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Business Computing, especially CIM

Robust Solution to the CLSP and the DLSP with Uncertain Demand and Online Information Base

Dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the subject of Economics and Business Sciences
(Dr. rer. pol.)
at the University of Paderborn

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Paderborn, June 2013

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To the memory of my mother,
who made me who I am.

To my grandfather,
who is the best example to me.

To my husband,
who makes me happy.

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1 Introduction

IBM corporation, which offers optimization software solutions, highlighted four key 2reasons why business managers and pla2nners should take a look at the state of the art in optimization technology:

- "*Changes in the economy and in business process management are making smart, agile economic planning and scenario comparison a necessity. ...*
- *The efficient use of resources has never been more critical in terms of impact on profitability. …*
- *Advances in computer hardware and optimization software have made it possible to evaluate large planning and scheduling problems that were too difficult for computers as recently as five years ago. …*
- *Advances in software technology are making optimization accessible to nontechnical planners, schedulers and managers who make the decisions in most organizations. …*" [1].

However, the optimization in production planning faces serious difficulties: the market data is typically uncertain, so strategic management teams have to estimate market needs and production environment conditions based on historical data or contract conditions.

In the presented work, we focus on medium-term production planning, in particular on lot sizing decisions under demand uncertainty. When choosing lot sizes, a manufacturer has to determine production amounts and production timing in order to optimize a goal function, e.g. minimizing the overall costs, while meeting demand requirements and satisfying existing capacity restrictions. Decisions in lot sizing directly affect the production system performance; therefore development and improvement of solution procedures for lot sizing problems is essentially important.

In case the demand uncertainty is neglected during production planning process, the obtained production plan may become extremely costly or even infeasible. Failures or delays in delivery to customers are highly undesirable and may cause penalties; demands that are unsatisfied in time can cause the loss of customers. For this reason the robustness of a production plan is a main requirement. Moreover, manufacturers require a production plan that is robust in the non-probabilistic way – remains feasible for each possible uncertain demand scenario. In addition, a guaranteed upper bound for the total costs can contribute significantly to the improvement of strategic planning.

However, data uncertainty is not the only aspect affecting the mathematical model of production; incomplete information about the market or the production planning system also significantly influences the mathematical model. When the total planning horizon is large enough, a manufacturer often deals with partly given market data: even though it is possible to estimate or create a prognosis of market behavior for the nearest future, information about demands in far outstanding periods is typically unavailable. Production planning problems with incomplete information about data also belong to the class of production planning problems under demand uncertainty.

Generalizing, each mathematical model that describes a system or a process only approximates the reality, and many data parameters that are used in mathematical models should be considered as uncertain values instead of fixed numbers. Consequently, there is a need of a solution immunized against data uncertainty in a wide variety of applications, both in industry and science, e.g. in engineering, biology or chemistry. For all scientific areas, however, the difficulties for modeling under uncertainty are basically the same:

- incorporation of the defined uncertainty set into the model;
- solution/optimization of the model under the given uncertainty and the determination of a robust solution.

The recent research efforts toward robustness in production planning tend to use the probabilistic interpretation of uncertainty, and few of them include case-studies necessary for computational evaluation and comparison of production planning models under demand uncertainty. Hence, computational evaluation and comparisons of solution approaches are required for production planning models with uncertain demand. In science, manufacturing processes are typically described with complex optimization models that include many integer variables; consequently, tractability of a robust model is also critical.

The presented research investigates two basic production planning models for lot sizing under demand uncertainty and contains seven main chapters. Chapter 2 “Problem Statement” discusses the uncertain Capacitated Lot Sizing Problem (CLSP) and the uncertain Discrete Lot

Sizing and Scheduling Problem (DLSP) models in detail, and highlights the main research goals. Chapter 3 “State of the Art” comprises fundamental concepts, developments and techniques that relate to the stated goals. Two subsections of the “State of the Art” are devoted to the existing production planning models and the methods for uncertainty treatment, respectively. Chapter 4 “Action points” emphasizes particular research issues that are not covered by the state of the art, but are required to achieve the research goals. In other words, this chapter defines the main research steps that have to be done. Subsequent chapter 5 “Methods” describes specific approaches to be applied, and explains each required implementation step in detail. The implementation and interpretation of the results are presented in chapter 6 “Results and discussion”, including the theoretical results as well as the computational examples performed for validation. Finally, conclusions and directions for further research are outlined in chapter 7 “Conclusions”.

2 Problem statement

Presented research investigates production planning systems with lot sizing that are affected by environmental uncertainty – uncertain demands. In particular, a manufacturer possesses information only on upper and lower bounds of potential demand, and no additional knowledge is provided. Not only production planning problems with uncertain demand are analyzed, but also cases with incomplete information about the market: if meaningful demand borders are available for the fixed planning horizon only, or if new information about demand comes into the system gradually over the time (online).

The main goal of the presented research is to construct and solve uncertain mathematical lot sizing models in a robust way. Robustness against demand uncertainty is the crucial point and is understood in a non-probabilistic way, meaning that the solution should stay feasible for any possible demand scenario. The constructed production plan should also reflect all real-life production system restrictions. In order to analyze the obtained results, several solution approaches dealing with uncertainty and a set of computational examples are considered.

The research focuses on two production planning problems, which are essential in manufacturing: Capacitated Lot Sizing Problem (CLSP) and Dynamic Lot Sizing and Scheduling Problem (DLSP). Both problems are described with corresponding mathematical optimization models, but since they have a different structure they are analyzed independently. The CLSP and the DLSP are well-known production planning problems, and detailed model descriptions are provided in chapter 3. Three subsections below describe the main components of the research problem: formulation of the CLSP and the DLSP with the specific structure and uncertain demands, main research goals.

2.1 Description of the uncertain Capacitated Lot Sizing Problem (CLSP)

In this section, the considered Capacitated Lot Sizing Problem (CLSP), its mathematical optimization model and the data uncertainty, which affects the model, are described.

The CLSP with several production machines, several products and several working slots during one planning period is considered; production in overtime slot is more costly. If production occurred, corresponding setup costs have to be paid. Production capacity of each machine is limited individually for each product and for the total production amount in a normal and an overtime slot. Actual maximal production pro period is calculated in accordance with productivity coefficients in normal and overtime slots. The maximal and minimal stock levels are specified by a manufacturer and should not be violated.

The CLSP with uncertain information about demand for all planning periods, except the first one, is considered. In particular, the interval demand uncertainty is considered: upper and lower bounds for possible demand values are known, but the exact value of demand is unavailable, see Figure 2.1. Therefore, the so-called "uncertainty interval" is defined for the demand, and it is assumed that the uncertain demand takes values outside of the uncertainty interval with a probability of zero (never).

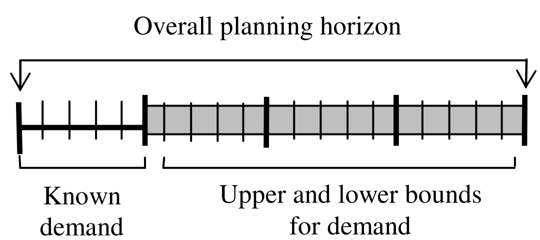


Figure 2.1. Interval uncertainty of demand

An exemplary case is when a company A expects the demand for the production in the next planning period equal to the value D. However, due to the contract signed between the company A and a customer the demand could be changed by ten percentage points from the mentioned value at the beginning of the planning period. Thereby, during the planning process, the demand uncertainty interval is $[0.9D, 1.1D]$. No additional information is available, such as probabilities or distribution function, only upper and lower bounds for the demand are provided. The manufacturer aims to satisfy each possible realization of the demand from the uncertainty set, while minimizing the production costs. The question is: how many units should the manufacturer produce? In other words, which production plan would be optimal in this case? For instance, if the company A chooses to produce $0.9D$ and the value of actual demand is higher, then A has to pay for the production in overtime slot or, even worse, the company is unable to satisfy the demand on time. If the company A chooses to produce $1.1D$ and the value of actual demand is less, then A has to pay additional holding or utilization costs.

If we consider the example above for several products and production machines, for a large planning horizon, taking all additional production system restrictions and costs into account, the problem becomes more complex.

If the model can be resolved when the production process has already started (e.g. when new information about the market comes into the system), the production planning is implemented under "folding horizon": the production plan is created not once, but several times based on the current state of the production planning system and the new market data, see Figure 2.2.

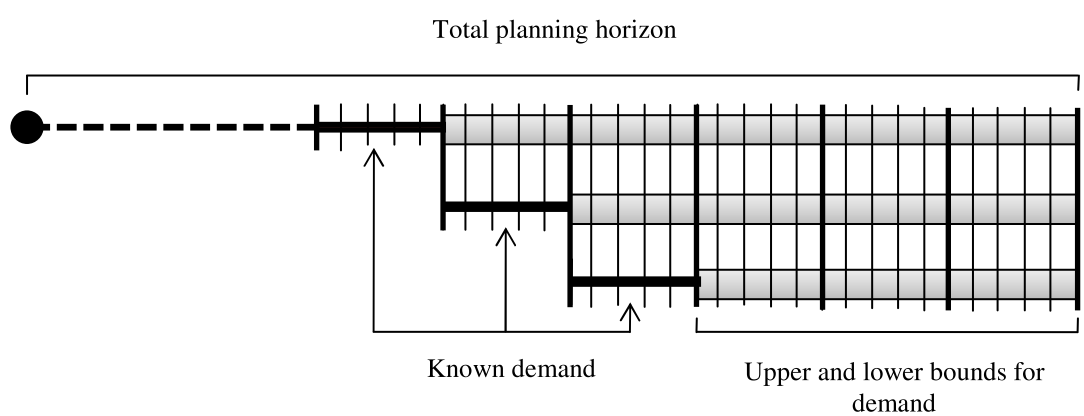


Figure 2.2. Folding horizon

Additionally, the research's aim is to analyze the case when not only the data uncertainty affects the mathematical model of production, but also incomplete information about the market. In this case, the demand is deterministic in the first planning period, defined by uncertainty intervals for the next few periods, but is totally unknown for distant periods, see Figure 2.3.

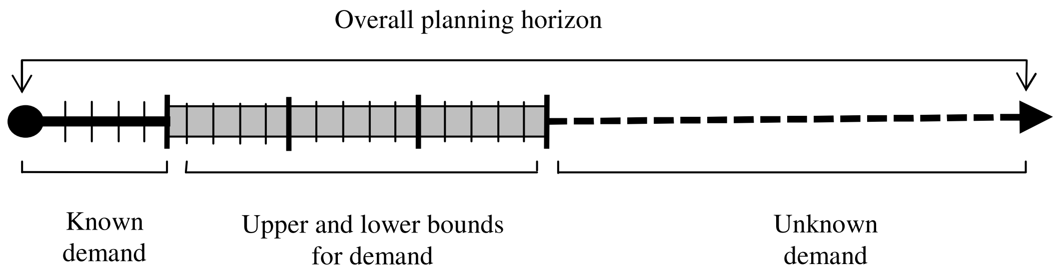


Figure 2.3. Incomplete information about demand for production

It is assumed that new information about market data becomes available over time, so the new production plan should be created according to the new market state and current product amounts in stock. Such a CLSP is referred to as a CLSP with an online information base, meaning that information about market comes into the production planning system in real time, for instance at the end of each planning period.

When information about the market is updated in an online manner, the production planning is implemented under "rolling horizon": first, the production plan is created for the planning periods with known information about demand; then, it is recalculated when the new information comes into the system, see Figure 2.4. The new production plan is based on the current state of the production planning system.

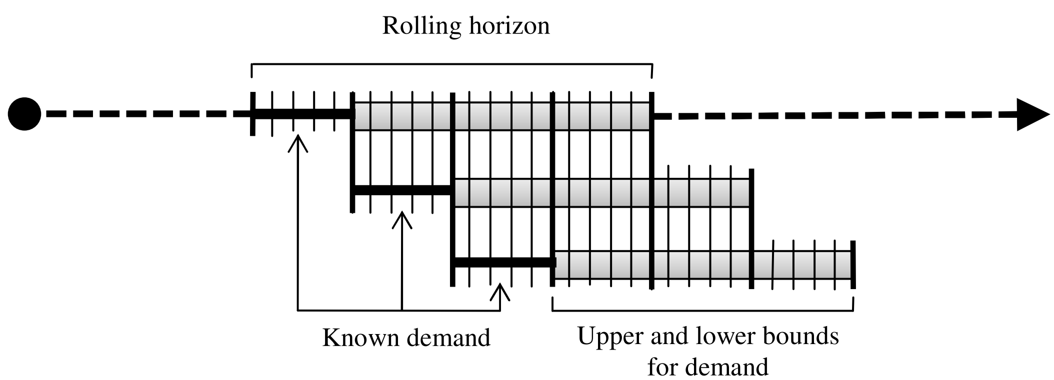


Figure 2.4. Rolling horizon

Let us exemplarily consider a manufacturing company that aims to optimize the summarized costs for the total planning horizon of one year, considering one planning period equal to one week. At the beginning, the exact value of the market demand is known only for the next week. For three following weeks it is possible to define the demand uncertainty interval.

Therefore, an information base of four weeks for the customer's demand is given in total; information for the rest of the year is not available. At the end of the first week, new information about the market comes into the system and an information base of four weeks is available again starting from the second planning period (from the second week). The total number of periods, for which an information base is given, is referred to as rolling horizon. In the considered example, the rolling horizon is equal to four weeks (four planning periods).

Planning under rolling horizon makes the optimization more complicated: typically, it is impossible to obtain the optimal production plan for the total planning horizon and it is non-trivial to find a feasible one.

The notation and the general mathematical model of uncertain CLSP with interval demand uncertainty are provided below. It is assumed that the interval demand uncertainty is defined as $[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*]$, where d_{jt}^* are given nominal values of the demand for each product and planning period, θ is the given uncertainty level. The CLSP with interval demand uncertainty belongs to the problems with uncertain data. It should be resolved at the end of each planning period given that an online information base or a folding horizon is considered.

Parameters:

$j = 1 \ldots M$	products,
$i = 1 \ldots K$	production machines,
$t = 1 \ldots N$	planning periods,

Data:

d_{jt}^*	nominal demand for product j in the planning period t (units),
θ	uncertainty level of demand,
NC	productivity coefficient in normal time slot,
OvC	productivity coefficient in overtime slot,
u_{ijt}	production capacity of machine i for product j in normal working time slot of period t (units),
w_{ijt}	production capacity of machine i for product j in overtime slot of period t (units),

U_{it}	total production capacity of machine i in normal working time slot of period t (units),
W_{it}	total production capacity of machine i in overtime slot of period t (units),
c_{ijt}	production costs (per unit) for product j in normal working time slot of period t using production machine i (\$),
ov_{ijt}	production costs (per unit) for product j in overtime slot of period t using production machine i (\$),
h_{jt}	holding costs for product j (per unit and per period) in period t (\$),
s_{ijt}	setup costs for machine i in normal working time slot of period t, when producing product j (\$),
sv_{ijt}	setup costs for machine i in overtime slot of period t, when producing product j (\$),
I_{j0}	initial stock of product j (units),
I_j^{min}	minimal stock of product j at the end of any period (units),
I_j^{max}	maximal stock of product j at the end of any period (units).

Decision variables of the CLSP model are the following:

x_{ijt}	quantity of product j to be produced in normal working time slot of period t using production machine i,
y_{ijt}	quantity of product j to be produced in overtime slot of period t using production machine i,
I_{jt}	stock of product j at the end of period t,
z_{ijt}	binary variable, which equals to 1 when $x_{ijt} \geq 0$ in period t and 0 otherwise,
zv_{ijt}	binary variable, which equals to 1 when $y_{ijt} \geq 0$ in period t and 0 otherwise.

The mathematical model, describing the production planning problem:

$$min\left(\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}x_{ijt} + ov_{ijt}y_{ijt} + s_{ijt}z_{ijt} + sv_{ijt}zv_{ijt}\right) + \sum_{t=1}^{N}\sum_{j=1}^{M} h_{jt}I_{jt}\right) \quad (2.1)$$

s.t.:

$$I_{j1} = I_{j0} + \sum_{i=1}^{K} \left(x_{ij1} + y_{ij1} \right) - d_{j1}, \forall\, j \in \{1 \dots M\} \tag{2.2}$$

$$I_{jt} = I_{j,t-1} + \sum_{i=1}^{K} \left(x_{ijt} + y_{ijt} \right) - d_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{2 \dots N\} \tag{2.3}$$

$$x_{ijt} \le u_{ijt} \cdot z_{ijt}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.4}$$

$$y_{ijt} \le w_{ijt} \cdot zv_{ijt}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.5}$$

$$\sum_{j=1}^{M} x_{ijt} \le U_{it}, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{2.6}$$

$$\sum_{j=1}^{M} y_{ijt} \le W_{it}, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{2.7}$$

$$I_j^{min} \le I_{jt} \le I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.8}$$

$$z_{ijt} \in \{0,1\}, zv_{ijt} \in \{0,1\}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.9}$$

$$x_{ijt} \ge 0, y_{ijt} \ge 0, \qquad i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.10}$$

$$d_{jt} \in \left[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^* \right], \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.11}$$

The objective function (2.1) minimizes the summarized (over the products, machines and periods) total costs of the manufacturer: production and setup costs in normal and overtime working slots and holding costs. Constraints (2.2), (2.3) are the so-called balance restrictions: they describe the connection between amounts produced, amounts sold and amounts that are in stock in each planning period. Constraints (2.4) and (2.5) fix the setup costs if production has occurred and, at the same time, verify capacity limits of one machine for each particular product. Constraints (2.6) and (2.7) are summarized over products capacity restrictions of one machine. Inequalities in (2.8) set the lower and upper bounds on stock. Constraints (2.9), (2.10) show the non-negative nature of variables. Uncertainty of the demand data is described by (2.11).

2.2 Description of the uncertain Discrete Lot sizing and Scheduling Problem (DLSP)

In this section, the Discrete Lot sizing and Scheduling Problem (DLSP), its mathematical optimization model and the data uncertainty, which affects the model, are described.

The DLSP with several production machines and several products is investigated. Whenever a machine switches from one type of product to another, corresponding setup costs have to be paid. As in all small buckets models (see chapter 3), the item type cannot be changed during a production period. A production machine can either use the full capacity of a production period or remain idle during that period. The maximal and minimal stock levels are specified by the manufacturer and should not be violated.

The main objective of the manufacturer is similar to the one claimed for the CLSP: minimization of the total production, holding and setup costs, while satisfying the customers' demand. However, unlike the previous problem, not only the length of lot sizes is important, but also the exact sequence of the production lots. Since a unique item is assigned to one machine in each planning period, the resulting sequence of item-machine-period assignments naturally defines the production schedule.

Analogically to the previous section, the DLSP with uncertain demand for all planning periods, except the first one, is investigated. In particular, the aim is to investigate the DLSP with interval demand uncertainty and the DLSP with an online information base (planning under rolling horizon). In order to avoid duplication, the reader is referred to section 2.1, e.g. to the Figure 2.1, Figure 2.3, Figure 2.4.

The notation and general mathematical model of uncertain DLSP with interval demand uncertainty are provided below. It is assumed, same as for the uncertain CLSP, that the interval uncertainty is defined as $[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*]$, where d_{jt}^* are given nominal values of the demand for each product and planning period, θ is the given uncertainty level. The DLSP with interval demand uncertainty belongs to the problems with uncertain data. It should be resolved at the end of each planning period given that an online information base or a folding horizon is considered.

Parameters:

$j = 1 \dots M$ products,

$i = 1 \dots K$ production machines,

$t = 1 \dots N$ planning periods,

Data:

d_{jt}^{*} nominal demand for product j in the planning period t (units),

θ uncertainty level of demand,

p_{ij} production speed of machine i for product j (units per period),

c_{ijt} production costs (per unit) for product j at normal working time slot of period t using production machine i (\$),

h_{jt} holding costs for product j (per unit and per period) in period t (\$),

s_{ijt} setup costs for machine i at normal working time slot of period t, when producing product j (\$),

E_{ij0} binary variable describing the initial state of machine i; it is equal to 1 when machine i is set up to produce product j and 0 otherwise,

I_{j0} initial stock of product j (units),

I_j^{min} minimal stock of product j at the end of any period (units),

I_j^{max} maximal stock of product j at the end of any period (units).

Decision variables of the DLSP model are:

I_{jt} stock of product j at the end of period t,

z_{ijt} binary variable that equals to 1 when production of product j on the production machine i occurred in period t and equals 0 otherwise.

A mathematical model of production process is the MIP problem:

$$min\left(\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}p_{ij}z_{ijt} + s_{ijt}\, max\left(0, z_{ijt} - z_{ij,t-1}\right)\right) + \sum_{t=1}^{N}\sum_{j=1}^{M} h_{jt}I_{jt}\right) \qquad (2.12)$$

s.t.:

$$I_{j1} = I_{j0} + \sum_{i=1}^{K} p_{ij} z_{ij1} - d_{j1}, \qquad \forall j \in \{1 \dots M\} \tag{2.13}$$

$$I_{jt} = I_{j,t-1} + \sum_{i=1}^{K} p_{ij} z_{ijt} - d_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{2 \dots N\} \tag{2.14}$$

$$\sum_{j=1}^{M} z_{ijt} \leq 1, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{2.15}$$

$$I_{jt} \geq I_j^{min}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.16}$$

$$I_{jt} \leq I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.17}$$

$$z_{ijt} \in \{0,1\}, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.18}$$

$$d_{jt} \in \left[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*\right], \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{2.19}$$

Here the objective function (2.12) minimizes the total costs of the manufacturer (summarized over the products, machines and periods): production and setup costs in normal and overtime working slots, and holding costs. Constrains (2.13), (2.14) are the so-called balance restrictions: they describe the connection between amounts produced, amounts sold and amounts that are in stock in each planning period. Constrain (2.15) states that at most one item can be produced on one machine per one planning period. Inequalities (2.16), (2.17) set the lower and upper bounds on stock. Constrain (2.18) shows the binary nature of decision variables. The uncertainty of the demand data is described by (2.19).

2.3 Main research goals

The crucial aim of the research is to solve the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) with robustness guarantees.

In a rapidly changing environment, the manufacturer needs to continuously adapt the production plan in accordance with the current market situation. Unavoidably, this leads to changes in lot sizes, stock volumes, machine loads, but most importantly, it leads to changes in the value of total cost. Due to this fact, manufacturing companies aim to immunize production planning systems against uncertainty of environment or, in other words, to make them robust.

The term “robustness” is used in the production planning as well as in other disciplines, but it could be defined in several different ways. In manufacturing, most frequently, the ability to deliver a product to the customer on time and on budget plays a crucial role. The manufacturing company needs a feasibility guarantee of the created production plan as well as an upper bound of total costs for any environment conditions. These two main concepts define the idea of robustness in production planning.

Hence, the robustness of a solution is understood in the presented research as an aggregate of the two following properties:

- a feasibility of the created production plan for any possible demand realization scenario from the uncertainty set;
- a possibility to guarantee the meaningful upper bound of total costs or fixed performance ratio of the solution algorithm.

One important expectation of the manufacturer from the robust production plan is the ability to satisfy each possible scenario of demand realization from the uncertainty set, meaning that even though the total manufacturing costs may change, the manufacturer is able to deliver the product to the customer in accordance with the schedule.

The feasibility of the created production plan gives the manufacturer certain advantages, since it guarantees the satisfaction of customers’ demands in any case. The increased quality of service for the production planning company directly leads to the competitive advantage in the market. That’s why the feasibility is a significant property of the robust production plan and is included into the objectives scope.

The second robustness criterion of the production plan is a guaranteed maximum value of the total costs. It has a significant impact on the quality of management in a manufacturing company: knowledge of the total investments in the production processes is especially important for the strategic planning team of a company, since it gives the opportunity to plan and manage other working directions in a more effective way. Typically, company profit depends on the value of the total costs and therefore can be estimated as well, so the production company strives to know the required inputs in advance and with certainty, although the exact values of the customers’ demand are unknown. Clearly, it is not a major concern if the total costs decrease, but the maximal possible value of the total investments should be guaranteed.

For the reasons mentioned above, it is a crucial aim of this work to provide the manufacturer a meaningful upper bound of the total costs along with the production plan.

To achieve the robustness goals, several subgoals are defined:

1. analyze the uncertain CLSP problem (2.1)-(2.11) in order to identify the influence of the demand scenario structure on the total value of the costs;
2. derive the performance guarantee for the strict online algorithm (no information about future demand is known) for the uncertain CLSP problem (2.1)-(2.11);
3. apply the Robust Optimization (RO) approach for the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) and evaluate the obtained results.

The first subgoal can be achieved through the determination of the worst case demand scenario – a realization of the demand from the uncertainty set, leading to the highest possible costs for the manufacturer. If the worst case of the demand scenario is determined, the deterministic CLSP can be solved for this particular case and the robustness goals will be achieved. Indeed, by solving the worst case, the upper bound on total costs for any demand scenario can be guaranteed. In addition, the worst case demand scenario is of particular interest for the situation when information is not just uncertain, but also incomplete (planning under rolling horizon).

Due to uncertainty and incomplete information about the future, it is typically impossible to find an optimal production plan. Thus, the major goal of this work is to generate a solution that is the closest to the best-possible one. If the uncertain CLSP (2.1)-(2.11) has a strict online information base, it is interesting to compare the production plan obtained from the online algorithm (planning without any demand knowledge) with the best possible production plan (planning with the complete demand knowledge). Calculating the ratio between the total cost values and comparing them for different demand scenarios, the competitive ratio of the online solution algorithm is defined. The competitive ratio provides the performance guarantee of the solution algorithm to the manufacturer and is used as a quality criterion.

The last subgoal concerns the Robust Optimization (RO) approach. The RO is widely used for solving uncertain Linear Programming (LP) models and, therefore, can be applied to the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) considered. However, the integrality of the decision variables seems to be an issue and should be investigated separately. An additional aim of this work is to compare the solution provided by the RO with solutions provided by other approaches. This task will be performed on the basis of a representative set of computational examples. Besides the differences in the value of the total costs, some other differences in solutions can be considered, such as the computational time, the needed hardware resources as well as the applicability of the method to different problem types. Using calculation experiments, the problem structure's influence on the solution can be analyzed (e.g. analysis of the solution quality dependence on the uncertainty level). Furthermore, it is possible to check if the defined worst-case of demand provides the highest total costs in a certain computational example.

3 State of the art

3.1 Production planning problems

3.1.1 Review of production planning problems

Production planning and inventory management problems serve as the typical application field of mathematical optimization models and algorithms. Along with the globalization, distinct growth of information technology and changes in production environments, new challenges, e.g. the construction and optimization of increasingly complex production planning models appear [1].

Decision support is crucial in production planning and is required on several levels: strategic, tactical or operational. This widely-used classification has its origins in the work of R. Anthony. He defines the strategic planning as *"the process of deciding on the objectives of the organization, the resources used to obtain these objectives, and the policies that are to govern the acquisition, use and disposition of these resources*" [2], the tactical planning as "*the process by which managers assure that these resources are obtained and used effectively and efficiently in the accomplishment of the organization's objectives*" [2] and the operational planning as "*the process of assuring that specific tasks are carried out effectively and efficiently*" [2].

Long-term (strategic) planning focuses on such decisions as the facility location or the entrance into new markets. Medium-term (tactical) planning includes resource planning, establishing production quantities or lot sizing. Short-term (operational) planning defines day-to-day scheduling of operations such as job sequencing [3].

The presented work focuses on medium-term planning, in particular on lot sizing decisions. These decisions directly affect the production system performance; therefore, development and improvement of the solution procedures in lot sizing is essential.

Lot sizing problems belong to the tactical level of planning: they have a medium planning horizon, use internal and external sources of information, exist under a medium degree of uncertainty and a medium degree of risk [4].

