f is C^α. $\square$

Lemma 85. *For $i \in \{1, \dots, n\}$, let E_i and F be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $F = \varprojlim F_j$ where F_j is a locally convex space with the limit maps $q_j : F \to F_j$ and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ with $\alpha := (\alpha_1, \dots, \alpha_n)$. Then $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map if and only if all the maps $f \circ q_j$ are C^α.*

Proof. After passing to an isomorphic locally convex space if necessary, we may assume that $F = \varprojlim F_j$ is realized as a closed vector subspace of $\prod_{j \in J} F_i$ (as usual). The asseration now follows from Lemma 72 and Lemma 84. $\square$

Lemma 86. *For $i \in \{1, \dots, n\}$, let E_i, F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i, and $(F_j)_{j \in J}$ be a family of locally convex spaces with direct product $F := \prod_{j \in J} F_j$. Let $\alpha := (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. If the topology on F is initial wit respect to the linear maps $\lambda_i : F \to F_i$. Then the topology on $C^\alpha(U, F)$ is initial wit respect to the maps $C^\alpha(U, \lambda_i)$.*

Proof. Abbreviate $U := U_1 \times \cdots \times U_n$. By definition, the topology on $C^\alpha(U,F)$ is initial with respect to the maps

$$d^\beta : C^\alpha(U,F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)_{c.o} \tag{4.17}$$

for all $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. By [18, Lemma 3.3], the compact-open topology on the space on the right-hand side in (4.17) is initial with respect to the maps

$$C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, \lambda) : C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F_i),$$

$$\gamma \mapsto \lambda_i \circ \gamma.$$

Hence by [18, Lemma B.4], the topology on $C^\alpha(U,F)$ is initial with respect to the maps

$$C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, \lambda_i) \circ d^\beta \tag{4.18}$$

Now the map in (4.18) coincides with $d^\beta \circ C^\alpha(U,\lambda_i)$ by the Chain Rule (Lemma 81), where

$$d^\beta : C^\alpha(U,F_i) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F_i). \tag{4.19}$$

The topology on $C^\alpha(U,F_i)$ being initial with respect to the d^β from (4.19), we deduce that the given topology on $C^\alpha(U,F)$ is initial wit respect to the maps $C^\alpha(U,\lambda_i)$, by [18, Lemma B.4]. □

Lemma 87. *Let $E_1,\ldots,E_n$ be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1,\ldots,n\}$ and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ with $\alpha := (\alpha_1,\ldots,\alpha_n)$. If $\lambda : F_1 \to F_2$ is a continuous linear map between locally convex spaces, then also*

$$C^\alpha(U_1 \times \cdots \times U_n, \lambda) : C^\alpha(U_1 \times \cdots \times U_n, F_1) \to C^\alpha(U_1 \times \cdots \times U_n, F_2),\ \gamma \mapsto \lambda \circ \gamma$$

is continuous linear.

Proof. Abbreviate $U := U_1 \times \cdots \times U_n$. The topology on F_1 is initial with respect to λ and $\mathrm{id}_{F_1} : F_1 \to F_1$. Hence by Lemmma 86, the topology on $C^\alpha(U,F_1)$ is initial with respect to $C^\alpha(U,\lambda)$ and $C^\alpha(U,\mathrm{id}_{F_1})$. In particular, $C^\alpha(U,\lambda)$ is continuous (and obviously it is linear). □

Lemma 88. *Let $E_1,\ldots,E_n$ be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1,\ldots,n\}$, $(F_j)_{j\in J}$ be a family of locally convex spaces with direct product $F := \prod_{j\in J} F_j$ and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ with $\alpha := (\alpha_1,\ldots,\alpha_n)$. Then*

$$C^\alpha(U_1 \times \cdots \times U_n, F) \cong \prod_{j\in J} C^\alpha(U_1 \times \cdots \times U_n, F_j).$$

Proof. Abbreviate $U := U_1 \times \cdots \times U_n$. Let $\mathrm{pr}_j : F \to F_j$ be the projection onto the $j - th$ component It follows from Lemma 72 that the map

$$\Phi : C^\alpha(U, \mathrm{pr}_j)_{j\in J} : C^\alpha(U, F) \to \prod_{j\in J} C^\alpha(U, F_j),\ \gamma \mapsto (\mathrm{pr}_j \circ \gamma)_{j\in J}$$

is a bijection.
Because the topology on F is initial with respect to the maps $\mathrm{pr}_j : F \to F_j$, Lemma 86 shows that the topology on $C^\alpha(U, F)$ is initial with respect to the maps $C^\alpha(U, \mathrm{pr}_j)$ for $j \in J$. Thus Φ is a homeomorphism. □

Lemma 89. *Let $E_1, \ldots, E_n, F$ be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for $i \in \{1, \ldots, n\}$, $(F_j)_{j\in J}$ be a family of locally convex spaces with $F := \varprojlim F_j$ and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ with $\alpha := (\alpha_1, \ldots, \alpha_n)$. Then*

$$C^\alpha(U_1 \times \cdots \times U_n, F) = \varprojlim C^\beta(U_1 \times \cdots \times U_n, F_j).$$

Proof. Abbreviate $U := U_1 \times \cdots \times U_n$. Let $q_l : F \to F_l$ be the limit maps and $q_{lj} : F_j \to F_l$ for $l \leq j$ be the bonding maps. We may assume that F is realized as a vector subspace of $\prod_{l\in L} F_l$ and $q_l := \mathrm{pr}_l|_F$. As a consequence of Lemma 85, the map

$$C^\alpha(U, F) \to \prod_{l\in L} C^\alpha(U, F_l),\ \gamma \mapsto (q_l \circ \gamma)_{l\in L}$$

co-restricts to a bijection

$$\Phi : C^\alpha(U, F) \to \varprojlim C^\alpha(U, F_l)$$

(Using the bonding maps $C^\alpha(U, q_{lj})$, $l \leq j$). Now Lemma 86 imply that Φ is a homeomorphism. □

Lemma 90. *For $i \in \{1, \ldots, n\}$, let E_i, F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ with $\alpha := (\alpha_1, \ldots, \alpha_n)$. Then*

$$C^\alpha(U_1 \times \cdots \times U_n, F) = \varprojlim_{\mathbb{N}_0^n \ni \beta \leq \alpha} C^\beta(U_1 \times \cdots \times U_n, F).$$

Proof. Abbreviate $U := U_1 \times \cdots \times U_n$. Since $C^\alpha(U, F) = \bigcap_{\mathbb{N}_0^n \ni \beta \leq \alpha} C^\beta(U, F)$, it is clear that $C^\alpha(U, F) = \varprojlim C^\beta(U, F)$ as a vector space, together with the inclusion maps $q_\beta : C^\alpha(U, F) \to C^\beta(U, F)$. Let τ be the initial topoloy $C^\alpha(U, F)$ with respect to the maps q_β. The topology on $C^\beta(U, F)$ being initial with respect to the maps

$d^\beta : C^\beta(U,F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)$. By [18, Lemma B.4], τ is also initial with respect to the maps

$$d^\beta \circ q_\beta : C^\alpha(U,F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F),\ \gamma \mapsto d^\beta \gamma,$$

and hence coincides with the compact-open C^α-topology on $C^\alpha(U,F)$. Hence $C^\alpha(U,F) = \varprojlim C^\beta(U,F)$ also as a topologicla vector space. $\square$

Lemma 91. *For $i \in \{1,\dots,n\}$, let E_i, F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ sucht that $\alpha_n \geq 1$ and $\alpha := (\alpha_1,\dots,\alpha_n)$. Abbreviate $U := U_1 \times \cdots \times U_n$. Then*

$$\Phi : C^\alpha(U,F) \to C^{(\alpha_1,\dots,\alpha_{n-1},0)}(U,F) \times C^{\alpha - e_n}(U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n), F),$$

$$f \mapsto (f, d^{(0,\dots,0,1)} f)$$

is a linear topological embedding with closed image.

Proof. The linearity is clear.
Because Φ is injective, it will be an embedding if we can show that the initial topology τ on $C^\alpha(U,F)$ with respect to Φ coincides with the compact-open C^α-topology ϑ. By transitivity of initial topologies, τ is initial with respect to the maps

$$d^\beta : C^\alpha(U,F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F),\ f \mapsto d^\beta f \tag{4.20}$$

for $\beta := (\beta_1,\dots,\beta_n) \in \mathbb{N}_0^n$ with $\beta_i \leq \alpha_i$ and $\beta_n = 0$ and the map

$$\varphi_\beta : C^\alpha(U,F) \to C(U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}} \times (E_n \times E_n)^{\beta_n}, F), \tag{4.21}$$

$$f \mapsto d^\beta(d^{e_n} f),$$

for $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha - e_n$.

To see that $\vartheta \subseteq \tau$, we show that τ makes the maps

$$d^\gamma : C^\alpha(U,F) \to C(U \times E_1^{\gamma_1} \times \cdots \times E_n^{\gamma_n}, F)$$

continuous for each $\gamma = (\gamma_1,\dots,\gamma_n) \in \mathbb{N}_0^n$ with $\gamma \leq \alpha$. If $\gamma_n = 0$, this is clear from (4.20), applied with $\beta := \gamma$. If $\gamma_n \geq 1$, define $\beta := \gamma - e_n$. Then φ_β from (4.21) is continuous . Also the map

$$h : U \times E_1^{\gamma_1} \times \cdots \times E_n^{\gamma_n} \to U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}} \times (E_n \times E_n)^{\beta_n},$$

$$(x_1, \ldots, x_n, w_1, \ldots, w_n) \mapsto$$

$$(x_1, \ldots, x_{n-1}, (x_n, w_{n,1}), w_1, \ldots w_{n-1}, (w_{n,2}, 0), \ldots, (w_{n,\gamma_n}, 0))$$

is continuos, where $w_n = (w_{n,1}, \ldots w_{n,\gamma_n})$. Note that

$$d^\gamma f(x, w_1, \ldots, w_n) = d^\beta(d^{e_n} f)(h(x, w_1, \ldots, w_n))$$

for $f \in C^\alpha(U, F)$. Thus $d^\gamma = C(h, F) \circ \varphi_\beta$, where

$$C(h, F) : C(U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}} \times (E_n \times E_n)^{\beta_n}, F) \to$$

$$C(U \times E_1^{\gamma_1} \times \cdots \times E_n^{\gamma_n}, F),$$

$$g \mapsto g \circ h$$

is continuous. Hence d^γ is continuous with respect to τ for all γ and thus $\vartheta \subseteq \tau$.

Also $\tau \subseteq \vartheta$ (and thus $\tau = \vartheta$): Becausw ϑ makes each of the maps d^β from (4.20) continuous , it only remains to show that ϑ makes φ_β freom (4.21) continuous for each $\beta \in \mathbb{N}_0^n$ such that $\beta \le \alpha - e_n$. This will follow from the formula

$$\varphi_\beta = C(h, F) \circ d^{\beta + e_n} + \sum_{j=1}^{\beta_n} C(h_j, F) \circ d^\beta \tag{4.22}$$

with $d^\beta : C^\alpha(U, F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)$,

$$d^{\beta+e_n} : C^\alpha(U, F) \to C(U \times E_1^{\beta_1} \times \cdots \times, E_{n-1}^{\beta_{n-1}}, E_n^{\beta_n+1}, F),$$

$$h : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}} \times (E_n \times E_n)^{\beta_n} \to$$

$$U \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}} \times E_n^{\beta_n+1},$$

$$(x_1, \ldots, x_{n-1}, (x_n, u_0), w_1, \ldots, w_{n-1}, (u_1, v_1), \ldots, (u_{\beta_n}, v_{\beta_n})) \mapsto$$

$$(x_1, \ldots, x_n, w_1, \ldots, w_{n-1}, (u_0, u_1, \ldots, u_{\beta_n}))$$

and

$$h_j : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}} \times (E_n \times E_n)^{\beta_n} \to U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n},$$

$$(x_1, \ldots, x_{n-1}, (x_n, u_0), w_1, \ldots, w_{n-1}, (u_1, v_1), \ldots, (u_{\beta_n}, v_{\beta_n})) \mapsto$$

$$(x_1, \ldots, x_n, w_1, \ldots, w_{n-1}, (u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_{\beta_n})).$$

It remains to prove (4.22). We first note that, because $d^{e_n} f(x_1, \ldots, x_n, u_0)$ is linear in u_0, we have

$$d^{e_n}(d^{e_n} f)(x_1, \ldots, x_{n-1}, (x_n.u_0), (0, u_1)) = d^{e_n} f(x_1, \ldots, x_n, u_1) \tag{4.23}$$

and hence

$$d^{2e_n}(d^{e_n} f)(x_1, \ldots, x_{n-1}, (x_n.u_0), (0, u_1), (0, u_2)) = 0. \tag{4.24}$$

We now write

$$((u_1, v_1), \ldots, (u_{\beta_n}, v_{\beta_n})) = ((u_1, 0) + (0, v_1), \ldots, (u_{\beta_n}, 0) + (0, v_{\beta_n}))$$

in the final argument of

$$d^\beta(d^{e_1} f)(x_1, \ldots, x_{n-1}, (x_n, u_0), w_1, \ldots, w_{n-1}, ((u_1, v_1), \ldots, (u_{\beta_n}, v_{\beta_n}))),$$

in which this function is symmetric β_n-linear. We expand into the sum of the 2^{β_n} corresponding contributions, omit the terms vanishing by (4.24) as they omit 2 or more contributions $(0, v_j)$, and rewrite those containing one contribution $(0, v_j)$ using (4.23). This gives (4.22).

The image of Φ is closed: Let $(g_i)_{i \in I}$ be a net in imΦ which converges, say to (f, g) with $f \in C^{(\alpha_1, \ldots, \alpha_{n-1}, 0)}(U, F)$ and $g \in C^{\alpha - e_n}(U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n), F)$. Let $f_i \in C^\alpha(U, F)$ with $\Phi(f_i) = g_i$. We claim that $d^{e_n} f(x, y)$ exists for $x = (x_1, \ldots, x_n) \in U^0$ and $y \in E_n$, and is given by

$$d^{e_n} f(x, y) = g(x_1, \ldots, x_{n-1}, (x_n, y)). \tag{4.25}$$

Because the righthand side of (4.25) makes sense and is a continuous also for $(x, y) \in U \times E_n$, we see that f is C^{e_n} with

$$d^{e_n} f = g \tag{4.26}$$

a $C^{\alpha - e_n}$-map. Hence f is C^α, by Lemma 77. Using (4.26), we see that $\Phi(f) = (f, d^{e_n} f) = (f, g)$, whence $(f, g) \in \mathrm{im}Phi$ and so imΦ is closed. It remains to verify the claim. Abbreviate $y^* := (0, y) \in E_1 \times \cdots \times E_n$. For fixed $t \in \mathbb{R} \setminus \{0\}$ such that $x + ty^* \in U^0$, the functions

$$[0, 1] \to F, s \mapsto d^{e_n} f_i(x + sty^*, y)$$

converge uniformly to $s \mapsto g(x + sty^*, y)$ (as $(x + [0, 1]ty^*) \times \{y\}$ is compact and $d^{e_n} f_i \to g$ uniformly on comact sets). The right-hand side of

$$f_i(x + ty^*) - f_i(x) = t \int_0^1 d^{e_n} f_i(x + sty^*, y)\, ds$$

therefore converges to $t\int_0^1 g(x+sty^*,y)\,ds$, and the Left-hand side converges to $f(x+ty^*)-f(x)$, which lies in F (whence also the weak integral exists in F, not only in a completion $\tilde{F}$). Thus

$$\frac{1}{t}f_i(x+ty^*)-f(x)=\int_0^1 g(x+sty,y)\,ds,$$

which converges to $\int_0^1 g(x,y)\,ds=g(x,y)$ as $t\to 0$, by continuity of $g(x+sty,y)$ in (s,t), the claim is established. □

Lemma 92. *Let $E_1,\dots,E_n$ and F be locally convex spaces, $U_i\subseteq E_i$ be a locally convex subset with dense interior for $i\in\{1,\dots,n\}$, $\alpha:=(\alpha_1,\dots,\alpha_n)$ such that $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$ and $\sigma\in S_n$ and $\alpha_\sigma:=(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(n)})$. If $f\colon U_1\times\cdots\times U_n\to F$ is a C^α-map, then the map*

$$\Phi_\sigma: C^\alpha(U_1\times\cdots\times U_n,F)\to C^{\alpha_\sigma}(U_{\sigma(1)}\times\cdots\times U_{\sigma(n)},F)$$

taking f to $\Phi_\sigma(f):=g$ as in (4.9) is an isomorphism of topological vector spaces.

Proof. For each $\beta\in\mathbb{N}_0^n$ such that $\beta\le\alpha$, we have $d^{\beta_\sigma}\circ\Phi_\sigma=C(h_\beta,F)\circ d^\beta$ by (4.10) where

$$h_\beta: U_{\sigma(1)}\times\cdots\times U_{\sigma(n)}\times E_{\sigma(1)}^{\beta_{\sigma(1)}}\times\cdots\times E_{\sigma(n)}^{\beta_{\sigma(n)}}\to U_1\times\cdots\times U_n\times E_1^{\beta_1}\times\cdots\times E_n^{\beta_n}$$

$$(x_{\sigma(1)},\dots,x_{\sigma(n)},w_{\sigma(1)},\dots,w_{\sigma(n)})\mapsto(x_1,\dots,x_n,w_1,\dots,w_n).$$

Since d^β and $C(h_\beta,F)$ are continuous, also $d^{\beta_\sigma}\circ\Phi_\sigma$ is continuous for each β as before, and hence Φ is continuous. The same argument show that $(\Phi_\sigma)^{-1}=\Phi_{\sigma^{-1}}$ is continuous. □

Lemma 93. *Let $\alpha:=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}_0\cup\{\infty\})^n$, $E_1,\dots,E_n$, $H_1,\dots,H_n$ and F be locally convex spaces, $U_i\subseteq E_i$, $P_i\subseteq H_i$ be locally convex subsets with dense interior for $i\in\{1,\dots,n\}$ and $g_i:P_i\to E_i$ be C^{α_i}-maps such that $g_i(P_i)\subseteq U_i$. Abbreviate $P:=P_1\times\cdots\times P_n$ and $U:=U_1\times\cdots\times U_n$. Then*

$$C^\alpha(g,F): C^\alpha(U,F)\to C^\alpha(P,F),\ f\mapsto f\circ g$$

is continuous and linear.

Proof. The linearity is clear. The topologies on $C^\alpha(P,F)$ and $C^\alpha(U,F)$ are initial with respect to the inclusion maps $i_\beta: C^\alpha(P,F)\to C^\beta(P,F)$ and $j_\beta: C^\alpha(U,F)\to C^\beta(U,F)$, respectively, for $\beta\in\mathbb{N}_0^n$ such that $\beta\le\alpha$. Since $i_\beta\circ C^\alpha(g,F)=C^\beta(g,F)\circ j_\beta$, it suffices

to show that each $C^\beta(g,F)$ is continuous. Hence $\alpha \in \mathbb{N}_0^n$ without loss of generality. Now the proof is by induction on $|\alpha|$.
$|\alpha| = 0$, then $C^\alpha(g,F) = C(g,F)$ is continuous.
Induction step: Assume that $|\alpha| \geq 1$. Thus $\alpha_j \geq 1$ for some $j \in \{1,\ldots,n\}$. Let $\sigma \in S_n$ be the permutation which interchanges j and n. Define α_σ,

$$\Phi_\sigma : C^\alpha(U,F) \to C^{\alpha_\sigma}(U_{\sigma(1)},\ldots,U_{\sigma(n)},F)$$

and an analogous isomorphism

$$\Psi_\sigma : C^\alpha(P,F) \to C^{\alpha_\sigma}(P_{\sigma(1)},\ldots,P_{\sigma(n)},F)$$

as in Lemma 92. Then

$$C^\alpha(g,F) = \Psi_\sigma^{-1} \circ C^{\alpha_\sigma}(g_{\sigma(1)} \times \cdots \times g_{\sigma(n)},F) \circ \Phi_\alpha,$$

and it suffices to show that $C^{\alpha_\sigma}(g_{\sigma(1)} \times \cdots \times g_{\sigma(n)},F)$ is continuous. Hence $\alpha_n \geq 1$ without loss of generality. By Lemma 91. $C^\alpha(g,F)$ will be continuous if $i_{\alpha-e_n} \circ C^\alpha(g,F)$ and $d^{e_n} \circ C^\alpha(g,F)$ are continuous (with $i_{\alpha-e_n}$ as at the beginning of the proof). Now $i_{\alpha-e_n} \circ C^\alpha(g,F) = C^{\alpha-e_n}(g,F) \circ j_{\alpha-e_n}$ is continuous by induction. Also for $f \in C^\alpha(P,F)$ and $(x_1,\ldots,x_{n-1},(x_n,y)) \in P_1 \times \cdots \times P_{n-1} \times (P_n \times H_n)$ we have

$$d^{e_n}(f \circ g)(x_1,\ldots,x_{n-1},(x_n,y)) = (d^{e_n} f)(g_1(x_1),\ldots,g_{n-1}(x_{n-1}),dg_n(x_n,y)),$$

i.e., $d^{e_n} \circ C^\alpha(g,F) = C^{\alpha-e_n}(g_1 \times \cdots \times g_{n-1} \times dg_n,F) \circ d^{e_n}$, which is continuous by induction. $\square$

Theorem 94. *Let F and E_i for $i \in \{1,\ldots,n\}$ be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \mathbb{N}$, $2 \leq j \leq n$, let $U := U_1 \times \cdots \times U_{j-1}$ and $V := U_j \times \cdots \times U_n$, $\gamma := (\alpha_1,\ldots,\alpha_{j-1})$ and $\eta := (\alpha_j,\ldots,\alpha_n)$. If $f\colon U \times V \to F$ is $C^{(\gamma,\eta)}$, then*

(a) *The map $f_x\colon V \to F, y \mapsto f(x,y)$ is C^η for each $x := (x_1,\ldots,x_{j-1}) \in U$.*
(b) *The map $f^\vee\colon U \to C^\eta(V,F)$, $x \mapsto f_x$ is C^γ.*
(c) *The mapping $\Phi\colon C^{(\gamma,\eta)}(U \times V,F) \to C^\gamma(U,C^\eta(V,F))$, $f \mapsto f^\vee$ is linear and a topological embedding.*

Proof.

(a) $\gamma_x\colon V \to F$ is C^η for all $x \in U$, by Lemma 67.

(b) We have

$$C^{\gamma}(U,C^{\eta}(V,F))=\varprojlim_{\substack{\zeta\in(\mathbb{N}_0)^{n-j+1},\\ \zeta\leq\eta}} C^{\gamma}(U,C^{\zeta}(V,F))$$

by Lemmas 89 and 90. It therefore suffices to prove the assertion when $\eta\in(\mathbb{N}_0)^{n-j+1}$ (cf. [4, Lemma 10.3]). We may assume that $\gamma\in(\mathbb{N}_0)^{n-j+1}$. The proof is by induction on $|\gamma|$.