Lot sizing is used in many production planning concepts, which are mostly implemented as a part of production planning software systems. For example, it is included in Hierarchical Production Planning (HPP), Just in Time Manufacturing (JIT), Material Requirements Planning (MRP-I), Manufacturing Resource Planning (MRP-II), Enterprise Resource Planning (ERP). This production planning software is widely used in industry, though the lot sizing problems are usually solved heuristically, which causes some criticism. For example, Pochet and Wolsey provide the following characteristic of the MRP systems: "*In summary, the myopic MRP decomposition scheme leads to important productivity and flexibility losses, two of the key levers in all manufacturing strategies, which is exactly the opposite of what is expected from a good planning system, and the opposite of what was initially expected from MRP systems. Indeed, the starting idea of MRP was to distinguish the dependent demand, which is computable, from the uncertain independent demand, for which forecasts are needed, with the objective of knowing when and how much is needed of each component, and thereby opening the way to a reduction of the global inventory levels*" [5].

A variety of lot sizing models is used in industry and science, but each model can be described by the following parameters and characteristics:

- planning horizon type and length of the planning periods;
- number of the production levels;
- demand type;
- number of resources (machines) and products;
- setup costs structure;
- backlogging (inventory shortage) possibility;
- presence or absence of capacity constraints.

By the type of planning horizon lot sizing models are classified into models with finite or infinite planning horizon, continuous or discrete time. Discrete time lot sizing models encompass “big bucket models” (sometimes called also “large bucket models”) and “small bucket models”. M. Salomon defines the differences between them as following: “… *'small' time bucket models allow for at most one item to be produced per period, while setups can be carried across period boundaries, whereas 'large' bucket models permit multiple items to be produced per period, but setups are disallowed to be carried over, even if production of a given product occurs in* *successive periods*" [6]. One also distinguish between folding and rolling planning horizons, where the model is solved not once, but at the end of each planning period, taking into account new information new information about state of the market and the production planning system. The difference is that in models with folding horizon information about the market or production planning system is uncertain, but available up to the end of planning horizon, whereas in the models with rolling horizon information about the market or production planning system is incomplete and is known only for the fixed number of periods.

The lot sizing model is referred to as one-level model, if it is assumed that products are manufactured directly from the row materials without any sub-assemblies, and it is referred to as multi-level otherwise.

Demand in lot sizing models can be static (constant), maintaining a fixed value over all planning periods, or dynamic, changing its value over planning periods. The demand is called deterministic if its value is known in advance or uncertain if its exact value is unknown. The uncertain demand is called probabilistic if its described with the help of some probabilistic concept or by probabilities.

Lot sizing problems can also be classified into one-resource or multi-resource and one-item or multi-item, depending on the available machines and number of the end products.

Setup costs may be taken into account using small bucket or big bucket models, or ignored completely. As it was described previously, setup costs can be also sequence-dependent or sequence-independent.

Some lot sizing models allow backlogging of the demand (shortening of the inventory) – satisfying the demand later for additional costs; such models are called lot sizing models with backlogging. If the lot sizing model is defined without backlogging (model with lost sales), then delivery to the customer after required date is not allowed, meaning that the order will be lost.

Finally, the capacitated lot sizing problems should be mentioned: the lot sizing problems, wherein the capacity of production and/or inventory is limited. If the production capacity is assumed to be infinite, then a model is called uncapacitated.

The most important lot sizing models, which can be called classical, are the following: the Economic Order Quantity model (EOQ), the Wagner-Within model (WW), the Economic Lot sizing and Scheduling Problem (ELSP), the Capacitated Lot sizing Problem (CLSP), the Discrete Lot sizing and Scheduling Problem (DLSP), the Continuous Setup Lot sizing Problem

(CSLP), the Proportional Lot sizing and Scheduling Problem (PLSP), and the General Lot sizing and Scheduling Problem (GLSP). The CLSP and the DLSP models are considered in more detail later on; the capacitated models review that includes description, properties, computational efficiency and solution algorithms for other lot sizing models can be found in [6], [3], [7].

The most important properties of the mentioned above models, except for the later-developed PLSP and GLSP, were summarized in the work of M. Salomon [6] and are presented in Table 3.1.

Table 3.1: Comparison between lot sizing models. (Source: M. Salomon [6])

	Time axis	Maximum # of items produced per period	Setup costs	Demand	Production quantity per period
EOQ	infinite, continuous	1	constant, per batch	constant	unrestricted
WW	finite, discrete	1	dynamic, per period	dynamic	unrestricted
ELSP	infinite, continuous	*	constant, per batch	constant	**
CLSP	finite, discrete	unrestricted	dynamic, per period	dynamic	less than or equal to capacity
CSLP	finite, discrete	1	dynamic, per batch	dynamic	less than or equal to capacity
DLSP	finite, discrete	1	dynamic, per batch	dynamic	zero or equal to capacity

* - maximum production of one item per unit of time,

** - production quantity equal to zero or equal to production rate.

All considered production planning problems are typically modeled as Linear Programming (LP) or Mixed-Integer Programming (MIP) models. Both the LP and the MIP problems are extensively used in Operations Research field. Their mathematical definitions, properties, computational status, solution methods and applications are discussed for example in [8], [9], [5] or [10].

3.1.2 Capacitated Lot Sizing Problem (CLSP)

The Capacitated Lot Sizing Problem (CLSP) in its classical formulation belongs to the class of one-level one-resource multi-item production planning problems with the finite planning horizon. The CLSP belongs to the class of big bucket models, so the planning periods range from days to weeks or sometimes months. The crucial aim of a manufacturer is to define the total production scope and sizes of production lots. Therefore, it is unnecessary to set the sequence of lots within a period, because the exact production order can be determined on the operational level. For the same reason, a machine is set up at the beginning of planning period, whether or not the item it produces has changed, and keeps producing the given item for the duration of the period.

The total production amount in each planning period must lie within the prescribed bounds, which are reflected by the capacity constraint in the corresponding mathematical model.

The main goal of the manufacturer is satisfying demands, while minimizing the total manufacturing costs. Backlogging of the demand (inventory shortage) is not allowed. Therefore, three types of costs are associated with CLSP: production, setup and holding costs. Production costs are proportional to the produced amounts and are dynamic - may vary from period to period. Setup costs are fixed and are sequence independent, so the production order does not influence the setup costs. Holding costs are proportional to the amounts in stock and are written off at the end of each planning period.

To summarize, the production plan created by solving the CLSP should determine the production amounts (lot sizes) of each product as well as the stocks in each planning period. The total costs should be minimized, while satisfying the demands. At the same time, the production sequence in each planning period should not be defined, and it can be determined by the manufacturer without any loss in costs. The notations and the mathematical formulation of the CLSP problem are presented below.

Parameters:

$j = 1 \dots M$ products,

$t = 1 \dots N$ planning periods.

Data:

d_{jt} nominal demand for product j in the planning period t,

p_{jt} capacity needed to produce one unit of product j in the planning period t,

U_t total production capacity in the planning period t,

c_{jt} production costs (per unit) for product j in the planning period t,

h_j holding costs for product j (per unit and per period),

s_j setup costs associated with production of product j,

I_{j0} initial stock of product j,

Decision variables of the CLSP model are the following:

x_{jt} quantity of product j (lot size) to be produced in the planning period t,

I_{jt} stock of product j at the end of planning period t,

z_{jt} binary variable, which equals to 1 when $x_{jt} \geq 0$ in period t and 0 otherwise,

The mathematical model, describing the production process is the MIP problem:

$$min \sum_{t=1}^{N} \sum_{j=1}^{M} \left(s_j z_{jt} + c_{jt} x_{jt} + h_j I_{jt} \right) \tag{3.1}$$

s.t.:

$$I_{jt} = I_{j,t-1} + x_{jt} - d_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{3.2}$$

$$x_{jt} \leq \left(\sum_{k=t}^{N} d_{jk} \right) z_{jt}, \qquad \forall\, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{3.3}$$

$$\sum_{j=1}^{M} p_{jt} x_{jt} \leq U_t, \qquad \forall t \in \{1 \dots N\} \tag{3.4}$$

$$z_{jt} \in \{0,1\}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{3.5}$$

$$x_{jt} \geq 0, I_{jt} \geq 0, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{3.6}$$

The objective function (3.1) minimizes the total costs of the manufacturer. Restriction (3.2) is the so-called balance restriction: it describes the connection between amounts produced, amounts sold and amounts in stock in each planning period. Restriction (3.3) takes into account the setup costs if production has occurred, whereas restriction (3.4) ensures capacity is not exceeded the limit. Finally, restriction (3.5) shows that the setup variable is binary, and restriction (3.6) shows the non-negative nature of the other decision variables.

In the 1980s, the CLSP problem was proved to belong to the class of NP-hard problems even for the single-item formulation. Later, it was shown that the multi-item CLSP problem is strictly NP-hard. Consequently, there is currently no efficient exact algorithm for CLSP. However, advances in computer hardware and optimization software allow solving even large CLSP problems that were too difficult for computers several years ago with an appropriate computational accuracy.

Based on the literature, solution methods for the CLSP models are classified into three main groups: exact methods, common-sense or specialized heuristics and the mathematical programming-based heuristics.

Further information about the CLSP models can be found in the works [6], [3], [7].

3.1.3 Discrete Lot sizing and Scheduling Problem (DLSP)

The Discrete Lot Sizing and Scheduling Problem (DLSP) in its classical formulation belongs to the class of one-level one-resource multi-item production planning problems with the finite planning horizon. The DLSP belongs to the class of small bucket models, so the planning periods range from minutes to hours or sometimes days.

In contrast to the CLSP model, production process is organized under the “all-or-nothing” assumption – production amount in each planning period equals to zero or to the full production capacity. Since a unique item is assigned to each planning period, the resulting sequence of item-period assignments naturally defines the production schedule.

Another important difference between the CLSP and the DLSP is the structure of setup costs. In the DLSP setup costs are fixed, but they only have to be paid if the type of produced item was changed. Production of the same item may be continued in the next planning period without the additional setup costs.

The main objective of the manufacturer is similar to the one in the CLSP: minimizing the total costs, while satisfying the customers' demand. Backlogging of the customers demand (inventory shortage) is not allowed and, therefore, three types of costs are associated with CLSP: production, setup and holding costs.

Production costs are proportional to the produced amounts and are dynamic - may vary from period to period. Holding costs are proportional to the amounts in stock and are written off at the end of each planning period.

To summarize, the production plan created by the DLSP should determine the production amounts (lot sizes) of each product as well as the sequence of lot sizes and the stock in each planning period. The total costs should be minimized, while demands are satisfied. The notations and the mathematical formulation of the DLSP problem are presented below.

Parameters:

$j = 1 \ldots M$ products,

$t = 1 \ldots N$ planning periods.

Data:

d_{jt} nominal demand for product j in the planning period t,

p_j production capacity available for product j in one planning period,

U_t total production capacity in the planning period t,

c_{jt} production costs (per unit) for product j in the planning period t,

h_j holding costs for product j (per unit and per period),

s_j setup costs associated with production of product j,

I_{j0} initial stock of product j.

Decision variables of the DLSP model are the following:

I_{jt} stock of product j at the end of planning period t,

z_{jt} binary variable that equals to 1 when production of product j occurred in planning period t and equals to 0 otherwise.

The mathematical model, describing the production process is the MIP problem:

$$min\sum_{t=1}^{N}\sum_{j=1}^{M}\left(s_j max\left(0, z_{jt}-z_{j,t-1}\right)+c_{jt}p_j z_{jt}+h_j I_{jt}\right) \tag{3.7}$$

s.t.:

$$I_{jt}=I_{j,t-1}+p_j z_{jt}-d_{jt}, \qquad \forall j\in\{1\ldots M\}, t\in\{1\ldots N\} \tag{3.8}$$

$$\sum_{j=1}^{M} z_{jt}\le 1, \qquad \forall t\in\{1\ldots N\} \tag{3.9}$$

$$z_{jt}\in\{0,1\}, \qquad \forall j\in\{1\ldots M\}, t\in\{1\ldots N\} \tag{3.10}$$

$$I_{jt}\ge 0, \qquad \forall j\in\{1\ldots M\}, t\in\{1\ldots N\} \tag{3.11}$$

Similarly to CLSP, the objective function (3.7) minimizes the total costs of the manufacturer; restriction (3.8) is the balance restriction. Inequalities (3.9) ensure that products of only one type are produced on a given machine in each planning period. Finally, restriction (3.10) shows that the decision variables z_{jt} are binary, while restriction (3.11) shows that the stock is non-negative.

In the 1990s, the DLSP problem was proved to belong to the class of NP-hard problems. Analogically to CLSP, solution methods for the DLSP models are classified into exact methods, common-sense or specialized heuristics and mathematical programming-based heuristics.

Further information about the DLSP models can be found in [11], [6], [3], [7].

3.2 Uncertainty treatment

3.2.1 Uncertainty in production planning

Uncertainty in production planning systems was classified by Ho [12] into two main groups; his classification is shown in the summarized form in Figure 3.1.

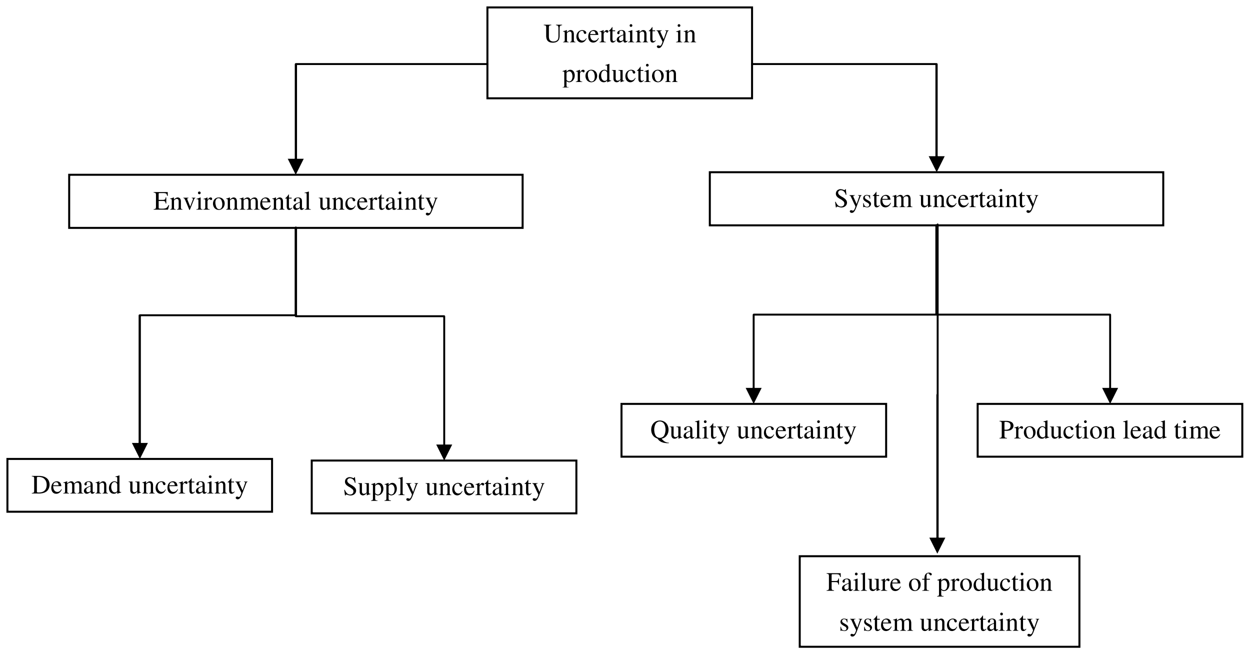


Figure 3.1. Classification of uncertainty in production planning

Giannoccaro and Pontrandolfo in [13] classified models that are currently used in production planning to treat uncertainty into four main groups: conceptual models, artificial intelligence-based models, analytical models and simulation models. Conceptual models utilize such methods as safety stocks or safety lead times, artificial intelligence-based models utilize genetic algorithms or fuzzy logic, analytical models include mathematical programming and deterministic approximations, whereas simulation models include heuristic methods or probability distributions.

According to the literature review by Mula et.al. [14], from the 87 reviewed models for production planning under uncertainty, 35 were the analytical models, 27 were based on the artificial intelligence, 16 were the simulation models and 9 were the conceptual models. The authors analyzed additionally a connection between the particular production planning issues and the corresponding modeling approaches that were used, and summarized the results in the table, see Table 3.2. The number of research papers that have used one or another method is indicated in the brackets.

Table 3.2: Classification scheme for models for production planning under uncertainty
Source: Mula et.al. [14]

Research topic	Number of citations
1. Aggregate planning	Artificial intelligence models (8) Simulation models (2)
2. Hierarchical production planning	Analytical models (3)
3. Material requirement planning	Conceptual models (9) Analytical models (6) Artificial intelligence models (4) Simulation models (10)
4. Capacity planning	Analytical models (4) Simulation models (1)
5. Manufacturing resource planning	Analytical models (7) Artificial intelligence models (5) Simulation models (2)
6. Inventory management	Analytical models (10) Artificial intelligence models (5)
7. Supply chain planning	Conceptual models (1) Analytical models (5) Artificial intelligence models (5)

The authors concluded the review by stating that the analytical modeling approach, and in particular the stochastic programming, is the most popular tool for uncertainty handling in production planning. However, if the production planning problem structure is complex or there are several types of uncertainty in the model, then the stochastic programming is typically replaced by the simulation or artificial intelligence-based approaches. The dynamic programming approach was used only for a few models and the corresponding studies were mainly theoretical [14].

Further information can be found in [15], [16], [17], [14].

3.2.2 Worst-case analysis and competitive analysis

Introduction to worst-case analysis and competitive analysis

The term "worst-case analysis" is typically associated with the circuit analysis, but may be also used in the analysis of algorithms. In the latter case, the goal of the worst-case analysis can be formulated as following: to find such a structure of a model input that the analyzed algorithm provides the worst possible model output. How "good" or "bad" the output is, typically is measured based on the model objective. If, for example, a LP minimization model is considered, the worst possible output will be defined as the highest possible value of the objective. In practice, the worst-case input for the considered model and the solution algorithm is typically hard to determine due to the high complexity of the system.

The worst-case analysis is applied for algorithms evaluation in different research areas: in finance [18], computer science [19], cellular manufacturing [20], discrete mathematics [21], etc. In production planning, to the best of our knowledge, worst-case analysis is mainly used for evaluating the heuristic solution approaches performance. For example, Axsäter analyzed four different heuristics for the uncapacitated dynamic lot sizing problem [22], whereas Bitran et al. analyzed heuristics, product aggregation, and partitioning of the planning horizon [23].

Competitive analysis is a research technique that is closely related to the worst-case analysis and is mostly applied to online planning problems. Main terms used in the competitive analysis are the following: online optimization problems (or simply online problem), online and offline algorithms, competitive ratio. An online problem is an optimization problem where decisions should be made without complete knowledge of the required information; the model input is received in an online manner and, correspondingly, the output should be also provided online. An offline algorithm is an optimal algorithm for a given optimization problem and its input. It is assumed that the offline algorithm has access to all of input data. The online algorithm constructs decision based only on the model inputs (data or events) from the past.

Origins of competitive analysis as well as its definition are described in the work of Borodin and El-Yaniv [24]: "*The traditional approach to studying online algorithms falls within the framework of distributional (or average-case) complexity, whereby one hypothesizes a distribution on events (event sequences) and studies the expected total cost (payoff) or expected cost (payoff) per event. During the past 10 years the interest in this subject has been renewed* *largely as a result of the approach of competitive analysis, whereby the quality of an online algorithm on each input sequence is measured by comparing its performance to that of an optimal offline algorithm, which is (for an online problem) an unrealizable algorithm that has full knowledge of the future. Competitive analysis thus falls within the framework of worst-case complexity*".

To provide the formal definition of the competitive ratio the notations used in [24] are kept: I denotes an input of an optimization model, $ALG(I)$ denotes the cost associated with the solution provided by an online algorithm ALG for the input I and $OPT(I)$ denotes the optimal cost for the input I (provided by the offline algorithm).

The online algorithm ALG is called c-competitive if exist such α and c ($\alpha, c \in \mathbb{R}$) that for any model input I the following holds:

$$ALG(I) \leq c \cdot OPT(I) + \alpha \tag{3.12}$$

If (3.12) stays true even for $\alpha = 0$, then the algorithm ALG is called strictly c-competitive.

The smallest c, for which an online algorithm ALG is c-competitive, is called the competitive ratio of this algorithm:

$$inf\,\{\text{c} \mid ALG(I) \leq c \cdot OPT(I) + \alpha\} \tag{3.13}$$

The smallest c, such that an online algorithm ALG is strictly c-competitive, is called the strict competitive ratio of ALG:

$$inf\,\{\text{c} \mid ALG(I) \leq c \cdot OPT(I)\} \tag{3.14}$$

In Game Theory online and offline players are typically considered instead of online and offline algorithms. However, it does not change the fundamental meaning.

Review of M. Wagner work "Online lot-sizing problems with ordering, holding and shortage costs"

In this section, a brief overview of M. Wagner research [25] is provided.

Two inventory management models are considered there: the inventory model describing perishable products with lost sales and the inventory model describing durable products with backlogging. Both models are affected by the demand uncertainty and are strictly online – the demand for the current planning period is revealed only at the end of the period, after the inventory ordering decisions have been already made. Both models use finite planning horizon and period-dependent cost structure.

The mathematical formulation of the models is reproduced below with the original notation, though it differs from the generally accepted.

Parameters:

$i = 1 \dots n$	planning periods,

Data:

d_i	demand in planning period i (units),
c_i	unit ordering cost in period i (\$),
s_i	unit inventory shortage costs in period i (\$),
h_i	unit inventory holding costs in period i (\$),
K_i	fixed ordering cost for placing an order in period i (\$),
I_0	initial stock (units).

Decision variables:

q_i	ordering quantity in period i (units).
I_i	stock in the end of period i,
$I_i^+ = \max(I_i, 0)$	positive inventory at the end of period i,
$I_i^- = \max(-I_i, 0)$	negative inventory at the end of period i.

The mathematical model, describing the inventory management problem for perishable products with lost sales:

$$\min_{\boldsymbol{q} \geq \boldsymbol{0}} \sum_{i=1}^{n} (c_i q_i + h_i (q_i - d_i)^+ + s_i (d_i - q_i)^+ + K_i \delta(q_i)) \tag{3.15}$$

The online optimization problem (3.15) minimizes the total costs over all ordering strategies $\boldsymbol{q}$, considering the fixed planning horizon of N periods. The production cannot be put in the stock, and if the demand exceeds product availability in any period, the excess demand is lost.

The mathematical model, describing the inventory management problem for durable products with backlogging is the following:

$$min\sum_{i=1}^{n}(c_i q_i + h_i I_i^+ + s_i I_i^- + K_i\delta(q_i)) \tag{3.16}$$

s.t.:

$$I_i = I_{i-1} + q_i - d_i, \qquad \forall i \in \{1 \dots n\} \tag{3.17}$$

$$q_i \geq 0, \qquad \forall i \in \{1 \dots n\} \tag{3.18}$$

The online optimization problem (3.16)-(3.18) minimizes the total costs over all ordering strategies $\boldsymbol{q}$, considering the fixed planning horizon of N periods and taking into account that the product can be held in the stock. The excess demand is backlogged for the future periods.

The main results of [25] are the sufficient condition for existence and formulas for the lower and upper bounds for competitive ratio; to derive them, the techniques of competitive analysis and the worst-case analysis were utilized.

The reasoning provided by Wagner is valid under three main assumptions [25]:

Assumption 1. $\boldsymbol{c}, \boldsymbol{K}, \boldsymbol{h}, \boldsymbol{s} > 0$.

Assumption 2. $\boldsymbol{d} \neq 0$.

Assumption 3. *There exists a period in which it is optimal to procure a positive quantity.*

To prove the main statements in [25], the known result from the linear-fractional programming was utilized:

Lemma 1. *If* $\{x: f'(x) + g > 0, x \geq 0\}$ *is non-empty, then the optimization problems*

$$\max_{x}\left(\frac{\boldsymbol{c}'\boldsymbol{x} + d}{\boldsymbol{f}'\boldsymbol{x} + g}\right) \qquad \qquad \max_{y,z}(\boldsymbol{c}'\boldsymbol{y} + dz)$$

s.t. *and* *s.t.*

$$\boldsymbol{f}'\boldsymbol{x} + g > 0 \qquad \qquad \boldsymbol{f}'\boldsymbol{y} + gz = 1$$

$$x \geq 0 \qquad \qquad \boldsymbol{y} \geq \boldsymbol{0}, z \geq 0$$

are equivalent [26].

For the inventory model with perishable products and lost sales the main result was formulated in the following theorem [25]:

Theorem 3.1. *In period* i, *if* $c_i \geq s_i$, *order* $q_i = 0$ *units and if* $c_i < s_i$, *order* q_i *units. The competitive ratio of this strategy is at most:*

$$\max_{i:c_i<s_i}\left\{\max\left\{\left(\frac{c_i + h_i}{K_i}\right)q_i + 1, \frac{s_i}{c_i}\right\}\right\}.$$

Furthermore, the competitive ratio of any algorithm is at most:

$$\max_{i:c_i<s_i}\left\{\frac{s_i}{c_i}\right\}.$$

To prove the statement of Theorem 3.1, its narrowed version for one planning period was formulated. The proof of the narrowed theorem includes the following steps:

- definition of competitive ratio (3.14) was expanded using the known representation of model costs (3.15);
- two different procurement strategies were analyzed: $q > d$ and $q \leq d$;
- for each procurement strategy the formula of the competitive ratio is a linear-fractional program, so the Lemma 1 was applied to get the corresponding LP program;
- for the LP problems the corresponding dual systems were constructed and solved;
- maximum of obtained dual solutions defines the value of competitive ratio.

After the narrowed version of Theorem 3.1 was proved, the special cases were analyzed: periods with zero demands and periods where the offline algorithm orders nothing. Summarizing the results, the statement of Theorem 3.1 was proved.

For the inventory model with durable products and backlogging the main result was formulated as the following theorem [25]:

Theorem 3.2. *For an arbitrary online strategy* $\boldsymbol{q} \geq \boldsymbol{0}$, $\boldsymbol{b'q} + K \leq 0$ *is a sufficient condition for the existence of a finite strict competitive ratio. Furthermore, if a strict finite ratio* ρ *exists, it satisfies:*

$$\boldsymbol{\max\left\{\max_{1\leq i\leq n}\left\{\frac{a_i}{\alpha_i}\right\}, \frac{b'q + K}{K'e}\right\} \leq \rho \leq \max_{1\leq i\leq n}\left\{\frac{a_i}{\alpha_i}\right\}}.$$

To prove the statement of Theorem 3.2, two subsets of the planning periods set were considered: $P = \{i: I_i \geq 0\}$ and $N = \{i: I_i \leq 0\}$. P and N denote respectively the periods with non-negative and non-positive inventory. Then two additional Lemmas were formulated and proved [25]:

Lemma 2. *For an arbitrary online strategy* $\boldsymbol{q} \geq 0$*, we can write the online costs as*

$$Z(\boldsymbol{d}) \leq \boldsymbol{a}'\boldsymbol{d} + \boldsymbol{b}'\boldsymbol{q} + \boldsymbol{K},$$

where

$$a_i = \sum_{\substack{j=i,\\ j\in N}}^{n} s_j - \sum_{\substack{j=i,\\ j\in P}}^{n} h_j,$$

$$b_i = c_i + \sum_{\substack{j=i,\\ j\in P}}^{n} h_j - \sum_{\substack{j=i,\\ j\in N}}^{n} s_j,$$

for $i = 1 \dots n$ *and* $\boldsymbol{K} = \sum_{i=1}^{n} K_i \delta(q_i)$.

Lemma 3. *The optimal offline cost of model (3.16)-(3.18) has the following lower and upper bounds:*

$$\boldsymbol{\alpha}'\boldsymbol{d} \leq Z^*(\boldsymbol{d}) \leq \boldsymbol{c}'\boldsymbol{d} + \boldsymbol{K}'\boldsymbol{e}, \forall \boldsymbol{d} \geq \boldsymbol{0},$$

where $\alpha = (\alpha_1 \dots \alpha_n)$ *is defined as:*

$$\alpha_i = min\left\{ \min_{1\leq j\leq i-1} \left\{ c_j + \sum_{k=j}^{i-1} h_k \right\}, c_i, \min_{i+1\leq j\leq n} \left\{ c_j + \sum_{k=i}^{j-1} s_k \right\}, \sum_{k=i}^{n} s_k \right\}, \qquad i = 1 \dots n$$

and can be interpreted as the minimum marginal cost of satisfying demand d_i *by considering (1) procuring in period* $j < i$ *and carrying the inventory to period* i*, (2) procuring in period* i*, (3) backlogging until period* $j > i$*, or (4) the cost of not satisfying the demand.*

The proof of Theorem 3.2 includes the following steps:

- definition of competitive ratio (3.14) was expanded using the results provided by Lemma 2 and Lemma 3;
- two linear-fractional programs were constructed for the lower and upper bound of the competitive ratio;
- Lemma 1 was applied to get the corresponding LP programs;
- the corresponding dual systems were constructed and solved for the LP problems.

As the result, the sufficient condition for the finite competitive ratio existence and the formula for lower and upper bounds for the competitive ratio were derived.

Additionally, the special “Make-to-Order” procurement strategy was analyzed in [25]. The considered strategy always had a negative stock; ordering occurred only if the sum of corresponding shortage costs was higher than the ordering cost. The upper and lower bounds of strict competitive ratio were provided. Moreover, if ordering and shortage costs were constant in all planning periods, the exact value of competitive ratio was calculated.

It should be noted that the work [25] may be considered as an extension of [27], wherein online inventory problems and corresponding competitive ratio were analyzed.

3.2.3 Sensitivity analysis

Saltelli et al. define sensitivity analysis as: "*The study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input*" [28].

Sensitivity analysis is a well-established scientific research area with a number of solution methods, which typically tries to solve the following question: if the optimal solution is found, up to which extent can one alter the model's parameters, so that the obtained solution remains optimal?

Sensitivity analysis is applicable to uncertain LP and MIP models, but data uncertainty is usually ignored in the first step and the solution is generated based on nominal data. Then such a magnitude of input data vector is determined that the obtained solution stays optimal. Therefore, one can only measure the robustness of the obtained solution, but cannot control it.

Sensitivity analysis studies deviations of model solutions and data with the help of perturbations and defines the sensitivity of the initial system.

Another question, which can be answered by sensitivity analysis (e.g. by parametric self-dual simplex method), is the following: if a model is infeasible, how can we change the model parameters to make it feasible?

Advantages of the sensitivity analysis were formulated by Rappaport in [29]: "

1. *In its applied organizational setting, sensitivity analysis may be broadly defined as a study to determine the responsiveness of the conclusions of an analysis to changes or errors in parameter values used in the analysis.*
2. *Sensitivity analysis tests the responsiveness of model results to possible changes in parameter values, and thereby offers valuable information for appraising the relative risk among alternative courses of action.*

3. *Sensitivity analysis can provide systematic guidelines for allocating scarce organizational resources to data collection and data refinement activities.*
4. *Under the sensitivity analysis approach, if the value of a decision is insensitive to estimated parameter variations, the decision to purchase no additional information can be made without introducing a statistical decision model.*
5. *If the value of a decision is sensitive to estimated parameter variations and the information decision is not obvious, a statistical decision model may be developed to guide the information decision.*“

Detailed information about sensitivity analysis can be found in [29], [30], [31], [8], [10] and [28].

3.2.4 Stochastic Optimization

Stochastic Optimization (SO) is used in Operation Research to optimize mathematical optimization models under uncertainty, given that some probabilistic information about the uncertainty is provided. Thus, the SO typically treats the uncertain parameters as the random variables with the known probability distributions.

The SO research area began to develop after the work of G. Dantzig [32], wherein several LP models under uncertainty were analyzed. This paper was also included in the list “Ten Most Influential Papers of Management Science’s First Fifty Years”, published in 2004 [33]. The SO approach is widely used in the different application areas, only a few of them are presented in the following list:

- inventory management [34], [35];
- production planning [36], [37];
- portfolio selection and financial markets [38], [39];
- supply chain management [40];
- power systems [41].

More examples of the SO applications can be found in [42], [36].

To provide a brief overview of the main ideas used in SO, the following basic terms are explained below: probabilistic model, chance constraints and multi-stage stochastic programming model.

Let us consider a deterministic optimization model: $min_{x\geq 0} F(x,d)$, where x is a decision variable, d is a data parameter. If d is uncertain, then the SO assumes that it can be considered as a random variable D with a known distribution. Typically, the distribution is estimated based on historical data. A probabilistic model associated with the initial optimization model minimizes the expected value of objective function. It can be written as:

$$\min_{x\geq 0}\{f(x) \coloneqq E[F(x,d)]\} \tag{3.19}$$

The probabilistic model (3.19) minimizes the objective on average, since by the law of large numbers, the average cost, for a sufficiently large number of trials, will converge to the expectation $E[F(x,d)]$.

However, it is usually quite difficult to find $E[F(x,d)]$ in a closed form and, therefore, it is difficult to find the solution of (3.19). In case the data uncertainty is characterized by the fixed number of possible scenarios with the given probabilities, the expected value $E[F(x,d)]$ can be written as:

$$E[F(x,d)] = \sum_{k=1}^{K} p_k F(x,d_k)$$

Here the expected value $E[F(x,d)]$ is represented by a weighted sum of K given data scenarios.

Let us consider the case, when the data uncertainty is defined by the uncertainty interval $d \in [d^{min}, d^{max}]$. Distribution of the corresponding random variable D over the uncertainty interval is unknown, but the expected value $E[D]$ is available. For this optimization model the worst case approach can be utilized: try to minimize the worst possible outcome of the model over all possible expected incomes, considering all possible data distributions over the uncertainty interval. The corresponding worst-case probabilistic model is the following:

$$\min_{x\geq 0} \sup_{\mathcal{H}\in\mathcal{M}} E_{\mathcal{H}}[F(x,D)], \tag{3.20}$$

where $\mathcal{M}$ is the set of all probability distributions on interval $[d^{min}, d^{max}]$ considering the given mean $E[D]$.

The solution of model (3.20) is typically more conservative than the solution of model (3.19), since it is optimal for the worst-case input scenario, while the solution of model (3.19) is optimal on average.

Another powerful tool, which is often used in the SO, is the chance constraint construction. The idea is to add the additional chance (or probabilistic) constrain to the initial optimization model (3.19), which will restrict the chance of model output exceeding some fixed value. In other words, the inequality $F(x,d) \leq \tau$ must stay true with some fixed probability α. Parameter $\tau > 0$ is usually called threshold of the model output, $\alpha \in [0,1]$ is called a significance level. The corresponding chance constraint is written as the following:

$$Pr\{F(x,d) > \tau\} \leq \alpha$$

or equivalently,

$$Pr\{F(x,d) \leq \tau\} \geq 1 - \alpha.$$

Augmenting the model (3.19) with the corresponding chance constrain, a solution that is optimal on average results. It guarantees that the costs will be below the fixed threshold value τ with the probability α.

The stochastic programming model (3.19) and the corresponding model with chance constraint describe a one planning period problem. If the problem has several planning periods, then multi-stage stochastic programming model is utilized: instead of the uncertain random variable D the random process D_t is considered, and the model (3.19) is extended for the several planning periods. The resulting model utilizes vectors instead of the single variables and is written as the following:

$$\min_{\boldsymbol{x} \geq 0}\{f(\boldsymbol{x}) \coloneqq E[F(\boldsymbol{x},\boldsymbol{d})]\}$$

Analogically to the one-period problem, if several sets of uncertain data scenarios have known probabilities, then the objective can be presented as their weighted sum.

The dynamic programming approach (Wagner-Within algorithm) can be also utilized for calculating the optimal decision. Dynamic programming equations include the expected values of uncertain data and typically cannot be solved in a closed form, so numerical methods are used for calculating the objective.

Great interest to the field of the SO in the past 60 years caused a lot of different solution methods and techniques used in close connection with the SO, e.g. Monte Carlo method, dual analysis, numerical methods, heuristics, scenario trees and sample average approximation. The fundamental concepts and solution methods of the stochastic programming and optimization can be found in [36], [43], [44].

3.2.5 Robust Optimization

Robust Optimization (RO) is a specific and relatively novel methodology for handling optimization problems with uncertain data. As opposed to SO, the RO approach is based on the worst-case analysis and does not require any probabilistic data, though it may be used if available.

The RO approach has its origins in the field of Robust Control, which appeared first in the works of Bode in the 1930s. Main development of the Robust Control took place in 1980s and 1990s and entailed many publications, see for instance [45], [46], [47], [48]. Robustness questions were also considered in the statistics, see for instance the work of Huber [49].

Modern RO is based mainly on convex optimization and robust linear programming and arose in works of Soyster [50], Ben-Tal, Nemirovski [51], [52] and El Ghaoui [53], [54]. Advances in computing technologies as well as development of fast interior point methods for semi-definite optimization [26] raised interest to the RO techniques. The rapid growth of interest in the field of RO is noted in 2000s and is still ongoing.

A solution that is immunized against data uncertainty is required in a wide variety of applications. There are many research papers showing up the use of the RO approach in different fields of study, such as:

- Finance and Portfolio Management [55], [56];
- Statistics, Learning and Estimation [57], [58];
- Inventory and Supply Chain Management [59], [60];
- Engineering: Structural Design [61], Circuit Design [62], Power Control in Wireless Channels [63];
- Control Theory [64], [65];

As far as the production planning is concerned, robustness goals are typically achieved in the probabilistic sense with the help of the SO, though during the last years several researches were done based on the RO techniques [66].

In the RO it is typically assumed that the model data uncertainty is defined by the uncertainty intervals or uncertainty sets - the totality of all conceivable values. RO provides a solution that stays feasible for each possible uncertainty realization, whereas SO provides robust solution in a probabilistic sense. A high-level summary of the RO methodology and an overview of related applications can be found in [67], [66], [68]. The RO paradigm is explained in details in the work of A. Ben-Tal, L. El Ghaoui and A. Nemirovski [69]. Several crucial definitions and concepts from this book are reproduced below and are used further in the presented research.

Let us remind the general representation of LP problem: $min_x\{c^T x + d: Ax \leq b\}$. The RO methodology is developed with the help of four definitions, they are reproduced below from [69].

Definition 1. *An uncertain LP problem is a collection of LP programs of a common structure:*

$$\{\min_x\{c^T x + d \,:\, Ax \leq b\} \,:\, (c, A, b) \in U\}, \tag{3.21}$$

with the data (c, A, b) varying in a prescribed uncertainty set U.

Typically, it is impossible to solve (3.21), because the number of LP problems in the collection above growth rapidly with the size of uncertainty set U and number of uncertain data parameters. In order to include uncertainty into the mathematical model, construction of the Robust Counterpart is proposed in RO, so it is unnecessary to solve the collection of LP programs (3.21). The RC is the worst-case oriented model and, therefore, provides the feasible solution for each possible data realization.

Definition 2. *Robust Counterpart (RC) of the uncertain LP problem (3.21) is the optimization problem:*

$$\min_x\{\sup_{(c,A,b)\in U} c^T x + d \,:\, Ax \leq b, \qquad \forall(c, A, b) \in U\}$$

or, equivalently, the optimization problem:

$$\min_{x,t}\{t \,:\, c^T x + d \leq t,\, Ax \leq b, \qquad \forall(c, A, b) \in U\} \tag{3.22}$$

Optimal solution of the RC (3.22) is called robust optimal solution of uncertain LP problem (3.21).

However, talking about RC and robust optimal solution for (3.21), it is assumed that three following assumptions are applied to the considered decision environment [69]:

Assumption 4. *All entries in the decision vector x represent "here and now" decisions: they should get specific numerical values as a result of solving the problem before the actual data "reveals itself".*

Assumption 5. *The decision maker is fully responsible for consequences of the decisions to be made when, and only when, the actual data is within the prespecified uncertainty set U.*

Assumption 6. *The constraints of the uncertain LP in question are "hard" – the decision maker cannot tolerate violations of constraints when the data is in U.*

It was shown in [69] that any uncertain LP model can be written as a model with certain objective and, correspondingly, it always has the RC with the certain objective. It was shown also that the RC for the LP with certain objective is constructed only by constrains transformation. Moreover, without loss of generality, the uncertainty set U can be considered as a direct product of sets $U_1 \dots U_i$, where i is the total number of constraints in original uncertain LP model and each U_k is closed and convex set.

The RC (3.22) is a semi-infinite LP model, which in general can be NP-hard, but it was proved in [52] that "*the RC of the uncertain LO problem with uncertainty set U is computationally tractable whenever the convex uncertainty set U itself is computationally tractable*". This is the case when the uncertainty set U is a direct product of uncertainty intervals. So if the uncertainty set U is polytope, then the RC is an explicit system of linear inequalities. Tractability of the RC was also established for other uncertainty set structures, e.g. for conic quadratic or semidefinite representations.

However, if the initial uncertain problem is not a LP model, but a MIP model, then the tractability of the corresponding RC is not guaranteed. Even though the RO methodology can be applied to the uncertain MIP models and the RC is constructed analogically to the LP case, the resulting RC would be a MIP model, which is more computationally complex.

If Assumption 5 and Assumption 6 are not strict, meaning that some of the model restrictions can be violated to some extent or the uncertain data may take values from the prescribed neighborhood of the uncertainty set U, then the RC may be transformed into the Globalized Robust Counterpart (GRC). Solution provided by the GRC guarantees that if the uncertain data fluctuates in the prescribed neighborhood of the uncertainty set U, then violation of the constraints has a fixed magnitude.

An extension of the RC also appears when the uncertain data come into the system not on one occasion, but step by step, so Assumption 4 may be extended. Dealing with a multi-stage decision making process, some model variables represent not “here and now” decisions, but rather “wait-and-see” decisions – they can be made after several process stages are finished. To make the system adaptable for the changing environment conditions, “wait-and-see” decision variables are allowed to depend on a part of the true data. The corresponding prescribed part of the true data defines the so-called “information base” for decision variables on each decision stage. If specific rules for the decision variables dependency on the information base are chosen, then the extended RC - Adjustable Robust Counterpart - is defined [69]:

Definition 3. *Adjustable Robust Counterpart (ARC) of the uncertain LP problem (3.21) is the RC (3.22), where the decision variables are depended on a prescribed part of the true data (information base):*

$$x_j = X_j\left(P_j\omega\right), \tag{3.23}$$

where P_j are prescribed matrixes defining information base and $X_j(\cdot)$ are decision rules to be chosen.

After solving the ARC model, the optimal robust decision rules (functions of the true data) are provided, whereas the robust optimal numerical values for the decision variables are given by the RC. The ARC of the uncertain LP problem (3.21) is less conservative than the corresponding RC, meaning decreased value of total costs. The reason is that the adjustable robust optimal solution is adaptable to the current state of the uncertain environment.

The crucial issue for the ARC is tractability of constructed model. Since optimization is done over decision rules, which depend on many real variables, it may appear that the solution algorithm is intractable. In order to solve the ARC in a polynomial time a careful choice of decision rules is required.

Definition 4. *Affinely Adjustable Robust Counterpart (AARC) of the uncertain LP problem (3.21) is the ARC, where the decision rules $X_j(\cdot)$ are affine functions.*

It was shown in [69] that the AARC in computational sense is not harder than the initial RC or, in other words, the constructed AARC has the same tractability status as the initial RC in case with fixed recourse. An uncertain LP problem with prescribed information base is called a problem with fixed recourse if the coefficients of every adjustable variable in the model are certain. However, analogically to the RC, if the initial uncertain problem does not belong to the

LP, but rather to the MIP, then tractability status of the corresponding AARC is not determined in general case.

In case of fixed recourse, the same reasoning may be applied as in the case of RC construction: if the uncertainty set U is a polytope, then the AARC represents an explicit system of linear inequalities. Analogically, if Assumption 5 and Assumption 6 are not strict, then the AARC may be transformed into Affine Adjustable Globalized Robust Counterpart (AAGRC).

The Robust Optimization includes a lot of other techniques, for instance Robust Counterpart Approximations of Scalar Chance Constraints, as well as considers other uncertainty models and sets structures, for instance uncertain conic quadratic problems. Detailed information on the RO approach can be found in [69].

4 Action points

The crucial aim of the presented research is to solve the uncertain CLSP (2.1)-(2.11) and the DLSP (2.12)-(2.19) with robustness guarantees.

The first subgoal is to analyze the uncertain CLSP problem (2.1)-(2.11) with the online information base (planning under the rolling horizon), and to find such a structure of a model input that the analyzed solution algorithm provides the worst possible model outcome. According to the state of the art, the formulated subgoal can be solved with the help of the worst-case analysis. However, the worst-case analysis is typically applied to the simple structures of uncapacitated inventory/production models, whereas the considered CLSP has capacity restrictions. Moreover, it has also several production slots and, therefore, specific cost structure. Since the CLSP (2.1)-(2.11) is quite complex, some additional assumptions are required to simplify the analysis. After determining the worst case demand scenario and solving the worst case, the upper bound on total costs for any possible demand scenario and, additionally, the competitive ratio of considered online algorithm can be respectively identified and guaranteed to the manufacturer.

In the work of M. Wagner [25], described in chapter 3 “State of the Art”, the sufficient condition for the existence of a finite competitive ratio and a formula for lower and upper bounds for the competitive ratio were derived for two specific inventory models: for perishable products with lost sales and for durable products with backlogging. Both models are affected by demand uncertainty and are strictly online. The second subgoal in the presented research is logically formulated as the following: to apply results of [25] for the production planning models and to extend them by considering models with capacity restrictions. The competitive analysis approach utilized in [25] will allow providing the performance guarantees of an online algorithm to the manufacturer.

The last research subgoal is to derive a robust solution for the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) with the help of the Robust Optimization (RO) concept. RO techniques provide a solution that is robust in the non-probabilistic way, and that is consistent with the stated main research aim. However, as outlined in chapter 3, there are just a few applications of the RO in the field of production planning (lot sizing); moreover, in these applications computational examples are rarely used. Another important issue is that RC and AARC models are widely used for solving uncertain LP problems, but may be intractable for the MIP problems. In that way, the integrality of decision variables in the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) should be included into RC and AARCs and investigated separately. Another action point considered is to compare the solution provided by the RO with solutions provided by other approaches (e.g. probabilistic ones), which cannot be done without a representative set of computational examples and a simulation. In particular, differences in the value of total costs, computational time, needed hardware resources as well as appliance of the method on different problem types should be compared. Using calculation experiments, the influence of the problem structure on the solution can be additionally analyzed, e.g. how the solution quality depends on the uncertainty level.

5 Methods

According to chapter 3 (section 3.2.1), four main models dealing with the uncertainty are used in production planning. However, most of them are inapplicable for the case outlined in this thesis or do not allow to meet the stated goals.

The structure of the uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19) is too complex to use the dynamic programming approach, which provides required robustness guarantees. Conceptual models, such as safety stocks, are typically unsuitable for the cases with incomplete information about market (online information base). Simulation models as well as stochastic optimization usually utilize heuristics or probabilistic data, and therefore provide feasible solution only with guaranteed probability.

In order to achieve the goals stated in the problem statement (chapter 2), it is proposed to utilize analytical approaches from the worst-case analysis and competitive analysis for the simplified versions of the CLSP (2.1)-(2.11). However, the Robust Optimization (RO) is considered as the main solution approach for the uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19).

5.1 Analytical approach for defining the worst-case of demand distribution

In the CLSP (2.1)-(2.11) the setup costs do not play an important role in the production planning process or do not exist at all and, therefore, they are excluded from the consideration. To simplify the reasoning, the case where the manufacturer has only one production machine and only one product to produce is analyzed; there is also no upper bound for the stock. As in the classical CLSP model, the objective function is a function of costs, which should be minimized. It includes the production costs at the normal and the overtime working time slots as well as the holding costs. The demand in the considered CLSP is an uncertain data parameter and is defined by the uncertainty interval:

$$[d_t^* - \theta d_t^*, d_t^* + \theta d_t^*], \qquad \forall t \in \{1 \dots N\},$$

where d_t^* are given nominal values of the demand and θ is the number, defining the uncertainty level.

The simplified CLSP model is the following:

$$min \sum_{t=1}^{N} (c_t x_t + ov_t y_t + h_t I_t) \tag{5.1}$$

s.t.:

$$I_t = I_{t-1} + x_t + y_t - d_t, \qquad \forall t \in \{1 \ldots N\} \tag{5.2}$$

$$x_t \leq w, \qquad \forall t \in \{1 \ldots N\} \tag{5.3}$$

$$x_t + y_j \leq k, \qquad \forall t \in \{1 \ldots N\} \tag{5.4}$$

$$x_t \geq 0, y_t \geq 0, I_t \geq 0, \qquad \forall t \in \{1 \ldots N\} \tag{5.5}$$

One of the crucial research aims is to provide a solution for a production planning problem with an uncertain demand; this solution should stay feasible for the worst-case demand scenario and be comparable with the optimal solution.

Two types of solving algorithms are considered:

1. The algorithm with the planning horizon $n = N$ ("offline"): in this case the demand values for all planning periods are known in advance. The mathematical model can be easily solved without the rolling or folding horizon, and as a result, the optimal solution is obtained.
2. The algorithm with the fixed planning horizon n ("online"), where $1 \leq n < N$: in this case the demand values are known for the next n planning periods only. The model is updated with the new market data and is iteratively solved at the end of each planning period. Consequently, the considered algorithm has the rolling horizon, and most-probably the non-optimal solution is obtained.

In order to identify the influence of the demand distribution on the behavior of the online and the offline algorithms, the technics from the Worst-Case Analysis (WCA) and the Competitive Analysis (CA) are used. The definitions of the problem instance and the competitive ratio are formulated for the CLSP problem (5.1)-(5.5) and the corresponding solving algorithms.

In a next step, the behavior of the online and the offline algorithm, depending on the demand distribution structure, is studied. Therefore, the theorem that describes the worst-case of demand scenario is formulated, meaning that the online algorithm provides the most expensive production plan in comparison with the offline solution. For this purpose, some additional definitions, assumptions and statements are required. To identify what aspects rouse the differences between the behavior of the online and the offline algorithms, the possible changes in the demand distribution are analyzed.

To verify the theorem statement, computational examples are implemented. The objective value and the competitive ratio of the online algorithm are computed for the several demand distributions based on the same problem instance.

Finally, the appliance of the analytical approach for defining the worst-case of the demand distribution should be analyzed for the production planning problems with other structures. For example, the extension of the theorem for the DLSP problem or the CLSP problem with backlogging and setup costs is of particular interest.

5.2 Analytical approach for deriving the upper and lower bounds of the competitive ratio

One of the research aims is to provide a robust production plan to the manufacturer, while the guaranteed difference between the obtained and the optimal solution is less than a fixed value. Ideally, the upper and the lower bound should be provided for the competitive ratio of the online algorithm.

The goals mentioned above refer to the approach described in the work of M. Wagner [25]. In his research, two inventory management models are considered: the mathematical model for perishable products with lost sales and the mathematical model for durable products with backlogging. To apply the technique for deriving the value of competitive ratio proposed in [25], firstly it is necessary to switch from the inventory management problems to the production planning problems. For this purpose in the inventory models (3.15) and (3.16)-(3.18), production amounts are considered instead of inventory orders, setup costs are considered instead of costs for placing an inventory order. Furthermore, the initial notations of

M. Wagner are used, though they differ from the generally accepted notations. The parameters, data and decision variables of the production planning model are presented below.

Parameters:

$i\ = 1 \ldots n$	planning periods,

Data:

d_i	demand in period i (units),
c_i	production costs (per unit) in period i (\$),
s_i	backlogging costs (per unit) in period i (\$),
h_i	holding costs per unit and per period (\$),
K_i	setup costs in period i, when producing product j (\$),
I_0	initial stock (units).

Decision variables:

q_i	quantity of product to be produced in period i (units).
I_i	stock at the end of period i,
$I_i^+ = max(I_i, 0)$	positive inventory at the end of period i,
$I_i^- = max(-I_i, 0)$	negative inventory at the end of period i.

The mathematical model, describing the production process for perishable products with lost sales, is the following:

$$\min_{\boldsymbol{q}\geq\boldsymbol{0}} \sum_{i=1}^{n} (c_i q_i + h_i(q_i - d_i)^+ + s_i(d_i - q_i)^+ + K_i\delta(q_i)) \tag{5.6}$$

The mathematical model, describing the production process for durable products with backlogging, is the following:

$$min \sum_{i=1}^{n} (c_i q_i + h_i I_i^+ + s_i I_i^- + K_i\delta(q_i)) \tag{5.7}$$

s.t.:

$$I_i = I_{i-1} + q_i - d_i, \qquad \forall i \in \{1 \dots n\} \tag{5.8}$$

$$q_i \geq 0, \qquad \forall i \in \{1 \dots n\} \tag{5.9}$$

Both models, perishable products with lost sales (5.6) and durable products with backlogging (5.7)-(5.9), do not include capacity restrictions. Hence, the capacity restrictions are added into the mathematical models; thereby, the results shown in the work of Wagner are extended. Assumption 1, Assumption 2 and Assumption 3 from the section 3.2.2 are assumed to be valid for the reasoning in the present thesis.

For the extended mathematical model (5.6), wherein the capacity restrictions are added, the analogue of Theorem 3.1 (see p.32) is formulated. The thesis follows the reasoning proposed in Wagner's work in order to derive and prove the formula for the maximum value of the competitive ratio and the maximum value of the strict competitive ratio. The extended results (for case with the capacity restrictions) are proved for the production planning problem with just one planning period; in a subsequent step, these results are generalized for multi-period problems. The main proof idea consists of the following steps:

- the upper and the lower bounds of the total costs are found for the offline and the online algorithms;
- the obtained bounds are plugged into the competitive ratio definition;
- with the help of Lemma 1 from the linear-fractional programming (see p. 31), a switch to the linear optimization problem occurs;
- the dual model for the linear optimization model is constructed and solved;
- the resulting solution defines upper and lower bounds for the competitive ratio; the bounds are further extended for the multi-period production planning problem.

For the extended mathematical model (5.7)-(5.9), wherein the capacity restrictions are added, the analogue of the Theorem 3.2 (see p. 32) is formulated. The thesis follows the reasoning proposed in the Wagner's work, and first proves the analogue of the Lemma 3 (see p. 33), which defines the upper and lower bounds for the offline costs. Since the mathematical model with the capacity restrictions has a different feasibility domain, the proof of the formulated lemma becomes more complex. The upper bound of the total offline costs is defined by the feasible production plan $q_i = L_i$ (for each planning period i), while the additional reasoning is required for deriving the lower bound of the total costs. First, the additional assumption should be formulated to restrict the maximum value of the total demand. Next, the linear optimization problem is defined for deriving the corresponding lower bound of the offline costs. To find a feasible solution, the inverted matrix of the LP problem is constructed and, based on this, the dual linear problem is defined. The solution of the dual problem is found by considering the special cases of the system. The proved lemma allows defining the sufficient condition of the competitive ratio existence for the mathematical model (5.7)-(5.9). Moreover, the upper and lower bound for the strict competitive ratio are derived. The corresponding theorem is proved based on the Lemma 1 from the linear-fractional optimization.

In order to verify the derived sufficient condition of the competitive ratio existence for the online algorithm, two computational examples are implemented. The sufficient condition is checked for the given numerical data and the two different production planning strategies.

Finally, the appliance of the analytical approach for deriving the upper and lower bounds of the competitive ratio is analyzed for the production planning problems with other structures. For example, a possibility to extend the theorem for the DLSP problem or the CLSP problem without backlogging is investigated in this thesis.

5.3 Robust Optimization approach

The Robust Optimization (RO) approach is specifically developed for mathematical models with uncertain data parameters and is in particular applicable, for linear optimization models. To utilize the advantages of the RO approach and to accomplish the specific goals, firstly the given data uncertainty is analyzed and the uncertain CLSP and the uncertain DLSP problems are related to an existing subclass of RO problems.