The case $\gamma=0$. If $\eta=0$, then the assertion follows from [11, Theorem 3.4.1]. If $\eta\neq0$, the topology on $C^{\eta}(V,F)$ is initial with respect to the maps

$$d^{(\beta_j,\beta_{j+1},\dots,\beta_n)}\colon C^{\eta}(V,F)\to C(V\times E_j^{\beta_j}\times\cdots\times E_n^{\beta_n},F)_{c.o},\ g\mapsto d^{(\beta_j,\beta_{j+1},\dots,\beta_n)}g,$$

for $\beta_i\in\mathbb{N}_0$ such that $\beta_i\leq\alpha_i$. Hence, we only need that $d^{(\beta_j,\beta_{j+1},\dots,\beta_n)}\circ f^{\vee}\colon U\to C(V\times E_j^{\beta_j}\times\cdots\times E_n^{\beta_n},F)_{c.o}$ is continuous for $\beta_i\in\{0,1,\dots,\alpha_i\}$. Now

$$\begin{aligned}&d^{(\beta_j,\beta_{j+1},\dots,\beta_n)}(f^{\vee}(x))=d^{(\beta_j,\beta_{j+1},\dots,\beta_n)}(f(x,\bullet))\\&=d^{(0,\dots,0,\beta_j,\beta_{j+1},\dots,\beta_n)}f(x,\bullet)=(d^{(0,\dots,0,\beta_j,\beta_{j+1},\dots,\beta_n)}f)^{\vee}(x).\end{aligned}$$

Thus $d^{(\beta_j,\beta_{j+1},\dots,\beta_n)}\circ f^{\vee}=(d^{(0,\dots,0,\beta_j,\beta_{j+1},\dots,\beta_n)}f)^{\vee}\colon U\to C(V\times E_j^{\beta_j}\times\cdots\times E_n^{\beta_n},F)_{c.o}$, which is continuous by induction. As a consequence, $f^{\vee}\colon U\to C^{\eta}(V,F)$ is continuous.
The case $\gamma\neq0$. Using Corollary 70, we may assume that, $\alpha_{j-1}\neq0$. If $\eta=0$, then $f^{\vee}\colon U\to C(V,F)$. Let $x\in U^0:=U_1^0\times\cdots\times U_{j-1}^0,\ z\in E_{j-1}$ and $z^*\in(\{0\})^{j-2}\times E_{j-1}$ be the element with final component z. Since $U_{j-1}\times V\to F$, $(u,v)\mapsto f(x_1,\dots,x_{j-2},u,v)$ is $C^{1,0}$, the proof of Theorem 59 show that

$$\frac{1}{t}(f^{\vee}(x+tz^*)-f^{\vee}(x))\to d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$$

in $C(V,F)$ as $t\to0$. Thus $d^{(0,\dots,0,1)}f^{\vee}(x,z)$ exists for all $x\in U^0,\ z\in E_{j-1}$ and is given by $d^{(0,\dots,0,1)}f^{\vee}(x,z)=d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$.
Now $U\to C(V,F),x\mapsto d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$ is a continuous function in all of U (by $\gamma=0$); so $f^{\vee}$ is $C^{(0,\dots,0,1)}$ on U, and $d^{(0,\dots,0,1)}f^{\vee}(x,z)=d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$. Because

$$h\colon(U\times E_{j-1})\times V\to F,\ ((x,z),y)\mapsto d^{(0,\dots,0,1,0\dots,0)}f(x,y,z)$$

is $C^{(\alpha_1,\dots,\alpha_{j-2},\alpha_{j-1}-1,\eta)}$ (see Lemma 76 and Corollary 70), by induction $d^{(0,\dots,0,1)}(f^{\vee})=h^{\vee}\colon U\times E_{j-1}\to C(V,F)$ is $C^{(\alpha_1,\dots,\alpha_{j-2},\alpha_{j-1}-1)}$. Moreover, $f^{\vee}$ is $C^{(\alpha_1,\dots,\alpha_{j-2},\alpha_{j-1}-1)}$ by induction. Hence Lemma 76 and Corollary 70 show that $f^{\vee}$ is C^{γ}.
If $\eta\neq0$, again by Corollary 70, we may assume that $\alpha_n\neq0$. By Lemmas 87 and 91

$$C^{\eta}(V,F)\to C^{\eta-e_n}(V,F)\times C^{(\alpha_j,\dots,\alpha_{n-1},\alpha_n-1)}(V\times E_n,F),\ \varphi\mapsto(\varphi,d^{(0,\dots,0,1)}\varphi)$$

is a linear topological embedding with closed image, $f^\vee\colon U \to C^{\eta-e_n}(V,F)$ will be C^γ if $f^\vee\colon U \to C(V,F)$ is C^γ (which holds by induction) and the map

$$h\colon U \to C^{(\alpha_j,\ldots,\alpha_{n-1},\alpha_n-1)}(V\times E_n,F),\ x\mapsto d^{(0,\ldots,0,1)}(f^\vee(x))$$

is C^γ. (see Lemma 84). For $x\in U,\ y\in V$ and $z\in E_n$, we have

$$h(x)(y,z) = d^{(0,\ldots,0,1)}(f^\vee(x))(y,z) = d^{(0,\ldots,0,1)}(f(x,\bullet))(y,z) = d^{(0,\ldots,0,1)}f(x,y,z),$$

thus $h = (d^{(0,\ldots,0,1)}f)^\vee$ for $d^{(0,\ldots,0,1)}f\colon U\times(V\times E_n)\to F$.
This function is $C^{(\gamma,\alpha_j,\ldots,\alpha_{n-1},\alpha_n-1)}$, by Lemma 76. Hence h is C^γ, by induction.

(c) The linearity of Φ is clear. For $y\in V$, the point evaluation $\lambda\colon C^\eta(V,F)\to F,\ \psi\mapsto\psi(y)$ is continuous linear. Hence, for $\beta_i\in\mathbb{N}_0,\ \beta_i\le\alpha_i,\ x\in U$ and $w\in E_1^{\beta_1}\times\cdots\times E_{j-1}^{\beta_{j-1}}$,

$$\begin{aligned}(d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}f^\vee)(x,w)(y) &= \lambda((d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}f^\vee)(x,w))\\ &= d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}(\lambda\circ f^\vee)(x,w)\\ &= d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}(f(\bullet,y)(x,w))\\ &= d^{(\beta_1,\beta_2,\ldots,\beta_{j-1},0,\ldots,0)}f(x,y,w),\end{aligned}$$

using that $(\lambda\circ f^\vee)(x) = \lambda(f^\vee(x)) = f^\vee(x)(y) = f(x,y)$. Hence

$$(d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}f^\vee)(x,w) = (d^{(\beta_1,\beta_2,\ldots,\beta_{j-1},0,\ldots,0)}f)(x,\bullet,w).$$

Hence by Schwarz' Theorem (Proposition 69), for $v\in E_j^{\beta_j}\times\cdots\times E_n^{\beta_n}$,

$$d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}((d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}f^\vee)(x,w))(y,v) = d^{(\beta_1,\beta_2,\ldots,\beta_n)}f(x,y,w,v).$$

Φ *is continuous at 0.* Let $W\subseteq C^\gamma(U,C^\eta(V,F))$ be a 0-neighbourhood. After shrinking W, without loss of generality

$$W = \bigcap_{\tau\le\beta}\{f\in C^\gamma(U,C^\eta(V,F))\colon d^\varsigma f(K_\varsigma)\subseteq P_\varsigma\}$$

for some $\beta=(\beta_1,\beta_2,\ldots,\beta_{j-1})\in\mathbb{N}_0^{j-1}$ with $\beta_i\le\alpha_i$, $\varsigma=(\tau_1,\tau_2,\ldots,\tau_{j-1})\in\mathbb{N}_0^{j-1}$ with $\tau_i\le\beta_i$, $K_\varsigma\subseteq U\times E_1^{\tau_1}\times\cdots\times E_{j-1}^{\tau_{j-1}}$ compact and $P_\varsigma\subseteq C^\eta(V,F)$ 0-neighbourhood (see Lemma 83). Using Lemma 83 again, after shrinking P_ς, we may assume that,

$$P_\varsigma = \bigcap_{\rho\le\delta}\{g\in C^\eta(V,F)\colon d^\rho g(K_{\varsigma,\rho})\subseteq P_{\varsigma,\rho}\}$$

with $\delta = (\delta_j, \ldots, \delta_n) \in \mathbb{N}_0^{n-j+1}$ such that $\delta_i \leq \alpha_i$, $\rho := (\tau_j, \tau_{j+1}, \ldots, \tau_n)$, $K_{\varsigma,\rho} \subseteq V \times E_j^{\tau_j} \times \cdots \times E_n^{\tau_n}$ compact and $P_{\varsigma,\rho} \subseteq F$ 0-neighbourhood. Then W is the set of all $f \in C^\gamma(U, C^\eta(V,F))$ such that $d^\rho(d^\varsigma f(x,w))(y,v) \in P_{\varsigma,\rho}$ for all $(x,w) \in K_\varsigma \subseteq U \times E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ and $(y,v) \in K_{\varsigma,\rho} \subseteq V \times E_j^{\tau_j} \times \cdots \times E_n^{\tau_n}$. The projections of $U \times E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ onto the factors $U, E_1^{\tau_1}, \ldots, E_{j-1}^{\tau_{j-1}}$ are continuous, hence the images $K_\varsigma^1, K_\varsigma^2, \ldots, K_\varsigma^j$ of K_ς under these projections are compact. After replacing K_ς by $K_\varsigma^1 \times K_\varsigma^2 \times \cdots \times K_\varsigma^j$, without loss of generality $K_\varsigma = K_\varsigma^1 \times K_\varsigma^2 \times \cdots \times K_\varsigma^j$. Likewise, without loss of generality $K_{\varsigma,\rho} = K_{\varsigma,\rho}^1 \times K_{\varsigma,\rho}^2 \times \cdots \times K_{\varsigma,\rho}^{n-j+2}$ with compact sets $K_{\varsigma,\rho}^1 \subseteq V$ and $K_{\varsigma,\rho}^2 \subseteq E_j^{\tau_j}, \ldots, K_{\varsigma,\rho}^{n-j+2} \subseteq E_n^{\tau_n}$.
Now if $f \in C^{\gamma,\eta}(U \times V, F)$, then $d^\rho(d^\varsigma f^\vee(x,w))(y,v) = d^{(\varsigma,\rho)} f(x,y,w,v)$. Hence $f^\vee \in W$ if and only if $d^{(\varsigma,\rho)} f(K_\varsigma^1 \times K_{\varsigma,\rho}^1 \times K_\varsigma^2 \times \cdots \times K_\varsigma^j \times K_{\varsigma,\rho}^2 \times \cdots \times K_{\varsigma,\rho}^{n-j+2}) \subseteq P_{\varsigma,\rho}$ for all $\varsigma \leq \beta$, $\rho \leq \delta$. This is a basic neighbourhood in $C^{(\gamma,\eta)}(U \times V, F)$ (see Lemma 83). Thus $\Phi^{-1}(W)$ is a 0-neighbourhood, whence Φ is continuous at 0, and hence Φ is continuous. It is clear that Φ is injective. To see that Φ is an embedding, it remains to show that $\Phi(W)$ is a 0-neighbourhood in $\mathrm{im}(\Phi)$ for each W in a basis of 0-neighbourhoods in $C^{\gamma,\eta}(U \times V, F)$. Let

$$W := \bigcap_{\substack{\varsigma \leq \beta, \\ \rho \leq \delta}} \{f \in C^{\gamma,\eta}(U \times V, F) : d^{(\varsigma,\rho)}(K_{\varsigma,\rho}) \subseteq P_{\varsigma,\rho}\}$$

for some $\beta \in \mathbb{N}_0^{j-1}$ with $\beta_i \leq \alpha_i$, $\delta \in \mathbb{N}_0^{n-j+1}$ with $\delta_i \leq \alpha_i$, compact sets $K_{\varsigma,\rho} \subseteq U \times V \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}$ and 0-neighbourhood $P_{\varsigma,\rho} \subseteq F$ where $(\varsigma,\rho) = (\tau_1, \ldots \tau_n)$, after increasing $K_{\varsigma,\rho}$, we may assume that $K_{\varsigma,\rho} = L_{\varsigma,\rho}^1 \times K_{\varsigma,\rho}^1 \times L_{\varsigma,\rho}^2 \times \cdots \times L_{\varsigma,\rho}^j \times K_{\varsigma,\rho}^2 \times \cdots \times K_{\varsigma,\rho}^{n-j+2}$ with compact sets $L_{\varsigma,\rho}^1 \subseteq U$, $K_{\varsigma,\rho}^1 \subseteq V$, $L_{\varsigma,\rho}^2 \times \cdots \times L_{\varsigma,\rho}^j \subseteq E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ and $K_{\varsigma,\rho}^2 \times \cdots \times K_{\varsigma,\rho}^{n-j+2} \subseteq E_j^{\tau_j} \times \cdots \times E_n^{\tau_n}$. Then $\Phi(W) := \{\varphi \in \mathrm{im}(\Phi) \colon d^\rho(d^\eta \varphi(x,w))(y,v) \in P_{\varsigma,\rho}\}$ for all ς and ρ, $x \in L_{\varsigma,\rho}^1$, $y \in K_{\varsigma,\rho}^1$, $w \in L_{\varsigma,\rho}^2 \times \cdots \times L_{\varsigma,\rho}^j$ and $v \in K_{\varsigma,\rho}^2 \times \cdots \times K_{\varsigma,\rho}^{n-j+2}$, which is a 0-neighbourhood in $\mathrm{im}(\Phi)$, by Lemma 83. $\square$

Lemma 95. *Let Q be a topological space and $i \in \{1, \ldots, n\}$, let E_i, F be locally convex spaces, $\tau_i \in \mathbb{N}$ and*

$$f \colon Q \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n} \to F$$

be a map such that $f(x, w_1, \ldots, w_{i-1}, \bullet, w_{i+1}, \ldots, w_n) \colon E_i^{\tau_i} \to F$ is symmetric τ_i-linear for all $x \in Q$ and $w_j \in E_j^{\tau_j}$ with $j \neq i$. Then f is continuous if and only if $g \colon Q \times E_1 \times \cdots \times E_n \to F$, $g(x, v_1, v_2, \ldots, v_n) := f(x, \underbrace{v_1, \ldots, v_1}_{\tau_1 - times}, \underbrace{v_2, \ldots, v_2}_{\tau_2 - times}, \ldots, \underbrace{v_n, \ldots, v_n}_{\tau_n - times})$ is continuous.

Proof. The continuity of g follows directly from the continuity of f. If, conversely, g is continuous, then the assertion follows by n applications of the Polarization Identity [8, Theorem A]. $\square$

Theorem 96. (Exponential Law for C^α-mappings). *Let F and E_i for $i \in \{1, \ldots, n\}$ be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and let $X_i := \{0\}$ if $\alpha_i = 0$, otherwise $X_i := E_i$. For $j \in \{2, \ldots, n\}$ define $U := U_1 \times \cdots \times U_{j-1}$, $V := U_j \times \cdots \times U_n$, $\gamma := (\alpha_1, \ldots, \alpha_{j-1})$, $\eta := (\alpha_j, \ldots, \alpha_n)$. Assume that V is locally compact or $U \times V \times X_1 \times X_2 \times \cdots \times X_n$ is a k-space. Then*

$$\Phi\colon C^{(\gamma,\eta)}(U \times V, F) \to C^\gamma(U, C^\eta(V, F)),\ f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Moreover, if $g\colon U \to C^\eta(V, F)$ is C^γ, then

$$g^\wedge\colon U \times V \to F,\ g^\wedge(x, y) := g(x)(y)$$

is $C^{(\gamma,\eta)}$.

Proof. We only need to show the final assertion. Indeed, given $g \in C^\gamma(U, C^\eta(V, F))$, the map $g^\wedge$ will be $C^{(\gamma,\eta)}$ and hence $g = (g^\wedge)^\vee = \Phi(g^\wedge)$. Thus Φ will be surjective. so, by Theorem 94, Φ will be an isomorphism of topological vector spaces.
Locally compact condition. For $x := (x_1, \ldots, x_{j-1}) \in U$ and $y := (y_j, \ldots, y_n) \in V$, $g^\wedge(x, y) = g(x)(y) = \varepsilon(g(x), y)$ where $\varepsilon\colon C^\eta(V, F) \times V \to F$, $(\psi, y) \mapsto \psi(y)$ is $C^{(\infty,\eta)}$ (Proposition 80). Hence $g^\wedge$ is $C^{(\gamma,\eta)}$ by Chain Rule for C^α-mappings (Lemma 81).
k-space condition. If $g\colon U \to C^\eta(V, F)$ is C^γ, define $g^\wedge\colon U \times V \to F$, $g^\wedge(x, y) = g(x)(y)$. For fixed $x \in U$, we have $g^\wedge(x, \bullet) = g(x)$ which is C^η, hence

$$\begin{aligned}(\breve{D}_j \cdots \breve{D}_n\, g^\wedge)(x, y) &= d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}(g(x))(y, w_j, \ldots, w_n)\\ &= (d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ g)(x)(y, w_j, \ldots, w_n)\end{aligned}$$

exists for $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$, $y \in V^0 := U_j^0 \times \cdots \times U_n^0$ and $w_i \in E_i^{\beta_i}$. Also,

$$(\breve{D}_j \cdots \breve{D}_n\, g^\wedge)(x, y) = (\varepsilon_{(y,w_j,\ldots,w_n)} \circ d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ g)(x),$$

where $\varepsilon_{(y,w_j,\ldots,w_n)}\colon C(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F) \to F$, $f \mapsto f(y, w_j, \ldots, w_n)$. For fixed $(y, w_j, \ldots, w_n)$, this is the function $\varepsilon_{(y,w_j,\ldots,w_n)} \circ d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ g$ of x, which is C^γ. Since $\varepsilon_{(y,w_j,\ldots,w_n)}$ and $d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}\colon C^\eta(V, F) \to C(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F)$ are continuous linear, we obtain the directional derivatives

$$\begin{aligned}
&(\breve{D}_1 \cdots \breve{D}_n\, g^\wedge)(x,y)\\
&= \varepsilon_{(y,w_j,\ldots,w_n)}(d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}(d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}g(x,w_1,\ldots,w_{j-1})))\\
&= d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}(d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}g(x,w_1,\ldots,w_{j-1}))(y,w_j,\ldots,w_n)\\
&= (d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ (d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}g))(x,w_1,\ldots,w_{j-1})(y,w_j,\ldots,w_n)\\
&= (d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ (d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}g))^\wedge((x,w_1,\ldots,w_{j-1}),(y,w_j,\ldots,w_n))
\end{aligned}$$

for $x \in U^0 := U_1^0 \times \cdots \times U_n^0$. To see that $g^\wedge$ is $C^{(\gamma,\eta)}$, it therefore suffices to show that $h\colon (d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ (d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}g))^\wedge\colon U \times E_1^{\beta_1} \times \cdots \times E_{j-1}^{\beta_{j-1}} \times V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n} \to F$ is continuous for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$. By Lemma 95, h will be continuous if we can show that

$$f\colon U \times X_1 \times \cdots \times X_{j-1} \times V \times X_j \times \cdots \times X_n \to F,$$

$$\begin{aligned}
&(x,w_1,\ldots,w_{j-1},y,w_j,\ldots,w_n)\\
&\mapsto h(x,\underbrace{w_1,\ldots,w_1}_{\beta_1-\text{times}},\underbrace{w_2,\ldots,w_2}_{\beta_2-\text{times}},\ldots,\underbrace{w_{j-1},\ldots,w_{j-1}}_{\beta_{j-1}-\text{times}},y,\underbrace{w_j,\ldots,w_j}_{\beta_j-\text{times}},\ldots,\underbrace{w_n,\ldots,w_n}_{\beta_n-\text{times}})
\end{aligned}$$

is continuous. Now $\psi\colon U \times X_1 \times \cdots \times X_{j-1} \to U \times E_1^{\beta_1} \times \cdots \times E_{j-1}^{\beta_{j-1}}$, $(x,w_1,\ldots,w_{j-1}) \mapsto (x,\underbrace{w_1,\ldots,w_1}_{\beta_1-\text{times}},\underbrace{w_2,\ldots,w_2}_{\beta_2-\text{times}},\ldots,\underbrace{w_{j-1},\ldots,w_{j-1}}_{\beta_{j-1}-\text{times}})$ is continuous and $\varphi\colon V \times X_j \times \cdots \times X_n \to V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$, $(y,w_j,\ldots,w_n) \mapsto (y,\underbrace{w_j,\ldots,w_j}_{\beta_j-\text{times}},\underbrace{w_{j+1},\ldots,w_{j+1}}_{\beta_{j+1}-\text{times}},\ldots,\underbrace{w_n,\ldots,w_n}_{\beta_n-\text{times}})$ is continuous and

$$\theta := C^0(\varphi,F) \circ d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}g \circ \psi\colon U \times X_1 \times \cdots \times X_{j-1} \to C^0(V \times Y, F)$$

is continuous. Since $U \times X_1 \times \cdots \times X_{j-1} \times V \times X_j \times \cdots \times X_n$ is a k-space by hypothesis, it follows that $\theta^\wedge\colon U \times X_1 \times \cdots \times X_{j-1} \times V \times X_j \times \cdots \times X_n \to F$ is continuous (see [20, Proposition B.15]). But $\theta^\wedge = f$, and thus f is continuous. □

4.4 The Exponential Law for C^α-mappings on manifolds

Definition 97. For $i \in \{1,\ldots,n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on a locally convex space, $\alpha := (\alpha_1,\ldots,\alpha_n)$ with $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and F be a locally convex space. A map $f\colon M_1 \times \cdots \times M_n \to F$ is called C^α if $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})\colon V_{\varphi_1} \times \cdots \times V_{\varphi_n} \to F$ is C^α for all charts $\varphi_i\colon U_{\varphi_i} \to V_{\varphi_i}$ of M_i. Then f is continuous in particular.

Remark 98. In the preceding situation, assume that $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$ is C^α for charts φ_i in a (not necessarily maximal) atlas of M_i, for $i \in \{1, \dots, n\}$. Then f is C^α, using the Chain Rule (Lemma 81). In paticular, a map f as in Definition 66 is C^α as defined there iff it is C^α in the sense of Definition 97.

Definition 99. If $M_1 \times \cdots \times M_n$ and N are smooth manifolds and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, we say that a map $f\colon M_1 \times \cdots \times M_n \to N$ is C^α if it is continuous and $\varphi \circ f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1}) : V_1 \times \cdots \times V_n \to V$ is C^α for all charts $\varphi : U \to V$ of M and charts $\varphi_i : U_i \to V_i$ of M_i such that $U_1 \times \cdots \times U_n \subseteq f^{-1}(U)$. Again using Lemma 81 we see that f is C^α if and only if for each $x = (x_1, \dots, x_n) \in M_1 \times \cdots \times M_n$, there exists a chart $\phi : U \to V$ of M with $f(x) \in U$ and charts $\varphi_i : U_i \to V_i$ of M_i with $x_i \in U_i$ such that $U_i \times \cdots \times U_n \subseteq f^{-1}(U)$ and $\varphi \circ f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1}) : V_1 \times \cdots \times V_n \to V$ is C^α.

Definition 100. In the situation of Definition 97, let $C^\alpha(M_1 \times \cdots \times M_n, F)$ be the space of all C^α-maps $f\colon M_1 \times \cdots \times M_n \to F$. Endow $C^\alpha(M_1 \times \cdots \times M_n, F)$ with the initial topology with respect to the maps $C^\alpha(M_1 \times \cdots \times M_n, F) \to C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F)$, $f \mapsto f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$, for φ_i in the maximal smooth atlas of M_i with $i = 1, \dots, n$.

Lemma 101. *Let F be a locally convex space. For $i \in \{1, \dots, n\}$, let M_i be a smooth manifold modelled on a locally convex space, $\mathcal{A}_i$ be the maximal smooth atlas for M_i with a chart $\varphi_i : U_{\varphi_i} \to V_{\varphi_i}$ and $\alpha := (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Then the map*

$$\Phi : C^\alpha(M_1 \times \cdots \times M_n, F) \cong \prod_{\varphi_i \in \mathcal{A}_i} C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F),\ f \mapsto f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$$

is an embedding with closed image.

Proof. It is clear that Φ is injective (and linear). The topology on $C^\alpha(M_1 \times \cdots \times M_n, F)$ being initial with respect to the maps $f \mapsto f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$, we deduce that Φ is a topological embedding. To see that $\operatorname{im}(\Phi)$ is closed, let $(g_\beta)_{\beta \in B}$ be a net in $\operatorname{im}(\Phi)$ which converges to some $g \in \prod_{\varphi_i \in \mathcal{A}_i} C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F)$. We have $g_\beta = (g_{\beta, \varphi_1, \dots, \varphi_n})_{\varphi_i \in \mathcal{A}_i}$ and $g = (g_{\varphi_1, \dots, \varphi_n})_{\varphi_i \in A_i}$. Then $g_{\beta, \varphi_1, \dots, \varphi_n} \to g_{\varphi_1, \dots, \varphi_n}$ in $C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F)$ and hence also pointwise. As a consequence,

$$g_{\varphi_1, \dots, \varphi_n} = g_{\psi_1, \dots, \psi_n} \circ (\psi_1, \dots, \psi_n) \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$$

on $\varphi_1(V_{\varphi_1} \cap V_{\psi_1}) \times \cdots \times \varphi_n(V_{\varphi_n} \cap V_{\psi_n})$, for all φ_i, $\psi_i \in \mathcal{A}_i$. Hence the map $f : M_1 \times \cdots \times M_n \to F$ is well-defined via $f(x) := g_{\varphi_1, \dots, \varphi_n}(\varphi_1(x), \dots, \varphi_n(x))$ for $x \in U_{\varphi_1} \times \cdots \times U_{\varphi_n}$. Because

$$f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1}) = g_{\varphi_1, \dots, \varphi_n} \tag{4.27}$$

is C^α, the map f is C^α and by (4.27), we have $\Phi(f) = g$. Hence $\operatorname{im}(\Phi)$ is closed. □

Lemma 102. *Let $M_1, \dots, M_n$ be smooth manifolds (possibly with rough boundary), F and X be locally convex spaces, $W \subseteq F$ be an open set and $f : W \to X$ be a smooth map. Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$. If $W = F$ or $M_1, \dots, M_n$ are compact, then*

$$C^\alpha(M_1 \times \cdots \times M_n, f) : C^\alpha(M_1 \times \cdots \times M_n, W) \to C^\alpha(M_1 \times \cdots \times, X), \gamma \mapsto f \circ \gamma$$

is a C^∞-map.