5.3.1 Robust Counterpart (RC)

The formal definition of the uncertain Linear Optimization (LO) problem and its Robust Counterpart (RC) that was given in [69] are cited in in the thesis by (3.21) and (3.22), see p. 39-39. In a simplified way, a RC is an optimization model that is constructed for an initial uncertain mathematical model and whose robust numerical solution is provided before true values of uncertain data are known.

The construction of Robust Counterparts (RCs) is described in detail in the first chapter of [69]. The authors propose the following algorithm for the RC construction, assuming that the uncertain LO problem (3.21) has the certain objective: *"… to get RC, we act as follows:*

- *preserve the original certain objective as it is, and*
- *replace every one of the original constraints*

$$(Ax)_i \leq b_i \;\Leftrightarrow\; a_i^T x \leq b_i$$

(a_i^T is i-th row in A) with its Robust Counterpart

$$a_i^T x \leq b_i \;\forall [a_i, b_i] \in U_i,$$

where U_i is the projection of U on the space of data of i-th constraint:

$$U_i = \{[a_i, b_i] : [A, b] \in U\}."$$

However, to get a RC in a tractable representation, some additional efforts are required. In first chapter of [69] it is stated that *"the RC of the uncertain LO problem with uncertainty set U is computationally tractable whenever the convex uncertainty set U itself is computationally tractable"*. It was also proven in [52] that the semi-infinite constraint of the uncertain LO system with the conic representation of the uncertainty set can be represented by a system of conic inequalities. The corollary from the formulated theorem states that if the uncertainty set is polyhedral, then a semi-infinite constraint can be presented as an explicit system of linear inequalities.

The uncertain demand d_{jt} in the CLSP (2.1)-(2.11) and the DLSP (2.12)-(2.19) models takes values from the corresponding uncertainty interval $\left[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*\right]$ in each planning period. Therefore, the uncertainty set the vector of uncertain demand $\boldsymbol{d}$ belongs to is a polyhedral, which definitely is a convex set. It may be concluded that the methodology of the RC construction is applicable to the considered uncertain production planning problems.

In order to construct the RC for the uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19) the following steps should be completed:

1. The uncertain models (2.1)-(2.11) and (2.12)-(2.19) should be written as the uncertain problems with the certain objectives and the semi-infinite constraints;
2. Each one of the original semi-infinite constraints should be replaced by the explicit system of inequalities;

3. The variables that do not represent the actual decisions (e.g. stock variables) should be eliminated.

The certain objective can be constructed by substituting the uncertain objective expression by an additional variable, and by restricting this variable to be equal or higher than the initial objective in the additional constraint of the system. Step 3 can be done by expressing redundant variables by the variables that represent the actual decisions.

To construct the RC in a solvable form as it is stated in step 2, every original model constraint is transformed to the system of linear inequalities:

$$f_0(\boldsymbol{a}) + \sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(f_{ijt}(\boldsymbol{a}) \times d_{jt}\right) \leq 0, d_{jt} \in \left[d_{jt}^{min}; d_{jt}^{max}\right]$$

$$\Updownarrow$$

$$\begin{cases} f_{ijt}(\boldsymbol{a}) \times d_{jt}^{min} \leq p_{jt} \\ f_{ijt}(\boldsymbol{a}) \times d_{jt}^{max} \leq p_{jt} \\ f_0(\boldsymbol{a}) + \sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M} u_{jt} \leq 0 \end{cases} \tag{5.10}$$

Here, $\boldsymbol{a}$ is the vector of the decisions variables, $f_0(\boldsymbol{a})$ and $f_{ijt}(\boldsymbol{a})$ are the given functions of $\boldsymbol{a}$, d_{jt} is the uncertain demand parameter belonging to the uncertainty interval $\left[d_{jt}^{min}; d_{jt}^{max}\right]$, p_{jt} is an additional variable.

The uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19) contain both, integer and binary variables; consequently, attention should be paid to the tractability of the constructed RCs. Remarks on a tractability of a RC for uncertain mixed-integer models were done in [69]: *"With no integer variables, the fact that the RC is an LO program straightforwardly implies tractability of the RC, while in the presence of integer variables no such conclusion can be made. Indeed, in the mixed integer case already the instances of the uncertain problem P typically are intractable, which, of course, implies intractability of the RC. In the case when the instances of P are tractable, the "fine structure" of the instances responsible for this rare phenomenon usually is destroyed when passing to the mixed-integer reformulation of the RC. There are some remarkable exceptions to this rule; however, in general the Uncertain Mixed-Integer LO is incomparably more complex computationally than the Uncertain LO with real variables."*

To check the tractability of the constructed RCs, the thesis proposes to implement the mathematical models in the optimization software and to test them on a numerical example.

5.3.2 Affinely Adjustable Robust Counterpart (AARC)

Adjustability of the production plan to the changing market situation provides a strict advantage and, therefore, is desirable. According to chapter 3, the Adjustable Robust Counterpart (ARC) provides a less conservative solution for an optimization problem and suits to the environment that allows decisions to be made along with the uncertain data appearance.

ARC aroused as a natural extension of the RC for uncertain optimization models, since some decision variables are allowed to get numerical values already after some part of uncertain data are known. So in the ARC the decision rules – functions of the uncertain data – are considered instead of the original decision variables. However, when switching to the decision rules, the tractability of the resulting optimization model becomes an issue. To make the model computationally effective, the class of affine functions is used for the decision rules representation.

In order to construct the AARC for the uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19), the following steps have to be completed:

1. The information base (a part of data on which the variables are allowed to be dependent) should be defined;
2. Corresponding decision variables of the initial model should be substituted by the affine decision rules;
3. The affine decision rules should be plugged into the mathematical model, which includes new decision variables, and should be transformed into the solvable form.

The information base for the ARC is defined in the definition of the ARC (3.23) by the matrixes P_j. If P_j equal to zero matrixes, meaning that the decision variables are independent from the actual market data, then the special case of ARC – the Robust Counterpart – is obtained. The completeness of the information base is in inverse proportion with the computational status of the constructed AARC. At the same time, the more complete the information base is, the better the gained solution is, in terms of proximity to the optimal solution.

In a next step, the information base B_t for the decision variables in the planning period t of the uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19) is defined by the actual demands obtained in previous planning periods: $B_t = \{1 \dots t\}$.

If the initial uncertain model has a vector $\boldsymbol{a}$ that is comprised of the decision variables, and the information base B_t is defined, then the decision variable a_t should be replaced by the affine decision rule (5.11) in order to construct the AARC:

$$a_t = \pi_{jt}^0 + \sum_{s \in B_t} \pi_{jt}^s \, d_{js} \tag{5.11}$$

The coefficients of the affine decision rules π_{jt}^0 and π_{jt}^s are the new decision variables of the AARC.

To get a computationally tractable representation of the AARC, the affine decision rules are plugged into the uncertain CLSP (2.1)-(2.11) and the uncertain DLSP (2.12)-(2.19) models. The same transformations as for the RC construction are implemented: getting the certain objective, representing each semi-infinite constraint by the system of linear inequalities, eliminating of the variables that describe the state of production planning system.

5.3.3 Features of the AARC for the CLSP

The objective function in the uncertain CLSP (2.1)-(2.11) is the minimization of the manufacturer's total costs. During the RC construction in this thesis, the reasoning was worst-case oriented: the aim was to minimize the maximal possible costs of the manufacturer on the prescribed uncertainty set of demand. The worst-case argumentation is maintained further on, and the AARC model that optimizes the manufacturer's costs for the worst-case demand scenario is constructed. The resulting AARC is referred to in this work as "the AARC WORST-CASE" for shortening.

One of the advantages of the RO approach is the fact that it can be applied not only for the worst-case scenario optimization, since the only requirement is the convexity of the objective function in the initial uncertain optimization model. Based on this fact, the AARC WORST-CASE model can be designed less conservative, and a weighted sum of several demand scenarios can be optimized instead of the worst-case. The objective function of the AARC WORST-CASE model is substituted by minimization of the weighted sum of several demand scenarios. Corresponding demand scenarios are proposed to have a given probabilities to occur, so we plug them into the objective with the corresponding weights. Such the AARC optimizes the weighted manufacturer's costs on several demand scenarios, so it will be called further in our work as "the AARC SCENARIOS" for shortening.

The established RO paradigm mainly discusses models with continuous decision variables (due to the tractability issues) and their robust counterparts. However, the initial CLSP (2.1)-(2.11) contain the binary variables z_{ijt} and zv_{ijt}, which we decided to leave unchanged in the mathematical model, even though it may affect the tractability status of the resulting AARC. To apply the classical RO approach for the AARC construction, we also ignored integrality of the decision variables in the uncertain CLSP (2.1)-(2.11): for the AARC construction we replaced the integer decision variables from the initial CLSP by the affine decision rules, whose coefficients are real numbers. Consequently, the resulting values of decision variables may appear to be non-integers, meaning that the obtained production plan should be additionally analyzed before the implementation stage. Rounding of the obtained non-integer solution values is typically used, however we address the integrality issue differently.

Both the AARC WORST-CASE and the AARC SCENARIOS models with real coefficients in the affine decision rules belong to the class of Mixed Integer Programming (MIP) problems (because of the binary nature of variables z_{ijt} and zv_{ijt}). They were implemented and solved using the IBM ILOG CPLEX Optimization Studio in order to calculate the affine decision rules. Hardware used: Intel(R) Core(TM) i5-450M 2.40 GHz processor and 4.00GB RAM.

5.3.4 Integrality issue of the AARC for the CLSP

The decision variables describing the production amounts and the amounts of product in stock in the uncertain CLSP (2.1)-(2.11) can take only integer values. Since in the AARC they are substituted by the affine decision rules, the corresponding affine decision rules have to take only non-negative integer values independently from the actual data realization.

In the present thesis, it is proposed to narrow the parametric family of linear functions used for the decision rules down to the linear functions with integer coefficients. For this purpose, the coefficients π_{jt}^{0} and π_{jt}^{s} in (5.11) are restricted to be integers. Thereby, the function of decision rules are forced to take only integer values for each possible demand scenario from the prescribed uncertainty set. The explanation is based on the following: the demand parameters take values which belong to the set of natural numbers with zero $\{\mathbb{N} \cup 0\}$; due to the new restriction all the coefficients of the affine decision rules belongs to the set of integers $\mathbb{Z}$. Then the closure property under addition and multiplication of the set of integers $\mathbb{Z}$ allows stating that the production amounts x_{ijt} and y_{ijt} will take only integer values for every demand scenario from the uncertainty set.

The new AARC WORST-CASE and AARC SCENARIOS model (with integer coefficients in affine decision rules) are MIP problems, which were implemented and solved using the IBM ILOG CPLEX Optimization Studio. The same hardware is used as for the AARCs with the real coefficients in the decision rules.

However, narrowing the parametric family of decision rules may affect the proximity of the obtained solutions to the optimal ones; in addition, the tractability of the resulting AARC may be affected. By switching to the integer coefficients in the linear functions of the decision rules, the number of the integer variables in the resulting AARC model is increased; it logically entails the higher computational complexity of the AARC.

To evaluate the changes in the AARC performance, the AARC with real coefficients in the decision rules and the AARC with integer coefficients in decision rules are compared using a numerical example.

5.3.5 Testing workflow for the AARC WORST-CASE and AARC SCENARIOS models. Simulation of uncertain demand

One of the research goals is the computational evaluation of the RO approach. To achieve this, a computational example and simulation of the demand scenarios are considered.

Production system parameters and market data for the initial uncertain CLSP (2.1)-(2.11) are provided by the operating manufacturing company. The numerical values are given for the nominal demands and the uncertainty level. However, in order to evaluate the performance of the AARC solution, the actual realizations of the demand are required. Thus, one may calculate the value of the total costs provided by the AARC production plan, the value of total costs provided by some probabilistic algorithm (assuming that additional probabilistic information is given) and the optimal value of the total costs for one particular demand scenario. Next, it can be evaluated how close the obtained values are to each other, and the "price of robustness" – difference in total costs between optimal and robust production plans – can be calculated.

Obviously, to make the comparison of the solution approaches fair enough, the comparison of costs for only one demand scenario is insufficient. One may desire to compare the differences in costs for a large number of demand scenarios and to compute the difference in costs on average. The maximal difference in costs as well as the time of computation is also of particular interest.

The constructed AARC models for the given CLSP were tested using the workflow presented in Figure 5.1.

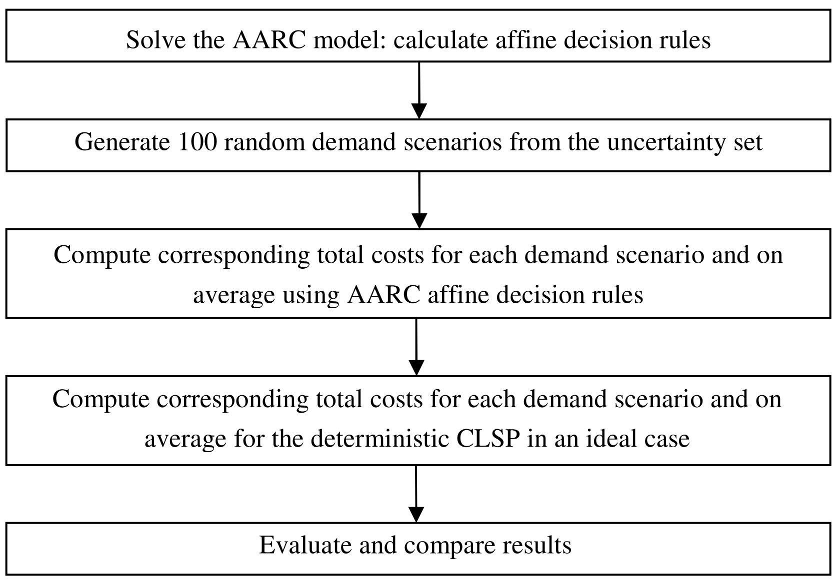


Figure 5.1. Testing workflow for the RO approach (CLSP)

When the first step of the workflow was completed, the difference in the computation time as well as in other solving parameters of the AARC WORST-CASE and the AARC SCENARIOS models were compared.

As a next step, a simulation of 100 demand scenarios is implemented. Three scenarios were fixed to the lowest possible demand, nominal demand and highest possible demand in all planning periods. For the calculation experiment, the fact that demands are integer numbers and allowed them to take any real values from the uncertainty interval was omitted. One of the generated demand scenarios is shown in Figure 5.2.

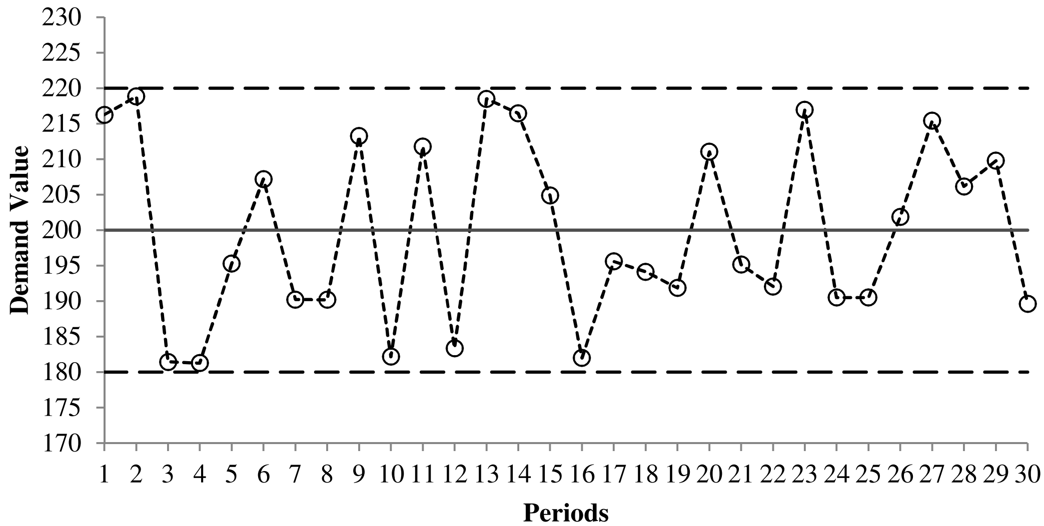


Figure 5.2. One of the generated demand scenarios for the product 2

The numeric values of the production amounts, stocks and total costs were calculated for the AARC WORST-CASE and the AARC SCENARIOS model solutions on each generated demand scenario and on average using the previously calculated AARC affine decision rules.

To identify the "price of robustness" of the AARCs solutions in a next step, the deterministic CLSP model (2.1)-(2.11) was solved for each generated demand scenario. All demands were assumed to be certain and known in advance, forming an ideal case. Accordingly, the obtained objective values are optimal for the corresponding demand scenarios. It is worth noting that the accuracy tolerance was set to 0.5% in deterministic CLSP to reduce the time of computation.

Several parameters were used to compare the obtained results: the absolute values of the objective functions, the maximal absolute and relative gaps between the optimal and the obtained solutions, and the relative gap on average over 100 generated demand scenarios. The computational accuracy was taken into account as well.

Since the integrality of decision variables was addressed in the initial CLSP by constructing decision rules of a special kind, it was intended to investigate the newly constructed AARC WORST-CASE and AARC SCENARIOS models (with integer coefficients in decision rules) on simulated demand scenarios and to compare the obtained results with the results of AARC models that use standard affine decision rules. For this purpose, the same workflow as shown in Figure 5.1 was used and, therefore, changes in performance of the AARC solution with switching to the narrowed affine decision rules could be compared.

Additionally, the testing workflow shown in Figure 5.1 was implemented for the newly constructed AARC WORST-CASE and AARC SCENARIOS models (with integer coefficients in decision rules) using the closest to the reality simulation: the demand took only integer values from the defined uncertainty interval. The demand was allowed to deviate from the nominal values d_{jt}^* by the defined uncertainty percentage, but not less than for 1 unit. The described simulation along with the given data represents the case-study of the RO application. The performance of the AARCs was evaluated by comparing it with the optimal and probabilistic approaches by analogy with the real values demand simulation.

5.3.6 Comparison of solution approaches. The influence of uncertainty level on the total costs value

As it was already mentioned, one may wish to compare the performance of different solution approaches on the same production planning problem instance and to calculate the "price of robustness" – difference in total costs between optimal and robust production plans.

The construction of a non-adjustable Robust Counterpart (RC) and the construction of a probabilistic model (assuming that additional probabilistic information is given) were considered as alternative solution approaches to the AARC. Additionally, all obtained solutions were compared with the optimal solution. The optimal solution for a production planning problem instance is a solution obtained from the deterministic CLSP model (2.1)-(2.11), which was solved for particular demand scenario. All demands were assumed to be certain and known in advance forming an ideal case.

To construct the probabilistic model, the demand was assumed to be uniformly distributed over the uncertainty interval. Due to the symmetry of the uncertainty intervals $\left[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*\right]$, the mean demand values are equal to the nominal demands d_{jt}^*. Given this additional probabilistic information, the mean value of the total costs was optimized: the mean demands d_{jt}^* were plugged into the CLSP model (2.1)-(2.11), and the resulting deterministic MIP problem was solved. It is worth to note that the deterministic CLSP model with the nominal demand values describes another possible production strategy – a strategy where a manufacturer simply ignores the data uncertainty.

To make a comparison of the approaches, 100 production planning problem instances, formed by the implemented demand simulation, were solved with the help of RC, AARC WORST-CASE, AARC SCENARIOS (both with real and integer coefficients in decision rules), probabilistic model and deterministic CLSP describing an ideal case. The absolute and relative gaps between the total costs values of the obtained solutions were computed and compared.

Another important aspect is the influence of the uncertainty level on the following factors: the performance of the solution approaches, the changes in total production costs and the price of robustness of the AARC solutions. For this purpose, the demand simulation was repeated for different levels of demand uncertainty θ: 5%, 10%, 20%, 30% and 50%. The obtained production planning problem instances were used to calculate the total costs values corresponding to the RC, the AARCs, the probabilistic model and the optimal solutions. The absolute and relative gaps between the total costs values of the obtained solutions were computed and compared for each uncertainty level.

Worth noting that the comparison of the AARC solutions with the RC solutions shows whether the adjustability of decision variables provides an actual advantage. However, the RC model became infeasible already for 21% of demand uncertainty; so the RC solution was calculated only for 5%, 10% and 20% of uncertainty level.

5.3.7 Features of the AARC for the DLSP

The crucial issue that hinders AARC construction for the uncertain DLSP (2.12)-(2.19) is the binary nature of the decision variables z_{ijt}. To adjust the decision variables and to get the resulting AARC computationally effective, the binary z_{ijt} has to be substituted by the linear functions of the decision rules (5.11). Obviously, such a replacement is not equivalent, since the range of linear function (5.11) is not equal to {0, 1}.

To overcome this challenge, it is proposed to construct the AARC for the DLSP (2.12)-(2.19) using the following steps:

1. Decision variables z_{ij1}, which are responsible for the production decisions in the next planning period, are not substituted by the linear decision rules in the constructed AARC; they are left unchanged in the model and keep their binary nature.
2. Decision variables z_{ijt} for $t > 1$, which are responsible for the production decisions in the further planning period, are substituted in the constructed AARC by the linear decision rules (5.11) that are restricted to take values in the interval [0, 1].
3. The production plan provided by the constructed AARC is formed under a folding planning horizon: the AARC is resolved at the end of each planning period, taking into account new market and system data.

The solutions z_{ij1} for the first planning period should be regarded as „the guide to action“ for the manufacturer. The values of decision variables z_{ijt} for $t > 1$ should be interpreted as the possibilities of production of product j on the production machine i in the planning period t or as the portion of planning period t, during which machine i should produce product j in order to satisfy market demand. The folding planning horizon means that instead of solving the AARC only once in the beginning of the production process, the AARC is resolved at the end of each planning period. Thereby, a new solution is generated at the end of each planning period, taking into account new information about demand and the current state of the production planning system.

The AARC, which takes into account aspects mentioned above, has one strict advantage: due to the adjustability of the variables, it is expected to be less conservative than the corresponding RC for the DLSP. However, the disadvantages of the constructed AARC are also serious. Firstly, the robustness of the AARC solution is not guaranteed any more, since the DLSP "all or nothing" assumption can be violated. The reason is that the decision rules (5.11) are allowed to take values on the interval [0, 1], making restriction (2.15) feasible for several non-zero decision rules values that corresponds to z_{ijt}. Thus, the situation wherein the manufacturer is not able to satisfy the customer demand due to the unavailable production capacity may occur. Secondly, the upper bound of total costs is not provided to the manufacturer for the same reason. The "all or nothing" assumption is relaxed; therefore, the total costs provided by the AARC for the worst-case of demand distribution are not accurate. Finally, the AARC contains binary variables, which reflect its computational status.

The AARC for the DLSP (2.12)-(2.19) belongs to the class of Mixed Integer Programming (MIP) problems (because of the binary nature of variables z_{ijt}) and is planned to be implemented and solved using the IBM ILOG CPLEX Optimization Studio. The following hardware is used: Intel(R) Core(TM) i5-450M 2.40 GHz processor and 4.00GB RAM.

5.3.8 Testing workflow for the AARC for the DLSP model. Simulation of uncertain demand

Analogically to the testing of the AARCs of the CLSP, a computational example and simulation of the demand scenarios are considered for the AARC of the DLSP. Production system parameters and market data for the initial uncertain DLSP (2.12)-(2.19) are provided by the operating manufacturing company.

The aim is to analyze the constructed AARC by doing the following:

- generate 20 demand scenarios (though 3 of them are fixed to the highest, nominal and lowest demand, all others are randomly generated),
- calculate and compare the production plans provided by the AARC, the RC and the optimal solution for each demand scenario;
- identify the influence of uncertainty level on performance of the solution algorithms.

The reader is referred to the section 5.3.5, wherein the method of demand generation and the method of solution comparison are described in detail.

The testing workflow, described in Figure 5.1, is applied for testing the resulting AARC for the DLSP (2.12)-(2.19). However, step 3, which proposes to compute the corresponding total costs for each demand scenario and on average using AARC affine decision rules, is done under the folding planning horizon. The algorithm that is used to calculate the total costs of AARC solution is shown in Figure 5.3.

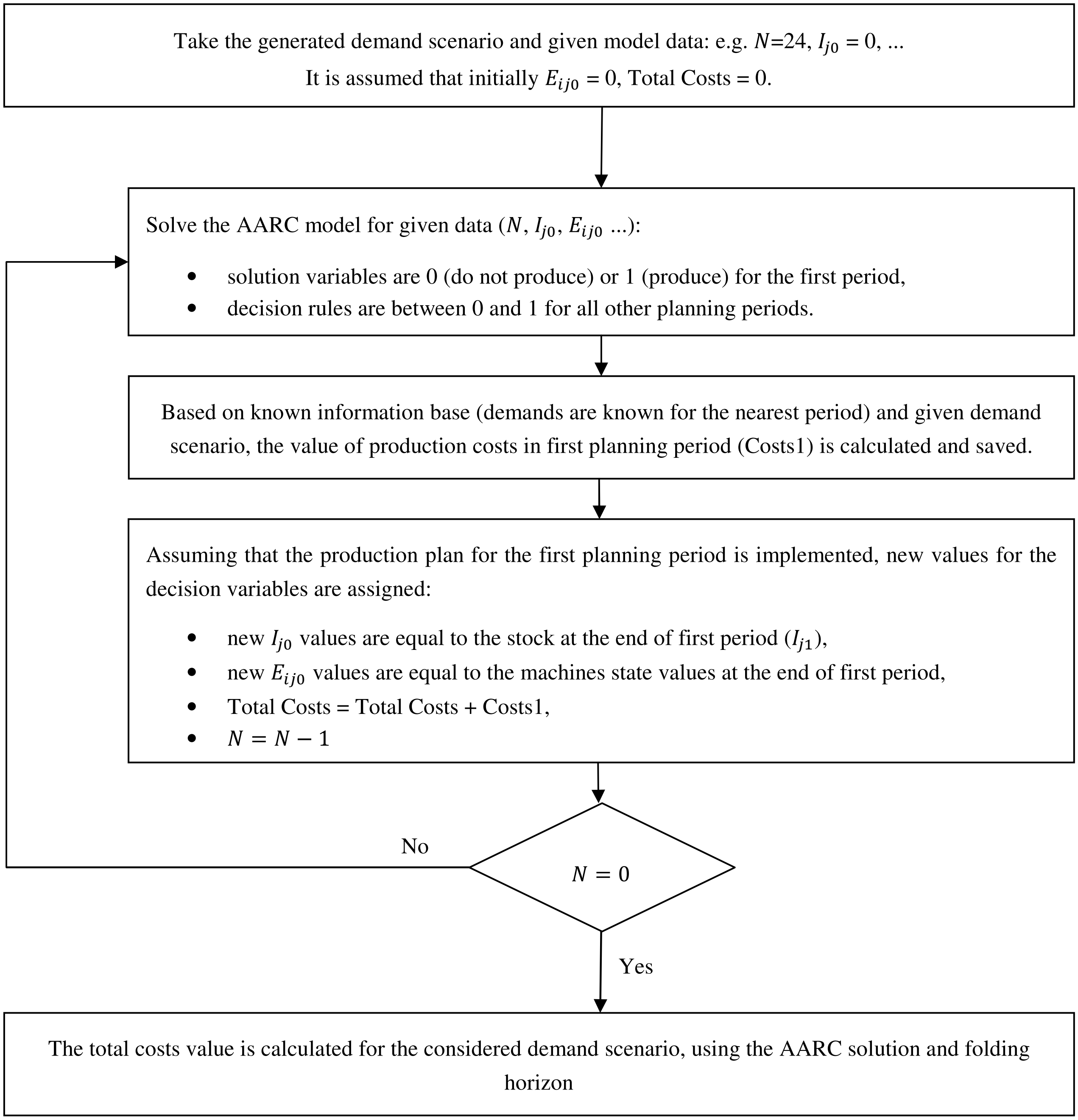


Figure 5.3. Testing workflow for the AARC of the DLSP

6 Implementation and results

6.1 Analytical approach for defining the worst-case of demand distribution

6.1.1 Important definitions and assumptions

Let us recall the stated subgoal (see section 5.1): to find such a demand distribution (demand takes values from the prescribed uncertainty interval) for the overall planning horizon N that an online algorithm provides the worst solution in comparison to the offline (optimal) solution. To compare the online and the offline algorithms, the technics of competitive analysis are used. The underlying idea is: do not consider the absolute behavior of the algorithm, but rather the ratio between the algorithm's behavior and the optimal behavior on the same problem instance. Required definitions are formulated below.

Definition 5. *The set of problem instances of the initial uncertain production planning problem is the set P of all production planning problems I that differ from each other only by the demand distribution (demand takes values from the prescribed uncertainty interval) over the planning periods, while all other model parameters stay the same.*

Definition 6. $cost_A(I)$ *denotes the cost of an algorithm A (the objective function value when the problem is solved by the algorithm A) on the problem instance I. The optimal objective value on the problem instance I is noted as* $cost_{opt}(I)$*. A competitive ratio of the algorithm A for the production planning problem P (minimizing the costs) is the number* $c \in \mathbb{R}$*, defined as:*

$$inf\ \left\{c \;\middle|\; cost_A(I) \le c * cost_{opt}(I), \forall I \in P\right\} \tag{6.1}$$

Based on the fact that all problem parameters except the unknown demand values are fixed, a difference in the solution quality between the online and the offline algorithms may only be caused by a possibility of production reallocation. The considered problem of the robust optimization is generally quite complex to analyze, subsequently several crucial assumptions are done in order to simplify the problem.

Definition 7. *Under the meaningful interval we understand the integral part* $\left\lfloor\frac{(ov_j-p_j)}{h_j}\right\rfloor$. *It defines the number of periods during which it will be cheaper to hold one production unit (considering production in the normal working time slot with lower production costs), rather than to produce this unit in the overtime slot.*

Assumption 7. *To simplify, it is assumed that the total number of planning periods is less than the corresponding meaningful interval:* $N \le \left\lfloor\frac{(ov_j-p_j)}{h_j}\right\rfloor$.

Assumption 8. *The total demand value summarized over all planning periods is fixed to the integral part of the total production capacity in normal production shift summarized over all planning periods:*

$$\sum_{t=1}^{N} d_t = \lfloor N * w \rfloor$$

Assumption 9. *The length of the overtime slot and the length of the normal working time slot are equal.*

To define the competitive ratio of an online algorithm A in terms of the competitive analysis, it is necessary to find:

$$inf\ \left\{c \,\middle|\, cost_A(I) \le c * cost_{opt}(I), \forall I \in P\right\}.$$

In turn, this is the same as to find:

$$sup_{I \in P} \left\{\frac{cost_A(I)}{cost_{opt}(I)}\right\}.$$

Statement 1. *If* a, b, n *are the positive integers and* $a \ \ge \ b$, *then* $\frac{a}{b} \ge \frac{a+n}{b+n}$.

Proof. Let's compare

$$\frac{a}{b} \quad \text{and} \quad \frac{a+n}{b+n}$$

This is the same as to compare:

$$\frac{a(b+n)}{b(b+n)} \quad \text{and} \quad \frac{b(a+n)}{b(b+n)}$$

$$ab + an \quad \text{and} \quad ab + bn$$

$$an \quad \text{and} \quad bn$$

$$a \quad \text{and} \quad b$$

By definition $a \geq b$, this means that:

$$\frac{a}{b} \geq \frac{a+n}{b+n}$$

■

6.1.2 Main theorem

The formulation and the proof of the theorem that describes the worst case of the demand distribution in terms of CA are provided below. The worst-case of the demand distribution defines also the value of the competitive ratio.

"The objective value while solving the model by the algorithm" is replaced by "the cost of the algorithm" for shortening. Similarly, the term “production shift” is defined as rescheduling of production from the overtime slot of one period to the normal time slot of one of the previous periods. Shifts allow to satisfy demand in time and to avoid the high costs of overtime production. The number of shifts made by an algorithm inside the meaningful interval reflects the algorithm effectiveness.

The next issue focused is the fact that the competitive ratio of an online algorithm directly depends on the difference between the number of shifts made by the online and the offline algorithm. To illustrate this point, the difference in total costs between the solutions provided by an online and an offline algorithm is considered, if the offline algorithm shifted one production unit more than the online algorithm. From the problem (5.1) it is known that the total costs are:

$$\sum_{t=1}^{N} (c_t x_t + ov_t y_t + h_t I_t)$$

For simplicity, a production planning system with fixed production and holding costs in all planning periods is considered. If the offline algorithm shifted from the planning period j to planning period k one production unit, while the online algorithm did not shift anything, then the difference in costs is the following:

$$cost_{online} - cost_{offline} = ov - (c + hL), \tag{6.2}$$

where $L = j - k$ is the number of periods that the shifted production unit was held in stock before delivery to the customer according to the offline production plan.

According to Assumption 7, the difference in costs (6.2) is positive. Next, the case when the number of units that were not shifted by the online algorithm increased to m is considered. Correspondingly, both $cost_{online}$ and $cost_{offline}$ increased: the costs of production in overtime slot increased by $ov \cdot (m-1)$ for the online algorithm; the additional holding costs equal to the $hL \cdot (m-1)$ appeared for the offline algorithm. At the same time, the difference in costs between the online and the offline solutions also increased to $m\big(ov-(c+hL)\big)$. Turning back to the competitive ratio definition (6.1), it means that the difference between the numerator and the denominator increased and the numerator is greater than the denominator. It immediately leads to the growth of the competitive ratio by the Statement 1.

All aspects considered, the aim is to find such a demand distribution that the difference in shifts made by the online and the offline algorithms is maximal. This demand distribution will define the maximal ratio between the online and the offline costs.

Theorem 6.1. *The following demand distribution realizes the worst-case of demand distribution and defines the competitive ratio for the online algorithm with the rolling horizon* n *out of* N *total periods, considering Assumption 7, Assumption 8 and Assumption 9:*

1. *If* $(N-n)$ *is even:*
   - *first* $\frac{(N-n)}{2}$ *periods have zero demand;*
   - *next* n *periods have average demand equal to the production capacity in the normal time slot;*
   - *last* $\frac{(N-n)}{2}$ *periods have the highest possible demand.*
2. *If* $(N-n)$ *is odd:*
   - *first* $\frac{(N-n+1)}{2}$ *periods have zero demand;*
   - *next* $n-1$ *periods have average demand equal to the production capacity in the normal time slot;*
   - *last* $\frac{(N-n+1)}{2}$ *periods have the highest possible demand.*

The graphical representation for one of the cases of the Theorem 6.1 is shown in Figure 6.1.

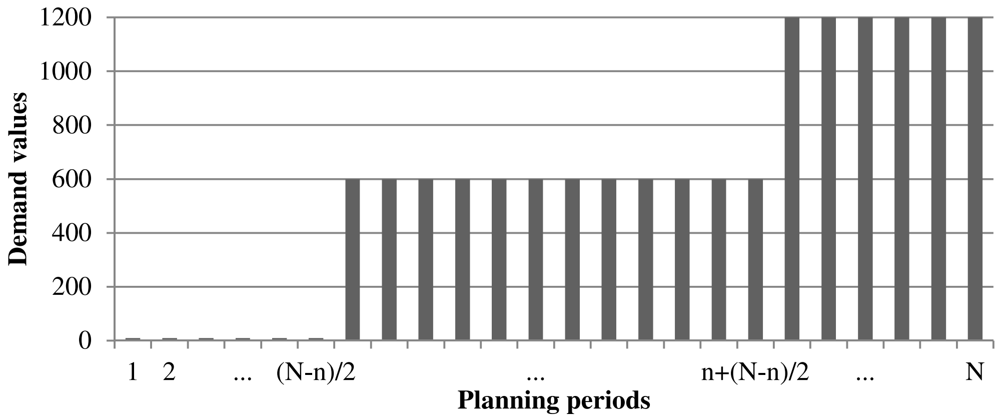


Figure 6.1. Illustration of the demand scenario from Theorem 6.1 in case $(N-n)$ *is even*

Proof. We will prove the Theorem 6.1 for the case when $(N-n)$ is even. The case when $(N-n)$ is odd can be proved by the parity of reasoning.

Obviously, the optimal production plan for the CLSP problem (5.1)-(5.5) with the demand distribution from Theorem 6.1 includes $\frac{(N-n)}{2}$ production shifts. Since the production in overtime is more expensive and we are working under Assumption 7, it is meaningful to avoid the overtime production and produce for the lower price (in the normal working time slots) in advance.

On the other hand, the online algorithm will not be able to make any production shifts: during the first $\frac{(N-n)}{2}$ and the next n planning periods the online algorithm put nothing in stock. The reason is that due to the current market knowledge and rolling horizon restriction, information about high demands in the last planning periods is not available for the online algorithm during first $\frac{(N-n)}{2}+n$ planning periods. When the first planning period with the maximal demand is taken into account by the online algorithm, there is no possibility to shift the production anymore, meaning there is no time available in the normal time slots of previous $n-1$ periods. The difference between the number of shifts made by the offline algorithm and the number of shifts made by the online algorithm is $\frac{(N-n)}{2}$.

In a next step, it is analyzed what will happen if the demand distribution is changed. The competitive ratio is defined by the fraction with the numerator $cost_{online}$ and the denominator $cost_{offline}$. Any change in the demand distribution from the Theorem 6.1 will lead to one of the following consequences:

1. $cost_{offline}$ increases and $cost_{online}$ stays the same; $cost_{offline}$ increases and $cost_{online}$ decreases. By the basic fraction properties, both situations will lead to the decreased value of the competitive ratio.
2. $cost_{online}$ and $cost_{offline}$ decrease simultaneously. Such a behavior may be obtained in practice in the following three situations:
   a. The number of shifts made by the online algorithm remains the same, while the number of shifts made by the offline algorithm decreases. This situation may be caused by a demand distribution that requires less production in overtime slot, if we will not consider shifting at all, than the demand distribution from the Theorem 6.1. Consequently, both $cost_{online}$ and $cost_{offline}$ decrease. At the same time, the number of shifts made by the offline algorithm decreases, so the difference between the number of shifts made by the offline algorithm and the number of shifts made by the online algorithm decreases. Correspondingly, the difference in costs $(cost_{offline} - cost_{online})$ and the competitive ratio also decreases.
   b. The number of shifts made by the online algorithm increases, while the number of shifts made by the offline algorithm stays the same. Consequently, the difference between numbers of shifts made by the offline algorithm and the number of shifts made by the online algorithm decreases and the competitive ratio also decreases.
   c. The number of shifts made by the online algorithm increases, while the number of shifts made by the offline algorithm decreases. Analogically to the previous cases, the difference between numbers of shifts made by the offline algorithm and the number of shifts made by the online algorithm decreases and the competitive ratio also decreases.
3. $cost_{online}$ and $cost_{offline}$ increase simultaneously. Such a behavior may be obtained in practice only if the number of shifts made by the offline algorithm increases. This inevitably leads to the increased number of shifts made by the online algorithm. The reason is that if additional periods where demand is higher than average appear, they will be obtained by the online algorithm with rolling horizon n. (Both algorithms for this particular case will behave the same, which leads to the increased value of total

costs for the production plans provided by the online and the offline algorithms. The competitive ratio will also decrease based on the Statement 1.

4. $cost_{online}$ increases, while $cost_{offline}$ stays the same; $cost_{online}$ decreases, while $cost_{offline}$ stays the same. Such situations cannot be obtained in practice, because any change in the demand distribution inevitably leads to the change in the $cost_{offline}$, since Assumption 8 is assumed.
5. $cost_{online}$ stays the same, while $cost_{offline}$ decreases. Such situations cannot be obtained in practice, because changes of the $cost_{offline}$ inevitably lead to changes of $cost_{online}$. Indeed, $cost_{offline}$ decreases only for the demand distribution that requires less production in overtime slot (if we will not consider shifting at all) than the demand distribution from the Theorem 6.1. For this situation, $cost_{online}$ decreases as well (see case 2b).
6. $cost_{online}$ increases, while $cost_{offline}$ decreases. Such situations cannot be obtained in practice, because no demand distribution increases the volume of non-shifted demand for the online algorithm. On the other hand, if the $cost_{online}$ has increased with the same value of non-shifted demand, then both, the online and the offline, algorithms reallocated higher production volume to the normal working slot. This case was considered in 3.

Thus, demand distribution in the theorem realizes the worst case, the Theorem 6.1 is proved.

■

6.1.3 Comments on the proof

The examples of the demand distributions that demonstrate the Theorem 6.1 proof are provided below:

1. $cost_{offline}$ increases and $cost_{online}$ stays the same.

   The demand distribution remains as indicated in the Theorem 6.1 for the planning periods 2 till N; the maximal demand is obtained in the first planning period and the minimal demand is obtained in the last period, see Figure 6.2.
2. $cost_{online}$ and $cost_{offline}$ decrease simultaneously.

a. The minimal demand is obtained in the first planning period, the average demand is obtained in the periods 2 till $N-1$ and the maximal demand is obtained in the last period, see Figure 6.3.

b. The minimal demand is obtained in the first planning period; the maximal and the minimal demand are alternating in further periods, see Figure 6.4.

c. The maximal demand is obtained in the first planning period; next periods are alternating minimal and maximal demand values, see Figure 6.5.

3. $cost_{online}$ and $cost_{offline}$ increase simultaneously.

The minimal demand occurs within the first $\left\lfloor\frac{N}{2}\right\rfloor$ periods and the maximal demand occurs in the next $\left\lfloor\frac{N}{2}\right\rfloor$ periods, see Figure 6.6.

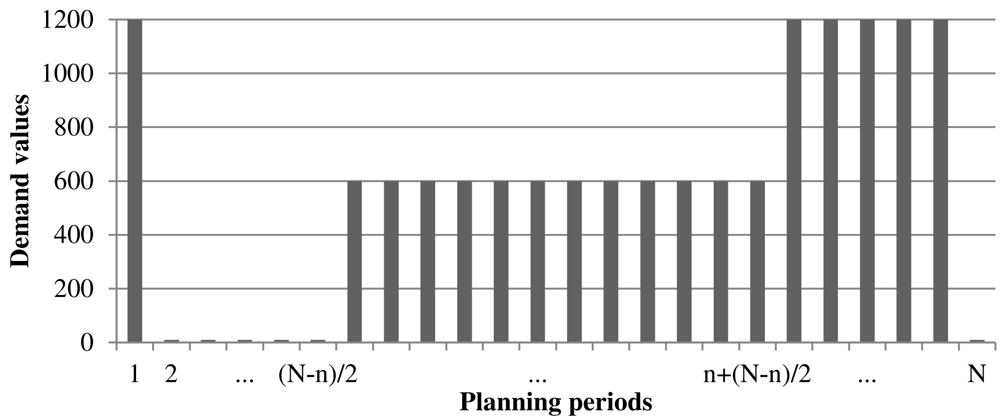


Figure 6.2. Illustration of the demand scenario for case 1 of the Theorem 6.1 proof

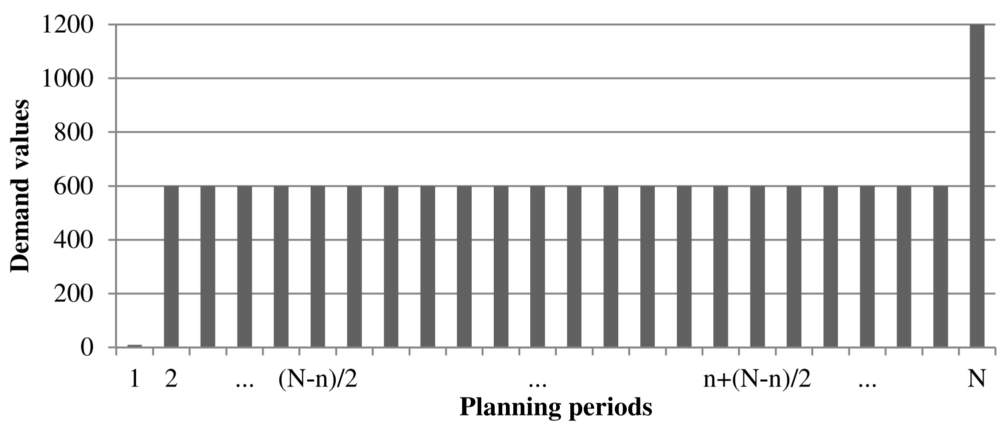


Figure 6.3. Illustration of the demand scenario for case 2a of the Theorem 6.1 proof

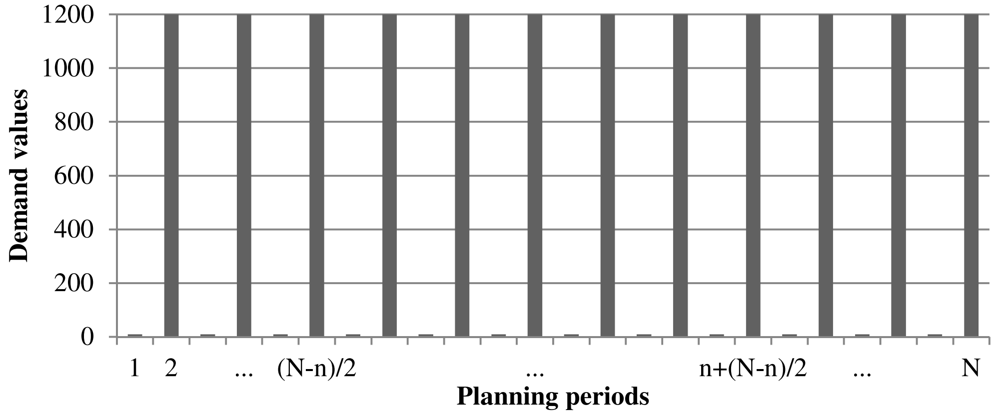


Figure 6.4. Illustration of the demand scenario for case 2b of the Theorem 6.1 proof

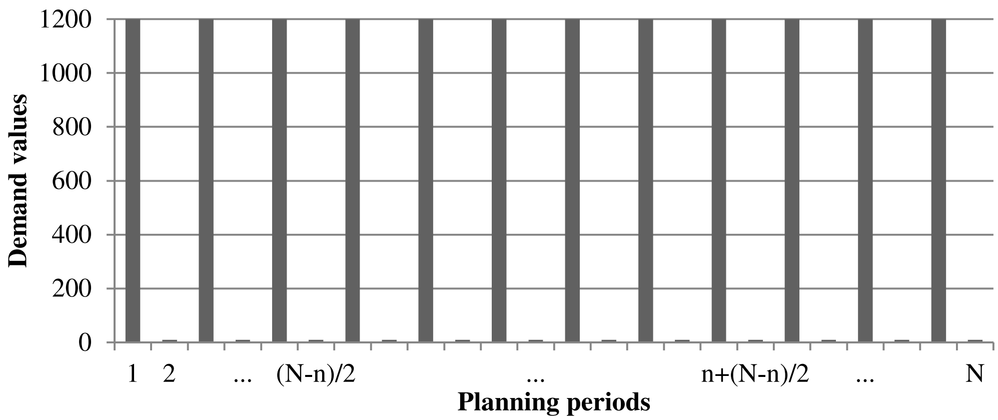


Figure 6.5. Illustration of the demand scenario for case 2c of the Theorem 6.1 proof

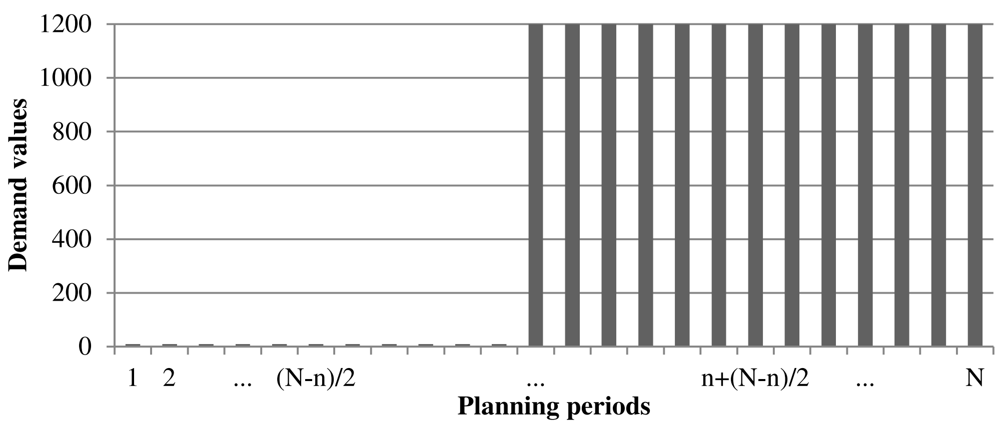


Figure 6.6. Illustration of the demand scenario for case 3 of the Theorem 6.1 proof

6.1.4 Computational example

To illustrate the obtained results, a computational experiment was implemented. The following data was used for the CLSP model (5.1)-(5.5):

$N = 24$	number of planning periods in overall planning horizon,
$n = 8$	number of planning periods in rolling horizon of the online algorithm,
$d_t^* = 600$	nominal demand in the planning period t (units),
$\theta = 0{,}9$	uncertainty level of demand (90%),
$D = 14400$	total summarized demand (units),
$u_t = 600$	production capacity in normal working time slot of period t (units),
$w_t = 840$	production capacity in overtime slot of period t (units),
$c_t = 100$	production costs (per unit) in normal working time slot of period t ($),
$ov_t = 150$	production costs (per unit) in overtime slot of period t ($),
$h_t = 2$	holding costs (per unit and per period) in period t ($),
$I_0 = 0$	initial stock of product j (units).

The mathematical model of the CLSP (5.1)-(5.5) was implemented in the IBM ILOG CPLEX Optimization Studio. The Optimization Programming Language (OPL) was used to describe the objective function and the restrictions of the model.

In order to verify the Theorem 6.1 statement, four different demand scenarios were considered and compared by the behavior of the online algorithm. For each demand scenario, the total value of costs was calculated for the production plan provided by the offline algorithm as well as for the production plan provided by the online algorithm.

Scenario 1 describes the case, where planning periods with the lowest possible demand are followed by planning periods with the highest possible demand, see Figure 6.7.

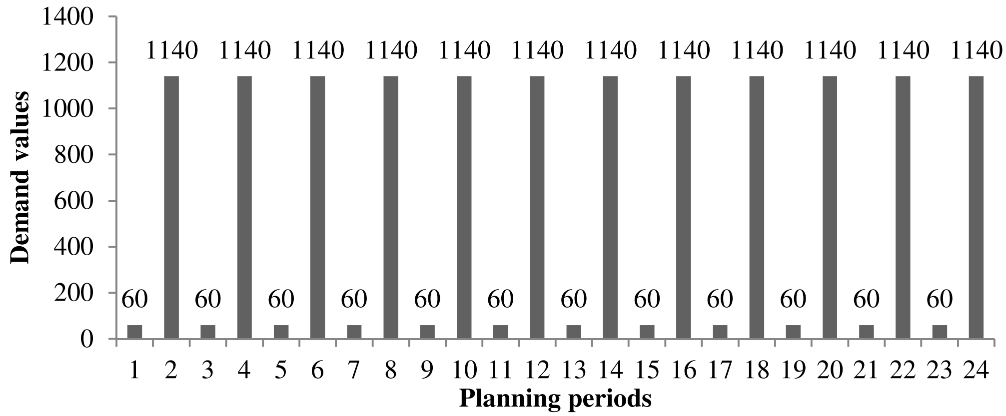


Figure 6.7. Scenario 1 of the demand realization

Since the online algorithm shifts the production from periods with high demand to periods with lower demand (with the help of the rolling horizon), the obtained value of the objective function will be exactly the same as in the offline case.

Scenario 2 includes the lowest possible demand in the first twelve planning periods and the highest possible demand in the last twelve planning periods, see Figure 6.8.

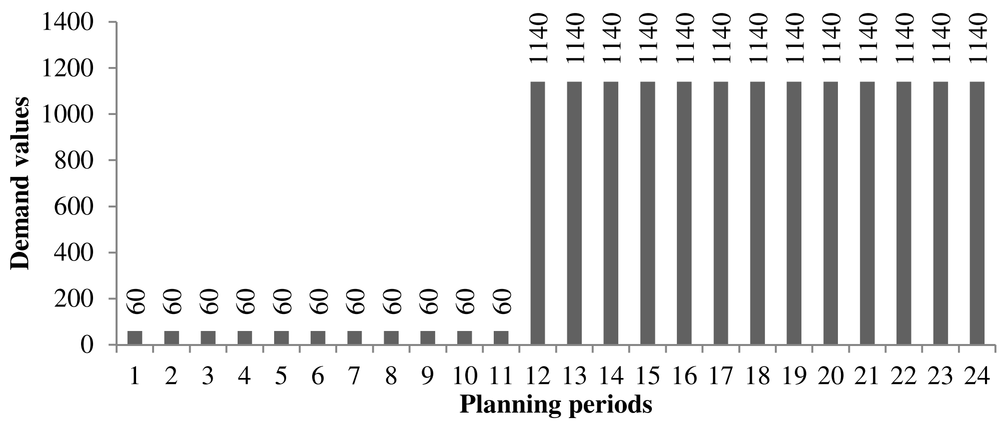


Figure 6.8. Scenario 2 of the demand realization

In this case, the online algorithm with the rolling horizon is not able to solve the problem as effectively as the offline algorithm does. In fact, the online algorithm will make only four shifts of high-costly overtime production, while the offline algorithm will make twelve.

Scenario 3 is defined by alternating blocks of six planning periods with alternating minimal and maximal demand, see Figure 6.9.

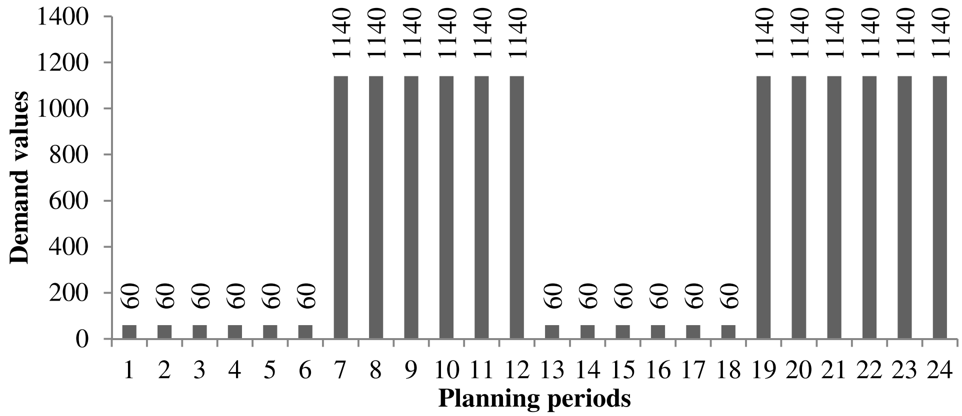


Figure 6.9. Scenario 3 of the demand realization

The online algorithm with the rolling horizon is not able to shift all highest demands in Scenario 3 and, therefore, has the overtime production in four planning periods. The offline algorithm completely eliminates production in overtime slots, so the total costs and the value of the competitive ratio differ for the offline and online algorithms.

Scenario 4 matches the demand distribution from the Theorem 6.1, see Figure 6.10.

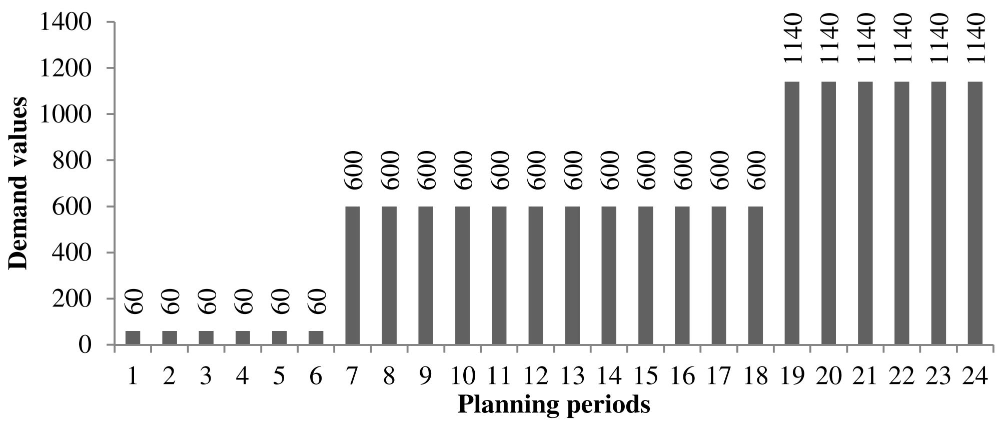


Figure 6.10. Scenario 4 of the demand realization

For Scenario 4, the online algorithm makes no shifts, while the offline algorithm is able to shift all overtime slots productions.