Proof. If $W = F$, we only need to show that $C^\alpha(M_1 \times \cdots \times M_n, F) \to C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, X)$, $\gamma \mapsto (f \circ \gamma) \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$ is smooth for all charts φ_i of M_i, by Lemma 101. As this map coincides with $C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, f)$ composed with the continuous linear map $\gamma \mapsto \gamma \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$, we may assume that $M_i = U_i$ is a locally convex subset with dense interior of a locally convex space E_i, for $i = 1, \dots, n$. Because $C^\alpha(U_1 \times \cdots \times U_n, X) = \varprojlim C^\beta(U_1 \times \cdots \times U_n, X)$ with the inclusion maps $i_\beta : C^\alpha(U_1 \times \cdots \times U_n, X) \to C^\beta(U_1 \times \cdots \times U_n, X)$ for $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$, (see Lemma 90), we only need too show that $i_\beta \circ C^\alpha(U_1 \times \cdots \times U_n, f)$ is smooth (see Lemma 85 and [4]). Now $i_\beta \circ C^\alpha(U_1 \times \cdots \times U_n, f) = C^\beta(U_1 \times \cdots \times U_n, f) \circ j_\beta$ with the continuous linear inclusion map $j_\beta : C^\alpha(U_1 \times \cdots \times U_n, F) \to C^\beta(U_1 \times \cdots \times U_n, F)$. Hence $\alpha \in \mathbb{N}_0^n$ without loss of generality.
The proof is by induction on $|\alpha|$. If $|\alpha| = 0$, then $C^\alpha(U_1 \times \cdots \times U_n, f) \to C(U_1 \times \cdots \times U_n, f)$ is smooth, see [21].
If $|\alpha| \geq 1$, there is $j \in \{1, \dots, n\}$ such that $\alpha_j \geq 1$. Let $\sigma \in S_n$ be the permutation which interchanges j and n. Define α_σ and $\Phi_\sigma : C^{\alpha_\sigma}(U_1 \times \cdots \times U_n, F) \to C^\alpha(U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}, F)$ and the analogous isomorphism $\Psi_\sigma : C^{\alpha_\sigma}(U_1 \times \cdots \times U_n, X) \to C^\alpha(U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}, X)$ as in Lemma 92. Then $C^\alpha(U_1 \times \cdots \times U_n, f) = \Psi_\sigma^{-1} \circ C^{\alpha_\sigma}(U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}, f) \circ \Phi_\sigma$. Hence $\alpha_n \geq 1$ without loss of generality, By Lemma 91 and 84, it now suffices to show that $i_{\alpha - e_n} \circ C\alpha(U_1 \times \cdots \times U_n, f)$ is smooth (which holds by induction as this map coincides with $C^{\alpha - e_n}(U_1 \times \cdots \times U_n) \circ j_{\alpha - e_n})$ and $d^{e_n} \circ C^\alpha(U_1 \times \cdots \times U_n, f)$ is smooth. Now $d^{e_n}(f \circ \gamma)(x_1, \dots, x_{n-1}, (x_n, y)) = df(\gamma(x_1, \cdots, x_n), d^{e_n}\gamma(x_1, \cdots, x_{n-1}, (x_n, y)))$, i.e.,

$$\begin{aligned} &d^{e_n} \circ C^\alpha(U_1 \times \cdots \times U_n, f) \\ &= C^{\alpha - e_n}(U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n), df) \circ (C^{\alpha - e_n}(h, F) \circ j_{\alpha - e_n}, d^{e_n}) \end{aligned} \tag{4.28}$$

with $h : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to U_1 \times \cdots \times U_n$, $h(x_1, \dots, x_{n-1}, (x_n, y)) \mapsto (x_1, \dots, x_n)$. By induction and Lemma 93, the right-hand side of (4.28) is smooth and hence also $d^{e_n} \circ C^\alpha(U_1 \times \cdots \times U_n, f)$ is smooth, as required.
We only sketch the proof of the case that $M_1, \dots, M_n$ are compact and $W \subseteq F$ is

an open set. Then $C(M_1 \times \cdots \times M_n, W)$ is open $C(M_1 \times \cdots \times M_n, W)$ and hence $C^\alpha(M_1 \times \cdots \times M_n, W)$ is open in $C^\alpha(M_1 \times \cdots \times M_n, F)$. One can show that Lemma 101 is also valied for non-maximal atlases. The asseration will therefore follow if we can show:
If E_i is a locally convex space, $U_i \subseteq E_i$ a locally convex set with dense interior and $V_i \subseteq U_i$ relatively open with compact closure $K_i := \bar{V}_i$ in U_i for $i \in \{1, \ldots, n\}$, then the map

$$\{\gamma \in C^\alpha(U_1 \times \cdots \times U_n, F) : \gamma(K_1 \times \cdots \times K_n) \subseteq W\} \to C^\alpha(V_1 \times \cdots \times V_n, X),$$

$$\gamma \mapsto f \circ \gamma|_{V_1 \times \cdots \times V_n}$$

is smooth. But this can be shown like the case $W = E$. $\square$

Proposition 103. *For $i \in \{1, \ldots, n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on a locally convex space, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and F be a locally convex space. Let $j \in \{2, \ldots, n\}$. Define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1, \ldots, \alpha_{j-1})$ and $\eta := (\alpha_j, \ldots, \alpha_n)$. For $x \in M$ and $f \in C^{(\gamma,\eta)}(M \times N, F)$, write $f^\wedge(x) := f_x := f(x, \bullet) : N \to F$. Then*

(a) *$f^\vee \in C^\gamma(M, C^\eta(N, F))$ for all $f \in C^{(\gamma,\eta)}(M \times N, F)$.*

(b) *The map*

$$\Phi \colon C^{(\gamma,\eta)}(M \times N, F) \to C^\gamma(M, C^\eta(N, F)),\ f \mapsto f^\vee$$

is linear and a topological embedding.

Proof. (a) For $x \in M$, it is clear that $f^\vee(x) = f(x, \bullet)$ is a C^η-map $N \to F$. It suffices to show that $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1}) \colon U_{\varphi_1} \times \cdots \times U_{\varphi_{j-1}} \to C^\eta(N, F)$ is C^γ for each chart $\varphi_k \colon U_{\varphi_k} \to V_{\varphi_k}$ of M, where $k \in \{1, \ldots, j-1\}$. For all $l \in \{j, \ldots, n\}$, let $\mathcal{A}_l$ be the maximal smooth atlas for M_l. Because the map

$$\Psi \colon C^\eta(N, F) \to \prod_{\substack{\varphi_l \in \mathcal{A}_l, \\ j \leq l \leq n}} C^\eta(U_{\varphi_j} \times \cdots \times U_{\varphi_n}, F),\ h \mapsto (h \circ (\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}))_{\substack{\varphi_l \in \mathcal{A}_l, \\ j \leq l \leq n}}$$

is a linear topological embedding with closed image $f^\vee \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})$ is C^γ if and only if $\Psi \circ f \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})$ is C^γ (see Lemma 101), which holds if all components are C^γ. Hence we only need that

$$\theta \colon V_{\varphi_1} \times \cdots \times V_{\varphi_{j-1}} \to C^\eta(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F),$$

$$x \mapsto f^\vee((\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})(x)) \circ (\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}) = (f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})^\vee)(x)$$

is C^γ. But $\theta = (f\circ(\varphi_1^{-1}\times\cdots\times\varphi_n^{-1}))^\vee$ where $f\circ(\varphi_1^{-1}\times\cdots\times\varphi_n^{-1})\colon V_{\varphi_1}\times\cdots\times V_{\varphi_n}\to F$ is $C^{(\gamma,\eta)}$, hence θ is C^γ by Theorem 94.

(b) It is clear that Φ is linear and injective. Because Ψ is linear and a topological embedding, also

$$C^\gamma(M,\Psi)\colon C^\gamma(M,C^\eta(N,F))\to C^\gamma(M,\prod_{\substack{\varphi_l\in\mathcal{A}_l,\\ j\leq l\leq n}} C^\eta(V_{\varphi_j}\times\cdots\times V_{\varphi_n},F)),\ f\mapsto\Psi\circ f$$

is a topological embedding, by Lemma 86. Let $P:=\prod_{\substack{\varphi_l\in\mathcal{A}_l,\\ j\leq l\leq n}} C^\eta(V_{\varphi_j}\times\cdots\times V_{\varphi_n},F)$ and $\mathcal{A}_k$ be the maximal smooth atlas for M_k where $k\in\{1,\ldots,j-1\}$. The map

$$\Xi\colon C^\gamma(M,P)\to\prod_{\substack{\varphi_k\in\mathcal{A}_k,\\ 1\leq k\leq j-1}} C^\gamma(V_{\varphi_1}\times\cdots\times V_{\varphi_{j-1}},P),\ f\mapsto(f\circ(\varphi_1^{-1}\times\cdots\times\varphi_{j-1}^{-1}))_{\substack{\varphi_k\in\mathcal{A}_k,\\ 1\leq k\leq j-1}}$$

is a linear topological embedding. Let

$$Q:=\prod_{\substack{\varphi_k\in\mathcal{A}_k,\\ 1\leq k\leq j-1}}\prod_{\substack{\varphi_l\in\mathcal{A}_l,\\ j\leq l\leq n}} C^\gamma(V_{\varphi_1}\times\cdots\times V_{\varphi_{j-1}},C^\eta(V_{\varphi_j}\times\cdots\times V_{\varphi_n},F)).$$

Using the isomorphism $\prod_{\substack{\varphi_k\in\mathcal{A}_k,\\ 1\leq k\leq j-1}} C^\gamma(V_{\varphi_1}\times\cdots\times V_{\varphi_{j-1}},P)\cong Q$, (see Lemma 88) we obtain a linear topological embedding

$$\Gamma:=\Xi\circ C^\gamma(M,\Psi)\colon C^\gamma(M,C^\eta(N,F))\to Q,$$

$$f\mapsto(C^\eta(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1},F)\circ\ f\ \circ(\varphi_1^{-1}\times\cdots\times\varphi_{j-1}^{-1}))_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\leq i\leq n}}$$

where $C^\eta(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1},F)\colon C^\eta(N,F)\to C^\eta(V_{\varphi_j}\times\cdots\times V_{\varphi_n},F),\ f\mapsto f\circ(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1})$. Also the map

$$\omega\colon C^{(\gamma,\eta)}(M\times N,F)\to\prod_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\leq i\leq n}} C^{(\gamma,\eta)}(V_{\varphi_1}\times\cdots\times V_{\varphi_n},F),$$

$$f\mapsto(f\circ(\varphi_1^{-1}\times\cdots\times\varphi_n^{-1}))_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\leq i\leq n}}$$

is a topological embedding, by Definition 100. Now we have the commutative diagramme.

$$\begin{array}{ccc}
C^{(\gamma,\eta)}(M\times N,F) & \xrightarrow{\Phi} & C^\gamma(M,C^\eta(N,F)) \\
\Big\downarrow{\scriptstyle\omega} & & \Big\downarrow{\scriptstyle\Gamma} \\
\prod_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\leq i\leq n}} C^{(\gamma,\eta)}(V_{\varphi_1}\times\cdots\times V_{\varphi_n},F) & \xrightarrow{\zeta} & Q
\end{array}$$

where ζ is the map $(f_{\varphi_1,\ldots,\varphi_n})_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\leq i\leq n}} \mapsto (f^\vee_{\varphi_1,\ldots,\varphi_n})_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\leq i\leq n}}$. Because the vertical arrows are topological embeddings and also the horizontal arrow at the bottom (by Lemma 57 and Theorem 94) is a topological embbeding, we deduce that the map Φ at the top has to be a topological embedding as well. □

Theorem 104. *For $i \in \{1,\ldots,n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on a locally convex space E_i, F be a locally convex space and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Let $X_i := \{0\}$ if $\alpha_i = 0$, otherwise $X_i := E_i$. For $j \in \{2,\ldots,n\}$ define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1,\ldots,\alpha_{j-1})$ and $\eta := (\alpha_j,\ldots,\alpha_n)$. Assume that N is locally compact or $M \times N \times X_1 \times X_2 \times \cdots \times X_n$ is a k-space. Then*

$$\Phi\colon C^{(\gamma,\eta)}(M\times N,F) \to C^\gamma(M,C^\eta(N,F)),\ f\mapsto f^\vee \tag{4.29}$$

is an isomorphism of topological vector spaces. Moreover, a map $g\colon M \to C^\eta(N,F)$ is C^γ if and only if

$$g^\wedge\colon M\times N\to F,\ g^\wedge(x,y) := g(x)(y)$$

is $C^{(\gamma,\eta)}$.

Proof. By Proposition 103, we only need to show that Φ is surjective. To this end, let $g \in C^\gamma(M,C^\eta(N,F))$ and define $f := g^\wedge\colon M\times N\to F$, $f(x,y) := g(x)(y)$. For all $i \in \{1,\ldots,n\}$, let $\varphi_i : U_{\varphi_i} \to V_{\varphi_i}$ be charts for M_i. Then

$$f\circ(\varphi_1^{-1}\times\cdots\times\varphi_n^{-1})\colon V_{\varphi_1}\times\cdots\times V_{\varphi_n}\to F,$$

$$(x_1,\ldots,x_n)\mapsto (C^\eta(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1},F)\circ g\circ(\varphi_1^{-1}\times\cdots\times\varphi_{j-1}^{-1}))^\wedge(x_1,\ldots,x_n)$$

with $C^\eta(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1},F)\colon C^\eta(N,F)\to C^\eta(V_{\varphi_j}\times\cdots\times V_{\varphi_n},F), h\mapsto h\circ(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1})$ continuous linear. Hence $C^\eta(\varphi_j^{-1}\times\cdots\times\varphi_n^{-1},F)\circ g\circ(\varphi_1^{-1}\times\cdots\times\varphi_{j-1}^{-1})\colon V_{\varphi_1}\times\cdots\times V_{\varphi_{j-1}}\to C^\eta(V_{\varphi_j}\times\cdots\times V_{\varphi_n},F)$ is C^γ. Hence $f\circ(\varphi_1^{-1}\times\cdots\times\varphi_n^{-1})$ is $C^{(\gamma,\eta)}$ by the exponential law (Theorem 96). Indeed:
Locally compact condition. For all $l \in \{j,\ldots,n\}$, if N is locally compact, then the open subset U_{φ_l} is locally compact and hence also the V_{φ_l}. Hence the Exponential Law (Theorem 96) applies.
k-space condition. $V_{\varphi_1}\times\cdots\times V_{\varphi_n}\times X_1\times X_2\times\cdots\times X_n$ is homeomorphic to the open subset $U_{\varphi_1}\times\cdots\times U_{\varphi_n}\times X_1\times X_2\times\cdots\times X_n$ of the k-space $M\times N\times X_1\times X_2\times\cdots\times X_n$ and hence a k-space. Again, the Exponential Law (Theorem 96) applies. □

Remark 105. The same conclusion holds in the following situations:

(a) $M_j, \dots, M_n$ are finite-dimensional manifolds without boundary, with smooth boundary or with corners (then N is a locally compact).
(b) $M_1, \dots, M_n$ are manifolds without boundary, with smooth boundary or with corners and $E_1 \times \cdots \times E_n \times X_1 \times \cdots \times X_n$ is a k-space.

Corollary 106. *For $i \in I := \{1, \dots, n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on a locally convex space E_i, F be a locally convex space and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \{2, \dots, n\}$ define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1, \dots, \alpha_{j-1})$ and $\eta := (\alpha_j, \dots, \alpha_n)$. Assume that* (a), (b) *or* (c) *is satisfied:*
(a) *For all $i \in I$, E_i is a metrizable.*
(b) *For all $i \in I$, M_i is manifold with corners and E_i is a hemicompact k-space.*
(c) *For all $i \in \{j, \dots, n\}$, M_i is a finite-dimensional manifold with corners.*
Then

$$\Phi\colon C^{(\gamma,\eta)}(M \times N, F) \to C^\gamma(M, C^\eta(N, F)),\ f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Moreover, a map $g\colon M \to C^\eta(N, F)$ is C^γ if and only if

$$g^\wedge\colon M \times N \to F,\ g^\wedge(x, y) := g(x)(y)$$

is $C^{(\gamma,\eta)}$.

Proof. Case $M_j \dots, M_n$ are finite-dimensional manifolds with corners. Let M_l be of dimension m_l for $l \in \{j, \dots, n\}$. Then each point of M_l has an open neighbourhood homeomorphic to an open subset V_l of $[0, \infty[^{m_l}$. Hence V_l is locally compact, thus M_l is locally compact. Thus Theorem 104 applies.
Case E_i is a metrizable. Then for all $i \in I$, each point $x_i \in M_i$ has an open neighbourhood $U_i \subseteq M_i$ homeomorphic to a subset $V_i \subseteq E_i$. Since $V_1 \times \cdots \times V_n$ is metrizable, it follows that $U_1 \times \cdots \times U_n \times E_1 \times \cdots \times E_n$ is metrizable and hence a k-space. Hence by Lemma 61 $M_1 \times \cdots \times M_n \times E_1 \times \cdots \times E_n$ is a k-space and Theorem 104 applies.
Case $E_1, \dots, E_n$ are k_ω-spaces, M_i is a manifold with corners. For all $x_i \in M_i$ there is an open neighbourhood $U_i \subseteq M_i$ homeomorphic to an open subset V_i of finite intersections of closed half-space in E_i. Hence $V_1 \times \cdots \times V_n \times E_1 \times \cdots \times E_n$ is an (relatively) open subset of a closed subset of $(E_1 \times \cdots \times E_n)^2$. The latter product is k_ω since $E_1, \dots, E_n$ are k_ω-spaces (see [22, Proposition 4.2(i)]), and hence a k-space. Since open subsets (and also closed subsets) of k-spaces are k-spaces, it follows that $V_1 \times \cdots \times V_n \times E_1 \times \cdots \times E_n$ is a k-space. Now Lemma 61 shows that $M_1 \times \cdots \times M_n \times E_1 \times \cdots \times E_n$ is a k-space, and thus Theorem 104 applies. $\square$

Remark 107. (a) For the case when each M_i is a manifold with corners and $(E_1 \times \cdots \times E_n)^2$ is a k-space, the conclusion can be proved like the result for k_ω-spaces in Corollary 106.

(b) Note that C^γ-maps $U_1 \times \cdots \times U_n \to F$ can be defined just as well if, for all $j \in \{1, \ldots, n\}$ with $\gamma_j = 0$, U_j is a Hausdorff topological space (rather than a subset of some locally convex space E_j). All results carry over to this situation (with obvious modifications).

(c) If F is a complex locally convex space, we obtain analogous results if E_j is a locally convex space over $\mathbb{K}_j \in \{\mathbb{R}, \mathbb{C}\}$ and all directional derivatives in the $j-th$ variable are considered as derivatives over the ground field $\mathbb{K}_j$. The corresponding maps could be called $C^\gamma_{\mathbb{K}_1,\ldots,\mathbb{K}_n}$-maps.

Chapter 5

Regular Lie groups and the Fundamental Theorem

In this chapter we discuss the C^k-regularity concept. After recalling some definitions and results (mainly from [32], [27], [17] and [21]), we shall introduce a version of the Fundamental Theorem for $\mathfrak{g}$-valued functions (Theorem 132).

Definition 108. The *Maurer–Cartan form* $\kappa_G \in \Omega^1(G, \mathfrak{g})$ is the unique left invariant $\mathfrak{g}$-valued 1-form on G with $\kappa_{G,1} = \mathrm{id}_{\mathfrak{g}}$, i.e., $\kappa_G(x_l) = x$ for each $x \in \mathfrak{g}$.

The logarithmic derivative of a map f can be described as a pull-back of the Maurer-Cartan form.

Definition 109. Let M be a smooth manifold (with boundary) and K a Lie group with Lie algebra $\mathfrak{k}$ and Maurer-Cartan form $\kappa_K \in \Omega^1(K, \mathfrak{k})$. For an element $f \in C^1(M, K)$ we call $\delta(f) := f^*\kappa_K =: f^{-1} \cdot df \in \Omega^1_{C^0}(M, \mathfrak{k})$ the *(left) logarithmic derivative* of f.

Remark 110. Let E be a locally convex space, M be a smooth finite dimensional manifold (possibly with boundary). We write $\Omega^1_{C^r}(M, E)$ for the space of E-valued 1-forms on M defining C^r-functions $TM \to E$. The space of E-valued smooth 1-forms will be denoted by $\Omega^1(M, E)$. We endow $\Omega^1_{C^r}(M, E)$ with the topology induced by the embedding

$$\Omega^1_{C^r}(M, E) \hookrightarrow C^r(TM, E),$$

where TM is the tangent bundle and $C^r(TM, E)$ is endowed with the compact open C^r-topology, so that $\Omega^1_{C^r}(M, E)$ is a closed subspace of $C^r(TM, E)$. The space $\Omega^1(M, E)$ is endowed with the topology induced by the diagonal embedding

$$\Omega^1(M, E) \hookrightarrow \prod_{r=1}^{\infty} \Omega^1_{C^r}(M, E).$$

The left logarithmic derivative
is a $\mathfrak{k}$-valued 1-form on M. For $k \in \mathbb{N}_0 \cup \{\infty\}$, we thus obtain a map

$$\delta : C^k(M, K) \to \Omega^1_{C^{k-2}}(M, \mathfrak{k})$$

satisfying the following lemma.

Lemma 111. *For $f, g \in C^k(M, G)$, the following assertions hold:*

(a) *The map $f^{-1} : M \to G$, $m \mapsto f(m)^{-1}$ is C^k with*

$$\delta(f^{-1}) = -\mathrm{Ad}(f)\delta(f).$$

(b) *We have the following product and quotient rules:*

$$\delta(fg) = \mathrm{Ad}(g)^{-1}\delta(f) + \delta(g).$$

and

$$\delta(fg^{-1}) = \mathrm{Ad}(g)(\delta(f) - \delta(g)).$$

From this it easily follows that

Lemma 112. *If M is connected and $f, g \in C^k(M, G)$, then*

$$\delta(f) = \delta(g) \quad \Longleftrightarrow \quad (\exists h \in G)\, g = \lambda_h \circ f.$$

In particular, $\delta(f) = \delta(g)$ and $f(m_0) = g(m_0)$ for some $m_0 \in M$ imply $f = g$.

Definition 113. (Integrability and local integrability). We call $\alpha \in \Omega^1_{C^0}(M, \mathfrak{k})$ *integrable* if there exists a C^1-function $f : M \to K$ with $\delta(f) = \alpha$. We say that α is *locally integrable* if each point $m \in M$ has an open neighbourhood U such that $\alpha|_U$ is integrable.

Remark 114. Using induction on k, we can prove: If $\alpha \in \Omega^1_{C^k}(M, \mathfrak{k})$ is integrable and $\alpha = \delta f$ with a C^1-function $f : M \to K$, then f is C^{k+1}.

In the following, we frequently abbreviate $I := [0, 1]$.

Definition 115. (Left product integral and left evolution). Let $\xi : I \to \mathrm{L}(G)$ be a continuous curve, defined on an interval $I \subseteq \mathbb{R}$. If $\gamma : I \to G$ is a C^1-curve such that $\delta(\gamma) = \xi$, we call γ a *left product integral* for ξ. If $\gamma(0) = 1$, we call γ the *left evolution* of ξ and write $\mathrm{Evol}_G(\xi) := \gamma$.

Definition 116. (**C^k-Regular Lie group**). Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A Lie group G with Lie algebra $\mathfrak{g}$ is called *C^k-regular*, if for each $\xi \in C^k(I, \mathfrak{g})$, the initial value problem

$$\gamma(0) = 1, \quad \delta(\gamma) = \xi \tag{5.1}$$

has a solution $\gamma = \gamma_\xi \in C^{k+1}(I, G)$, and the evolution map

$$\mathrm{evol}_G : C^k(I, \mathfrak{g}) \to G, \ \xi \mapsto \gamma_\xi(1)$$

is smooth. We recall from Lemma 112 that the solutions of (5.1) are unique whenever they exist. If G is C^k-regular, we write

$$\mathrm{Evol}_G : C^k(I, \mathfrak{g}) \to C^{k+1}(I, G), \ \xi \mapsto \gamma_\xi$$

for the corresponding map on the level of Lie group-valued curves.

The group G is called regular if it is C^∞-regular.

Proposition 117. *Let G be a connected, simply connected real Lie group and H be a regular Lie group. Then every continuous Lie algebra homomorphism $\psi : \mathrm{L}(G) \to \mathrm{L}(H)$ integrates to a smooth group homomorphism $\varphi : G \to H$ such that $\mathrm{L}(\varphi) = \psi$.*

Proof. For the proof we refer to [21]. □

Remark 118. Proposition 117 implies: If $\mathfrak{g}$ is a locally convex, Mackey complete topological Lie algebra, then there is (up to isomorphism) at most one simply connected, regular Lie group G with $\mathrm{L}(G) \cong \mathfrak{g}$.