For the online case the model was solved under the rolling horizon:

1. The production plan is constructed for the nearest eight planning periods;
2. The created production plan is implemented for the nearest period;
3. The planning horizon is moved by one planning period, and the mathematical model is updated with the new demand data;

4. The stock at the end of the first planning period and new demand data are used as the input data for the new model.

The value of the objective function as well as the value of the competitive ratio is presented in Table 6.1 for the online and offline algorithm for each considered demand scenario.

Table 6.1: Comparison of the offline and online algorithms on four generated demand scenarios.

	Objective function value of the offline algorithm ($)	Objective function value of the offline algorithm ($)	Relative gap between the objectives of the offline and the online algorithms (%)	Competitive ratio value for the online algorithm
Scenario 1	1454136	1454136	0	1
Scenario 2	1616168	1759192	8,85	1,0885
Scenario 3	1587848	1657000	4,36	1,0436
Scenario 4	1557288	1852890	19	1,1898

The solution of the online algorithm is worse than the optimal one for three out of four demand distributions: the relative percentage gap between the objectives of the online and offline algorithms is positive. The corresponding value of the competitive ratio is higher than the one for Scenario 2, Scenario 3 and Scenario 4.

Despite the fact that not every assumption was satisfied (e.g. Assumption 9), the statement of the Theorem 6.1 is confirmed. The highest value of the competitive ratio of the online algorithm with the rolling horizon is obtained in case of the demand distribution described in Theorem 6.1. It should be noted that the difference in the costs between solutions provided by the online and offline algorithm reaches 19% in the worst-case scenario.

The formulated theorem and the computational example were already published in [70].

6.1.5 Capabilities and limitations of the approach

Theorem 6.1 is formulated based on the worst-case analysis of the CLSP problem (5.1)-(5.5) without backlogging and setup costs. Table 6.2 shows applicability of the analytical approach for defining the worst-case of demand distribution for production planning problems with other structures.

Table 6.2: Applicability of the analytical approach for defining the worst-case of demand distribution

Production planning model structure	CLSP without BG* without SC**	CLSP without BG* with SC**	CLSP with BG* without SC**	CLSP with BG* with SC**	DLSP
Applicability of the analytical approach for defining the worst-case	Yes	Limited	No	No	No

BG* - backlogging, SC** - setup costs.

The analytical approach for defining the worst-case of demand distribution may be applied to CLSP problems without backlogging and with a specific structure of setup costs: until Assumption 7 used in our analysis stays valid.

For DLSP and CLSP with other production system structures, considered analytical approach cannot be applied. Changes in the production system structure lead to the changes in the structure of the goal function and restrictions. The whole model becomes complex and interrelation between demand distribution and the values of different production costs become not evident any more.

Nonetheless, the worst-case demand for the CLSP model (5.1)-(5.5) is described for the length of the rolling horizon from 1 to $N-1$ periods by Theorem 6.1. Hence, the method is applicable for strict online problems, where demand information is known only for the current planning period, and for the online problems, where nearly all information about the future is available.

Obviously, Assumption 7, Assumption 8, and Assumption 9 strictly limit the described approach. Even though the problem can be generalized relatively easily and Assumption 9 can be eliminated, two other assumptions are crucial for the considered analysis. An additional future work in this direction can be done. However, since the application of the analytical approach is limited, future research should be mostly considered as a base for a better problem understanding.

Some other approaches are required in order to analyze the worst-case of demand distribution for the DLSP and more complicated CLSP structures.

6.2 Analytical approach for deriving the value of competitive ratio

6.2.1 Perishable products with lost sales

Formulation of the CLSP model

We consider the mathematical model (5.6), which describes inventory management for perishable products with lost sales. In order to add the capacity restrictions, the highest possible production amount in planning period i is fixed by L_i units. The mathematical model for perishable products with lost sales with the capacity restriction is:

$$\min_{\mathbf{0} \le \boldsymbol{q} \le \boldsymbol{L}} \sum_{i=1}^{n} (c_i q_i + h_i (q_i - d_i)^+ + s_i (d_i - q_i)^+ + K_i \delta(q_i)) \tag{6.3}$$

The assumption that the production process is strictly online is still valid, meaning that the exact value of the demand for the current planning period reveals only at the end of the current period. All other assumptions from the work of Wagner [25] are assumed to be valid as well (Assumption 1, Assumption 2 and Assumption 3 from p.31).

Deriving a competitive ratio

To derive the value of the competitive ratio, the reasoning from the Wagner's work is repeated. The formula, which defines the value of the competitive ratio for the production planning model with perishable products and lost sales, was given in Theorem 3.1 of Wagner's work (see p. 32). The new extended formulation of the theorem is given below.

Theorem 6.2. *We consider the following production planning strategy: in period* i*, if* $c_i \geq s_i$*, order* $q_i = 0$ *units and if* $c_i < s_i$*, order* $q_i \leq L_i$ *units. The competitive ratio of this strategy is at most:*

$$\max_{i:c_i<s_i}\left\{\max\left\{\frac{K_i-(s_i-c_i)q_i}{K_i-(s_i-c_i)L_i},\left(\frac{c_i+h_i}{K_i}\right)q_i+1,\frac{s_i}{c_i}\right\}\right\}.$$

The maximum value of the strict competitive ratio is:

$$\max_{i:c_i<s_i}\left\{\max\left\{\frac{K_i}{K_i-(s_i-c_i)L_i},\left(\frac{c_i+h_i}{K_i}\right)L_i+1,\frac{s_i}{c_i}\right\}\right\}.$$

Furthermore, the competitive ratio of any algorithm is at least $\max\limits_{i:c_i<s_i}(s_i/c_i)$*.*

The considered production strategy is conditioned on Assumption 3, which implies that the production period with $c_i < s_i$ does exist. Otherwise, the optimal production strategy is backlogging in all planning periods, both in the online and offline case. To prove the Theorem 6.2, its special case of a single period model with unknown demand d is considered for a start.

Theorem 6.3. *If* $c \geq s$*, it is optimal (the strict competitive ratio equals to 1) to order zero units. If* $c < s$*, the strict competitive ratio of ordering* q *units is equal to:*

$$\max\left\{\frac{K-(s-c)q}{K-(s-c)L};\ \frac{q(c+h)}{K}+1;\ \frac{s}{c}\right\}.$$

The maximum value of the strict competitive ratio is:

$$\max\left\{\frac{K}{K-(s-c)L};\ \frac{L(c+h)}{K}+1;\ \frac{s}{c}\right\}.$$

Proof. For the case $c \geq s$ the proof is obvious: since the cost structure is known in advance for the online and offline algorithms, the provided solution is to produce zero units in both cases. Therefore, the competitive ratio of the online algorithm is equal to one.

However, an additional argumentation is required to prove the Theorem 6.3 statement for $c < s$. Analogically to the proof provided in paper [25], the competitive ratio value is defined by:

$$\sup_{d\geq 0}\left(\frac{Z(d)}{Z^*(d)}\right), \tag{6.4}$$

where $Z(d)$ is the value of the objective function in the online case and $Z^*(d)$ is the value of the objective function in the offline case. Three possible cases determining the structure of $Z(d)$ and $Z^*(d)$ are considered, noting the online player's production quantity by q:

1. The demand exceeds the maximal order quantity: $0 \le q \le L < d$.

   The best production strategy for the offline algorithm is to produce the highest possible amount, which equals to L. The objective value of such a production strategy defines the structure of the denominator in (6.4), and the following lower bound on the competitive ratio is determined:

$$\rho = \sup_{d\ge 0}\left(\frac{Z(d)}{Z^*(d)}\right) \ge \sup_{d\ge 0}\left(\frac{Z(d)}{cL + s(d-L) + K}\right)$$

   According to the Assumption 3, $cd + K < sd$. So the demand value is bounded: $d > K/(s-c)$. Based on that, the upper bound for the competitive ratio is derived:

$$\rho = \sup_{d>K/(s-c)}\left(\frac{Z(d)}{cL + s(d-L) + K}\right) \le \sup_{d\ge 0}\left(\frac{Z(d)}{cL + s(d-L) + K}\right)$$

   Regarding to the fact that the case $0 \le q \le L < d$ is considered, the online costs are defined as:

$$Z(d) = sd + (c-s)q + K$$

   This value is put into the numerator of (6.4), which results the formula for the competitive ratio:

$$\rho = \sup_{d\ge 0}\left(\frac{sd + (c-s)q + K}{sd + (c-s)L + K}\right) \tag{6.5}$$

   By Lemma 1 from the linear-fractional programming (see p. 31), the optimization problem (6.5) is equal to the following linear program:

$$\max_{y,z}\left(sy + \big((c-s)q + K\big)z\right)$$

s.t.:

$$sy + \big((c-s)L + K\big)z = 1$$

$$z \ge 0, \qquad y \ge 0$$

   The dual linear system is the following:

$$\min \alpha$$

s.t.:

$$\alpha s \ge s$$

$$\alpha\big((c-s)L + K\big) \ge (c-s)q + K$$

The competitive ratio (6.4) is equal to the optimal dual solution:

$$max\left\{1, \frac{K-(s-c)q}{K-(s-c)L}\right\} = \frac{K-(s-c)q}{K-(s-c)L} \tag{6.6}$$

2. The demand is less than the production capacity: $0 < d \leq L$.

The capacity restriction for demand value becomes meaningless for the offline algorithm: it is always possible to produce enough to satisfy the demand. The optimal production amount in the offline case equals to d units. Analogically to the case 1, the total costs of the offline algorithm are plugged into the denominator of (6.4) and the competitive ratio is the following:

$$\rho = \sup_{0<d\leq L}\left(\frac{Z(d)}{cd+K}\right)$$

In order to deduce the total costs of the online algorithm, two subcases are analyzed:

a) The production amount in the online case is less than the demand value: $0 < d < q \leq L$.

The value of the competitive ratio is the following:

$$\rho = \sup_{0<d\leq L}\left(\frac{-hd+(c+h)q+K}{cd+K}\right)$$

Based on Lemma 1 from the linear-fractional programming, a switch to the linear programming model and the corresponding dual model is implemented. The derived competitive ratio coincides with the solution of the dual system and equals to:

$$\frac{q(c+h)}{K+1} \tag{6.7}$$

However, since $q \leq L$, the maximal possible value of competitive ratio equals to the following:

$$\frac{L(c+h)}{K+1} \tag{6.8}$$

b) The production amount in the online case is higher than the demand value:

$$q \leq d < L$$

The value of the competitive ratio is the following:

$$\rho = \sup_{0<d<L}\left(\frac{sd+(c-s)q+K}{cd+K}\right)$$

Based on Lemma 1 from the linear-fractional programming, a switch to the linear programming model and the corresponding dual model follows. The derived competitive ratio coincides with the solution of the dual system and equals to:

$$\frac{s}{c} \tag{6.9}$$

Summarizing cases 1, 2a and 2b, the total value of the competitive ratio is defined as the maximal value among (6.6), (6.7) and (6.9):

$$max\left\{\frac{K-(s-c)q}{K-(s-c)L};\frac{q(c+h)}{K}+1;\frac{s}{c}\right\}$$

Taking into account (6.8), the total competitive ratio is always less than:

$$max\left\{\frac{K}{K-(s-c)L};\frac{L(c+h)}{K}+1;\frac{s}{c}\right\}$$

Thus, Theorem 6.3 is proved.

■

To prove the statement of Theorem 6.2, the total costs of the online algorithm and the total costs of the offline algorithm are represented as the sums of costs in each planning period:

$$Z(d)=\sum_{i=1}^{N} Z_i(d_i),$$

$$Z^*(d)=\sum_{i=1}^{N} Z_i^*(d_i)$$

Next, the set $S=\{i: c_i<s_i\}$ is defined. By Theorem 6.3 for each planning period $i\in S$, the following inequality is fulfilled:

$$Z_i(d_i)\leq\ max\left\{\frac{K-(s-c)q}{K-(s-c)L};\frac{q(c+h)}{K}+1;\frac{s}{c}\right\}\ Z_i^*(d_i)$$

To prove the statement of Theorem 6.2, the statement of Theorem 6.3 is utilized; additionally the cases where demand is equal to zero or the offline algorithm produces nothing are analyzed. The further proof exactly repeats the proof from the M. Wagner's work [25].

6.2.2 Durable products with backlogged demand

Formulation of the CLSP model

The mathematical model (5.7)-(5.9), which describes durable products with backlogging, is considered. In order to add the capacity restrictions, the highest possible production amount in planning period i is fixed by L_i units.

The mathematical model describing the production process for durable products with backlogging with the capacity restriction looks like:

$$min\sum_{i=1}^{n}(c_i q_i + h_i I_i^+ + s_i I_i^- + K_i \delta(q_i)) \tag{6.10}$$

s.t.:

$$I_i = I_{i-1} + q_i - d_i, \qquad \forall i \in \{1 \dots n\} \tag{6.11}$$

$$q_i \leq L_i, \qquad \forall i \in \{1 \dots n\} \tag{6.12}$$

$$q_i \geq 0, \qquad \forall i \in \{1 \dots n\} \tag{6.13}$$

The assumption that the production process is strictly online, meaning that the exact value of the demand for the current planning period reveals only at the end of the current period, still applies. All other assumptions from the work of Wagner are considered to be valid as well (Assumption 1, Assumption 2 and Assumption 3 from p. 31).

Deriving a competitive ratio

To derive the value of competitive ratio, the reasoning from Wagner's work is repeated. Two subsets of the planning periods set are considered: $P = \{i: I_i \geq 0\}$ and $N = \{i: I_i \leq 0\}$. P and N denote the periods with respectively non-negative and non-positive inventory. If the inventory equals to zero in the planning period i, we assign i arbitrary to P or N. The statement of Lemma 2 (see p. 33) stays true for the model (6.10)-(6.13), since the additional capacity restriction does not influence the representation of the objective function.

The aim is to identify the upper and lower bounds of the total costs for the offline algorithm and to formulate the analog of Lemma 3 (see p. 33) extended for the existing capacity restriction. In order to achieve this, an additional assumption is formulated.

Assumption 10. *The total demand, summarized over all planning periods, is assumed to be less that the total production capacity:*

$$\sum_{i=1}^{n} d_i \le \sum_{i=1}^{n} L_i$$

Lemma 4. *The optimal costs provided by the offline algorithm for the production planning model (6.10)-(6.13) have the following lower and upper bounds:*

$$\boldsymbol{\alpha}'\boldsymbol{d} \le Z^*(\boldsymbol{d}) \le \boldsymbol{c}'\boldsymbol{L} + \boldsymbol{K}'\boldsymbol{e}, \forall \boldsymbol{d} \ge \boldsymbol{0},$$

where the bold symbols denote the vectors and α_i *is defined as following:*

$$\alpha_i = min\left\{s_i, \frac{min(c_i)}{m}, h_i, \frac{min(c_i)}{n-m}\right\}, \qquad \forall i = \{1 \dots n\}$$

Proof: The aim is to define the lower and upper bounds for the optimal offline cost. Since the capacity restriction exists in the model, the production strategy $q_i = d_i$ is not always a feasible solution. However, due to Assumption 10, the production strategy $q_i = L_i$ in period i is a feasible offline solution, so we have the upper bound for the offline costs:

$$Z^*(\boldsymbol{d}) \le \boldsymbol{c}'\boldsymbol{L} + \boldsymbol{K}'\boldsymbol{e}$$

To begin deriving a valuable lower bound of the offline costs, the fixed setup costs are removed:

$$Z^*(\boldsymbol{d}) \ge \min_{\boldsymbol{q} \ge 0} \sum_{i=1}^{n} (c_i q_i + s_i I_i^- + h_i I_i^+) =$$

$$\min_{\boldsymbol{q} \ge 0} \sum_{i=1}^{n} \left(c_i q_i + s_i\, max\left\{\sum_{j=1}^{i} (d_j - q_j), 0\right\} + h_i\, max\left\{\sum_{j=1}^{i} (q_j - d_j), 0\right\}\right)$$

By introducing the additional variables $p_i \ge 0, r_i \ge 0,\ \forall i \in \{1 \dots n\}$ and using the fact that $s > 0,\ h > 0$, the lower bound is determined as the following linear program:

$$\min_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} \ (\boldsymbol{s}'\boldsymbol{p} + \boldsymbol{c}'\boldsymbol{q} + \boldsymbol{h}'\boldsymbol{r}) \tag{6.14}$$

s.t.:

$$p_i + \sum_{j=1}^{i} q_j \ge \sum_{j=1}^{i} d_j, \qquad \forall i \in N \tag{6.15}$$

$$r_i - \sum_{j=1}^{i} q_j \ge - \sum_{j=1}^{i} d_j, \qquad \forall i \in P \tag{6.16}$$

$$q_i \leq L_i, \qquad \forall i \in \{1 \dots n\} \tag{6.17}$$

$$\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \geq 0 \tag{6.18}$$

The size of the subset N is denoted by m ($dim\ \{N\} = m$), which is the number of periods with a non-negative stock. Planning periods belonging to the set N are denoted by $n_1 \dots n_m$. Thereby, the size of the subset P, which coincides with the numbers of periods with the non-negative stock, equals to $(n - m)$: $dim\ \{P\} = n - m$. Planning periods belonging to the set P are denoted by $p_1 \dots p_{n-m}$.

The mathematical model (6.14)-(6.18) with the new notations is rewritten as:

$$\min_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} (\boldsymbol{s}'\boldsymbol{p} + \boldsymbol{c}'\boldsymbol{q} + \boldsymbol{h}'\boldsymbol{r}) \tag{6.19}$$

s.t.:

$$p_{n_1} + \sum_{j=1}^{n_1} q_j \geq \sum_{j=1}^{n_1} d_j \tag{6.20}$$

$$\dots$$

$$p_{n_m} + \sum_{j=1}^{n_m} q_j \geq \sum_{j=1}^{n_m} d_j \tag{6.21}$$

$$r_{p_1} - \sum_{j=1}^{p_1} q_j \geq -\sum_{j=1}^{p_1} d_j \tag{6.22}$$

$$\dots$$

$$r_{p_{n-m}} - \sum_{j=1}^{p_{n-m}} q_j \geq -\sum_{j=1}^{p_{n-m}} d_j \tag{6.23}$$

$$-q_1 \geq -L_1 \tag{6.24}$$

$$\dots$$

$$-q_n \geq -L_n \tag{6.25}$$

$$\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \geq 0 \tag{6.26}$$

The system consists of the objective function and $2n$ inequalities (without the last group of constraints defining the non-negative nature of the variables): m inequalities (6.20)-(6.21) with the variables p_i and q_i, $(n - m)$ inequalities (6.22)-(6.23) with the variables r_i and q_i, n inequalities (6.24)-(6.25) with the variables q_i and the parameters L_i.

In the next step, the matrix of the linear system (6.19)-(6.26) is constructed (see Matrix 1) and the inverted matrix for the dual system is deduced. The matrix of the system (6.19)-(6.26) has the size $2n \times 2n$, so the inverted matrix correspondingly has the same size.

$$
A = \begin{pmatrix}
\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} &
\begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} &
\begin{bmatrix} -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & -1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & -1 & -1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}
\end{pmatrix}
$$

Matrix 1. Matrix of the linear system (6.19)-(6.26)

The first logical group of columns in Matrix 1 is constructed by the coefficients in front of the vector $\boldsymbol{p}$; it correspondingly comprises m sub-columns. The second logical group of columns in Matrix 1 is defined by the coefficients in front of the vector $\boldsymbol{r}$; it correspondingly consists of $(n-m)$ sub-columns. Finally, the last logical group of columns in Matrix 1 is constructed by the coefficients in front of the vector $\boldsymbol{q}$; it correspondingly comprises n sub-columns.

In order to decide where in the last column ones, minus ones and zeros should stay, the set of planning periods was written in the following way: $\{p_1\ n_1\ p_2\ n_2\ n_3 \dots n_m\ p_{n-m}\}$, where n_i is the element i of set N, p_j is the element j of set P. The concrete sequence of the elements n_i and p_i may be different; just one of the possible combinations for the demonstration purposes was considered.

The matrix of the dual linear system for the system (6.19)-(6.26) is constructed by inverting the initial Matrix 1, see Matrix 2.

$$A' = \begin{pmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ 0 & -1 & -1 & \cdots & -1 \\ 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & -1 & \cdots & -1 \\ 0 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \end{pmatrix}$$

Matrix 2. Inverted matrix of the linear system (6.19)-(6.26)

The dual system of the system (6.19)-(6.26) has the dual variables x_i, y_j, z_t, where $i \in N, j \in P, t \in \{1 \dots n\}$ and is written as the following based on Matrix 2:

$$max \left(\sum_{\substack{i=1 \\ i \in N}}^{n} \sum_{k=1}^{i} d_k x_i - \sum_{\substack{j=1 \\ j \in P}}^{n} \sum_{k=1}^{j} d_k y_j + \boldsymbol{Lz} \right)$$

s.t.:

$$x_i \le s_i, \qquad \forall i \in N$$

$$y_j \le h_j, \qquad \forall j \in P$$

$$\sum_{\substack{i=1 \\ i \in N}}^{n} x_i - \sum_{\substack{j=1 \\ j \in P}}^{n} y_j + z_1 \le c_1$$

$$\sum_{\substack{i=2 \\ i \in N}}^{n} x_i - \sum_{\substack{j=2 \\ j \in P}}^{n} y_j + z_2 \le c_2$$

...

$$\sum_{\substack{i=n-1\\i\in N}}^{n} x_i - \sum_{\substack{j=n-1\\j\in P}}^{n} y_j + z_{n-1} \le c_{n-1}$$

$$x_{n,n\in N} - y_{n,n\in P} + z_n \le c_n$$

$$\boldsymbol{z} \ge 0$$

Making the unification, the following dual system results:

$$max\left(\sum_{\substack{i=1\\i\in N}}^{n}\sum_{k=1}^{i} d_k x_i - \sum_{\substack{j=1\\j\in P}}^{n}\sum_{k=1}^{j} d_k y_j + \boldsymbol{Lz}\right)$$

s.t.:

$$x_i \le s_i, \qquad \forall i \in N$$

$$y_j \le h_j, \qquad \forall j \in P$$

$$\sum_{\substack{i=k\\i\in N}}^{n} x_k - \sum_{\substack{j=k\\j\in P}}^{n} y_j + z_k \le c_k, \qquad \forall k \in \{1 \dots n\}$$

$$\boldsymbol{z} \ge 0$$

Denoting $\boldsymbol{\gamma} = \boldsymbol{x}, \boldsymbol{\beta} = -\boldsymbol{y}, \boldsymbol{\tau} = \boldsymbol{z}$ and rewriting the system in new variables, the following model can be concluded:

$$max\left(\sum_{\substack{i=1\\i\in N}}^{n}\sum_{k=1}^{i} d_k \gamma_i - \sum_{\substack{j=1\\j\in P}}^{n}\sum_{k=1}^{j} d_k \beta_j + \boldsymbol{L\tau}\right) \tag{6.27}$$

s.t.:

$$\gamma_i \le s_i, \qquad \forall i \in N \tag{6.28}$$

$$\beta_j \le h_j, \qquad \forall j \in P \tag{6.29}$$

$$\sum_{\substack{i=k\\i\in N}}^{n} \gamma_k - \sum_{\substack{j=k\\j\in P}}^{n} \beta_j + \tau_k \le c_k, \qquad \forall k \in \{1 \dots n\} \tag{6.30}$$

$$\boldsymbol{\tau} \ge 0 \tag{6.31}$$

In order to find feasible solutions for the dual problem (6.27)-(6.31), several special cases were considered:

1. If $\boldsymbol{\beta}, \boldsymbol{\tau} = 0$, the model (6.27)-(6.31) is the following:

$$max \sum_{\substack{i=1 \\ i \in N}}^{n} \sum_{k=1}^{i} d_k \gamma_i$$

s.t.:

$$x_i \leq s_i, \qquad \forall i \in N$$

$$\sum_{\substack{i=k \\ i \in N}}^{n} \gamma_k \leq c_k, \qquad \forall k \in \{1 \dots n\}$$

The feasible solution of the model is defined by the following expression:

$$\gamma_i = min\left(s_i, \frac{min(c_i)}{m}\right),$$

where m is the size of the subset N.

2. If $\boldsymbol{\beta}, \boldsymbol{\gamma} = 0$, the model (6.27)-(6.31) is the following:

$$max\, \boldsymbol{L\tau}$$

s.t.:

$$\tau_k \leq c_k, \qquad \forall k \in \{1 \dots n\}$$

$$\boldsymbol{\tau} \geq 0$$

The feasible solution of the model is $\tau_i = c_i$.

3. If $\boldsymbol{\tau}, \boldsymbol{\gamma} = 0$, the model (6.27)-(6.31) is the following:

$$max \sum_{\substack{j=1 \\ j \in P}}^{n} \sum_{k=1}^{j} d_k \beta_j$$

s.t.:

$$\beta_j \leq h_j, \qquad \forall j \in P$$

$$\sum_{\substack{j=k \\ j \in P}}^{n} \beta_j \leq c_k, \qquad \forall k \in \{1 \dots n\}$$

The feasible solution of the model is defined by the following expression:

$$\beta_i = min\left(h_i, \frac{min(c_i)}{n-m}\right),$$

where $(n-m)$ is the size of the subset P.

Summarizing the three cases considered and using the weak duality, it can be stated that $\gamma' d, \beta' d, \tau' d$ are the lower bounds for $Z^*(\boldsymbol{d})$. Consequently, $\boldsymbol{\alpha'd}$ is also the lower bound for $Z^*(\boldsymbol{d})$, where α_i is defined by the following expression:

$$\alpha_i = min\ \{\gamma_i, \beta_i, \tau_i\} = min\left\{s_i, \frac{min\ c_i}{m}, h_i, \frac{min\ c_i}{n-m}\right\}$$

Thus, Lemma 4 is proved.

■

The sufficient condition for the existence of a finite competitive ratio for an arbitrary online production strategy as well as the set of the lower and upper bounds for the competitive ratio are presented in the theorem below.

Theorem 6.4. *For an arbitrary online strategy* $\boldsymbol{q} \geq 0$*, inequality* $\boldsymbol{b'q} + K \leq 0$ *is a sufficient condition for the existence of a finite strict competitive ratio. Furthermore, if a strict finite ratio ρ exists, it satisfies the following inequality:*

$$\frac{\boldsymbol{a'L} + \boldsymbol{b'q} + K}{\boldsymbol{c'L} + \boldsymbol{K'e}} \leq \rho \leq \max_{1\leq i\leq n}\left\{\frac{a_i}{\alpha_i}\right\}.$$

Proof: Utilizing the lower bound from Lemma 4, Lemma 3 and Assumption 2 that $\boldsymbol{d} \neq 0$, the upper bound for the competitive ratio is the following:

$$\rho \leq \sup_{\substack{0\leq \boldsymbol{d}\leq \boldsymbol{L} \\ d\neq 0}}\left(\frac{\boldsymbol{a'd} + \boldsymbol{b'q} + K}{\boldsymbol{\alpha'd}}\right).$$

With the help of Lemma 1 from the linear-fractional programming, the equal linear program is constructed:

$$\max_{y,z}(\boldsymbol{a'y} + (\boldsymbol{b'q} + K)z)$$

s.t.:

$$\boldsymbol{a'y} = 1$$

$$\boldsymbol{y} \geq 0, z \geq 0$$

The dual program is:

$$min\, \beta$$

s.t.:

$$\beta \boldsymbol{\alpha} \geq \boldsymbol{a}$$

$$0 \geq \boldsymbol{b}'\boldsymbol{q} + K$$

The dual model is feasible when $\boldsymbol{b}'\boldsymbol{q} + K \leq 0$. This implies that the competitive ratio is at maximum:

$$\max_{1 \leq i \leq n} \left\{ \frac{a_i}{\alpha_i} \right\}$$

The analysis performed above is repeated for the upper bound from Lemma 4. As a result, the following lower bound for the competitive ratio is constructed:

$$\rho \geq \sup_{\substack{0 \leq d \leq L \\ d \neq 0}} \left(\frac{\boldsymbol{a}'\boldsymbol{d} + \boldsymbol{b}'\boldsymbol{q} + K}{\boldsymbol{c}'\boldsymbol{L} + \boldsymbol{K}'\boldsymbol{e}} \right)$$

Obviously, the higher the nominator is, the higher the value of the competitive ratio is. Therefore, the lower bound for the competitive ratio equals to:

$$\frac{\boldsymbol{a}'\boldsymbol{L} + \boldsymbol{b}'\boldsymbol{q} + K}{\boldsymbol{c}'\boldsymbol{L} + \boldsymbol{K}'\boldsymbol{e}}$$

Thus, Theorem 6.4 is proved.