Proposition 119. *Let M be a finite-dimensional smooth manifold and E a locally convex space. Then $C^k(M, E)$ is a locally convex space, and the evaluation map $\varepsilon : C^k(M, E) \times M \to E$ is $C^{\infty,k}$. If E is Mackey complete, then $C^k(M, E)$ is Mackey complete.*

Proof. All the spaces $C(T^nM, T^nE)_c$ are locally convex. Therefore the corresponding product topology is locally convex, and hence $C^k(M, E)$ is a locally convex space.

The continuity of the evaluation map follows from the continuity of the evaluation map for the compact-open topology because the topology on $C^k(M, E)$ is finer. Next we observe that directional derivatives exist and lead to a map

$$d\,\mathrm{ev} : C^k(M, E)^2 \times T(M) \to E, \quad ((f, \xi), v_m) \mapsto \xi(m) + T_m(f)v_m$$

whose continuity follows from the first step, applied to the evaluation map of $C^k(TM, E)$. Hence ev is C^1, and iteration of this argument yields C^k.

In view of Proposition 58, we have

$$C^\infty(I, C^k(M, E)) \cong C^{\infty,k}(I \times M, E) \cong C^k(M, C^\infty(I, E)),$$

and if E is Mackey complete, then we have an integration map

$$C^k(M, C^\infty(I, E)) \to C^k(M, E), \quad \xi \mapsto \int_0^1 \bullet \; dt \circ \xi,$$

which implies that each C^k-curve with values in $C^k(M, E)$ has a Riemann integral, i.e., that $C^k(M, E)$ is Mackey complete. $\square$

Theorem 120. *Let M be a C^∞-manifold, K be a Lie group with Lie algebra $\mathfrak{k}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. If M is compact, then $C^k(M, K)$ carries a Lie group structure for which any $\mathfrak{k}$-chart (ϕ_K, U_K) of K yields a $C^k(M, \mathfrak{k})$-chart (ϕ, U) with*

$$U := \left\{ f \in C^k(M, K) : f(M) \subseteq U_K \right\}, \quad \phi(f) := \phi_K \circ f,$$

and the evaluation map of $\varepsilon : C^k(M, K) \times M \to K$ is $C^{\infty,k}$. The corresponding Lie algebra is $C^k(M, \mathfrak{k})$.

Proof. For the existence of the Lie group structure with the given charts we refer to [15]. The evaluation map ε is C^k on $U \times M$ for each domain U as above, because $V := C^k(M, \phi_K(U_K))$ is open in $C^k(M, \mathfrak{k})$ and the evaluation map of $C^k(M, \mathfrak{k})$ is C^k, verified in Proposition 119.

If $f \in C^k(M, K)$ is arbitrary, then $\varepsilon(f\phi^{-1}(g), x) = f(x)\phi^{-1}(g)(x)$ is C^k in $(g, x) \in V \times M$, whence ε is C^k on $fU \times M$. $\square$

Lemma 121. *Let G be a Lie group modelled on a locally convex space E, M be a compact manifold (possibly with boundary) and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the evaluation map*

$$\varepsilon : C^k(M, G) \times M \to G$$

is $C^{\infty,k}$.

Proof. It suffices to show that each $\gamma \in C^k(M, G)$ has an open neighbourhood $W \subseteq C^k(M, G)$ such that $\varepsilon|_{W \times M}$ is $C^{\infty,k}$. Let $\varphi : U \to V \subseteq E$ be a chart for G around $1 \in G$ such that $C^k(M, U)$ is open in $C^k(M, G)$ and $\varphi_* : C^k(M, U) \to C^k(M, V) \subseteq C^k(M, E)$ is a chart of $C^k(M, G)$. Then $W := \gamma \,.\, C^k(M, U)$ is an open neighbourhood of γ in $C^k(M, G)$. By the Chain Rule 1 (Lemma 43), $\varepsilon|_{W \times M}$ will be $C^{\infty,k}$ if we we can show that the map

$$C^k(M, U) \times M \to G, \; (\eta, x) \mapsto \varepsilon(\gamma \,.\, \eta, x) = \gamma(x)\eta(x) = \mu(\gamma(x), \varepsilon(\eta, x))$$

is $C^{\infty,k}$, where $\mu : G \times G \to G$ is the the group multiplication which is smooth, $\gamma(x)$ is C^k in x and $C^{\infty,k}$ in (η, x). By the Chain Rule 2 (Lemma 44), we only need to show that

$$\varepsilon_U : C^k(M,U) \times M \to U \subseteq G,\ (\eta,x) \mapsto \eta(x)$$

is $C^{\infty,k}$. Now we have a commutative diagram

$$\begin{array}{ccc} C^k(M,U) \times M & \xrightarrow{\varepsilon_U} & U \\ \Big\downarrow{\scriptstyle \varphi_* \times \mathrm{id}_m} & & \Big\downarrow{\scriptstyle \varphi} \\ C^k(M,V) \times M & \xrightarrow{\tilde{\varepsilon}} & V \end{array}$$

where $\tilde{\varepsilon} : C^k(M,V) \times M \to V$ is a $C^{\infty,k}$-map as a restriction of the $C^{\infty,k}$-map $C^k(M,E) \times M \to E$, $(\eta,x) \mapsto \eta(x)$ (see Proposition 42). The vertical arrows being charts, it followa that ε_U is $C^{\infty,k}$.

□

Proposition 122. *Let G be a Lie group, N be a manifold, M be a compact manifold (both possibly with boundary) and $r,k \in \mathbb{N}_0 \cup \{\infty\}$. Then a map*

$$f : N \to C^k(M,G)$$

is C^r if and only if

$$f^\wedge : N \times M \to G$$

is $C^{r,k}$.

Proof. Let $f : N \to C^k(M,G)$ is C^r. Then $f^\wedge(x,y) := f(x)(y) = \varepsilon(f(x),y)$ where $\varepsilon : C^k(M,G) \times M \to G$, $(\gamma,y) \mapsto \gamma(y)$ is $C^{\infty,k}$, by Lemma 121. Thus by Chain Rule 1 (Lemma 43), $f^\wedge$ is $C^{r,k}$.
Conversely, assume that $g := f^\wedge : N \times M \to G$ is a $C^{r,k}$-map. Then the map $g^\vee = (f^\wedge)^\vee = f$ is C^r if we can show that each $x_0 \in N$ has an open neighbourhood $W \subseteq N$ such that $g^\vee|_W$ is C^r. To achieve this, let $\varphi : U \to V \subseteq E$ be a chart of G around 1. The set $P := \{(x,y) \in N \times M : g(x,y)\, g(x_0,y)^{-1} \in U\}$ is open in $N \times M$ and contains $\{x_0\} \times M$. Because $\{x_0\}$ and M are compact, the Wallace Lemma (see [11, 3.2.10]) provides an open neighbourhood $W \subseteq N$ of x_0 such that $W \times M \subseteq P$. The map

$$h : W \times M \to U \subseteq G,\ (x,y) \mapsto g(x,y)\, g(x_0,y)^{-1}$$

is $C^{r,k}$ by Chain Rules 1 and 2 (Lemmas 43 and 44), because $g(x,y)$, $g(x_0,y)$ are $C^{r,k}$ in (x,y) and $h(x,y) = \nu(g(x,y), g(x_0,y))$ where $\nu : G \times G \to G, (a,b) \mapsto ab^{-1}$ is smooth. We claim that

$$h^\vee : W \to C^k(M,U), x \mapsto h(x,\bullet)$$

is C^r. If this is true, then also $g^\vee|_W$ is C^r, because $g^\vee(x) = h^\vee(x) \,.\, \gamma = (\rho_\gamma \circ h^\vee)(x)$ with $\gamma := g(x_0, \bullet) \in C^k(M, G)$. Using that the right translation $\rho_\gamma : C^k(M,G) \to C^k(M,G)$, $\eta \mapsto \eta \,.\, \gamma$ is smooth. To prove the claim, consider the commutative diagram

$$\begin{tikzcd} W \arrow[r, "h^\vee"] \arrow[dr, "(\varphi \circ h|_{W\times M})^\vee"'] & C^k(M,U) \arrow[d, "\varphi_*"'] \\ & C^k(M,V) \end{tikzcd}$$

where $\varphi \circ h|_{W\times M} \colon W \times M \to V \subseteq E$ is $C^{r,k}$ by definition of $C^{r,k}$-maps between manifolds and $(\varphi \circ h|_{W\times M})^\vee$ is C^r by the Vector-Valued Exponential Law in locally compact case (Theorem 59). Thus $h^\vee = (\varphi_*)^{-1} \circ (\varphi \circ h|_{W\times M})^\vee$ is C^r as well. $\square$

Lemma 123. *([17, Lemma 2.2]) A map $f : M \to C^{k+1}(I, G)$ is C^r if and only if f is C^r as a map to $C(I,G)$ and $D \circ f : M \to C^k(I, TG)$ is C^r, where $D : C^{k+1}(I,G) \to C^k(I,TG)$, $\gamma \mapsto \gamma'$.*

Proposition 124. *([17, Theorem A]) Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G is a C^k-regular, then the map*

$$\mathrm{Evol}_G : C^k(I, \mathfrak{g}) \to C^{k+1}(I, G)$$

is smooth.

Lemma 125. *Let $k \geq 2$. For each $f \in C^k(M, G)$, the 1-form $\alpha := \delta(f)$ satisfies the Maurer-Cartan equation*

$$d\alpha + \frac{1}{2}\left[\alpha, \alpha\right] = 0.$$

Proof. First we show that $\kappa_G = \delta(\mathrm{id}_G)$ satisfies the Maurer-Cartan equation. It suffices to evaluate $\mathrm{d}\alpha$ on left invariant vector fields x_l, y_l, where $x, y \in \mathfrak{g}$. Since $\kappa_G(z_l)$ constant, for each $z \in \mathfrak{g}$, we have

$$\begin{aligned} \mathrm{d}\kappa_G(x_l, y_l) &= x_l \kappa_G(y_l) - y_l \kappa_G(x_l) - \kappa_G([x_l, y_l]) = -\kappa_G([x,y]_l) = -\left[x, y\right] \\ &= -\frac{1}{2}\left[\kappa_G, \kappa_G\right](x_l, y_l). \end{aligned}$$

Therefore $\alpha = f^*\kappa_G$ satisfies

$$\mathrm{d}\alpha = f^* \mathrm{d}\kappa_G = -\frac{1}{2} f^* \left[\kappa_G, \kappa_G\right] = -\frac{1}{2}\left[f^*\kappa_G, f^*\kappa_G\right] = -\frac{1}{2}\left[\alpha, \alpha\right],$$

which is the Maurer-Cartan equation. $\square$

Remark 126. Assume that G is C^k-regular. For $\xi \in C^k(I, \mathfrak{g}), 0 \leq s \leq 1$, and $\eta(t) := \gamma_\xi(st)$, we have $\delta(\eta)(t) = s\xi(st)$. Therefore we obtain with $S : C^k(I, \mathfrak{g}) \times I \to C^k(I, \mathfrak{g})$, $\quad S(\xi, s)(t) := s\xi(st)$ the relation

$$\mathrm{Evol}_G(\xi)(s) = \gamma_\xi(s) = \mathrm{evol}_G(S(\xi, s)).$$

Lemma 127. *([21]) If G is C^k-regular, $x \in \mathfrak{g}$ and $\xi \in C^k([0,1], \mathfrak{g})$, then the initial value problem*

$$\eta'(t) = [\eta(t), \xi(t)], \quad \eta(0) = x$$

has a unique solution $\eta : [0,1] \to \mathfrak{g}$ given by

$$\eta(t) = \mathrm{Ad}(\gamma_\xi(t))^{-1} x.$$

Lemma 128. *Consider a $\mathfrak{g}$-valued 1-form on I^2 of class C^1,*

$$\alpha = v\mathrm{d}x + w\mathrm{d}y \in \Omega^1_{C^1}(I^2, \mathfrak{g}) \text{ with } v, w \in C^1(I^2, \mathfrak{g}).$$

(a) *α satisfies the Maurer-Cartan equation if and only if*

$$\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x} = [v, w]. \tag{5.2}$$

(b) *Suppose that α satisfies the Maurer-Cartan equation.*

   i. Assume that G is C^k-regular for some $k \in \mathbb{N}_0 \cup \{\infty\}$ and α of class C^k. If $f : I^2 \to G$ is C^2 with $\delta(f)(\partial_y) = w$ and $\delta(f)(\partial_x)(x, 0) = v(x, 0)$ for all $x \in I$, then $\delta(f) = \alpha$.

   ii. Assume that G is C^k-regular for some $k \in \mathbb{N}_0 \cup \{\infty\}$ and α of class C^{k+2}. Then the C^2-function $f : I^2 \to G$ defined by

$$f(x, 0) := \gamma_{v(\bullet, 0)}(x) \text{ and } f(x, y) := f(x, 0) \cdot \gamma_{w(x, \bullet)}(y)$$

   satisfies $\delta(f) = \alpha$.

Proof. (a) To evaluate the Maurer-Cartan equation for α, we first observe that

$$\frac{1}{2}[\alpha, \alpha](\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \left[\alpha(\frac{\partial}{\partial x}), \alpha(\frac{\partial}{\partial y})\right] = [v, w]$$

and obtain

$$\begin{aligned} \mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] &= \frac{\partial v}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial w}{\partial x}\mathrm{d}x \wedge \mathrm{d}y + [v, w]\,\mathrm{d}x \wedge \mathrm{d}y \\ &= (\frac{\partial w}{\partial x} - \frac{\partial v}{\partial y} + [v, w])\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

(b) *i.* We have

$$\delta f = \hat{v}\mathrm{d}x + w\mathrm{d}y \text{ with } \hat{v}(x,0) = v(x,0) \text{ for } x \in I.$$

The Maurer-Cartan equation for δf reads

$$\frac{\partial \hat{v}}{\partial y} - \frac{\partial w}{\partial x} = [\hat{v}, w]\,,$$

so that subtraction of this equation from (5.2) leads to

$$\frac{\partial (v - \hat{v})}{\partial y} = [v - \hat{v}, w]\,.$$

As $(v - \hat{v})(x,0) = 0$, the uniqueness assertion of Lemma 127, applied to $\eta_x(t) := (v - \hat{v})(x,t)$, implies that $(v - \hat{v})(x,y) = 0$ for all $x, y \in I$, hence that $v = \hat{v}$, which means that $\delta(f) = v\mathrm{d}x + w\mathrm{d}y = \alpha$.

ii. Because $v(\bullet, 0) \in C^{k+2}(I, \mathfrak{g})$ and G is C^{k+2}-regular, we have $\gamma_{v(\bullet,0)} \in C^{k+3}(I,G)$. Hence $I^2 \to G$, $(x,y) \mapsto \gamma_{v(\bullet,0)}(x)$ is a C^{k+3}-map and hence C^2. By Proposition 58, the map

$$w^\vee : I \to C^k(I, \mathfrak{g}),\ w^\vee(x)(y) := w(x,y)$$

is C^2, since w is C^{k+2} and hence $C^{2,k}$. Since

$$\mathrm{Evol}_G : C^k(I, \mathfrak{g}) \to C^{k+1}(I, G)$$

is smooth by Proposition 124, it follows that

$$\mathrm{Evol}_G \circ w^\vee : I \to C^{k+1}(I, G),\ x \mapsto \mathrm{Evol}_G(w^\vee(x)) = \gamma_{w(x,\bullet)}$$

is C^2. Hence $(\mathrm{Evol}_G \circ w^\vee)^\wedge : I \times I \to G,\ (x,y) \mapsto \gamma_{w(x,\bullet)}(y)$ is $C^{2,k+1}$ by Proposition 58. We can also consider $w^\vee$ as a C^1-map to $C^{k+1}(I, \mathfrak{g})$. Since G is also C^{k+1}-regular, arguing as before we see that

$$\mathrm{Evol}_G \circ w^\vee : I \to C^{k+2}(I, G)$$

is C^1, whence $(\mathrm{Evol}_G \circ w^\vee)^\wedge$ is $C^{1,k+2}$ using Proposition 122. Being $C^{2,k+1}$ (hence $C^{2,1}$) and $C^{1,k+2}$ (hence $C^{1,2}$), the map $(\mathrm{Evol}_G \circ w^\vee)^\wedge$ is C^2 in particular. Hence

$$f : I^2 \to G, f(x,y) := \gamma_{v(\bullet,0)}(x)\gamma_{w(x,\bullet)}(y)$$

is C^2. Now (i) shows that $\delta(f) = \alpha$.

□

Lemma 129. *Let $k \geq 2$, U be a convex subset of the locally convex space E with $U^\circ \neq \emptyset$, G a C^{k-2}-regular Lie group with Lie algebra $\mathfrak{g}$ and $\alpha \in \Omega^1_{C^k}(U, \mathfrak{g})$ be a C^k-differential form satisfying the Maurer-Cartan equation. Then α is integrable.*

Proof. We may w.l.o.g. assume that $x_0 = 0 \in U$. For $x \in U$ we consider the C^k-curve

$$\xi_x : I \to \mathfrak{g},\ t \mapsto \alpha_{tx}(x).$$

The map

$$U \times I \to \mathfrak{g},\ (x,t) \mapsto \xi_x(t)$$

is C^k hence $C^{2,k-2}$. Therefore the map $U \to C^{k-2}(I, \mathfrak{g})$, $x \mapsto \xi_x$ is C^2. Hence the function

$$f : U \to G,\ x \mapsto \mathrm{evol}(\xi_x)$$

is C^2.

First we show that $f(sx) = \gamma_{\xi_x}(s)$ holds for each $s \in I$. We have

$$S(s, \xi_x)(t) = s\xi_x(st) = \alpha_{stx}(sx) = \xi_{sx}(t)$$

and hence $f(sx) = \gamma_{\xi_x}(s)$, by Remark 126.

For $x, x+h \in U$, we consider the smooth map

$$\beta : I \times I \to U,\ (s,t) \mapsto t(x+sh)$$

and the C^2-function $F := f \circ \beta$. Then the preceding considerations imply $F(s,0) = f(0) = 1$,

$$\begin{aligned}\frac{\partial F}{\partial t}(s,t) &= \frac{d}{dt} f(t(x+sh)) = \frac{d}{dt}\gamma_{\xi_{x+sh}}(t) = F(s,t)\xi_{x+sh}(t) \\ &= F(s,t)\alpha_{t(x+sh)}(x+sh) = F(s,t)(\beta^*\alpha)_{(s,t)}(\frac{\partial}{\partial t}).\end{aligned}$$

Also, $\frac{\partial F}{\partial s}(s,0) = 0 = (\beta^*\alpha)_{(s,0)}(\frac{\partial}{\partial s})$.

As we have seen in Lemma 128(b), these relations lead to

$$\delta(F) = \beta^*\alpha \text{ on } I \times I.$$

We therefore obtain

$$\frac{d}{ds} f(x+sh) = \frac{\partial F}{\partial s}(s,1) = F(s,1)\alpha_{x+sh}(h) = f(x+sh)\alpha_{x+sh}(h),$$

and for $s = 0$ this leads to $T_x(f)(h) = f(x)\alpha_x(h)$, so that $\delta(f) = \alpha$.

$\square$

Proposition 130. *Let M be a connected manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and $\alpha \in \Omega^1(M, \mathfrak{g})$ a continuous 1-form. If α is locally integrable, then there exists a connected covering $q : \widehat{M} \to M$ such that $q^*\alpha$ is integrable. If, in addition, M is simply connected, then α is integrable.*

Proof. For the proof we refer to [21]. □

Definition 131. (a) For each locally integrable $\alpha \in \Omega^1_{C^k}(M, \mathfrak{g})$, the homomorphism

$$\mathrm{per}_\alpha^{m_0} : \pi_1(M, m_0) \to G,\ [\gamma] \mapsto \mathrm{evol}_G(\gamma^*\alpha),$$

for each piecewise smooth loop $\gamma : I \to M$ in m_0, is called the *period homomorphism of α with respect to m_0.*

(b) We write

$$\mathrm{MC}(M, \mathfrak{g}) := \left\{\alpha \in \Omega^1_{C^1}(M, \mathfrak{g}) \,:\, d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\right\}$$

for the set of solutions of the *Maurer–Cartan equation.*

Theorem 132. (Fundamental Theorem for $\mathfrak{g}$-valued functions).

Let M be a smooth manifold (possibly with boundary and modelled on a locally convex space), and G be a Lie group with a Lie algebra $\mathfrak{g}$. Then the following assertions hold:

(a) *If $k \geq 2$, G is C^{k-2}-regular and $\alpha \in \Omega^1_{C^k}(M, \mathfrak{g})$ satisfies the Maurer-Cartan equation, then α is locally integrable.*
(b) *If M is 1-connected and $\alpha \in \Omega^1_{C^0}(M, \mathfrak{g})$ is locally integrable, then it is integrable.*
(c) *Suppose that M is connected, fix $m_0 \in M$ and let $\alpha \in \mathrm{MC}(M, \mathfrak{g})$ such that α is locally integrable. Using piecewise smooth representatives of homotopy classes, we obtain a well-defined group homomorphism*

$$\mathrm{per}_\alpha^{m_0} : \pi_1(M, m_0) \to G,\ [\gamma] \mapsto \mathrm{evol}_G(\gamma^*\alpha),$$

and α is integrable if and only if this homomorphism is trivial.

Proof. (a) If α satisfies the Maurer-Cartan equation, then Lemma 129 implies its local integrability, provided G is C^{k-2}-regular.
(b) Proposition 130.
(c) For the proof we refer to [21].

□

Remark 133. If M is one-dimensional, then each $\mathfrak{g}$-valued 2-form on M vanishes, so that $[\alpha, \beta] = 0 = d\alpha$ for $\alpha, \beta \in \Omega^1_{C^1}(M, \mathfrak{g})$. Therefore all 1-forms $\alpha \in \Omega^1_{C^1}(M, \mathfrak{g})$ trivially satisfy the Maurer-Cartan equation.

Lemma 134. *Let M be a finite-dimensional manifold, V be a locally convex topological vector space and $\gamma : [0,1] \to M$ be a C^{s+1}-path with $s \in \mathbb{N}_0 \cup \{\infty\}$. Then*

$$\psi : \Omega^1_{C^s}(M,V) \to C^s([0,1],V),\ \omega \mapsto \gamma^*(\omega)$$

is a smooth map.

Proof. The evaluation map $\varepsilon : C^s(TM,V) \times TM \to V$, $(g,\omega) \mapsto g(\omega)$ is $C^{\infty,s}$, and

$$\psi^\wedge(\omega,t) = \gamma^*(\omega)(t) = \omega(\gamma'(t)) = \varepsilon(\omega,\gamma'(t)).$$

Thus $\psi^\wedge$ is $C^{\infty,s}$ by Chain Rule 1 (Lemma 43). Hence ψ is C^∞. $\square$

Lemma 135. *Let M be a compact manifold, N be a locally convex manifold, K be a Lie group with a Lie algebra $\mathfrak{k}$ and $i : C^r(M,K) \to C(M,K)$ be the inclusion map with $r,s \in \mathbb{N}_0 \cup \{\infty\}$, $r \geq 1$. A map $f : N \to C^r(M,K)$ is C^s if and only if $i \circ f : N \to C(M,K)$ is C^s and $\delta \circ f : N \to \Omega^1_{C^{r-1}}(M,\mathfrak{k})$ is C^s.*

Proof. It is well-known that i is a smooth homomorphism of groups. Also

$$\delta : C^r(M,K) \to \Omega^1_{C^{r-1}}(M,\mathfrak{k})$$

is smooth (see [36, Proposition A.4]). Hence if f is C^s, then also the compositions $i \circ f$ and $\delta \circ f$ are C^s.
Conversely, assume that $i \circ f$ and $\delta \circ f$ are C^s. Let $\varphi : U \to V$ be a chart for K around 1, such that $\varphi_* := C^r(M,\varphi) : C^r(M,U) \to C^r(M,V)$ is a chart for $C^r(M,K)$ and $C(M,\varphi)$ a chart for $C(M,K)$. Because $i \circ f$ is continuous, after replacing N by an open neighbourhood of a given point n of N, we may assume that $f(N)f^{-1}(n) \subseteq C(M,U)$. It suffices to show that $g : N \to C^r(M,K)$, $x \mapsto f(x)f(n)^{-1}$ is C^s.
Let $\pi : TM \to M$ be canonical map. Now note that $i \circ g = \rho_c \circ i \circ f$ is C^s, where we abbreviated $c := f(n)^{-1}$ and the right translation $\rho_c : C(M,K) \to C(M,K)$, $\gamma \mapsto \gamma c$ is a smooth map. Furthermore, $\delta \circ g$ is C^s. Indeed, $g(x) = f(x)f(n)^{-1}$ where $f(x)$, $f(n)^{-1} \in C^r(M,K)$. Hence $\delta(g(x)) = \mathrm{Ad}(f(n)).(\delta(f(x)) - \delta(f(n)^{-1}))$, and $\delta(f(n)^{-1})$ is independent of x, hence C^s in x. Also $\mathrm{Ad}(f(n)).\delta(f(x))$ is C^s in x, because $\delta \circ f : N \to \Omega^1_{C^{r-1}}(M,\mathfrak{k}) \subseteq C^{r-1}(TM,\mathfrak{k})$ is assumed C^s and $(\mathrm{Ad}(f(n)).\omega).(v) = \mathrm{Ad}(f(n)(\pi(v)))\omega(v) = h_*(\omega)(v)$, where $\omega \in \Omega^1_{C^{r-1}}(M,\mathfrak{k})$, $v \in TM$ and $h : TM \times \mathfrak{k} \to \mathfrak{k}$, $h(v,w) := \mathrm{Ad}(f(n)(\pi(v)))w$ is a C^r-function and linear in ω, entailing that $h_* : C^{r-1}(TM,\mathfrak{k}) \to C^{r-1}(TM,\mathfrak{k})$, $h_*(w)(v,w) := h(v,\omega(v))$ is continuous linear, hence C^s. Hence $f(N) \subseteq C^r(M,U)$ without loss of generality. Since $i \circ f$ is C^s, the map $\varphi_* \circ i \circ f : N \to C(M,V)$ is C^s. We have $(\varphi_* \circ f)(N) \subseteq C^r(M,V)$. We show that

$\varphi_* \circ f : N \to C^r(M,V)$ is C^s. As a tool, consider the set $P := \{(x,y) \in K \times K : xy \in U\}$ which is open in $U \times U$ and contains $\{1\} \times U$. Thus $Q := (\varphi \times \varphi)(P)$ is open in $V \times V$ and contains $\{0\} \times V$. The map $\nu : Q \to V$, $\nu(x,y) := \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$ is smooth. Also the map $\theta : V \times E \to E$, $(x,u) \mapsto d\nu(x,0;0,u)$ is smooth, and we have $d\varphi(x.v) = \theta(\varphi(x), d\varphi(v))$ for $x \in U$, $v \in T_1K = \mathfrak{k}$. It is known that the map $(j,d) : C^r(M,\mathfrak{k}) \to C(M,\mathfrak{k}) \times C^{r-1}(TM,\mathfrak{k}), \gamma \mapsto (\gamma, d\gamma)$ is a linear topological embedding with closed image. Hence $\varphi_* \circ f$ will be C^s if $j \circ \varphi_* \circ f$ is C^s and $\psi := d \circ \varphi_* \circ f : N \to C^{r-1}(TM,\mathfrak{k})$ is C^s. Now $j \circ \varphi_* \circ f = \varphi_* \circ i \circ f$ is C^s as just observed. By the Exponential Law (Proposition 58), ψ will be C^s if $\psi^\wedge : N \times TM \to \mathfrak{k}$ is $C^{s,r-1}$. But

$$\begin{aligned}
\psi^\wedge(x,v) &= d(\varphi \circ f(x))(v) \\
&= (d\varphi \circ T(f(x)))(v) \\
&= d\varphi(\pi(v).\delta(f(x))(v)) \\
&= \theta(\varphi(\pi(v)), d\varphi((\delta f(x))(v)))
\end{aligned}$$

and θ is C^∞, $\varphi(\pi(v))$ is C^∞ in (x,v), hence $C^{s,r-1}$ in (x,v), $d\varphi$ is C^∞ and $(\delta f(x))(v)$ is $C^{s,r-1}$ in (x,v) by the Exponential Law (Proposition 58). Thus $\psi^\wedge$ is indeed $C^{s,r-1}$, by Chain Rule 2 (Lemma 44). □

Proposition 136. *Let N be a locally convex manifold, M a connected finite-dimensional manifold and K a C^{s-1}-regular Lie group. Then a function $f : N \times M \to K$ is $C^{r,s}$ if and only if*

(a) *there exists a point $m_0 \in M$ such that $f^{m_0} : N \to K$, $n \mapsto f(n,m_0)$ is C^r, and*
(b) *the functions $f_n : M \to K$, $m \mapsto f(n,m)$ are C^s and*
$F : N \to \Omega^1_{C^{s-1}}(M,\mathfrak{k})$, $n \mapsto \delta(f_n)$ is C^r.

Proof. If f is a $C^{r,s}$-map, then the map f^{m_0} is C^r and each f_n is C^s. Since $\Omega^1_{C^{s-1}}(M,\mathfrak{k})$ is a closed vector subspace of $C^{s-1}(TM,\mathfrak{k})$, it only remains to show that the map $F : N \to C^{s-1}(TM,\mathfrak{k})$ is C^r. By Proposition 58, it suffices to show that

$$F^\wedge : N \times TM \to \mathfrak{k},\ (n,v) \mapsto \delta(f_n)v = \kappa_K(T(f_n)v)$$

is $C^{r,s-1}$.
Now the Maurer-Cartan form κ_K is a smooth map $TK \to \mathfrak{k}$ and the map

$$N \times TM \to TK,\ (n,v) \mapsto T(f_n)(v)$$

is a $C^{r,s-1}$-map (cf. Lemma 36). In view of Lemma 44 the assertion follows.
Because M can be covered by compact submanifolds L with boundary and the Pullbacks

$\Omega^1_{C^{s-1}}(M,\mathfrak{k}) \to \Omega^1_{C^{s-1}}(L,\mathfrak{k})$ induced by inclusion are continuous linear, we may assume that M is compact for the proof of the conclusion. We first show that f^m is C^r for each $m \in M$. Pick a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$. Then

$$\begin{aligned} f^m(n) &= f_n(m) = f_n(m_0)\mathrm{evol}_K(\delta(f_n \circ \gamma)) = f_n(m_0)\mathrm{evol}_K(\gamma^*\delta(f_n)) \\ &= f^{m_0}(n)\mathrm{evol}_K(\gamma^* F(n)). \end{aligned}$$

Since f^{m_0} and F are C^r, the smoothness of evol_K and the smoothness of

$$\gamma^* : \Omega^1_{C^{s-1}}(M,\mathfrak{k}) \to C^{s-1}([0,1],\mathfrak{k})$$

(see Lemma 134) imply that f^m is C^r. Now we show that f is $C^{r,s}$. Let $m \in M$ and choose a chart (ϕ, U) of M for which $\phi(U)$ is convex with $\phi(m) = 0$. We have to show that the map

$$h : N \times \phi(U) \to K,\ (n,x) \mapsto f(n, \phi^{-1}(x))$$

is $C^{r,0}$. For $\gamma_x(t) := tx$, $0 \le t \le 1$, we have

$$\begin{aligned} h(n,x) &= h(n,\gamma_x(1)) = h(n,0)\mathrm{evol}_K(\delta(f_n \circ \phi^{-1} \circ \gamma_x)) \\ &= f^m(n)\mathrm{evol}_K(\gamma_x^*(\phi^{-1})^* F(n)). \end{aligned}$$

Since f^m and F are C^r-maps and evol_K is smooth and

$$(\phi^{-1})^* : \Omega^1_{C^{s-1}}(U,\mathfrak{k}) \to \Omega^1_{C^{s-1}}(\phi(U),\mathfrak{k})$$

is a topological linear isomorphism, in view of the Chain Rule 1 (Lemma 43) it suffices to show that the map

$$\Omega^1_{C^{s-1}}(\phi(U),\mathfrak{k}) \times \phi(U) \to C^{s-1}([0,1],\mathfrak{k}),\ (\alpha, x) \mapsto \gamma_x^*\alpha$$

is $C^{r,0}$. In view of Theorem 94, this follows from the fact that the map

$$\Omega^1_{C^{s-1}}(\phi(U),\mathfrak{k}) \times (\phi(U) \times [0,1]) \to \mathfrak{k},\ (\alpha, x, t) \mapsto \gamma_x^*\alpha(t) = \alpha_{tx}(x),$$

is $C^{\infty,s-1}$ and hence $C^{\infty,0,s-1}$ (as a function of three variables). □

Lemma 137. *Let M be a connected finite-dimensional smooth manifold (possibly with boundary) and K a C^k-regular Lie group with Lie algebra $\mathfrak{k}$.*

(a) *If $\gamma : [0,1] \to M$ is a piecewise smooth curve, then the map*

$$\Omega^1_{C^k}(M,\mathfrak{k}) \to K,\ \alpha \mapsto \mathrm{evol}_K(\gamma^*\alpha)$$

is smooth.

(b) *Let* (φ, U) *be a chart of* M *for which* $\varphi(U)$ *is a convex 0-neighbourhood and* $\gamma_x(t) := \varphi^{-1}(t\varphi(x))$ *for* $x \in U, t \in [0,1]$. *Then the map*

$$\Omega^1_{C^{k+l}}(M, \mathfrak{k}) \times U \to K,\ (\alpha, x) \mapsto \mathrm{evol}_K(\gamma_x^*\alpha)$$

is $C^{\infty,l}$.

Proof. (a) This follows from the smoothness of evol_K and the fact that for each smooth path $\eta : [0,1] \to M$ the map

$$\Omega^1_{C^k}(M, \mathfrak{k}) \to C^k([0,1], \mathfrak{k}),\ \alpha \mapsto \eta^*\alpha = \alpha \circ \eta'$$

is smooth (see Lemma 134).

(b) We may assume that $M = U = \phi(U)$ and $\phi = \mathrm{id}_U$. Since K is C^k-regular, we only need to show that the map

$$\Omega^1_{C^{k+l}}(U, \mathfrak{k}) \times U \to C^k([0,1], \mathfrak{k}),\ (\alpha, x) \mapsto \gamma_x^*\alpha$$

is $C^{\infty,l}$. By the linearity in the first argument, we only need to show that the map is $C^{0,l}$. By the Exponential Law for C^α maps (Proposition 103), we only need to show that

$$\Omega^1_{C^{k+l}}(U, \mathfrak{k}) \times U \times [0,1] \to \mathfrak{k},\ (\alpha, x, t) \mapsto (\gamma_x^*\alpha)_t = \alpha_{\gamma_x(t)}\gamma_x'(t),$$

is $C^{\infty,l,k}$ as a function of 3 variables, which holds if it is $C^{\infty,l+k}$ as a function of the 2 variables $(\alpha, (x,t))$. But $\alpha_{\gamma_x(t)}\gamma_x'(t) = \alpha(xt, x) = \varepsilon(\alpha, (xt, x))$ is $C^{\infty,l+k}$, like $\varepsilon : C^{l+k}(TU, \mathfrak{k}) \times TU \to \mathfrak{k}$.

□

Chapter 6

The mapping group as an infinite-dimensional Lie group

In this chapter we study Lie group structures on groups of the form $C^k(M, K)$, where M is a non-compact smooth manifold and K is a, possibly infinite-dimensional, Lie group. All finite-dimensional manifolds considered in this chapter are assumed to be paracompact, without further mentions.

6.1 Lie group structure on mapping groups

Proposition 138. *Let M be a connected finite-dimensional smooth manifold and K a regular Lie group. Assume that the group $G := C^k(M, K)$ carries a Lie group structure which is compatible with evaluations in the sense that $\mathfrak{g} := C^k(M, \mathfrak{k})$ is the corresponding Lie algebra and all point evaluations $\mathrm{ev}_m : G \to K, m \in M$, are smooth with*

$$\mathrm{L}(\mathrm{ev}_m) = \mathrm{ev}_m : \mathfrak{g} \to \mathfrak{k}.$$

Then the following holds:

(a) *The evaluation map $\mathrm{ev} : G \times M \to K, (f, m) \mapsto f(m)$ is $C^{\infty,k}$.*

(b) *If N is a locally convex C^r-manifold and $f : N \to G$ is C^r, then $f^\wedge : N \times M \to K$ is $C^{r,k}$.*

(c) *If, in addition, G is C^{r-3}-regular, where $r \geq 3$, then a map $f : N \to G$ is C^r if and only if the corresponding map $f^\wedge : N \times M \to K$ is $C^{r,k}$.*

Proof. (a) Let $N \subseteq M$ be a compact submanifold (possibly with boundary). Then $C^k(N, K)$ carries the structure of a regular Lie group (see [21]). Let $q_G : \tilde{G}_0 \to G_0$

denote the universal covering of the identity component G_0 of G. Consider the continuous homomorphism of Lie algebras

$$\psi : \mathrm{L}(G) = C^k(M, \mathfrak{k}) \to C^k(N, \mathfrak{k}), \quad f \mapsto f\mid_N .$$

In view of the regularity of $C^k(N, K)$, there exists a unique morphism of Lie groups

$$\tilde{\varphi} : \tilde{G}_0 \to C^k(N, K) \quad \text{with} \quad \mathrm{L}(\tilde{\varphi}) = \psi.$$

Then, for each $n \in N$, the homomorphism $\mathrm{ev}_n \circ \tilde{\varphi} : \tilde{G}_0 \to K$ is smooth with differential $\mathrm{L}(\mathrm{ev}_n \circ \tilde{\varphi}) = \mathrm{ev}_n$, so that $\mathrm{ev}_n \circ \tilde{\varphi} = \mathrm{ev}_n \circ q_G$. We conclude that

$$\ker q_G \subseteq \ker \tilde{\varphi},$$

and hence that $\tilde{\varphi}$ factors to the restriction map $\rho : C^k(M, K)_0 \to C^k(N, K)$, i.e., $\tilde{\varphi} = \rho \circ q_G$. In particular, the restriction map $C^k(M, K) \to C^k(N, K)$ is a smooth homomorphism of Lie groups. Since $\varepsilon : C^k(N, K) \times N \to K$ is a $C^{\infty,k}$-map, by Theorem 120, (a) follows.

(b) If f is C^r, then $f^\wedge = \mathrm{ev} \circ (f \times \mathrm{id}_M)$ is $C^{r,k}$, using that ev is $C^{\infty,k}$ by (a).

(c) We may w.l.o.g assume that N is 1-connected. If $f^\wedge$ is $C^{r,k}$, we define $\beta \in \Omega^1_{C^{r-1}}(N, \mathfrak{g})$ by

$$\beta(\nu)(m) = \kappa_K(T(f^\wedge(\cdot, m))(\nu)),$$

which is a $C^{r-1,k}$-map $TN \times M \to \mathfrak{k}$.

We claim that β satisfies the Maurer-Cartan equation. Since the evaluation map $\mathrm{ev}_m : \mathfrak{g} \to \mathfrak{k}$ is a continuous homomorphism of Lie algebras, and the corresponding maps $(\mathrm{ev}_m)_* : \Omega^2_{C^{r-2}}(N, \mathfrak{g}) \to \Omega^2_{C^{r-2}}(N, \mathfrak{k})$, $\omega \mapsto \mathrm{ev}_m \circ \omega$ separate the points, for $m \in M$ it follows that β satisfies the Maurer-Cartan equation, using that $\beta(\nu)(m) = \delta f^\wedge(\cdot, m)(\nu)$.

Fix a point $n_0 \in N$. The Fundamental Theorem (Theorem 132) implies the existence of a unique C^r-map $h : N \to G$ with $h(n_0) = f(n_0)$ and $\delta(h) = \beta$. Then

$$\delta(\mathrm{ev}_m \circ h) = \mathrm{ev}_m \circ \delta(h) = \mathrm{ev}_m \circ \beta = \delta(\mathrm{ev}_m \circ f),$$

so Lemma 112, applied to K-valued functions, yields $\mathrm{ev}_m \circ h = \mathrm{ev}_m \circ f$ for each m, which leads to $h = f$. This proves that f is a C^r-map.

□

Example 139. If M is a compact manifold (possibly with boundary), then the ordinary Lie group structure on $G := C^k(M, K)$ is compatible with evaluations. To identify

$T_1(G)$ with $C^k(M, \mathfrak{k})$, pick a chart $\varphi : U \to V \subseteq \mathfrak{k}$ of K around 1 such that $\varphi(1) = 0$ and $d\varphi|_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$. Then $\psi := d(\varphi_*)|_{T_1 G} \colon T_1 G \to C^k(M, \mathfrak{k})$ is a suitable isomorphism (cf. [15]).

Note that ψ^{-1} is the map $C^k(M, \mathfrak{k}) \to T_1 G$, $\gamma \mapsto \frac{d}{dt}|_{t=0}(\varphi^{-1} \circ t\gamma)$. If K has a smooth exponential function, then ψ^{-1} coincides with the map $\gamma \mapsto \frac{d}{dt}|_{t=0}(\exp_K \circ(t\gamma))$, because the smooth map $(\varphi \circ \exp_K)_* : C^k(M, \mathfrak{k}) \to C^k(M, \mathfrak{k})$, $\gamma \mapsto \varphi \circ \exp_K \circ \gamma$ satisfies $d(\varphi \circ \exp_K)_*(0, \bullet) = \mathrm{id}$ and thus

$$\begin{aligned} \frac{d}{dt}|_{t=0}(\exp_K \circ(t\gamma)) &= \frac{d}{dt}|_{t=0}(\varphi^{-1} \circ \varphi \circ \exp_K \circ(t\gamma)) \\ &= d(\varphi^{-1})_*(0, d(\varphi \circ \exp_K)_*(\gamma)) \\ &= \psi^{-1}. \end{aligned}$$

Remark 140. If K is regular and M as in Proposition 138, then a Lie group structure on $G := C^k(M, K)$ compatible with evaluations is unique whenever it exists. In fact, assume that there is another structure $\tilde{G}$. Let $f : \tilde{G} \to G$ and $g : G \to G$ be the maps $x \mapsto x$. Because g is smooth, the map $f^\wedge = g^\wedge$ is $C^{\infty,k}$ by Proposition 138(b) and hence f is smooth by Proposition 138(c). Likewise, f^{-1} is smooth and thus $\tilde{G} = G$.

Proposition 141. *If K is a C^{k-1}-regular Lie group, M a connected finite-dimensional smooth manifold and $k \geq 2$, then the map*

$$\delta : C^k_*(M, K) \to \Omega^1_{C^{k-1}}(M, \mathfrak{k})$$

is a topological embedding. Let $\mathrm{Evol}_K := \delta^{-1} : \mathrm{im}(\delta) \to C^k_*(M, K)$ denote its inverse. Then δ is an isomorphism of topological groups if we endow $\mathrm{im}(\delta)$ with the group structure defined by

$$\alpha * \beta := \beta + \mathrm{Ad}(\mathrm{Evol}_K(\beta))^{-1} \cdot \alpha \tag{6.1}$$

and

$$\alpha^{-1} := -\mathrm{Ad}(\mathrm{Evol}_K(\alpha)) \cdot \alpha. \tag{6.2}$$

Proof. By definition of the topology on $C^k(M, K)$, the tangent map induces a continuous group homomorphism

$$T : C^k(M, K) \to C^{k-1}(TM, TK), \quad f \mapsto T(f).$$

Let $\kappa_K : TK \mapsto \mathfrak{k}$ denote the (left) Maurer-Cartan form of K. Since $\delta(f) = f^*\kappa_K = \kappa_K \circ T(f)$, it follows that the composition

$$C^k(M, K) \to C^{k-1}(T(M), T(K)) \to C^{k-1}(T(M), \mathfrak{k}), \quad f \mapsto T(f) \mapsto \delta(f)$$

is continuous.

Next we show that δ is an embedding. Consider $\alpha = \delta(f)$ with $f \in C^k_*(M, K)$, i.e., $f(m_0) = 1$ holds for the base point $m_0 \in M$. To reconstruct f from α, since M is connected, we can find for $m \in M$ a smooth path $\gamma : [0, 1] \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$. Then $\delta(f \circ \gamma) = \gamma^*\delta(f) = \gamma^*\alpha$ implies $f(m) = \mathrm{evol}_K(\gamma^*\alpha)$.

We now choose an open neighbourhood U of m and a chart (φ, U) of M such that $\varphi(U)$ is convex with $\varphi(m) = 0$. For each $x \in U$ define $\gamma_x : [0, 1] \to U, \quad \gamma_x(t) := \varphi^{-1}(t\varphi(x))$. Then

$$\delta(f(m)^{-1}(f \circ \gamma_x)) = \delta(f \circ \gamma_x) = \gamma_x^*\,\delta f = \gamma_x^*\,\alpha$$

implies that $f(m)^{-1}f(x) = \mathrm{evol}_K(\gamma_x^*\,\alpha)$ and hence

$$f(x) = f(m).\,\mathrm{evol}_K(\gamma_x^*\,\alpha)$$

From Lemma 137, we immediately derive that the map

$$\Omega^1_{C^{k-1}}(M, \mathfrak{k}) \times U \to K, (\alpha, x) \mapsto \mathrm{evol}_K(\gamma^*\alpha) \cdot \mathrm{evol}_K(\gamma_x^*\alpha)$$

is continuous so that the corresponding map $\Omega^1_{C^{k-1}}(M, \mathfrak{k}) \mapsto C^0(U, K)$ is continuous. We conclude that the map

$$\delta(C^k_*(M, k)) \to C^0(U, K), \quad \delta(f) \mapsto f\mid_U$$

is continuous. We finally observe that for each open covering $M = \bigcup_{j \in J} U_j$, the restriction maps to U_j lead to a topological embedding $C^0_*(M, K) \hookrightarrow \prod_{j \in J} C^0(U_j, K)$. Hence

$$\delta(C^k_*(M, \mathfrak{k})) \to C^0(M, K), \quad \delta(f) \mapsto f$$

is continuous.

Now, we show by induction that

$$\theta_j : \delta(C^k_*(M, \mathfrak{k})) \to C^j(M, K), \quad \delta(f) \mapsto f$$

is continuous for $j = 0, \ldots, k$. The topology on $C^j(M, K)$ is initial with respect to inclusion $C^j(M, K) \to C^0(M, K)$ and the map $T : C^j(M, K) \to C^{j-1}(TM, TK)$. Because $\mathrm{incl} \circ \theta_j = \theta_0$ is continuous, the map θ_j will be continuous if we can show that also $T \circ \theta_j$ is continuous. Let m be the continuous group multiplication of $C^{j-1}(TM, TK)$. We have $Tf = f \cdot \delta f = f \cdot \alpha$ for $\alpha = \delta f$ and thus $T\theta_j(\alpha) = \theta_{j-1}(\alpha) \cdot \alpha$ inside $C^{j-1}(TU, TK)$. Because the inclusion $\Omega^1_{C^{k-1}}(U, \mathfrak{k}) \hookrightarrow C^{j-1}(TU, TK)$ is continuous, also $T \circ \theta_j = m \circ (\theta_{j-1} \times \mathrm{incl})$ is continuous (since θ_{j-1} is continuous by induction). $\square$

Theorem 142. *Let $s, k \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq s+1$, M be a connected finite-dimensional smooth manifold (with boundary), $m_0 \in M$ and K a C^s-regular Lie group. Assume that the subset $\delta(C^k_*(M,K))$ is a smooth submanifold of $\Omega^1_{C^{k-1}}(M,\mathfrak{k})$. Endow $C^k_*(M,K)$ with the smooth manifold structure for which $\delta : C^k_*(M,K) \to \operatorname{im}(\delta)$ is a diffeomorphism and endow*

$$C^k(M,K) \cong K \ltimes C^k_*(M,K)$$

with the product manifold structure. Assume that L_j for $j \in J$ are compact submanifolds (with boundary) of M with $m_0 \in L_j$ whose interiors L_j° cover M, and such that

$$\delta_j : C^k_*(L_j,K) \to \Omega^1_{C^{k-1}}(L_j,\mathfrak{k})$$

is an embedding of smooth manifolds onto a submanifold of $\Omega^1_{C^{k-1}}(L_j,\mathfrak{k})$. Then the following assertions hold:

(a) *For each $r \in \mathbb{N}_0 \cup \{\infty\}$ and locally convex C^r-manifold N, a map $f : N \times M \to K$ is $C^{r,k}$ if and only if for all $n \in N$, $f_n : M \to K$, $m \mapsto f(n,m)$ are C^k and the corresponding map*

$$f^\vee : N \to C^k(M,K), \quad n \mapsto f_n$$

is C^r.

(b) *K acts smoothly by conjugation on $C^k_*(M,K)$, and $C^k(M,K)$ carries a C^s-regular Lie group structure compatible with evaluations.*

Proof. (a) Let m_0 be the base point of M. According to Proposition 136, $f : N \times M \to K$ is $C^{r,k}$ if and only if f^{m_0} is C^r, all the maps f_n are C^k, and $\delta \circ f^\vee : N \to \Omega^1_{C^{k-1}}(M,\mathfrak{k})$ is C^r. In view of our definition of the manifold structure on $C^k_*(M,K)$, the latter condition is equivalent to the C^r-property for the map $N \to C^k_*(M,K)$, $\quad n \mapsto f_n(m_0)^{-1} f_n = f^{m_0}(n)^{-1} f_n$. Since the evaluation in m_0 coincides with the projection

$$G := C^k(M,K) \cong K \ltimes C^k_*(M,K) \to K,$$

we see that f is $C^{r,k}$ if and only if all the maps f_n are C^k and $f^\vee$ is C^r.