■

6.2.3 Computational example

An example of the "backlog-up-to" decision policy was considered in the work [25] and the formula for the strict competitive ratio was derived.

In this section, the aim is to consider the example of a production planning system, where the backlogging is high-priced and therefore undesirable. Moreover, two different production strategies that do not use backlogging are focused. The following data was used for the CLSP model (6.10)-(6.13):

$N = 12$ number of planning periods in the overtime planning horizon,

$D = 120$ production capacity in one planning period and the maximal possible demand (units),

$c_i = 100$ production costs (per unit) in the period i ($):

c_i in odd planning periods	c_i in even planning periods
5	10

$k = 1$ setup costs ($),

$s = 100000$ backlogging costs per unit ($),

$h = 1$ holding costs per unit and per period ($),

$I_0 = 0$ initial stock (units),

The decision variables are:

q_i production amount in the planning period i (units),

$\delta(q_i) \in \{0,1\}$ binary setup variable that is equal to one if the production is obtained in planning period i.

Production strategy 1

The following production strategy is considered:

- Production of $min\ (D, D - I^+)$ in odd planning periods;
- Production of $max\ (0, D - I^+)$ in even planning periods.

To calculate the upper and the lower bounds for the competitive ratio, firstly the necessary condition of the competitive ratio existence from Theorem 6.4 is checked. The value of $\boldsymbol{b'q} + K$ is calculated:

$$\boldsymbol{b'q} + K = \sum_{i=1}^{n}\left(\left(c_i + \sum_{j=i}^{n} h_j\right)q_i + K_i\delta(q_i)\right) = \sum_{i=1}^{n}\big((c_i + (n-i)h)q_i + K\delta(q_i)\big)$$

$$\leq \sum_{i=1}^{n}\big((c_i + (n-i)h)D + K\big) = D\sum_{i=1}^{n}(c_i + (n-i)h + K) \leq$$

$$\leq D\sum_{i=1}^{n}(c_i+(n-i)h)+nK$$

Since $D > 0, nK > 0$, it is enough to identify the sign of the following sum in order to check the necessary condition of the competitive ratio existence:

$$\sum_{i=1}^{n}(c_i+(n-i)h)=\sum_{i=1}^{n}c_i+\sum_{i=1}^{n}(n-i)h=\frac{n}{2}(c_1+c_2)+\left(\sum_{i=1}^{n-1}i\right)h>0$$

The necessary condition of the competitive ratio existence from Theorem 6.4 is unsatisfied, since $\boldsymbol{b'q}+K\leq 0$. Therefore, the upper and lower bounds for the competitive ratio do not exist.

Production strategy 2

In a next step, a static production strategy is considered, e.g. the production in each planning period equals to the half of the highest possible demand value: $q_i = D/2$.

According to Assumption 10, the total demand value is less than nD. The sign of $\boldsymbol{b'q}+K$ is checked in two corner cases: when the demand always takes the highest possible value and the lowest possible value in all planning periods.

If the demand takes the highest possible value D in each planning period, while $D/2$ units are produced, then the backlogged demand value is $I^- = D/2$ and all planning periods belong to the subset N. For this particular demand realization, the $\boldsymbol{b'q}+K$ value equals to the following:

$$\boldsymbol{b'q}+K=\sum_{i=1}^{n}\left(\left(c_i-\sum_{\substack{j=i,\\ j\in N}}^{n}s_j\right)q_i+K_i\delta(q_i)\right)=\sum_{i=1}^{n}\left(\left(c_i-\sum_{j=i}^{n}s_j\right)\frac{D}{2}+K\right)=$$

$$=\frac{D}{2}\sum_{i=1}^{n}\left(c_i-\sum_{j=i}^{n}s_j\right)+nK\leq 0$$

Obviously, $\boldsymbol{b'q}+K\leq 0$, since the punishment for the unsatisfied demand has a much higher order of magnitude. According to Theorem 6.4, the finite value of the competitive ratio exists for the scenario with the highest demand.

If the demand takes the lowest possible value in the each planning period, while $D/2$ units are produced, then the amount in stock is always positive and all planning periods belong to the subset P. According to Assumption 2 $\boldsymbol{d} \neq 0$, so the lowest possible value of the total demand equals to 1. For this particular demand realization, the $\boldsymbol{b}'\boldsymbol{q} + K$ value equals to the following:

$$\boldsymbol{b}'\boldsymbol{q} + K = \sum_{i=1}^{n}\left(\left(c_i + \sum_{j=i}^{n} h_j\right)q_i + K_i\delta(q_i)\right) = \sum_{i=1}^{n}\big((c_i + (n-i)h)q_i + K\delta(q_i)\big)$$

$$= \sum_{i=1}^{n}\left((c_i + (n-i)h)\frac{D}{2} + K\right) = \frac{D}{2}\sum_{i=1}^{n}(c_i + (n-i)h + K)$$

$$= D\sum_{i=1}^{n}(c_i + (n-i)h) + nK \geq 0$$

Obviously, $\boldsymbol{b}'\boldsymbol{q} + K \geq 0$, since all sum components are positive. According to Theorem 6.4, finite value of the competitive ratio does not exist for the scenario with the lowest values of the demand.

To summarize, the formula for the finite competitive ratio cannot be derived using such a production strategy, because it is impossible to verify the sign of $\boldsymbol{b}'\boldsymbol{q} + K$ in a general case.

6.2.4 Capabilities and limitations of the approach

The considered solution approach is applicable to the two different production planning models: perishable products with lost sales (6.3) and durable products with backlogging (6.10)-(6.13). Switching to the terminology established in the production planning, the method for deriving the upper and lower bounds of the competitive ratio is applicable to the CLSP problems with backlogging and setup costs. At the same time, for production planning problems with different model structures, e.g. for DLSP model, a new reasoning is required, see Table 6.3.

It should be noted that the analytical approach for deriving the value of the competitive ratio is applicable only for the strict online problems (no actual demand values are known during the production process). The additional market information, e.g. in production planning models with the rolling or folding horizon, provides no advantages.

Table 6.3: Applicability of the analytical approach for deriving the value of competitive ratio

Production planning model structure	CLSP without BG* without SC**	CLSP without BG* with SC**	CLSP with BG* without SC**	CLSP with BG* with SC**	DLSP
Applicability of the analytical approach for deriving the value of competitive ratio	No	No	Yes	Yes	No

BG* - backlogging, SC** - setup costs.

The analytical approach for deriving the value of the competitive ratio may be applied to the CLSP with the backlogging but without any setup costs. The models that have some additional restrictions or the models where Assumption 10 is not satisfied can be investigated by the analytical approach for deriving the value of the competitive ratio as well. However, the argumentation and transformations have to be considered very carefully, since the new feasible domain of the mathematical model is defined. The existence of the corresponding sufficient condition and the upper and the lower borders for the competitive ratio is not automatically guaranteed for other models.

In addition, production planning problems with non-negative-only stocks are difficult to analyze, and are not in the focus of Wagner's paper [25]. As it can be noticed from the examples, it is impossible to derive the value of the competitive ratio if the production strategy does not propose all stocks negative. If the backlogging is prohibited or the production strategy proposes some periods with positive inventory, the sufficient condition for the finite competitive ratio ($\boldsymbol{b}'\boldsymbol{q} + K \leq 0$) is unsatisfied. If the set N is empty, the vector $\boldsymbol{b}$ becomes automatically greater than zero and neither the required condition, nor the rest of the analysis does hold. This is the reason why a positive-stock-only model cannot be analyzed by the analytical approach for deriving the value of the competitive ratio; different solution techniques are required for the non-negative-only stock. In fact, in the related research [27], it was shown that the finite performance ratio (similar to the competitive ratio) does not exist for any production planning problem that strictly maintains positive inventory.

6.3 Robust Optimization approach for the CLSP[1]

6.3.1 Robust Counterpart (RC)

Let us recall the formulation of the CLSP mathematical model with several producing machines and several products, where several production slots during one planning period are taken into account. Production in overtime slot, indeed, is more costly.

By definition, the robust counterpart of the uncertain CLSP described above will be an optimization model with the following objective:

$$min \left[\sup_{d_{jt}\in\left[d_{jt}^{*}-\theta d_{jt}^{*},d_{jt}^{*}+\theta d_{jt}^{*}\right]} \left(\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}x_{ijt}+ov_{ijt}y_{ijt}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right) + \sum_{t=1}^{N}\sum_{j=1}^{M} h_{jt}I_{jt} \right) \right] \tag{6.32}$$

This objective is augmented by constraints (2.2)-(2.10) that stay true for any $d_{jt} \in \left[d_{jt}^{*} - \theta d_{jt}^{*}, d_{jt}^{*} + \theta d_{jt}^{*}\right]$, $\forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$.

According to the RO paradigm, the RC model should have a certain objective. In order to determine it, objective (6.32) is equivalently rewritten by introducing the extra variable F and the additional restriction:

$$min\, F$$

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}x_{ijt}+ov_{ijt}y_{ijt}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right) + \sum_{t=1}^{N}\sum_{j=1}^{M} h_{jt}I_{jt} \le F$$

Adding the new objective and additional restriction to the (2.2)-(2.10) and forcing the restrictions to stay true for any $d_{jt} \in \left[d_{jt}^{*} - \theta d_{jt}^{*}, d_{jt}^{*} + \theta d_{jt}^{*}\right]$, $\forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$, the following Robust Counterpart can be deduced for the initial CLSP problem:

$$min\, F \tag{6.33}$$

s.t.:

[1] The author thanks Prof. Dr. Nemirovski for his helpful consultations, valuable comments and effective cooperation provided during the scientific visit to the Georgia Institute of Technology.

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}x_{ijt}+ov_{ijt}y_{ijt}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+\sum_{t=1}^{N}\sum_{j=1}^{M}h_{jt}I_{jt}\leq F \tag{6.34}$$

$$I_{j1}=I_{j0}+\sum_{i=1}^{K}\left(x_{ij1}+y_{ij1}\right)-d_{j1},\forall\, j\in\{1\dots M\} \tag{6.35}$$

$$I_{jt}=I_{j,t-1}+\sum_{i=1}^{K}\left(x_{ijt}+y_{ijt}\right)-d_{jt},\qquad \forall j\in\{1\dots M\},t\in\{2\dots N\} \tag{6.36}$$

$$x_{ijt}\leq u_{ijt}\cdot z_{ijt},\qquad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.37}$$

$$y_{ijt}\leq w_{ijt}\cdot zv_{ijt},\qquad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.38}$$

$$\sum_{j=1}^{M}x_{ijt}\leq U_{it},\qquad \forall i\in\{1\dots K\},t\in\{1\dots N\} \tag{6.39}$$

$$\sum_{j=1}^{M}y_{ijt}\leq W_{it},\qquad \forall i\in\{1\dots K\},t\in\{1\dots N\} \tag{6.40}$$

$$I_j^{min}\leq I_{jt}\leq I_j^{max},\qquad \forall j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.41}$$

$$z_{ijt}\in\{0,1\},zv_{ijt}\in\{0,1\},\qquad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.42}$$

$$x_{ijt}\geq 0,y_{ijt}\geq 0,\qquad i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.43}$$

$$d_{jt}\in\left[d_{jt}^{*}-\theta d_{jt}^{*},d_{jt}^{*}+\theta d_{jt}^{*}\right],\qquad \forall j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.44}$$

However, the RC should be rewritten in a solvable form. The objective function is preserved as it is, and each original constraint is replaced by the system of linear inequalities using the transformation (5.10).

The transformation (5.10) is directly applicable for constraints (6.34), (6.37)-(6.41), while equality constraints (6.35)-(6.36) of the initial RC firstly should be replaced equivalently by two inequalities. Functions $f_0(\boldsymbol{a})$ and $f_{ijt}(\boldsymbol{a})$, which are mentioned in the transformation (5.10), are formed for each constraint by combining the terms that respectively contain the uncertain demand d_{jt} or do not. Introducing the additional variables p_{jt} when needed, each constraint is transformed to the system of linear inequalities.

Additionally, the decision variable I_{jt} is expressed in terms of $I_{j0}, x_{ijt}, y_{ijt}, d_{jt}$ and is eliminated from the RC:

$$I_{jt} = I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

The construction of the RC may be done not only for the symmetric interval uncertainty, but also for any convex uncertainty set.

For shortening, the following notation is used:

$$d_{jt}^{min} = d_{jt}^{*} - \theta d_{jt}^{*}$$

$$d_{jt}^{max} = d_{jt}^{*} + \theta d_{jt}^{*}$$

Putting all system of inequalities described above together and augmenting the resulting system of linear constraints with the original objective to be minimized, the resulting model is:

$$min\, F \tag{6.45}$$

s.t.:

$$\sum_{i=1}^{K} \sum_{t=1}^{N} \sum_{j=1}^{M} \left(c_{ijt} x_{ijt} + ov_{ijt} y_{ijt} + s_{ijt} z_{ijt} + sv_{ijt} zv_{ijt}\right) + \sum_{t=1}^{N} \sum_{j=1}^{M} h_{jt} p_{jt} \le F \tag{6.46}$$

$$I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}^{min} \le p_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.47}$$

$$I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}^{max} \le p_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.48}$$

$$x_{ijt} \le u_{ijt} \cdot z_{ijt}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.49}$$

$$y_{ijt} \le w_{ijt} \cdot zv_{ijt}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.50}$$

$$\sum_{j=1}^{M} x_{ijt} \leq U_{it}, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{6.51}$$

$$\sum_{j=1}^{M} y_{ijt} \leq W_{it}, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{6.52}$$

$$I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}^{min} \geq I_j^{min}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.53}$$

$$I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}^{max} \geq I_j^{min}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.54}$$

$$I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}^{min} \leq I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.55}$$

$$I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(x_{ijt} + y_{ijt}\right) - \sum_{r=1}^{t} d_{jt}^{max} \leq I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.56}$$

$$z_{ijt} \in \{0,1\}, zv_{ijt} \in \{0,1\}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.57}$$

$$x_{ijt} \geq 0, y_{ijt} \geq 0, \qquad i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.58}$$

Since the decision variables x_{ijt}, y_{ijt} and the demand d_{jt} are non-negative integers, inequalities (6.48), (6.53) and (6.56) are redundant as they are weaker than inequalities (6.47), (6.54) and (6.55) respectively. Thereby, restrictions (6.46) and (6.47) can be combined, since there is no need in additional variable p_{jt} any more.

The resulting system of the RC for the initial uncertain CLSP problem is the following:

$$min\, F \tag{6.59}$$

s.t.:

$$\sum_{i=1}^{K} \sum_{t=1}^{N} \sum_{j=1}^{M} \left(c_{ijt} x_{ijt} + ov_{ijt} y_{ijt} + s_{ijt} z_{ijt} + sv_{ijt} zv_{ijt}\right) +$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(x_{ijt}+y_{ijt}\right)-\sum_{r=1}^{t}d_{jt}^{min}\right)\right)\leq F \tag{6.60}$$

$$x_{ijt}\leq u_{ijt}\cdot z_{ijt},\qquad \forall i\in\{1\dots K\}, j\in\{1\dots M\}, t\in\{1\dots N\} \tag{6.61}$$

$$y_{ijt}\leq w_{ijt}\cdot zv_{ijt},\qquad \forall i\in\{1\dots K\}, j\in\{1\dots M\}, t\in\{1\dots N\} \tag{6.62}$$

$$\sum_{j=1}^{M}x_{ijt}\leq U_{it},\qquad \forall i\in\{1\dots K\}, t\in\{1\dots N\} \tag{6.63}$$

$$\sum_{j=1}^{M}y_{ijt}\leq W_{it},\qquad \forall i\in\{1\dots K\}, t\in\{1\dots N\} \tag{6.64}$$

$$I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(x_{ijt}+y_{ijt}\right)-\sum_{r=1}^{t}d_{jt}^{min}\geq I_j^{min},\qquad \forall j\in\{1\dots M\}, t\in\{1\dots N\} \tag{6.65}$$

$$I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(x_{ijt}+y_{ijt}\right)-\sum_{r=1}^{t}d_{jt}^{min}\leq I_j^{max},\qquad \forall j\in\{1\dots M\}, t\in\{1\dots N\} \tag{6.66}$$

$$z_{ijt}\in\{0,1\}, zv_{ijt}\in\{0,1\},\qquad \forall i\in\{1\dots K\}, j\in\{1\dots M\}, t\in\{1\dots N\} \tag{6.67}$$

$$x_{ijt}\geq 0, y_{ijt}\geq 0,\qquad i\in\{1\dots K\}, j\in\{1\dots M\}, t\in\{1\dots N\} \tag{6.68}$$

The created RC belongs to the class of Mixed-Integer Programming (MIP) problems and includes real, integer and binary variables. It can be solved by using any appropriate optimization software. Because of the existing binary and mixed integer variables the tractability issues of the constructed model should be considered.

6.3.2 Affinely Adjustable Robust Counterpart (AARC)

In the following, the initial CLSP model (2.1)-(2.11) and it's RC (6.59)-(6.68) are considered. To make the transformations more specific, the symmetric demand uncertainty is considered, but the AARC can analogically be constructed for other types of uncertainty. The positive nominal demands d_{jt}^{*} are given for all planning periods in advance and positive θ is a given uncertainty level. So $d_{jt}^{min} = d_{jt}^{*} - \theta d_{jt}^{*}$, $d_{jt}^{max} = d_{jt}^{*} + \theta d_{jt}^{*}$ and the uncertain demand belongs to the newly defined uncertainty interval:

$$d_{jt} \in \left[d_{jt}^{*} - \theta d_{jt}^{*}, d_{jt}^{*} + \theta d_{jt}^{*}, \right], \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

To construct the Affine Adjustable Robust Counterparts (AARCs) for the uncertain CLSP (2.1)-(2.11) model, firstly it must be defined which model variables are allowed to depend on the prescribed amount of the actual market data. Recalling the previous deliberations, the decision variables are the following:

x_{ijt} quantity of product j to be produced in normal working time slot of period t using production machine i,

y_{ijt} quantity of product j to be produced in overtime slot of period t using production machine i,

I_{jt} stock of product j at the end of period t,

z_{ijt} binary variable, which equals to 1 when $x_{ijt} \geq 0$ in period t and 0 otherwise,

zv_{ijt} binary variable, which equals to 1 when $y_{ijt} \geq 0$ in period t and 0 otherwise.

The decision variables I_{jt} defining the stock value in each planning period can be expressed by the decision variables x_{ijt}, y_{ijt}, the initial stock and the actual demand values; therefore, they may be eliminated. The binary variables z_{ijt} and zv_{ijt} remain unchanged in the mathematical model, even though it may affect the tractability status of the resulting AARC.

Applying the methodology for the AARC construction, other decision variables are allowed to depend on the prescribed data amount; the decision-making policy is restricted to the affine decision rules in order to achieve a computational tractability:

$$x_{ijt} = \pi_{ijt}^{0} + \sum_{s \in B_t} \pi_{ijt}^{s} \, d_{js},$$

$$y_{ijt} = \omega_{ijt}^{0} + \sum_{s \in B_t} \omega_{ijt}^{s} \, d_{js}$$

Recalling the created RC (6.59)-(6.68) of the initial CLSP, the affine decision rules are plugged into the model, but the uncertain demand d_{jt} is not replaced by its corner values:

$$\min F$$

s.t.:

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}\left(\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\right)+ov_{ijt}\left(\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}\right)+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+$$

$$\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{r=1}^{t}\sum_{i=1}^{K}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}+\sum_{s=1}^{t}(\pi_{ijr}^{s}+\omega_{ijr}^{s})d_{js}\right)-\sum_{r=1}^{t}d_{jr}\right)\right)\leq F$$

$$\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\leq u_{ijt}\times z_{ijt},\qquad\forall i\in\{1\ldots K\},j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}\leq w_{ijt}\times zv_{ijt},\qquad\forall i\in\{1\ldots K\},j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$\sum_{j=1}^{M}\left(\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\right)\leq U_{it},\qquad\forall i\in\{1\ldots K\},t\in\{1\ldots N\}$$

$$\sum_{j=1}^{M}\left(\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}\right)\leq W_{it},\qquad\forall i\in\{1\ldots K\},t\in\{1\ldots N\}$$

$$I_{j0}+\sum_{r=1}^{t}\sum_{i=1}^{K}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}+\sum_{\substack{r\leq t:\\ s\in[1\ldots t]}}\left(\pi_{ijr}^{s}+\omega_{ijr}^{s}\right)d_{js}\right)-\sum_{r=1}^{t}d_{jr}\geq I_{j}^{min},$$

$$\forall j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$I_{j0}+\sum_{r=1}^{t}\sum_{i=1}^{K}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}+\sum_{s=1}^{t}(\pi_{ijr}^{s}+\omega_{ijr}^{s})d_{js}\right)-\sum_{r=1}^{t}d_{jr}\leq I_{j}^{max},$$

$$\forall j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$z_{ijt}\in\{0,1\},zv_{ijt}\in\{0,1\},\qquad\forall i\in\{1\ldots K\},j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\geq 0,\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}\geq 0,\qquad\forall i\in\{1\ldots K\},j\in\{1\ldots M\},t\in\{1\ldots N\}$$

The obtained model becomes complex to understand. To make it less sophisticated, similar terms and separate terms containing uncertain demand d_{jt} are respectively combined. In the first inequality, the production and setup costs are grouped separately from the holding costs for simplicity:

$$min\, F$$

s.t.:

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0} + ov_{ijt}\omega_{ijt}^{0} + s_{ijt}z_{ijt} + sv_{ijt}zv_{ijt}\right)$$

$$+\sum_{s=1}^{N}\sum_{j=1}^{M}\left(\sum_{i=1}^{K}\sum_{\substack{t:\\ s\in[1\ldots t]}}\left(c_{ijt}\pi_{ijt}^{s} + ov_{ijt}\omega_{ijt}^{s}\right)\right)d_{js}$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0} + \omega_{ijr}^{0}\right) + \sum_{s=1}^{t}\left(\sum_{i=1}^{K}\sum_{\substack{r\le t:\\ s\in[1\ldots r]}}\left(\pi_{ijr}^{s} + \omega_{ijr}^{s}\right) - 1\right)d_{js}\right)\right) \le F$$

$$\pi_{ijt}^{0} + \sum_{s=1}^{t}\pi_{ijt}^{s}d_{js} \le u_{ijt} \times z_{ijt}, \qquad \forall i \in \{1 \ldots K\}, j \in \{1 \ldots M\}, t \in \{1 \ldots N\}$$

$$\omega_{ijt}^{0} + \sum_{s=1}^{t}\omega_{ijt}^{s}d_{js} \le w_{ijt} \times zv_{ijt}, \qquad \forall i \in \{1 \ldots K\}, j \in \{1 \ldots M\}, t \in \{1 \ldots N\}$$

$$\sum_{j=1}^{M}\left(\pi_{ijt}^{0} + \sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\right) \le U_{it}, \qquad \forall i \in \{1 \ldots K\}, t \in \{1 \ldots N\}$$

$$\sum_{j=1}^{M}\left(\omega_{ijt}^{0} + \sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}\right) \le W_{it}, \qquad \forall i \in \{1 \ldots K\}, t \in \{1 \ldots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0} + \omega_{ijr}^{0}\right) + \sum_{s=1}^{t}\left(\sum_{i=1}^{K}\sum_{\substack{r\le t:\\ s\in[1\ldots r]}}\left(\pi_{ijr}^{s} + \omega_{ijr}^{s}\right) - 1\right)d_{js} \ge I_{j}^{min},$$

$$\forall j \in \{1 \ldots M\}, t \in \{1 \ldots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0} + \omega_{ijr}^{0}\right) + \sum_{s=1}^{t}\left(\sum_{i=1}^{K}\sum_{\substack{r\le t:\\ s\in[1\ldots r]}}\left(\pi_{ijr}^{s} + \omega_{ijr}^{s}\right) - 1\right)d_{js} \le I_{j}^{max},$$

$$\forall j \in \{1 \ldots M\}, t \in \{1 \ldots N\}$$

$$z_{ijt} \in \{0,1\}, zv_{ijt} \in \{0,1\}, \qquad \forall i \in \{1 \ldots K\}, j \in \{1 \ldots M\}, t \in \{1 \ldots N\}$$

$$\pi^0_{ijt} + \sum_{s=1}^{t} \pi^s_{ijt} d_{js} \geq 0, \omega^0_{ijt} + \sum_{s=1}^{t} \omega^s_{ijt} d_{js} \geq 0, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

In order to eliminate the unknown demand values d_{jt} and to guarantee the feasibility of the solution for any demand scenario, the maximum principle from the Convex Optimization is utilized [26]. The constraint in the general form is as follows:

$$\sum_{t=1}^{T} d_{jt} x_{ijt} \leq L,$$

where x_{ijt} is a decision variable, L is the known data and d_{jt} is the uncertain parameter. The following equivalences can be written down:

$$\sum_{t=1}^{T} d_{jt} x_{ijt} \leq L, \forall d_{jt} \in \left[d^*_{jt}(1-\theta), d^*_{jt}(1+\theta)\right]$$

$$\Updownarrow$$

$$\sum_{t:x_{ijt}<0} d^*_{jt}(1-\theta) x_{ijt} + \sum_{t:x_{ijt}>0} d^*_{jt}(1+\theta) x_{ijt} \leq L \qquad (6.69)$$

$$\Updownarrow$$

$$\sum_{t=1}^{T} d^*_{jt}\, x_{ijt} + \theta \sum_{t=1}^{T} d^*_{jt} \left|x_{ijt}\right| \leq L$$

Each constraint of the previously created AARC model can analogically be transformed using (6.69).