(b) For the evaluation map $f = \operatorname{ev} : C^k(M,K) \times M \to K$, we have $\operatorname{ev}^\vee = \operatorname{id}_G$ with $G := C^k(M,K)$, and $\operatorname{ev}_g = g$ for each $g \in G$. Hence (a) implies that ev is $C^{\infty,k}$.

In view of Proposition 141, δ is an isomorphism of topological groups if $\operatorname{im}(\delta)$ is endowed with the group structure (6.1). We now show that the operations (6.1) and (6.2) are smooth with respect to the submanifold structure on $\operatorname{im}(\delta)$.

The Lie group structure: To see that $C^k_*(M,K)$ is a Lie group, it suffices to show that the map

$$\theta : \mathrm{im}(\delta) \times \mathrm{im}(\delta) \to \Omega^1_{C^{k-1}}(M, \mathfrak{k}), \quad (\alpha, \beta) \mapsto \mathrm{Ad}(\mathrm{Evol}_K(\alpha)) \cdot \beta$$

is smooth. Let $(L_j)_{j\in J}$ be a family of compact submanifolds (with boundary) of M whose interiors L_j° cover M, as described in the theorem. Then

$$\Omega^1_{C^{k-1}}(M, \mathfrak{k}) \to \prod_{j\in J} \Omega^1_{C^{k-1}}(L_j, \mathfrak{k}),\ \alpha \mapsto (\alpha\,|_{TL_j})_{j\in J}$$

is linear and a topological embedding with closed image. Let

$$\rho_j : \Omega^1_{C^{k-1}}(M, \mathfrak{k}) \to \Omega^1_{C^{k-1}}(L_j, \mathfrak{k}),\ \alpha \mapsto \alpha\,|_{TL_j}$$

be the restriction map. Then θ will be smooth if we can show that $\rho_j \circ \theta$ is smooth for each $j \in J$. Now by the assumption and using the Lie group structure on $C^k_*(L_j, K)$, the map $\theta_j : \mathrm{im}(\delta_j) \times \mathrm{im}(\delta_j) \to \Omega^1_{C^{k-1}}(L_j, \mathfrak{k})$ analogous to θ is smooth.
Consider the commutative diagram, in which ψ is the restriction map

$$\begin{array}{ccc} \mathrm{im}(\delta) \times \mathrm{im}(\delta) & \xrightarrow{\theta} & \Omega^1_{C^{k-1}}(M, \mathfrak{k}) \\ \big\downarrow \psi & & \big\downarrow \rho_j \\ \mathrm{im}(\delta_j) \times \mathrm{im}(\delta_j) & \xrightarrow{\theta_j} & \Omega^1_{C^{k-1}}(L_j, \mathfrak{k}) \end{array}$$

In the above diagram $\rho_j \circ \theta = \theta_j \circ \psi$ is smooth, thus θ is smooth.

To see that $C^k(M, K) = K \ltimes C^k_*(M, K)$ is a Lie group, it remains to show that the action

$$\sigma : K \times C^k_*(M, K) \to C^k_*(M, K), (g, \gamma) \mapsto g\gamma g^{-1}$$

is smooth. This holds if and only if $\delta \circ \sigma$ is smooth. Now for $g \in K$, $\gamma \in C^k_*(M, K)$.

$$\delta(\sigma(\gamma)) = \delta(\gamma g^{-1}) = \mathrm{Ad}(g^{-1})^{-1}\delta(\gamma) + \underbrace{\delta(g^{-1})}_{=0} = \mathrm{Ad}(g)\delta(\gamma)$$

(considering g as a constant path in $C^k(M, K)$). Equivalently, writing $\delta(\gamma) = \alpha$, we thus have to show that

$$K \times \mathrm{im}(\delta) \to \mathrm{im}(\delta),\ (g, \alpha) \mapsto \mathrm{Ad}(g).\alpha$$

is smooth. This follows if

$$\tau : K \times C^{k-1}(TM, \mathfrak{k}) \to C^{k-1}(TM, \mathfrak{k}),\ (g, \gamma) \mapsto \mathrm{Ad}(g).\gamma$$

is smooth. Now $\tau^\wedge(g,\gamma,v) = \mathrm{Ad}(g)\gamma(v) = \mathrm{Ad}(g)\varepsilon(\gamma,v)$ is $C^{\infty,\infty,k-1}$ in (g,γ,v), by the Chain Rule for C^α-maps (Lemma 81), with evaluation $\varepsilon : C^{k-1}(TM,\mathfrak{k}) \times TM \to \mathfrak{k}$ which is $C^{\infty,k-1}$ (Proposition 42). Hence $\tau^\wedge$ is $C^{\infty,k-1}$ in $((g,\gamma),v)$ and thus $\tau = (\tau^\wedge)^\vee$ is C^∞ indeed.

If M is compact, then the Lie group structure on $C^k_*(M,K)$ coincides with the ordinary one. Indeed, write $C^k_*(M,K)_{ord}$ for the latter.
Also, write $f : C^k_*(M,K)_{ord} \to C^k_*(M,K)$, $g : C^k_*(M,K)_{ord} \to C^k_*(M,K)_{ord}$ and $h : C^k_*(M,K) \to C^k_*(M,K)$ for the maps given by $\gamma \mapsto \gamma$. Since h is smooth, $h^\wedge = f^\wedge$ is $C^{\infty,k}$ by (a). Hence f is smooth, by (a). Likewise, g is smooth, whence $g^\wedge = (f^{-1})^\wedge$ is $C^{\infty,k}$ (see Proposition 122). Hence f^{-1} is smooth, by Proposition 122. Thus f is isomorphism and thus $C^k_*(M,K)_{ord} = C^k_*(M,K)$. To emphasize the dependence on M, we occasionally write δ_M instead of δ. If M_1 with $m_1 \in M_1$ has properties analogous to M and $f : M_1 \to M$ is a smooth map with $f(m_1) = m_0$, then

$$f^* : C^k(M,K) \to C^k(M_1,K),\ \gamma \mapsto \gamma \circ f$$

is a smooth homomorphism of Lie groups and the diagram

$$\begin{array}{ccc} C^k(M,K) & \xrightarrow{\delta_M} & \Omega^1_{C^{k-1}}(M,\mathfrak{k}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ C^k(M_1,K) & \xrightarrow{\delta_{M_1}} & \Omega^1_{C^{k-1}}(M_1,\mathfrak{k}) \end{array} \tag{6.3}$$

commutes, where we also use the continuous linear (and hence smooth) map

$$f^* : \Omega^1_{C^{k-1}}(M,\mathfrak{k}) \to \Omega^1_{C^{k-1}}(M_1,\mathfrak{k}),\ \omega \mapsto f^*\omega.$$

Indeed (6.3) commutes because

$$f^*(\delta_M(\gamma)) = f^*(\gamma^*(\kappa_K)) = (\gamma \circ f)^*(\kappa_K) = (f^*(\gamma))^*(\kappa_K) = \delta_{M_1}(f^*(\gamma))$$

using the Maurer-Cartan form κ_K on K. Since δ_M and δ_{M_1} are isomorphisms onto their images, and f^* on the left-hand side of (6.3) is a group homomorphism, also the smooth map

$$f^* : \mathrm{im}(\delta_M) \to \mathrm{im}(\delta_{M_1})$$

is a homomorphism of groups.

The Lie algebra: We first determine the Lie algebra of $G_* := C^k_*(M,K)$ in the special case $M = [0,1]$. We know that

$$\delta : G_* \to C^{k-1}([0,1],\mathfrak{k}),\ \gamma \mapsto \delta\gamma$$

is an isomorphism of Lie groups. Also, it is known that $\mathrm{L}(\delta) \circ \psi^{-1} = \mathrm{d}\delta|_{T_1 G} \circ \psi^{-1}$ is the map $\gamma \mapsto \gamma'$, where $\psi := d(\varphi_*)|_{T_1 G}$ is the ususal isomorphism between $T_1 G$ and the Lie algebra $C^k_*([0,1],\mathfrak{k})$ (see [17]). Hence

$$\mathrm{L}(\delta) \circ \psi^{-1} : C^k_*([0,1],\mathfrak{k}) \to C^{k-1}([0,1],\mathfrak{k}),\ \gamma \mapsto \gamma'$$

is an isomorphism of topological Lie algebras, if the pointwise Lie bracket is used on the left-hand side.

General case: For general M, We first determine the tangent space $T_0(\mathrm{im}(\delta))$ to see the Lie algebra of this group. Let $\eta : I \to \mathrm{im}(\delta)$ be a C^k-curve with $\eta(0) = 0$ and $\beta := \eta'(0)$. Then

$$1 = \mathrm{per}^{m_0}_{\eta(t)}(\gamma) = \mathrm{evol}_K(\gamma^* \eta(t))$$

for each smooth loop γ in m_0 and each $t \in I$. Taking the derivative in $t = 0$, we get:

$$0 = T_0(\mathrm{evol}_K)(\gamma^* \beta) = \int_0^1 \gamma^* \beta = \int_\gamma \beta$$

(see [17]). Hence all periods of β vanish, so that β is exact. If, conversely, $\beta \in \Omega^1_{C^{k-1}}(M,\mathfrak{k})$ is an exact 1-form, then $\beta = \mathrm{d}f$ for some $f \in C^k_*(M,\mathfrak{k})$. We show that the curve

$$\alpha : [0,1] \to \mathrm{im}(\delta),\ t \mapsto \delta(\exp_K \circ (tf))$$

satisfies $\alpha'(0) = \beta$. For $x \in M$ and $v \in T_x M$, choose a smooth path γ in M from m_0 to x, such that $\gamma'(1) = v$. Then

$$\begin{aligned}
\alpha'(0)(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \delta(\exp_K \circ (tf))(v) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \delta(\exp_K \circ (tf))(\gamma'(1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma^*(\delta(\exp_K \circ tf))(1) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \delta_{[0,1]}(\exp_K \circ t(f \circ \gamma))(1) \\
&= \mathrm{d}\delta_{[0,1]}(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp_K \circ t(f \circ \gamma)))(1) = \mathrm{d}\delta_{[0,1]}(\psi^{-1}(f \circ \gamma))(1) \\
&= (f \circ \gamma)'(1) = \mathrm{d}f(\gamma'(1)) = \mathrm{d}f(v) = \beta(v)
\end{aligned}$$

thus $\alpha'(0) = \beta$. This shows that

$$T_0(\mathrm{im}(\delta)) = \mathrm{d}C^k_*(M,\mathfrak{k}) \cong C^k_*(M,\mathfrak{k})$$

as a topological vector space (apply Proposition 141 to the Lie group $(\mathfrak{k},+)$). By the preceeding, the map

$$\mathrm{d} : C^k_*(M,\mathfrak{k}) \to T_0(\mathrm{im}(\delta))$$

is an isomorphism of topological vector spaces. We now show that d is a homomorphism (hence an isomorphism) of Lie algebras if $C^k_*(M,\mathfrak{k})$ is endowed with the pointwise Lie bracket. We already know this if $M=[0,1]$. In the general case, note that the maps

$$\gamma^* : T_0(\mathrm{im}(\delta)) \to C^{k-1}([0,1],\mathfrak{k}),\ \omega \mapsto \gamma^*(\omega)$$

separate points (for γ ranging through the set of all smooth paths in M starting in m_0). Moreover, γ^* is a Lie algebra homomorphism, as it is the tangent map at 0 of the analogous smooth group homomorphism

$$\gamma^* : \mathrm{im}(\delta) \to C^{k-1}([0,1],\mathfrak{k}).$$

It therefore suffices to show that $\gamma^* \circ \mathrm{d}$ is a Lie algebra homomorphism for each γ. But

$$(\gamma^* \circ \mathrm{d})(f) = \gamma^*(\mathrm{d}f) = \mathrm{d}f \circ \gamma' = (f\circ\gamma)' = (\gamma^*(f))'$$

for $f \in C^k_*(M,\mathfrak{k})$, where $\gamma^* : C^k_*(M,\mathfrak{k}) \to C^k_*([0,1],\mathfrak{k})$ is a Lie algebra homomorphism and so is

$$C^k_*([0,1],\mathfrak{k}) \to C^{k-1}_*([0,1],\mathfrak{k}),\ f \mapsto f',$$

by the special case of $[0,1]$. Hence $\gamma^* \circ \mathrm{d}$ is a Lie algebra homomorphism. Consider the map

$$\Psi : C^k_*(M,\mathfrak{k}) \to T_1(G_*),\ f \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp_K \circ (tf)).$$

By the Chain Rule and the preceding, we have

$$T_1\delta\Psi(f) = T_1\delta\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp_K \circ tf) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\delta(t \mapsto \exp_K \circ tf) = \mathrm{d}f$$

for $f \in C^k_*(M,\mathfrak{k})$, i.e., $T_1(\delta)\circ\Psi = \mathrm{d}$. Since $T_1(\delta)$ and d are isomorphisms of topological Lie algebras, also Ψ is an isomorphism of topological Lie algebras. We mention that the maps $\mathrm{L}(\mathrm{ev}_x)$, for $\mathrm{ev}_x : C^k_*(M,K) \to K,\ f \mapsto f(x)$, separate points on $\mathrm{L}(G_*)$ [1]. It suffices to show that the maps

$$\mathrm{L}(\mathrm{ev}_x) \circ \Psi : C^k_*(M,\mathfrak{k}) \to \mathfrak{k}$$

separate points on $C^k_*(M,\mathfrak{k})$. This follows if we can establish

$$\mathrm{L}(\mathrm{ev}_x) \circ \Psi = \varepsilon_x \tag{6.4}$$

[1] ev_x is smooth because $x \in L_n$ for some n and the restriction map $C^k_*(M,K) \to C^k_*(L_n,K)$ is smooth, as well as evaluation at x on $C^k_*(L_n,K)$.

with $\varepsilon_x : C^k_*(M,\mathfrak{k}) \to \mathfrak{k}$, $f \mapsto f(x)$. But indeed, using the Chain Rule twice,

$$\begin{aligned}
\mathrm{L}(\mathrm{ev}_x)\Psi(f) &= \mathrm{L}(\mathrm{ev}_x)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp_K \circ tf)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{ev}_x(\exp_K \circ tf)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp_K(tf(x))\\
&= T_0\exp_K f(x) = f(x) = \varepsilon_x(f).
\end{aligned}$$

Let $i : K \to C^k(M,K)$ be the map taking $g \in K$ to the constant function $x \mapsto g$. Then

$$G = C^k(M,K) = C^k_*(M,K) \ltimes i(K)$$

internally, entailing that

$$\mathrm{L}(C^k(M,K)) = \mathrm{L}(C^k_*(M,K)) \ltimes \mathrm{L}(i(K))$$

internally. Hence

$$H : C^k(M,\mathfrak{k}) = C^k_*(M,\mathfrak{k}) \ltimes \mathfrak{k} \to \mathrm{L}(C^k(M,K)),\ \eta = \gamma + v \mapsto \Psi(\gamma) + \mathrm{L}(i)(v) \tag{6.5}$$

(for $\gamma = \eta - \eta(m_0) \in C^k_*(M,\mathfrak{k})$, $v = \eta(m_0) \in \mathfrak{k}$) is an isomorphism of topological vector spaces.

Consider the evaluation maps [1] $\mathrm{ev}_x : C^k(M,K) \to K$. Then the maps $\mathrm{L}(\mathrm{ev}_x)$ separate points on $\mathrm{L}(G)$. Indeed, $\ker(\mathrm{L}(\mathrm{ev}_{m_0})) = \mathrm{L}(C^k_*(M,K))$ because $G = G_* \rtimes K$ with ev_{m_0} the projection onto K. It therefore only remains to check that the maps $\mathrm{L}(\mathrm{ev}_x)$ separate points on $\mathrm{L}(C^k_*(M,K))$. But this has been already checked.

Since each $\mathrm{L}(\mathrm{ev}_x)$ is a Lie algebra homomorphism, H will be a Lie algebra homomorphism (hence an isomorphism) if we can show that $\mathrm{L}(\mathrm{ev}_x) \circ H$ is a Lie algebra homomorphism for each $x \in M$. The restriction of this map to $C^k_*(M,\mathfrak{k})$ is $\mathrm{L}(\mathrm{ev}_x) \circ \Psi$, hence a Lie algebra homomorphism. Moreover, the restriction to the constant functions corresponds to $\mathrm{L}(\mathrm{ev}_x) \circ \mathrm{L}(i)$ on $\mathfrak{k}$, and hence is a Lie algebra homomorphism. Because $C^k(M,\mathfrak{k}) = C^k_*(M,\mathfrak{k}) \rtimes \mathfrak{k}$, it only remains to show that

$$\mathrm{L}(\mathrm{ev}_x)H([\gamma,v]) = [\mathrm{L}(\mathrm{ev}_x)H(\gamma), \mathrm{L}(\mathrm{ev}_x)H(v)] \tag{6.6}$$

for $\gamma \in C^k_*(M,\mathfrak{k})$, $v \in \mathfrak{k}$. The left-hand side of (6.6) is $\mathrm{L}(\mathrm{ev}_x)\Psi([\gamma,v]) = \varepsilon_x([\gamma,v]) = [\gamma(x),v]$, (using (6.4)). The right-hand side of (6.6) is

$$[\mathrm{L}(\mathrm{ev}_x)\Psi(\gamma), \mathrm{L}(\mathrm{ev}_x)\mathrm{L}(i)(v)] = [\varepsilon_x(\gamma), \mathrm{L}(\underbrace{\mathrm{ev}_x \circ i}_{=\mathrm{id}})(v)] = [\gamma(x), v]$$

[1]These are smooth because ev_{m_0} is smooth and $\mathrm{ev}_x(f) = \mathrm{ev}_x(f\, i(\mathrm{ev}_{m_0}(f))^{-1})\mathrm{ev}_{m_0}(f)$, using the smooth evaluation map ev_x on $C^k_*(M,K)$ on the right-hand side.

as well. Hence H is an isomorphism of topological Lie algebras. Identifying $C^k(M,\mathfrak{k})$ with $\mathrm{L}(G)$ via H, the map $\mathrm{L}(\mathrm{ev}_x)$ corresponds to point evaluation

$$\delta_x : C^k(M,\mathfrak{k}) \to \mathfrak{k}, f \mapsto f(x),$$

i.e.,

$$\mathrm{L}(\mathrm{ev}_x) \circ H = \delta_x. \tag{6.7}$$

In fact, it suffices to show that both sides of (6.7) coincide on both $C^k_*(M,\mathfrak{k})$ and $\mathfrak{k}$. For $\gamma \in C^k_*(M,\mathfrak{k})$, we have $\mathrm{L}(\mathrm{ev}_x)H(\gamma) = \mathrm{L}(\mathrm{ev}_x)\Psi(\gamma) = \gamma(x)$ indeed. For $v \in \mathfrak{k}$, we have $\mathrm{L}(\mathrm{ev}_x)H(v) = \mathrm{L}(\mathrm{ev}_x)\mathrm{L}(i)(v) = v$ as well.

Remark 143. (a) The restriction maps $\rho_n : C^k(M,K) \to C^k(L_n,K)$, $\gamma \mapsto \gamma|_{L_n}$ are smooth, because their restrictions ρ'_n to $C^k_*(M,K)$ are smooth and $\rho_n = {\rho'}_n \times \mathrm{id}_K$ if we identify $C^k(M,K)$ with $C^k_*(M,K) \rtimes K$ and $C^k(L_n,K)$ with $C^k_*(L_n,K) \rtimes K$. Now a map $f : N \to G$ from a manifold N to G is smooth if and only if $\rho_n \circ f : N \to C^k(L_n,K)$ is smooth for each n. In fact, assume that $\rho_n \circ f$ is smooth. Then $n \mapsto f(m_0) = (\rho_n \circ f)(m_0)$ is smooth, and after replacing f with $n \mapsto f(x)f(x)(m_0)^{-1}$, we may assume that $\mathrm{im}(f) \subseteq G_*$. $\delta_n \circ \rho_n \circ f = i_n^* \circ \delta \circ f$ is smooth, where $i_n : L_n \to M$ is the inclusion map and $i_n^* : \Omega^1_{C^{k-1}}(M,\mathfrak{k}) \to \Omega^1_{C^{k-1}}(L_n,\mathfrak{k})$. Since $\Omega^1_{C^{k-1}}(M,\mathfrak{k}) = \varprojlim \Omega^1_{C^{k-1}}(L_n,\mathfrak{k})$ with the limit maps i_n^*, it follows that $\delta \circ f$ is smooth as a map to $\Omega^1_{C^{k-1}}(M,\mathfrak{k})$ and hence also smooth as a map to the submanifold $\mathrm{im}(\delta)$. Hence $f = \delta^{-1} \circ (\delta \circ f)$ is smooth as well. As a consequence, $C^k(M,g) : C^k(M,\mathfrak{k}) \to C^k(M,K)$, $\gamma \mapsto g \circ \gamma$ is smooth for each smooth map $g : \mathfrak{k} \to K$, because $C^k(L_n,g)$ is smooth (cf. [15]) and $\rho_n \circ C^k(M,g) = C^k(L_n,g) \circ \rho_n$.

(b) If $l \in \mathbb{N}_0 \cup \{\infty\}$ and a map $\theta : M \to C^l(I,K)$ is C^k, then $\theta^* : I \to C^k(M,K)$, $\theta^*(t)(x) = \theta(x)(t)$ is C^l.
Because the point evaluation $\mathrm{ev}_t : C^l(I,K) \to K$, $\gamma \mapsto \gamma(t)$ is smooth, we have $\theta^*(t) = \mathrm{ev}_t \circ \theta \in C^k(M,K)$ for each $t \in I$. By (a), θ^* will be C^l if we can show that $\rho_n \circ \theta^*$ is C^l for each n. But $(\rho_n \circ \theta^*) = \theta^*(t)|_{L_n} = (\theta|_{L_n})^*$, where $(\theta|_{L_n})^* : I \to C^k(L_n,K)$ is C^l, as follows by two applications of Proposition 122.

Regularity: To verify the C^s-regularity of G, let us show first that each $\gamma \in C^s(I,\mathfrak{g})$ has an evolution $\mathrm{Evol}_G(\gamma)$. Identifying $C^s(I,C^k(M,\mathfrak{k}))$ with $C^s(I,\mathfrak{g})$ via the isomorphism $C^s(I,H)$, we consider γ as a C^s-map $I \to C^k(M,\mathfrak{k})$. Then $\gamma^* : M \to C^s(I,\mathfrak{k})$, $\gamma^*(x)(t) := \gamma(t)(x)$ is C^k, using the Exponential Law (Theorem 59) twice. Hence $\mathrm{Evol}_K \circ \gamma^* : M \to C^{s+1}(I,K)$ is C^k, and therefore $\eta := (\mathrm{Evol}_K \circ \gamma^*)^* : I \to C^k(M,K)$, $\eta(t)(x) := (\mathrm{Evol}_K \circ \gamma^*)(x)(t)$ is C^{s+1} (see Remark (143)(b)). We claim that η is the evolution of γ. Indeed, $\eta(0)(x) = \mathrm{Evol}_K(\gamma^*(x))(0) = 1$ for all $x \in M$, whence $\eta(0) = 1$. To see that $\delta\eta = \gamma$, we only need to show that $(\delta\eta)(t)(x) = \gamma(t)(x) = \gamma^*(x)(t)$ for all $x \in M$ and $t \in [0,1]$, i.e., $\mathrm{ev}_x \circ \delta\eta = \gamma^*(x)$. However, recalling that $\mathrm{L}(\mathrm{ev}_x) = \mathrm{ev}_x$, we have

$$\begin{aligned} \mathrm{ev}_x \circ \delta\eta &= \mathrm{L}\,\mathrm{ev}_x \circ \delta\,\eta \\ &= \delta(\mathrm{ev}_x \circ \eta) \\ &= \delta(t \mapsto \mathrm{Evol}_K(\gamma^*(x))(t)) \\ &= \delta(\mathrm{Evol}_K(\gamma^*(x))) = \gamma^*(x). \end{aligned}$$

Thus $\mathrm{Evol}_G(\gamma) = \eta$. In particular, $\mathrm{evol}_G(\gamma) = \mathrm{Evol}_G(\gamma)(1) = \eta(1)$ is the map $M \to K$ taking x to $\mathrm{Evol}_K(\gamma^*(x))(1) = \mathrm{evol}_K(\gamma^*(x))$. Thus $\mathrm{evol}_G(\gamma) = \mathrm{evol}_K \circ \gamma^*$, i.e.,

$$\mathrm{evol}_G = C^k(M, \mathrm{evol}_K) \circ \Phi,$$

where $C^k(M, \mathrm{evol}_K)$ is smooth by Remark 143 (a) and

$$\Phi : C^s(I, C^k(M, \mathfrak{k})) \to C^k(M, C^s(I, \mathfrak{k})),\ \gamma \mapsto \gamma^*$$

is an isomorphism of topological vector spaces by the Exponential Law (Theorem 59). Thus evol_G is smooth, which completes the proof. □

Corollary 144. *If M is a one-dimensional 1-connected real manifold (with boundary), $k, s \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq s+1$ and K a C^s-regular Lie group, then the group $C^k_*(M,K)$ carries a unique C^s-regular Lie group structure for which*

$$\delta : C^k_*(M,K) \to \Omega^1_{C^{k-1}}(M, \mathfrak{k}) \cong C^k_*(M, \mathfrak{k})$$

is a C^∞-diffeomorphism. Also, $C^k(M,K) \cong C^k_*(M,K) \rtimes K$ carries the structure of a C^s-regular Lie group compatible with evaluations and the compact-open C^k-topology.

Proof. We may assume that $M = \mathbb{R}$, $M = [0,1]$ or $M = [0,1[$.
Take $L_n = [-n,n]$, $L_n = [0,1]$ and $L_n = [0, 1-\frac{1}{n}]$, respectively. Then $\mathrm{im}(\delta_n) = \Omega^1_{C^{k-1}}(L_n, \mathfrak{k})$ and Theorem 142 applies. □

6.2 Iterative constructions

Lemma 145. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Lie groups, $\phi_{nm} : G_m \to G_n$ morphisms of Lie groups defining an inverse system, $G := \varprojlim G_n$ the corresponding topological projective limit group and $\phi_n : G \to G_n$ the canonical maps. Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and assume that G carries a Lie group structure with the following properties:*

(a) *A map $f : M \to G$ of a smooth manifold M with values in G is C^r if and only if all the maps $f_n := \phi_n \circ f$ are C^r.*

(b) $\mathrm{L}(G) \cong \varprojlim \mathrm{L}(G_n)$ *as topological Lie algebras, with respect to the projective system defined by the morphisms* $\mathrm{L}(\phi_{mn}) : \mathrm{L}(G_n) \to \mathrm{L}(G_m)$.

Then the map

$$\Psi : C^r(M, G) \cong \varprojlim C^r(M, G_n), \quad f \mapsto (f_n)_{n \in \mathbb{N}}$$

is an isomorphism of topological groups.

Proof. First we note that our assumptions imply that

$$TG \cong \mathrm{L}(G) \rtimes G \cong \varprojlim (\mathrm{L}(G_n) \rtimes G_n) \cong \varprojlim T(G_n)$$

as topological groups. Moreover, writing $|\mathrm{L}(G)|$ for the topological vector space underlying $\mathrm{L}(G)$, considered as an abelian Lie algebra, we have

$$\mathrm{L}(TG) \cong |\mathrm{L}(G)| \rtimes \mathrm{L}(G) \cong \varprojlim (|\mathrm{L}(G_n)| \rtimes \mathrm{L}(G_n)) \cong \varprojlim \mathrm{L}(TG_n),$$

so that the Lie group TG inherits all properties assumed for G. Hence we may iterate this argument to obtain

$$T^k G \cong \varprojlim T^k G_n$$

for each k

We thus have homeomorphisms

$$C(T^k M, T^k G)_{c.o.} \cong \varprojlim C(T^k M, T^k G_n)_{c.o.},$$

which lead to a topological embedding

$$\begin{aligned} C^r(M, G) \hookrightarrow \prod_{k \in \mathbb{N}_0} C(T^k M, T^k G)_{c.o.} &\cong \prod_{k \in \mathbb{N}_0} \varprojlim C(T^k M, T^k G_n)_{c.o.} \\ &\cong \varprojlim \prod_{k \in \mathbb{N}_0} C(T^k M, T^k G_n)_{c.o.}, \end{aligned}$$

entailing that Ψ is a topological isomorphism. □

For compact manifolds N and M, a Lie group $C^{k,r}(N \times M, K)$ can be defined similarly to the classical construction of $C^k(N, K)$.

Lemma 146. *If K is a locally convex Lie group and N and M are compact manifolds (possibly with boundary), then the map*

$$\Phi : C^r(N, C^k(M, K)) \to C^{r,k}(N \times M, K), \quad f \mapsto f^\wedge$$

is an isomorphism of Lie groups.

Proof. The bijectivity of Φ follows from Proposition 122. To see that Φ is an isomorphism of Lie groups, let (ϕ, U) be a $\mathfrak{k}$-chart of K with $\phi(1) = 0$. Then $C^k(M, U)$ is an open identity neighbourhood, so that $C^r(N, C^k(M, U))$ is an open identity neighbourhood, and so is $C^{r,k}(N \times M, U)$. That Φ restricts to a diffeomorphism

$$C^r(N, C^k(M, U)) \to C^{r,k}(N \times M, U)$$

now follows from Proposition 58 which asserts that

$$C^r(N, C^k(M, \mathfrak{k})) \to C^{r,k}(N \times M, \mathfrak{k}), \quad f \mapsto f^\wedge$$

is an isomorphism of topological vector spaces, hence restricts to diffeomorphisms on open subsets. $\square$

A Lie group structure on $C^{r,k}(N \times M, K)$ compatible with evaluations is defined analogously to the case of $C^r(N, K)$.

Theorem 147. *Let K be a Lie group and N and M finite-dimensional smooth manifolds. We assume that $G := C^k(M, K)$ carries a C^s-regular Lie group structure compatible with evaluations and the compact-open C^k-topology. Let $r \in \mathbb{N}_0 \cup \{\infty\}$ with $r - 3 \geq s$. If $C^r(N, G)$ carries a C^s-regular Lie group structure compatible with evaluations and the compact-open C^r-topology, then $C^{r,k}(N \times M, K)$ carries a C^s-regular Lie group structure compatible with evaluations. Moreover, the canonical map*

$$\Phi : C^{r,k}(N \times M, K) \to C^r(N, G), \quad f \mapsto f^\vee$$

is an isomorphism of Lie groups.

Proof. In view of Proposition 138, the map Φ is a bijective group homomorphism. First we show that it is an isomorphism of topological groups.

Let $M = \bigcup_n M_n$ be an exhaustion of M by compact submanifolds M_n with boundary satisfying $M_n \subseteq M_{n+1}^0$. Then our definition of the group topology implies that

$$G = C^k(M, K) \cong \varprojlim C^k(M_n, K)$$

as a topological group. Put $G_n := C^k(M_n, K)$ and recall from Proposition 120 that it carries a Lie group structure compatible with evaluations. We also have the isomorphism of topological Lie algebras

$$\mathrm{L}(G) = C^k(M, \mathfrak{k}) \cong \varprojlim \mathrm{L}(G_n) \cong \varprojlim C^k(M_n, \mathfrak{k}).$$

Now let $(N_m)_{m\in\mathbb{N}}$ be an exhaustion of N by compact submanifolds with boundary. Then Lemmas 145 and 146 lead to the following isomorphisms of topological groups:

$$\begin{aligned} C^r(N,G) &\cong \varprojlim C^r(N,G_n) = \varprojlim C^r(N,C^k(M_n,K)) \\ &\cong \varprojlim_n \varprojlim_m C^r(N_m,C^k(M_n,K)) \cong \varprojlim_n \varprojlim_m C^{r,k}(N_m \times M_n,K) \\ &\cong C^{r,k}(N\times M,K). \end{aligned}$$

The preceding isomorphism leads to a C^s-regular Lie group structure on the topological group $C^{r,k}(N\times M,K)$. To see that this Lie group structure is compatible with evaluations, we first observe that $\mathrm{ev}_{(n,m)} = \mathrm{ev}_m \circ \mathrm{ev}_n \circ \Phi$, where

$$\mathrm{ev}_m \circ \mathrm{ev}_n : C^r(N,C^k(M,K)) \to K$$

is smooth. Now

$$\begin{aligned} \mathrm{L}(C^{r,k}(N\times M,K)) \xrightarrow[\cong]{\mathrm{L}(\Phi)} \mathrm{L}(C^r(N,G)) &\cong C^r(N,L(G)) \\ &\cong C^r(N,C^k(M,\mathfrak{k})) \cong C^{r,k}(N\times M,\mathfrak{k}). \end{aligned}$$

The map $\mathrm{L}(\mathrm{ev}_n)$ corresponds to $\mathrm{ev}_n : C^r(N,\mathrm{L}(G)) \to \mathrm{L}(G)$. Also, identifying $\mathrm{L}(G)$ with $C^k(M,\mathfrak{k})$, $\mathrm{L}(\mathrm{ev}_m)$ corresponds to $\mathrm{ev}_m : C^k(M,\mathfrak{k}) \to \mathfrak{k}$. Thus $\mathrm{L}(\mathrm{ev}_m \circ \mathrm{ev}_n) = \mathrm{ev}_m \circ \mathrm{ev}_n$ on $C^r(N,C^k(M,\mathfrak{k}))$, which corresponds to $\mathrm{ev}_{(n,m)}$ on $C^{r,k}(N\times M,\mathfrak{k})$. □

Example 148. Let $k,r,s \in \mathbb{N}_0 \cup \{\infty\}$ with $k \geq s+1$ and $r \geq s+3$. Then

$$C^{r,k}(\mathbb{R}\times\mathbb{R},K)$$

admits a C^s-regular Lie group structure compatible with evaluation and the compact-open $C^{r,k}$-topology. In fact, $G := C^k(\mathbb{R},K)$ admits a C^s-regular Lie group structure compatible with evaluations and the compact-open C^k-topology, by Corollary 144. Hence $C^r(\mathbb{R},G)$ admits a C^s-regular Lie group structure compatible with the evaluations and the compact-open C^r-topology, by Corollary 144. The assertion now follows from Theorem 147.

Remark 149. Continuing by induction, one could create Lie groups of the form $C^\alpha(\mathbb{R}^n\times M,K)$ with $\alpha \in \mathbb{N}^{n+1}$ if K is C^l-regular with sufficiently small l.

The following problem remains:
Can $C^k(M\times\mathbb{R}^n,K)$ be made a Lie group if $n \geq 1$ and $\dim(M\times\mathbb{R}^n) \geq 2$?

Chapter 7

Lie group structures on weighted function spaces

The notions of weighted spaces of continuous functions were first introduced by L. Nachbin [28] , further investigations have been made by Bierstedt [5], Summers [38], Prolla [35], and other authors. In this chapter, we study Lie group structures on weighted spaces of continuous functions of the form $CV_{(0)}(X, \mathfrak{g})$, where X is a completely regular Hausdorff space and $\mathfrak{g}$ is a Lie algebra.

7.1 Weighted function spaces

In this section, we assemble some basic material concerning weighted spaces. Let X be a completely regular Hausdorff space and E be a locally convex space. Recall that a subset B of E is said to be bounded if for every neighbourhood N of 0 there exists $\epsilon > 0$ such that $B \subseteq \epsilon N$. A function $f : X \to E$ is said to vanish at infinity if for each neighbourhood N of origin in E there exists a compact subset K of X such that $f(x) \in N$ for all x in $X \setminus K$, the complement of the set K in X. Then we define

$$C_b(X, E) = \{f \in C(X, E) : f(X) \text{ is bounded in } E\}, \text{ where } f(X) = \{f(x) : x \in X\},$$
$$C_0(X, E) = \{f \in C(X, E) : f \text{ vanishes at infinity on } X\}.$$

Definition 150. A set V of weights [1] on X is called a Nachbin family or a system of weights iff

(a) For every $x \in X$ there is a $v \in V$ such that $v(x) > 0$,

[1] A function $v : X \to [0, \infty[$ is called a weight if it is upper semicontinuous.

(b) For $\lambda > 0\, v_1, v_2 \in V$, there is a $v \in V$ such that $\lambda v_1 \leq v$ and $\lambda v_2 \leq v$ (pointwise).

We define the weighted spaces $CV(X, E)$ and $CV_0(X, E)$ of E-valued continuous functions on X with respect to a given Nachbin family V as follows:

$$CV(X, E) = \{f \in C(X, E) : fv \text{ is bounded for every } v \in V\},$$
$$CV_0(X, E) = \{f \in C(X, E) : fv \text{ vanishes at infinity for every } v \in V\}.$$

We will write $CV_{(0)}(X, E)$ to mean $CV(X, E)$ (resp. $CV_0(X, E)$). When $E = \mathbb{K}$, we write simply $CV(X)$ instead of $CV(X, \mathbb{K})$ and $CV_0(X)$ instead of $CV_0(X, \mathbb{K})$.

The seminorms P_v, where $P_v(f) = \|f\|_{P,v} := \sup\{v(x)P((f(x))) : x \in X\}$, generate a Hausdorff locally convex topology on each of these spaces for P ranging through the continuous seminorms on E and $v \in V$. This topology is called the weighted topology, and $CV(X, E)$ and $CV_0(X, E)$ endowed with this topology is called the weighted space of vector-valued continuous functions. If $E = \mathbb{K}$ and $P = |.|$, we also write $\|.\|_v$ instead of P_v.

Remark 151. $CV(X, E)$ and $CV_0(X, E)$ are vector spaces with the pointwise linear operations and $CV_0(X, E)$ is a closed vector subspace of $CV(X, E)$.

Definition 152. (a) A Nachbin family is called admissible if $\forall x \in X,\ \exists \gamma \in CV(X, \mathbb{R})$ such that $\gamma(x) \neq 0$.

(b) A Nachbin family is called strongly admissible if $\forall x \in X,\ \exists \gamma \in CV_0(X, \mathbb{R})$ such that $\gamma(x) \neq 0$.

Definition 153. If V, W are two Nachbin families on X, we say $V \leq W$ iff for every $v \in V$ there is a $w \in W$ such that $v \leq w$. In this case $CW(X, E)$ is continuously embedded in $CV(X, E)$. W and V are called equivalent ($W \sim V$) if $W \leq V$ and $V \leq W$ holds.

Remark 154. For each $f \in CV(X, E)$, the collection of $\mathcal{N}(v, U) = \{g \in CV(X, E) : (v(g - f))(X) \subseteq U\}$, where $f \in CV(X, E)$, $v \in V$ and U is a neighbourhood of 0 in E, is a base of neighbourhoods of f in the weighted space $CV(X, E)$.

Lemma 155. *Let X be a topological space, V a Nachbin family with $1 \in V$ and E a normed space, then the set*

$$CV(X, U) = \{f \in CV(X, E) : \operatorname{im}(f) + B_\epsilon^E(0) \subseteq U \text{ for some } \epsilon > 0\}$$

is an open 0-neighbourhood of $CV(X, E)$ for every open 0-neighbourhood U of E.

Remark 156. For each $f \in CV_0(X,E)$, the collection of $B_{\epsilon,v}(f) = \{g \in CV_0(X,E) : P_v(f-g) < \epsilon\}$, where $f \in CV_0(X,E)$, $v \in V$ and $\epsilon > 0$, is a base of open neighbourhoods of f in the weighted space $CV_0(X,E)$.

Proposition 157. *([40]) Let X be a Hausdorff topological space, E and F be normed spaces over $\mathbb{K}$, $U \subseteq X$ be an open 0-neighbourhood, V be a set of weights of X with $1_U \in V$, and $f : U \to F$ be $\mathbb{K}$-analytic.*

(a) *If $\mathbb{K} = \mathbb{C}$, then*

$$CV(E,f) : CV(X,U) \to CV(X,F), \gamma \mapsto f \circ \gamma$$

is complex analytic.

(b) *If $\mathbb{K} = \mathbb{R}$ and $f : U \to F$ admits a complex analytic extension $\tilde{f} : \tilde{U} \to F_{\mathbb{C}}$ to an open subset $\tilde{U} \subseteq E_{\mathbb{C}}$, then $CV(E,f)$ is real analytic, such an extension always exists by definition of real analyticity.*

7.2 Weighted topological Lie algebras

Definition 158. (a) A locally convex algebra is an algebra E endowed with a locally convex topology such that the multiplication of E separately continuous [1].

(b) E is said to be a topological algebra with $\mathcal{B}$-hypocontinuous multiplication (or that E is $\mathcal{B}$-hypotopological), where $\mathcal{B}$ is a family of subsets of E, if E is equipped with both left and right $\mathcal{B}$-hypocontinuous multiplication [2]. By means of the seminorms, this is equivalent to: for every $P \in \mathbb{P}$, where $\mathbb{P}$ is a set of seminorms on E defining its topology and $B \in \mathcal{B}$, there exist $P' \in \mathbb{P}$ and $M > 0$ such that

$$\max(P(xy), P(yx)) \leq MP'(y),\ x \in B,\ y \in E.$$

In case $\mathcal{B}$ is the set of all bounded subsets of E, we just say that E has hypocontinuous multiplication or is hypotopological.

Remark 159. ([9]) Every continuous bilinear map is hypocontinuous. The converse is in general false.

Proposition 160. *([34]) Let E be a locally convex algebra.*

[1] In the case of topological Lie algebras, we shall however assume that the Lie bracket is jointly continuous.

[2] Multiplication in E is said to be left (right) $\mathcal{B}$-hypocontinuous if for each 0-neighbourhood U in E and $B \subseteq \mathcal{B}$, there exists a 0-neighbourhood U' such that $BU' \subseteq U$ (resp. $U'B \subseteq U$).

(a) *$C(X,E)$ is an algebra for every completely regular space X if and only if E has continuous multiplication.*
(b) *If E is hypotopological, then $C_b(X,E)$ is an algebra for every Hausdorff completely regular space X.*

Proposition 161. *([33])*
(a) *$CV_{(0)}(X)$ is a locally convex algebra iff for every $g \in CV_{(0)}(X)$, $|g|\,\mathcal{W} \le \mathcal{W}$, i.e.,*

$$\forall v \in V,\ \exists v' \in V : |g|\, v \le v'.$$

(b) *$CV_{(0)}(X)$ has continuous product if and only if $V \le VV$.*

The next proposition describes a condition on the weights that makes $CV(X,\mathfrak{g})$ a topological Lie algebra.

Proposition 162. *Let X be a completely regular Hausdorff space and V be a Nachbin family on X. If $V \le V \cdot V$, then $CV(X,\mathfrak{g})$ is topological Lie algebra, for every locally convex topological Lie algebra $\mathfrak{g}$.*

Proof. Let $\gamma,\ \eta \in CV(X,\mathfrak{g})$. Given $v \in V$ and $P \in \mathbb{P}$, by the hypothesis $V \le V \cdot V$ there exists a weight $w \in V$ such that $v \le w \cdot w$.

Because $[\cdot,\cdot]$ is continuous there exists $Q \in \mathbb{P}$ such that $P([v,w]) \le Q(v)Q(w),\ \forall v, w \in \mathfrak{g}$. Hence

$$\begin{aligned} v(x)P([\gamma,\eta](x)) &\le w(x)w(x)P([\gamma(x),\eta(x)] \\ &\le w(x)w(x)Q(\gamma(x))Q(\eta(x)) \\ &= \underbrace{w(x)Q(\gamma(x))}_{\le Q_w(\gamma)}\underbrace{w(x)Q(\eta(x))}_{\le Q_w(\eta)} \end{aligned}$$

Hence $v(x)P([\gamma,\eta](x)) \le Q_w(\gamma)Q_w(\eta)$, independently of x. Passing to the supremum over x on the left-hand side, we obtain

$$P_v([\gamma,\eta]) \le Q_w(\gamma)Q_w(\eta) < \infty. \tag{7.1}$$

Because $P_v([\gamma,\eta]) < \infty$ for all $v \in V$ and all $P \in \mathbb{P}$, we have $[\gamma,\eta] \in CV(X,\mathfrak{g})$. Moreover $P_v([\gamma,\eta]) \le Q_w(\gamma)Q_w(\eta)$ by (7.1), hence the bilinear map

$$CV(X,\mathfrak{g}) \times CV(X,\mathfrak{g}) \to CV(X,\mathfrak{g}),\ (\gamma,\eta) \mapsto [\gamma,\eta]$$

is continuous at $(0,0)$ and thus continuous. The assertion follows. $\square$

Lemma 163. *Let X be a completely regular Hausdorff space and V be a Nachbin family on X. If $V \leq V \cdot V$, then $CV_0(X, \mathfrak{g})$ is a Lie subalgebra of $CV(X, \mathfrak{g})$, moreover $CV_0(X, \mathfrak{g})$ is an ideal of $CV(X, \mathfrak{g})$.*

Proof. Let $\gamma \in CV(X, \mathfrak{g})$ and $\eta \in CV_0(X, \mathfrak{g})$. Given $v \in V$ and $P \in \mathbb{P}$, there exists a weight $w \in V$ such that $v \leq w \cdot w$ and there exists a seminorm $Q \in \mathbb{P}$ such that $P([v, w]) \leq Q(v)Q(w)$, $\forall v, w \in \mathfrak{g}$. Now given $\epsilon > 0$, there exists a compact set $K \subseteq X$ such that

$$w(x)Q(\eta(x)) \leq \frac{\epsilon}{Q_w(\gamma) + 1} \tag{7.2}$$

for all $x \in X \backslash K$. Then, for $x \in X \backslash K$, by (7.2) we have

$$\begin{aligned} v(x)P([\gamma, \eta](x)) &\leq w(x)w(x)P([\gamma(x), \eta(x)]) \\ &\leq w(x)w(x)Q(\gamma(x))Q(\eta(x)) \\ &= \underbrace{w(x)Q(\gamma(x))}_{\leq Q_w(\gamma)} \underbrace{w(x)Q(\eta(x))}_{\leq \frac{\epsilon}{Q_w(\gamma)+1}} \\ &\leq \frac{\epsilon Q_w(\gamma)}{Q_w(\gamma) + 1} \\ &\leq \epsilon. \end{aligned}$$

Hence $v(x)P([\gamma, \eta](x))$ vanishes at infinity. Thus $[\gamma, \eta] \in CV_0(X, \mathfrak{g})$. Hence $CV_0(X, \mathfrak{g})$ is an ideal of $CV(X, \mathfrak{g})$. $\square$

7.3 The Lie group structure on $CV(X, H)_\circ$

Let H be any Banach Lie group with Lie algebra $\mathfrak{h}$. We show that if $1 \in V$, then $P := \{\gamma \in CV(X, \mathfrak{h}) : \|\gamma\|_\infty < \epsilon\}$ is a 0-neighbourhood. And if $\exp_H|_{B^{\mathfrak{h}}_\epsilon(0)}$ is a diffeomorphism onto an open 1-neighbourhood, then the map

$$\Phi : P \to H^X, \gamma \mapsto \exp_H \circ \gamma$$

is injective, hence $\Phi(P)$ can be made a manifold diffeomorphic to the open set $P \subseteq CV(X, \mathfrak{h})$, thus by the standard arguments $\langle \exp_H \circ \gamma : \gamma \in P(\text{ or } \mathfrak{h}) \rangle$ is a Lie group.

Let $W := \Phi(Q)$ and $Q := \{\gamma \in CV(X, \mathfrak{h}) : \|\gamma\|_\infty < \delta\}$ where δ is so small that

$$B^{\mathfrak{h}}_\delta(0) * B^{\mathfrak{h}}_\delta(0) \subseteq B^{\mathfrak{h}}_{\epsilon'}(0) \text{ with } \epsilon' < \epsilon, \tag{7.3}$$

where $*$ is the Baker-Campbell-Hausdorff multiplication.

Now $\exp_H(-x) = \exp_H(x)^{-1}$. Thus W is symmetric and (7.3) ensures that $W \cdot W \subseteq P$. Let $G := \langle \exp_G \circ \gamma : \gamma \in CV(X,\mathfrak{h})\rangle \subseteq H^X$. Then

$$CV(X,\mathfrak{h}) = \cup_{n=1}^{\infty} nQ$$

and $\exp_H \circ(n\gamma) = (\exp_H \circ\gamma)^n \in \langle \exp_H \circ\gamma : \gamma \in Q\rangle = \langle W\rangle$ thus,

$$\langle W\rangle = \langle \exp_H \circ\gamma : \gamma \in CV(X,\mathfrak{h})\rangle =: CV(X,H)_0 = G.$$

In particular, $\Phi(P) \subseteq \langle W\rangle$.

We want to apply Proposition 16 to create a Lie group structure on G. Let us check conditions (a) and (b) of Proposition 16.

Inversion is analytic on W. The continuous linear map

$$CV(X,\mathfrak{h}) \to CV(X,\mathfrak{h}),\ \gamma \mapsto -\gamma$$

restricts to the analytic self-map

$$i : Q \to Q,\ \gamma \mapsto -\gamma$$

of Q. Since $\Phi(-\gamma) = \Phi(\gamma)^{-1}$, the inversion map $G \to G$ restricts to a self-map

$$j : W \to W,\ \gamma \mapsto \gamma^{-1}$$

and $j \circ \Phi|_Q = \Phi \circ i$. We want to see that this map is analytic. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "j"] & W \\ Q \arrow[u, "\Phi|_Q"] \arrow[r, "i"] & Q. \arrow[u, "\Phi|_Q"'] \end{tikzcd}$$

Since i is analytic, also $j = \Phi \circ i \circ (\Phi|_Q)^{-1}$ is analytic.

Multiplication is analytic on W. The multiplication map

$$G \times G \to G$$

restricts to a map

$$W \times W \to G.$$

Now $CV(X, B_\delta^{\mathfrak{h}}(0)) \times CV(X, B_\delta^{\mathfrak{h}}(0)) \cong CV(X, B_\delta^{\mathfrak{h}}(0) \times B_\delta^{\mathfrak{h}}(0))$ and $m := C(X,\nu) : CV(X, B_\delta^{\mathfrak{h}}(0) \times B_\delta^{\mathfrak{h}}(0)) \to P$ is an analytic mapping by Proposition 157, where ν :

$B_\delta^{\mathfrak{h}}(0) \times B_\delta^{\mathfrak{h}}(0) \to B_{\epsilon'}^{\mathfrak{h}}(0)$, $\quad \nu(a,b) := a * b$ is the Baker-Campbell-Hausdorff multiplication. Since $Q = CV(X, B_\delta^{\mathfrak{h}}(0))$, we obtain a commutative diagram

$$\begin{array}{ccc} W \times W & \xrightarrow{\mu} & G \\ \uparrow{\scriptstyle \Phi|_Q \times \Phi|_Q} & & \uparrow{\scriptstyle \Phi} \\ Q \times Q & \xrightarrow{m} & P \end{array}$$

and $\mu(W \times W) \subseteq \Phi(P)$. Therefore the map $W \times W \to G$ induced by group multiplication is analytic.

7.4 Weighted Lie algebras and continuous product

Lemma 164. *Let X be a Hausdorff topological space. Assume that for each $x_0 \in X$, there exists a continuous function $h : X \to \mathbb{R}$ such that $h(x_0) \neq 0$ and $h \in C_0(X, \mathbb{R})$. Then X is locally compact.*

Proof. Since $h \in C_0(X, \mathbb{R})$, there exists a compact set $K \subseteq X$ such that $|h(x)| < \epsilon := |h(x_0)|/2$ for all $x \in X \backslash K$. Now $U := \{x \in X : |h(x) - h(x_0)| \leq \epsilon\}$ is a closed neighbourhood of x_0 in X. For $x \in U$, we have $|h(x)| \geq |h(x_0)| - |h(x) - h(x_0)| \geq 2\epsilon - \epsilon = \epsilon$. Hence $U \subseteq K$ and thus U is compact. Since U is a neighbourhood of x_0, X is locally compact.

□

Lemma 165. *Let X be a Hausdorff topological space. Assume that $V \subseteq C(X, [0, \infty[)$ and for each $x_0 \in X$, there exists a function $h \in CV_0(X, \mathbb{R})$ such that $h(x_0) \neq 0$. Then X is locally compact.*

Proof. By Definition 150 (a), there exists $v \in V$ such that $v(x_0) > 0$. Then $v \cdot h \in C_0(X, \mathbb{R})$ and $(v \cdot h)(x_0) = v(x_0) \cdot h(x_0) \neq 0$. Hence, by Lemma 164, X is a locally compact. □

Lemma 166. (a) *If X is a completely regular Hausdorff space and V is an admissible Nachbin family, then for each $x_0 \in X$ and each neighbourhood $U \subseteq X$ of x_0, there exists $\gamma \in CV(X, \mathbb{R})$ such that $\gamma(x_0) \neq 0$ and $\gamma|_{X \backslash U} \neq 0$.*

(b) *If X is a locally compact Hausdorff space and V is an admissible Nachbin family, then V is a strongly admissible Nachbin family.*

Proof. (a) Since X is a completely regular Hausdorff space, there exists $h \in C(X, [0, 1])$ such that $h(x_0) \neq 1$ and $h|_{X \backslash U} = 0$.

By dmissibility, there exists $\gamma \in CV(X,\mathbb{R})$ such that $\gamma(x_0) \neq 0$. Then $h \cdot \gamma \in CV(X,\mathbb{R})$. But $(h \cdot \gamma)(x_0) = h(x_0)\gamma(x_0) \neq 0$ and $h \cdot \gamma|_{X\backslash U} = 0$.

(b) Let $x_0 \in X$. Because X is locally compact, there exists a compact neighbourhood $U \subseteq X$ of x_0.
Since every locally compact space is completely regular, by (a) we find $\gamma \in CV(X,\mathbb{R})$ such that $\gamma|_{X\backslash U} = 0$ and $\gamma(x_0) \neq 0$.
But $K := U$ is compact and $v \cdot \gamma|_{X\backslash K} = 0$, hence $v(x)\,|\gamma(x)| \leq \epsilon$ for each ϵ and for each $x \in X\backslash K$. Thus $\gamma \in CV_0(X,\mathbb{R})$. Hence V is strongly admissible Nachbin family.

□

Lemma 167. *Let X be a completely regular space and V be an admissible Nachbin family. Assume that $v \in V$ and $v_1, \ldots, v_n \in V$ satisfy*

$$\|\gamma_1 \cdots \gamma_n\|_v \leq \|\gamma_1\|_{v_1} \cdots \|\gamma_n\|_{v_n}, \text{ for all } \gamma_1, \ldots, \gamma_n \in CV(X).$$

Then

$$v(x) \leq v_1(x) \cdots v_n(x), \text{ for all } x \in X. \tag{7.4}$$

Proof. If $v(x) = 0$, then (7.4) is clear. Now assume that $v(x) > 0$. Let ϵ be arbitrary. Since $v_1, \ldots, v_n$ are upper semicontinuous, there exists a neighbourhood $U \subseteq X$ of x such that $v_j(y) \leq v_j(x) + \epsilon$ for all $y \in U$.

Since V is an admissible Nachbin family and X is a completely regular space, Lemma 166 shows that we find $\gamma \in CV(X,\mathbb{R})$ such that $\gamma|_{X\backslash U} = 0$ and $\gamma(x) \neq 0$. Without loss of generality $\mathrm{im}(\gamma) \subseteq [0,1]$ and $\gamma(x) = 1$. For each j, we have

$$\begin{aligned}\|\gamma\|_{v_j} &= \sup\{v_j(y)\,|\gamma(y)| : y \in X\}\\ &= \sup\{v_j(y)\,|\gamma(y)| : y \in U\}\\ &\leq v_j(x) + \epsilon.\end{aligned}$$

Then

$$\|\overbrace{\gamma \cdots \gamma}^{j \text{ copies of } \gamma}\|_v \geq v(x)\,|\gamma_1(x) \cdots \gamma_n(x)| = v(x)$$

and

$$\begin{aligned}\|\gamma \cdots \gamma\|_v &\leq \|\gamma\|_{v_1} \cdots \|\gamma\|_{v_n}\\ &\leq (v_1(x) + \epsilon) \cdots (v_n(x) + \epsilon)\end{aligned}$$

Because ϵ was arbitrary, we get

$$v(x) \leq v_1(x) \cdots v_n(x).$$

□

Lemma 168. *Let $\alpha : E \to F$ be a continuous linear map between locally convex spaces. Then $\alpha \circ \gamma \in CV_{(0)}(X, F)$ for each $\gamma \in CV_{(0)}(X, E)$, and the map*

$$CV_{(0)}(X, \alpha) : CV_{(0)}(X, E) \to CV_{(0)}(X, F),\ \gamma \mapsto \alpha \circ \gamma$$

is continuous and linear.

Proof. Linearity is obvious. Now for each continuous seminorm P on F, there exists continuous seminorm Q on E such that $P(\alpha(x)) \leq Q(x)$ because α is continuous linear. Then for each $v \in V,\ x \in X$ and $\gamma \in CV(X, E)$

$$v(x)P(\alpha(\gamma(x))) \leq v(x)Q(\gamma(x))$$

hence $\|\alpha \circ \gamma\|_{P,v} \leq \|\alpha\|_{Q,v} < \infty$ and $\alpha \circ \gamma \in CV(X, F)$. Thus $CV(X, \alpha)$ is continuous. Now $v(x)P(\alpha(\gamma(x))) \leq v(x)Q(\gamma(x))$ shows that $v \cdot (\alpha \circ \gamma) \in C_0(X, F)$ if $v.\gamma \in C_0(X, E)$, hence $\alpha \circ \gamma \in CV_0(X, F)$ if $\gamma \in CV_0(X, E)$. □

Lemma 169. *Let E be a locally convex space, X be a topological space, V a Nachbin family and $0 \neq w \in E$. Then the map*

$$\Phi : CV(X, \mathbb{R}) \to CV(X, E),\ \gamma \mapsto \gamma \cdot w$$

is linear and a topological embedding. Furthermore,

$$\mathrm{im}(\Phi) = \{\gamma \in CV(X, E) : (\forall x \in X)\gamma(x) \in \mathbb{R}w\}.$$

Proof. If P is a continuous seminorm on E and $v \in V$, then

$$v(x)P(\gamma(x)w) = v(x)\left|\gamma(x)\right| P(w) \leq \|\gamma\|_v \cdot P(w),$$

thus $\gamma \cdot w \in CV(X, E)$ and $\|\Phi(\gamma)\|_{P,v} = \|\gamma \cdot w\|_{P,v} \leq P(w) \|\gamma\|_v$, hence the linear map Φ is continuous.

By Hahn-Banach Theorem, there exists $\lambda \in E'$ such that $\lambda(w) \neq 0$. W.l.o.g $\lambda(w) = 1$. Then $CV(X, \lambda) : CV(X, E) \to CV(X, \mathbb{R})$ is continuous and linear.

$(CV(X, \lambda) \cdot \Phi)(\gamma) = CV(X, \lambda)(\gamma \cdot w) = \lambda \circ (\gamma \cdot w)$. Here $(\lambda \circ \gamma \cdot w)(x) = \lambda(\gamma(x)w) = \gamma(x)\lambda(w) = \gamma(x)$. Thus $\lambda \circ (\gamma \cdot w) = \gamma$, hence $CV(X, \lambda) \circ \Phi = \mathrm{id}_{CV(X,\mathbb{R})}$, thus Φ is injective and $\Phi^{-1} = CV(X,\lambda)|_{\mathrm{im}\Phi}$ is continuous, thus Φ is a homeomorphism onto its image, i.e., a topological embedding.

Now let $\gamma \in CV(X,E)$ and assume that $\gamma(x) \in \mathbb{R}w$ for each $x \in X$. Since $w \neq 0$, we have $\gamma(x) = \eta(x)w$ with a unique real number $\eta(x) \in \mathbb{R}$. Then $\eta = CV(X,\lambda)(\gamma)$, hence $\eta \in CV(X,\mathbb{R})$. Hence $\gamma = \Phi(\eta)$ is in the image of Φ. □

We can prove the following lemma in the same way.

Lemma 170. *Let E be a locally convex space, X be a topological space, V a Nachbin family and $0 \neq w \in E$. Then the map*

$$\Phi : CV_0(X,\mathbb{R}) \to CV_0(X,E),\ \gamma \mapsto \gamma \cdot w$$

is linear and a topological embedding. Furthermore,

$$\mathrm{im}(\Phi) = \{\gamma \in CV_0(X,E) : (\forall x \in X)\gamma(x) \in \mathbb{R}w\}.$$

Theorem 171. *Let $n \in \mathbb{N}$ with $n \geq 2$, X be a topological space, $V \subseteq [0,\infty[^X$ a Nachbin family and $\beta : E_1 \times \cdots \times E_n \to F$ a continuous n-linear map between locally convex spaces, such that $\beta \neq 0$. Consider the conditions:*

(a) $V \leq \underbrace{V \cdot V \cdots V}_{n \text{ factors.}}$

(b) $\beta \circ (\gamma_1,\ldots,\gamma_n) \in CV(X,F)$ *for all* $\gamma_j \in CV(X,E_j)$ *for* $j = 1,\ldots,n$ *and*

$$CV(X,\beta) : CV(X,E_1) \times \cdots \times CV(X,E_n) \to CV(X,F),$$

$$(\gamma_1,\ldots,\gamma_n) \mapsto \beta \circ (\gamma_1,\ldots,\gamma_n)$$

is continuous.

(c) $\beta \circ (\gamma_1,\ldots,\gamma_n) \in CV_0(X,F)$ *for all* $\gamma_j \in CV_0(X,E_j)$ *for* $j = 1,\ldots,n$ *and*

$$CV_0(X,\beta) : CV_0(X,E_1) \times \cdots \times CV_0(X,E_n) \to CV_0(X,F),$$

$$(\gamma_1,\ldots,\gamma_n) \mapsto \beta \circ (\gamma_1,\ldots,\gamma_n)$$

is continuous.

Then (a) $\Rightarrow$ (b) *and* (a) $\Rightarrow$ (c). *If X is a completely regular Hausdorff space and V is an admissible Nachbin family, then* (b) $\Rightarrow$ (a).

If X is a completely regular Hausdorff space and V is a strongly admissible Nachbin family, then (c) $\Rightarrow$ (a).

Proof.

(a) $\Rightarrow$ (b): Let P be a continuous seminorm on F and $v \in V$.

By hypothesis, there exist $v_1, \ldots, v_n \in V$ such that $v \leq v_1 v_2 \cdots v_n$.

Because the n-linear map β is continuous, there exist continuous seminorms Q_j on E_j for $j \in \{1, \ldots, n\}$ such that

$$P(\beta(w_1, \ldots, w_n)) \leq Q_1(w_1) \cdot Q_2(w_2) \cdots Q_n(w_n).$$

Let $\gamma_j \in CV(X, E_j)$ for $j \in \{1, \ldots, n\}$. For $x \in X$, estimate

$$\begin{aligned} v(x)P((\beta \circ (\gamma_1, \ldots, \gamma_n))(x)) &\leq v_1(x)v_2(x) \cdots v_n(x)Q_1(\gamma_1(x))Q_2(\gamma_2(x)) \cdots Q_n(\gamma_n(x)) \\ &= \underbrace{v_1(x)Q_1(\gamma_1(x))}_{\leq \|\gamma_1\|_{Q_1,v_1}} \underbrace{v_2(x)Q_2(\gamma_2(x))}_{\leq \|\gamma_2\|_{Q_2,v_2}} \cdots \underbrace{v_n(x)Q_n(\gamma_n(x))}_{\leq \|\gamma_n\|_{Q_n,v_n}} \\ &\leq \|\gamma_1\|_{Q_1,v_1} \|\gamma_2\|_{Q_2,v_2} \cdots \|\gamma_n\|_{Q_n,v_n} \\ &< \infty. \end{aligned}$$

Thus $\beta \circ (\gamma_1, \ldots, \gamma_n) \in CV(X, F)$ and $\|\beta \circ (\gamma_1, \ldots, \gamma_n)\|_{P,v} \leq \|\gamma_1\|_{Q_1,v_1} \cdots \|\gamma_n\|_{Q_n,v_n}$, hence the n-linear map $CV(X, \beta)$ is continuous.

(a) $\Rightarrow$ (c): In the same way as (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Let X be a completely regular Hausdorff space and assume that V admissible. Because $\beta \neq 0$, there exist $0 \neq w_j \in E_j$ such that $w := \beta(w_1, \ldots, w_n) \neq 0$. Let

$$\Phi_j : CV(X, \mathbb{R}) \to CV(X, E_j),\ \gamma \mapsto \gamma \cdot w_j,$$

which is a continuous linear map by Lemma 169.

Let $\Phi : CV(X, \mathbb{R}) \to CV(X, F),\ \gamma \mapsto \gamma \cdot w$. Then Φ is a linear topological embedding by Lemma 169. Moreover,

$$\begin{aligned} (\beta \circ (\Phi_1(\gamma_1), \ldots, \Phi_n(\gamma_n)))(x) &= \beta(\gamma_1(x)w_1, \ldots, \gamma_n(x)w_n) \\ &= \gamma_1(x) \cdots \gamma_n(x)\beta(w_1, \ldots, w_n). \end{aligned}$$

Hence $(\gamma_1 \cdots \gamma_n) \cdot w = \beta \circ (\Phi_1(\gamma_1), \ldots, \Phi_n(\gamma_n)) \in CV(X, F)$ by hypothesis and by Lemma 169, $\gamma_1 \cdots \gamma_n \in CV(X, \mathbb{R})$.

Now let $\mu : \mathbb{R}^n \to \mathbb{R},\ (r_1, \ldots, r_n) \mapsto r_1 \cdots r_n$. By the preceding,

$$CV(X, \mu) : CV(X, \mathbb{R})^n \to CV(X, \mathbb{R}),\ (\gamma_1, \ldots, \gamma_n) \mapsto \mu \circ (\gamma_1, \ldots, \gamma_n)$$

is defined, and $CV(X, \beta) \circ (\Phi_1 \times \cdots \times \Phi_n) = \Phi \circ CV(X, \mu)$. Since $CV(X, \beta) \circ (\Phi_1 \times \cdots \times \Phi_n)$ is continuous, and Φ a topological embedding, it follows that also $CV(X, \mu)$ is continuous. Hence, for the proof (b) $\Rightarrow$ (a), without loss of generality $E_1 = \cdots = E_n =$

$F = \mathbb{R}$ and $\beta = \mu$. Thus assume $\gamma_1 \cdots \gamma_n \in CV(X, \mathbb{R})$ for all $\gamma_1, \ldots, \gamma_n \in CV(X, \mathbb{R})$ and asume that

$$CV(X, \mu) : CV(X, \mathbb{R})^n \to CV(X, \mathbb{R}),\ (\gamma_1, \ldots, \gamma_n) \mapsto \gamma_1 \cdots \gamma_n$$

is continuous. Let $v \in V$. We have to show there exist $v_1, \ldots, v_n \in V$ such that $v \leq v_1 \cdots v_n$. Because $CV(X, \mu)$ is continuous, there exist continuous seminorms Q_j on $CV(X, \mathbb{R})$ for $j = 1, \ldots, n$ such that

$$\|\gamma_1 \cdots \gamma_n\| \leq Q_1(\gamma_1) \cdots Q_n(\gamma_n). \tag{7.5}$$

Since V is Nachbin family, for each j, there exist $v_j \in V$ such that $Q_j \leq \|\cdot\|_{v_j}$. Now (7.5) implies that

$$\|\gamma_1 \cdots \gamma_n\|_v \leq \|\gamma_1\|_{v_1} \cdots \|\gamma_n\|_{v_n}.$$

Hence Lemma 167 shows that

$$v(x) \leq v_1(x) \cdots v_n(x),\ \text{for all } x \in X. \tag{7.6}$$

The proof of "(c) $\Rightarrow$ (a)" is similar. $\square$

Applying Theorem 171, we obtain:

Corollary 172. *If $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a locally convex topological Lie algebra, X a Hausdorff topological space and V any Nachbin family on X such that $V \leq VV$, then $CV(X, \mathfrak{g})$ is a locally convex topological Lie algebra with the Lie bracket*

$$[\cdot,\cdot] : CV(X, \mathfrak{g}) \times CV(X, \mathfrak{g}) \to CV(X, \mathfrak{g}),\ (\gamma, \eta) \mapsto [\gamma, \eta]$$

with

$$[\gamma, \eta](x) := [\gamma(x), \eta(x)]_{\mathfrak{g}}.$$

Proof. Taking $\beta := [\cdot,\cdot]_{\mathfrak{g}}$ in Theorem 171, the assertion follows. $\square$

Write $\mathfrak{g}^n$ for the n-th term of the descending central series of a Lie algebra $\mathfrak{g}$. By the definition of the Lie bracket in Corollary 172, we obtain:

Corollary 173. *If $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a nilpotent locally convex topological Lie algebra with $\mathfrak{g}^n = \{0\}$, X any Hausdorff topological space and V any Nachbin family on X such that $V \leq VV$, then $CV(X, \mathfrak{g})$ is a nilpotent locally convex topological Lie algebra with $CV(X, \mathfrak{g})^n = \{0\}$.*

The Lie group structure on $CV(X,\mathfrak{g})$

If $CV(X,\mathfrak{g})$ is a nilpotent locally convex topological Lie algebra and $CV(X,\mathfrak{g})^n = 0$, then the Baker-Campbell-Hausdorff series (BCH-series) is a finite sum of terms involving at most $n-1$ brackets and being finite, the series converges on all of $CV(X,\mathfrak{g}) \times CV(X,\mathfrak{g})$, thus

$$\begin{aligned}\gamma * \eta &= \gamma + \sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} (-1)^k \frac{(\mathrm{ad}\gamma)^{p_1}(\mathrm{ad}\eta)^{q_1}\cdots(\mathrm{ad}\gamma)^{p_k}(\mathrm{ad}\eta)^{p_k}(\mathrm{ad}\gamma)^m}{(k+1)(q_1+\cdots+q_k+1)p_1!q_1!\cdots p_k!q_k!m!}\eta \\ &= \gamma+\eta+\frac{1}{2}[\gamma,\eta]+\frac{1}{12}[\gamma,[\gamma,\eta]]+\frac{1}{12}[\eta,[\eta,\gamma]]-\frac{1}{24}[\gamma,[\eta,[\gamma,\eta]]]+\cdots\end{aligned}$$

can be defined for all $\gamma,\eta \in CV(X,\mathfrak{g})$. This is a continuous function

$$CV(X,\mathfrak{g}) \times CV(X,\mathfrak{g}) \to CV(X,\mathfrak{g})$$

in the variables (γ_1,γ_2) and a polynomial. Hence this is an analytic function of (γ,η). It is known that the Baker-Campbell-Hausdorff formula defines a group structure on any nilpotent Lie algebra [21]. By the preceding, the group multiplication is analytic. The inversion is the continuous linear map

$$CV(X,\mathfrak{g}) \to CV(X,\mathfrak{g}),\ \gamma \mapsto -\gamma$$

and therefore analytic as well. Thus $(CV(X,\mathfrak{g}),*)$ is an analytic Lie group with Lie algebra $(CV(X,\mathfrak{g}),[.,.])$.

Remark 174. One can show this does not work any more in general if $\mathfrak{g}$ is solvable. In this case for finite-dimensional $\mathfrak{g}$, it is still possible to make $\mathfrak{g}$ a Lie group $G=(\mathfrak{g},\mu)$, where $\mu : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is the analytic group multiplication (so G diffeomorphic to the vector space $\mathfrak{g}$).

Thus $CV(X,\mathfrak{g})$ is a topological Lie algebra. But we can not make this a Lie group using the multiplication

$$CV(X,\mu) : CV(X,\mathfrak{g}) \times CV(X,\mathfrak{g}) \to C(X,\mathfrak{g}),\ (\gamma,\eta) \mapsto \mu\circ(\gamma,\eta),$$

because the latter may take values outside $CV(X,\mathfrak{g})$.

Example 175. Let $X=\mathbb{R}$. $V=\{av_\alpha : \alpha<0,\ a>0\}$ is a Nachbin family, where $v_\alpha(x) := (1+|x|)^\alpha$. if

$$\gamma(x) := \begin{cases} \log x & \text{if } x\geq 1,\\ 0 & \text{if } x \leq 1,\end{cases}$$

then $\gamma \in CV(\mathbb{R})$, using that a continuous function η is in $CV(\mathbb{R})$ if and only if $\eta = O(|x|^\alpha)$ as $x \to \infty$, for all $\alpha > 0$. Let $G = \mathbb{R}^2$ as a manifold, which is a group with analytic multiplication $\mu((a,b),(c,d)) := (a+e^b \cdot c, b+d)$ and analytic inversion $i(a,b) := (-ae^{-b}, -b)$. We have then that $\tau(x) = (1,0)$ defines a function $\tau \in CV(\mathbb{R}, \mathbb{R} \times \mathbb{R})$ and

$$\sigma(x) := \begin{cases} (0, \log x) & \text{if } x \geq 1, \\ (0,0) & \text{if } x \leq 1, \end{cases}$$

defines a function $\sigma \in CV(\mathbb{R}, \mathbb{R} \times \mathbb{R})$. For these functions and for $x \geq 1$

$$(\mu \circ (\sigma, \tau))(x) = \mu(\sigma(x), \tau(x)) = \mu((0, \log x), (1,0)) = (e^{\log x}, \log x) = (\underbrace{x}_{\text{is not } O(\sqrt{x})}, \log x),$$

so $\mu \circ (\sigma, \tau) \notin CV(\mathbb{R}, \mathbb{R} \times \mathbb{R})$, $\|\mu \circ (\gamma, \eta)\|_{v_{-\frac{1}{2}}} = \infty$.

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