To simplify the notation, additional variables $\alpha_{js}, \beta_{js}, \gamma^s_{ijt}, \varepsilon^s_{ijt}, \xi^s_{jt}, \eta^s_{jt}$ are defined:

$$\alpha_{js} = \sum_{i=1}^{K} \sum_{t=s}^{N} \left(c_{ijt} \pi^s_{ijt} + ov_{ijt} \omega^s_{ijt}\right), \quad \forall\, j \in \{1 \dots M\}, s \in \{1 \dots N\}$$

$$-\beta_{js} \leq \alpha_{js} \leq \beta_{js}, \quad \forall\, j \in \{1 \dots M\}, s \in \{1 \dots N\}$$

$$-\gamma^s_{ijt} \leq \pi^s_{ijt} \leq \gamma^s_{ijt}, \qquad \forall\, i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\}, s \in \{1 \dots t\}$$

$$-\varepsilon^s_{ijt} \leq \omega^s_{ijt} \leq \varepsilon^s_{ijt}, \qquad \forall\, i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\}, s \in \{1 \dots t\}$$

$$\sum_{i=1}^{K} \sum_{r=s}^{t} \left(\pi^s_{ijr} + \omega^s_{ijr}\right) - \xi^s_{jt} = 1, \quad \forall\, j \in \{1 \dots M\}, 1 \leq s \leq t \leq N$$

$$-\eta^s_{jt} \leq \xi^s_{jt} \leq \eta^s_{jt}, \quad \forall\, j \in \{1 \dots M\}, 1 \leq s \leq t \leq N$$

Utilizing the equivalencies mentioned above, the final version of the AARC for the initial CLSP model (2.1)-(2.11) is deduced:

$$\min_{\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\varepsilon},\boldsymbol{\xi},\boldsymbol{\eta}} F \tag{6.70}$$

s.t.:

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0} + ov_{ijt}\omega_{ijt}^{0} + s_{ijt}z_{ijt} + sv_{ijt}zv_{ijt}\right) + \sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*} + \theta\sum_{s=1}^{N}\sum_{j=1}^{M}\beta_{js}d_{js}^{*} + \sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0} + \omega_{ijr}^{0}\right) + \sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*} + \theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\right)\right) \le F \tag{6.71}$$

$$\pi_{ijt}^{0} + \sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}^{*} + \theta\sum_{s=1}^{t}\gamma_{ijt}^{s}d_{js}^{*} \le u_{ijt} \times z_{ijt}, \quad \forall i \in \{1 \ldots K\}, j \in \{1 \ldots M\}, t \in \{1 \ldots N\} \tag{6.72}$$

$$\omega_{ijt}^{0} + \sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}^{*} + \theta\sum_{s=1}^{t}\varepsilon_{ijt}^{s}d_{js}^{*} \le w_{ijt} \times zv_{ijt}, \quad \forall i \in \{1 \ldots K\}, j \in \{1 \ldots M\}, t \in \{1 \ldots N\} \tag{6.73}$$

$$\sum_{j=1}^{M}\left(\pi_{ijt}^{0} + \sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}^{*} + \theta\sum_{s=1}^{t}\gamma_{ijt}^{s}d_{js}^{*}\right) \le U_{it}, \quad \forall i \in \{1 \ldots K\}, t \in \{1 \ldots N\} \tag{6.74}$$

$$\sum_{j=1}^{M}\left(\omega_{ijt}^{0} + \sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}^{*} + \theta\sum_{s=1}^{t}\varepsilon_{ijt}^{s}d_{js}^{*}\right) \le W_{it}, \quad \forall i \in \{1 \ldots K\}, t \in \{1 \ldots N\} \tag{6.75}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0} + \omega_{ijr}^{0}\right) + \sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*} - \theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*} \ge I_{j}^{min}, \quad \forall j \in \{1 \ldots M\}, t \in \{1 \ldots N\} \tag{6.76}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0} + \omega_{ijr}^{0}\right) + \sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*} + \theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*} \le I_{j}^{max}, \quad \forall j \in \{1 \ldots M\}, t \in \{1 \ldots N\} \tag{6.77}$$

$$z_{ijt} \in \{0,1\}, zv_{ijt} \in \{0,1\}, \quad \forall i \in \{1 \ldots K\}, j \in \{1 \ldots M\}, t \in \{1 \ldots N\} \tag{6.78}$$

$$\pi_{ijt}^0 + \sum_{s=1}^{t} \pi_{ijt}^s d_{js}^* - \theta \sum_{s=1}^{t} \gamma_{ijt}^s d_{js}^* \geq 0, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \qquad (6.79)$$

$$\omega_{ijt}^0 + \sum_{s=1}^{t} \omega_{ijt}^s d_{js}^* + \theta \sum_{s=1}^{t} \varepsilon_{ijt}^s d_{js}^* \geq 0, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \qquad (6.80)$$

The constructed model optimizes the worst-case of demand; therefore, it is further referred as the AARC WORST-CASE for shortening.

6.3.3 AARC with several demand scenarios

One of the advantages of the robust optimization approach is the fact that it can be applied not only for optimization of the worst-case scenario: mandatory requirement is a convexity of the objective. Based on this fact, the AARC model (6.70)-(6.80) can be designed less conservative; a weighted sum of several demand scenarios can be optimized instead of the worst-case.

It is assumed that the following three scenarios have the same probability to occur: the demand takes the lowest values, the demand takes the nominal values, and, finally, the demand takes the highest values in all planning periods. Two additional variables are introduced:

L total costs value when the lowest possible demand values in all planning periods,

A total costs value when the nominal demand values in all planning periods.

The objective function of the AARC WORST-CASE model is substituted by the weighted sum. Since all three scenarios are equally probable to occur, these are considered with the equal weights of 1/3:

$$\min_{\boldsymbol{\pi}, \boldsymbol{L}, \boldsymbol{A}, \boldsymbol{F}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\varepsilon}, \boldsymbol{\xi}, \boldsymbol{\eta}} \left(\frac{1}{3} \cdot L + \frac{1}{3} \cdot A + \frac{1}{3} \cdot F \right)$$

The corresponding restrictions are:

$$\sum_{t=1}^{N} \sum_{i=1}^{K} \sum_{j=1}^{M} \left(c_{ijt} \pi_{ijt}^0 + ov_{ijt} \omega_{ijt}^0 + s_{ijt} z_{ijt} + sv_{ijt} zv_{ijt} \right) + \sum_{s=1}^{N} \sum_{j=1}^{M} \alpha_{js} d_{js}^* - \theta \sum_{s=1}^{N} \sum_{j=1}^{M} \beta_{js} d_{js}^*$$
$$+ \sum_{t=1}^{N} \sum_{j=1}^{M} \left(h_{jt} \left(I_{j0} + \sum_{i=1}^{K} \sum_{r=1}^{t} \left(\pi_{ijr}^0 + \omega_{ijr}^0 \right) + \sum_{s=1}^{t} \xi_{jt}^s \, d_{js}^* - \theta \sum_{s=1}^{t} \eta_{jt}^s \, d_{js}^* \right) \right) \leq L$$

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0}+ov_{ijt}\omega_{ijt}^{0}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+\sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*}$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}\right)\right)\leq \mathrm{A}$$

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0}+ov_{ijt}\omega_{ijt}^{0}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+\sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*}+\theta\sum_{s=1}^{N}\sum_{j=1}^{M}\beta_{js}d_{js}^{*}$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}+\theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\right)\right)\leq F$$

Combining the objective function written above and the new constraints together, and augmenting the resulting system of the linear constraints with restrictions (6.72)-(6.80), the following model is resulting:

$$\min_{\boldsymbol{\pi},\mathbf{L},\mathbf{A}\boldsymbol{F},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\varepsilon},\boldsymbol{\xi},\boldsymbol{\eta}}\left(\frac{1}{3}\cdot L+\frac{1}{3}\cdot A+\frac{1}{3}\cdot F\right) \tag{6.81}$$

s.t.:

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0}+ov_{ijt}\omega_{ijt}^{0}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+\sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*}$$

$$-\theta\sum_{s=1}^{N}\sum_{j=1}^{M}\beta_{js}d_{js}^{*} \tag{6.82}$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}-\theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\right)\right)\leq L$$

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0}+ov_{ijt}\omega_{ijt}^{0}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+\sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*}$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}\right)\right)\leq A \tag{6.83}$$

$$\sum_{t=1}^{N}\sum_{i=1}^{K}\sum_{j=1}^{M}\left(c_{ijt}\pi_{ijt}^{0}+ov_{ijt}\omega_{ijt}^{0}+s_{ijt}z_{ijt}+sv_{ijt}zv_{ijt}\right)+\sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*}$$

$$+\,\theta\sum_{s=1}^{N}\sum_{j=1}^{M}\beta_{js}d_{js}^{*} \tag{6.84}$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}+\theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\right)\right)\leq F$$

$$\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}^{*}+\theta\sum_{s=1}^{t}\gamma_{ijt}^{s}d_{js}^{*}\leq u_{ijt}\times z_{ijt},\quad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.85}$$

$$\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}^{*}+\theta\sum_{s=1}^{t}\varepsilon_{ijt}^{s}d_{js}^{*}\leq w_{ijt}\times zv_{ijt},\quad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.86}$$

$$\sum_{j=1}^{M}\left(\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}^{*}+\theta\sum_{s=1}^{t}\gamma_{ijt}^{s}d_{js}^{*}\right)\leq U_{it},\qquad \forall i\in\{1\dots K\},t\in\{1\dots N\} \tag{6.87}$$

$$\sum_{j=1}^{M}\left(\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}^{*}+\theta\sum_{s=1}^{t}\varepsilon_{ijt}^{s}d_{js}^{*}\right)\leq W_{it},\qquad \forall i\in\{1\dots K\},t\in\{1\dots N\} \tag{6.88}$$

$$I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}-\theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\geq I_{j}^{min},\quad \forall j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.89}$$

$$I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}\left(\pi_{ijr}^{0}+\omega_{ijr}^{0}\right)+\sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*}+\theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\leq I_{j}^{max},\quad \forall j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.90}$$

$$z_{ijt}\in\{0,1\},zv_{ijt}\in\{0,1\},\qquad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.91}$$

$$\pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}^{*}-\theta\sum_{s=1}^{t}\gamma_{ijt}^{s}d_{js}^{*}\geq 0,\qquad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.92}$$

$$\omega_{ijt}^{0}+\sum_{s=1}^{t}\omega_{ijt}^{s}d_{js}^{*}+\theta\sum_{s=1}^{t}\varepsilon_{ijt}^{s}d_{js}^{*}\geq 0,\qquad \forall i\in\{1\dots K\},j\in\{1\dots M\},t\in\{1\dots N\} \tag{6.93}$$

The constructed model optimizes the weighted sum of the total costs for several demand scenarios; therefore, it is further referred to as the AARC SCENARIOS for shortening.

6.3.4 Computational examples for the RC

Example 1. The RC with rolling horizon

In this section, the RC is tested for the CLSP with incomplete information about the demand when planning under rolling horizon[2]. The considered production problem has 8 planning periods, one machine and one product. The holding costs are high in the last period and can be considered as utilization costs; there are no setup costs. Capacity restriction only exists for production; stock volume is not limited.

The production system parameters and market data for the considered example are indicated below:

$j = 1$	products,
$i = 1$	production machines,
$n = 4$	number of periods in rolling horizon,
$t = 1 \dots N, N = 8$	planning periods,
$d_t^* = 50$	nominal demand in the planning period t (units),
$\theta = 0{,}2$	uncertainty level of demand (20%),
$U = 100$	total production capacity of machine (units),
c_t	production costs (per unit) for product j at normal working time slot of period t using production machine i ($):

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
100	150	100	150	100	150	100	150

h_t holding costs (per unit) in period t ($):

h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
2	2	2	2	2	2	2	300

$I_{j0} = 0$ initial stock of product j (units).

[2] The example was published in [71]

The initial CLSP model (2.1)-(2.11) and it's RC (6.59)-(6.68) are considered for the indicated data and rolling horizon of 4 planning periods; the demand for the nearest period is assumed to be deterministic, see Figure 2.4.

In order to compare the results of RO approach, it is proposed to consider an ideal case (all demands are known in advance) and a probabilistic model for the uncertain CLSP. To construct the probabilistic model, it is proposed that demand has the union distribution and, therefore, all possible demand values have equal probabilities, since no information about the demand is available except the uncertainty interval. Based on that, the mean value of the demand can be calculated and used for the calculation of the expected value of costs.

To evaluate the RC solution, the following workflow is used:

- consider several demand scenarios;
- solve the RC model and the probabilistic model for each scenario under the rolling horizon;
- solve the deterministic mathematical model for each scenario (ideal case);
- compare the obtained total costs.

The workflow was implemented for four demand scenarios, see Table 6.4: the lowest possible demand in all planning periods; the highest possible demand in all planning periods; the case when the demand alternates between the lowest and the highest possible values; and, finally, the case described in Theorem 6.1.

Table 6.4: Four demand scenarios for testing the RC model (Example 1)

	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6	Period 7	Period 8
Scenario 1	40	40	40	40	40	40	40	40
Scenario 2	60	60	60	60	60	60	60	60
Scenario 3	40	60	40	60	40	60	40	60
Scenario 4	40	40	50	50	50	50	60	60

According to the proposed workflow, the RC, the probabilistic model and the deterministic model were solved, and the differences in total costs were determined, see Figure 6.11.

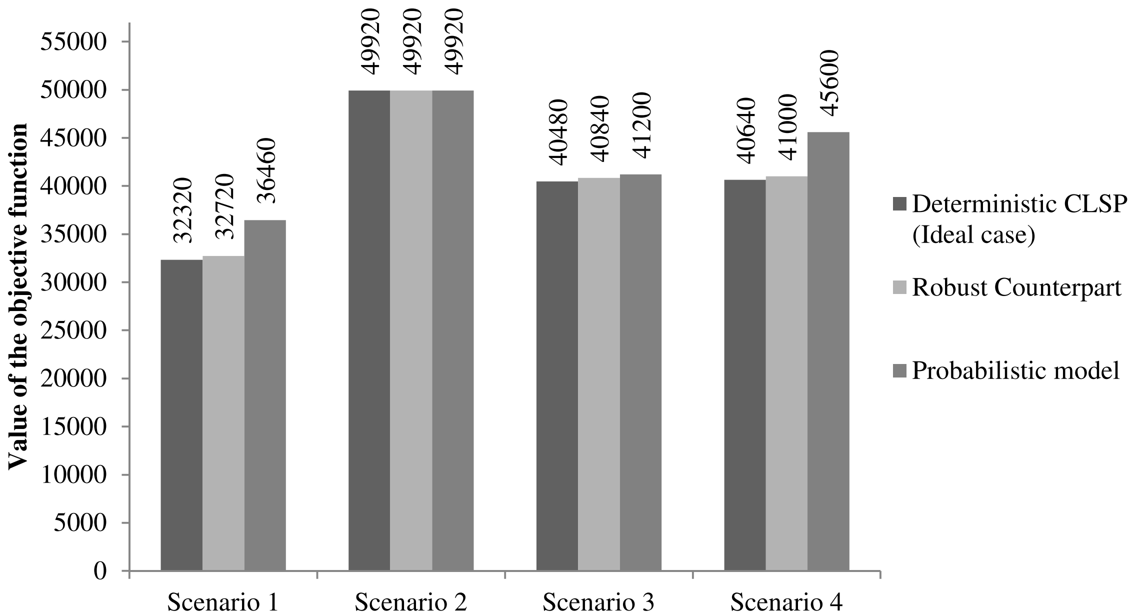


Figure 6.11. Comparison of the total costs between the RC, the probabilistic and the ideal model solutions on four demand scenarios (Example 1)

As it can be noticed, in 3 out of 4 demand scenarios the production plan proposed by the probabilistic model was more costly than the RC solution. The percentage differences between the costs associated with the RC, the probabilistic model and the optimal costs are presented in Table 6.5.

Table 6.5: Percentage differences between the costs associated with the RC, the probabilistic model and the optimal costs (Example 1)

	Percentage difference between costs associated with the RC and optimal costs	Percentage difference between costs associated with the probabilistic model and optimal costs	Percentage difference between costs associated with the probabilistic model and the RC
Scenario 1	1,24	12,81	11,6
Scenario 2	0	0	0
Scenario 3	0,889	1,78	0,9
Scenario 4	0,886	12,21	11,3

Summarizing, in the considered example, the highest difference in costs between the RC solution and the probabilistic solution (over four demand scenarios) is 11,6%, whereas the highest difference in costs between the RC and the optimal solution is less than 1,24%.

Example 2. The RC with folding horizon

In this section, the RC is tested for the CLSP with uncertain information about the demand when planning under a folding horizon[3]. The considered production problem has 7 planning periods with normal and overtime working slots, one machine and two products. The holding costs are high in the last period and can be considered as utilization costs; there are no setup costs. The capacity restriction only exists for the production; the stock volume is not limited. Backlogging (satisfaction of demand later than required) is allowed for some punishment; therefore, the stock variable can take negative values. In the last planning period, unsatisfied demand is highly undesirable, so the corresponding punishment is higher. Demand uncertainty is defined by the lower and upper bound for each product and each planning period. Production system parameters and market data for the considered example are indicated below:

$j = 2$ products,

$i = 1$ production machines,

$t = 1 \dots N, N = 7$ planning periods,

d_{jt}^{min} lower bound of demand for product j in planning period t (units):

	d_{j1}^{min}	d_{j2}^{min}	d_{j3}^{min}	d_{j4}^{min}	d_{j5}^{min}	d_{j6}^{min}	d_{j7}^{min}
$j = 1$	47	48	53	64	68	57	49
$j = 2$	32	31	23	15	20	27	32

d_{jt}^{max} upper bound of demand for product j in planning period t (units):

	d_{j1}^{max}	d_{j2}^{max}	d_{j3}^{max}	d_{j4}^{max}	d_{j5}^{max}	d_{j6}^{max}	d_{j7}^{max}
$j = 1$	63	66	71	86	92	77	67
$j = 2$	40	37	29	19	24	33	40

[3] The example was published in [72]

u_{jt} production capacity for product j in normal working time slot of period t (units):

	u_{j1}	u_{j2}	u_{j3}	u_{j4}	u_{j5}	u_{j6}	u_{j7}
$j=1$	70	70	70	70	70	70	70
$j=2$	25	25	25	25	25	25	25

w_{jt} production capacity for product j in overtime slot of period t (units):

	w_{j1}	w_{j2}	w_{j3}	w_{j4}	w_{j5}	w_{j6}	w_{j7}
$j=1$	20	20	20	20	20	20	20
$j=2$	12	12	12	12	12	12	12

c_{jt} production costs (per unit) for product j in normal working time slot of period t using production machine i ($):

	c_{j1}	c_{j2}	c_{j3}	c_{j4}	c_{j5}	c_{j6}	c_{j7}
$j=1$	100	120	100	120	100	120	100
$j=2$	70	50	70	50	70	50	70

ov_{jt} production costs (per unit) for product j in overtime slot of period t using production machine i ($):

	ov_{j1}	ov_{j2}	ov_{j3}	ov_{j4}	ov_{j5}	ov_{j6}	ov_{j7}
$j=1$	150	180	150	180	150	180	150
$j=2$	100	70	100	70	100	70	100

p_{jt} punishment for backlogging (per unit) in period t ($):

	p_{j1}	p_{j2}	p_{j3}	p_{j4}	p_{j5}	p_{j6}	p_{j7}
$j=1$	200	200	200	200	200	200	800
$j=2$	130	130	130	130	130	130	500

h_{jt} holding costs (per unit) in period t ($):

	h_{j1}	h_{j2}	h_{j3}	h_{j4}	h_{j5}	h_{j6}	h_{j7}
$j=1$	2	2	2	2	2	2	300
$j=2$	3	3	3	3	3	3	100

$s_{jt} = 3$ setup costs in normal working time slot of period t, when producing product j ($),

$sv_{jt} = 3$ setup costs in overtime slot of period t, when producing product j ($),

$I_{j0} = 0$ initial stock of product j (units).

For the first product, the upper and lower bounds of the demand are defined by 15% uncertainty level (taking into account rounding to an integer), whereas for the second product the demand uncertainty level is 10%. In Figure 6.12, the borders for the demand are shown graphically. The growth of the demand for the first product is obtained up to the fifth planning period, but then customer's demand starts to decrease. Vice versa, demand for the second product decreases in the first periods and increases at the end of the planning horizon.

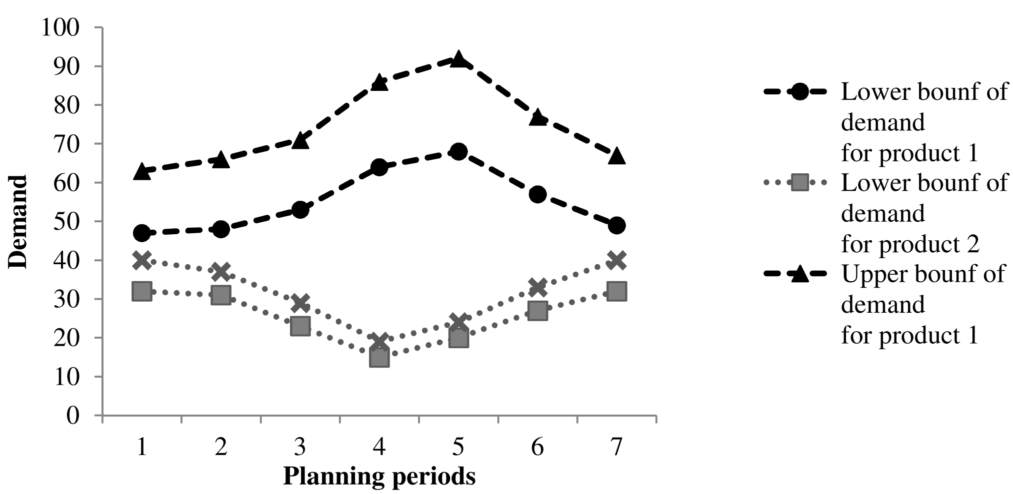


Figure 6.12. Lower and upper borders for the demand

The initial CLSP model (2.1)-(2.11) and it's RC (6.59)-(6.68) are considered with the indicated data and folding planning horizon. Both models are extended for the backlogging; demand for the nearest period is assumed to be deterministic, see Figure 2.2.

Completely by analogy with the considered Example 1, and in order to evaluate the constructed RC, an ideal case (all demands are known in advance) and a probabilistic model for uncertain CLSP were considered and tested. The testing workflow coincides with the one proposed in Example 1 and it was implemented for four demand scenarios, see Table 6.6.

Table 6.6: Four demand scenarios for testing the RC model (Example 2)

		Planning periods						
		1	2	3	4	5	6	7
Scenario 1	j=1	47	48	53	64	68	57	49
	j=2	32	31	23	15	20	27	32
Scenario 2	j=1	63	66	71	86	92	77	67
	j=2	40	37	29	19	24	33	40
Scenario 3	j=1	47	66	53	86	68	77	49
	j=2	32	37	23	19	20	33	32
Scenario 4	j=1	47	48	53	75	92	77	67
	j=2	32	31	23	17	24	33	40

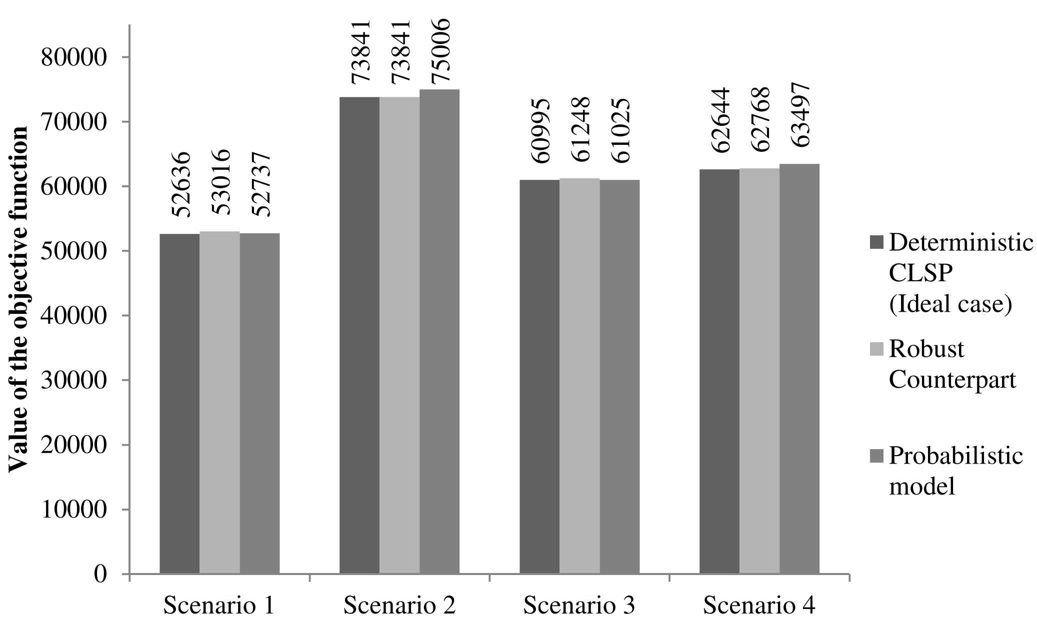


Figure 6.13. Comparison of the total costs between the RC, the probabilistic and the ideal model solutions on four demand scenarios (Example 2)

According to the proposed workflow, the RC, the probabilistic model and the deterministic model were solved and the differences in total costs were determined, see Figure 6.13.

It can be noticed that in two demand scenarios the production plan proposed by the probabilistic model was more costly than the RC solution, and in other two scenarios vice versa. The percentage differences between the costs associated with the RC, the probabilistic model and the optimal costs are presented in Table 6.7.

Table 6.7: Percentage differences between the costs associated with the RC, the probabilistic model and the optimal costs (Example 2)

	Percentage difference between costs associated with the RC and optimal costs	Percentage difference between costs associated with the probabilistic model and optimal costs	Percentage difference between costs associated with the probabilistic model and the RC
Scenario 1	0,72	0,19	-0,53
Scenario 2	0	1,58	1,58
Scenario 3	0,41	0,05	-0,36
Scenario 4	0,20	1,36	1,16

For the four presented scenarios of demand realization, the solution provided by the probabilistic approach was better than the solution provided by the RC in two cases; the maximal difference is 0.53%. However, in two scenarios where the RC solution was better, the maximal difference is three times higher and equals to 1.58%.

Another important difference between the robust and probabilistic approach is the fact that the robust optimization approach guarantees that the total costs will not exceed the value of 73841$ (in worst-case) for any possible demand realization. The expected value of the total costs equals to 62243$ for the probabilistic model, but in the worst case the total costs are 75006$, which is 20.5% higher. Moreover, if the production planning problem without backlogging is considered, the solution provided by the probabilistic model may become infeasible.

6.3.5 Computational examples for the AARC, simulation and analysis of results

In this section, the aim is to test the RO solution approach on the computational example, in particular the AARC models. The production system parameters and the market data indicated below are provided by the operating manufacturing company:

$j = 1 \dots M, M = 10$ products,

$i = 1 \dots K, K = 2$ production machines,

$t = 1 \dots N, N = 30$ planning periods,

d_{jt}^* nominal demand for product j in planning period t (units):

d_{1t}^*	d_{2t}^*	d_{3t}^*	d_{4t}^*	d_{5t}^*	d_{6t}^*	d_{7t}^*	d_{8t}^*	d_{9t}^*	d_{10t}^*
2	200	10	10	30	30	60	60	100	100

$\theta = 0{,}1$ uncertainty level of demand (10%),

$NC = 0{,}9$ productivity coefficient in normal time slot,

$OvC = 0{,}75$ productivity coefficient in overtime slot,

$u_{ijt} = NC \cdot 480$ production capacity of machine i for product j in normal working time slot of period t (units),

$w_{ijt} = OvC \cdot 240$ production capacity of machine i for product j in overtime slot of period t (units),

$U_{it} = NC \cdot 480$ total production capacity of machine i in normal working time slot of period t (units),

$W_{it} = OvC \cdot 240$ total production capacity of machine i in overtime slot of period t (units),

c_{ijt} production costs (per unit) for product j in normal working time slot of period t using production machine i (\$):

c_{ijt} on weekdays ($t \bmod 7 \in [1;5]$)	c_{ijt} on Saturdays ($t \bmod 7 = 6$)	c_{ijt} on Sundays ($t \bmod 7 = 0$)
6	7,5	9

ov_{ijt} — production costs (per unit) for product j in overtime slot of period t using production machine i ($):

ov_{ijt} on weekdays (t mod 7 ∈ [1; 5])	ov_{ijt} on Saturdays (t mod 7 = 6)	ov_{ijt} on Sundays (t mod 7 = 0)
7,5	7,5	9

$h_{jt} = 0{,}6$ — holding costs for product j (per unit and per period) in period t ($),

$s_{ijt} = 5 \cdot c_{ijt}$ — setup costs for machine i in normal working time slot of period t, when producing product j ($),

$sv_{ijt} = 5 \cdot ov_{ijt}$ — setup costs for machine i in overtime slot of period t, when producing product j ($),

$I_{j0} = 2 \cdot d^*_{j1}$ — initial stock of product j (units),

$I_j^{min} = 2 \cdot d^*_{j1}$ — minimal stock of product j at the end of any period (units),

$I_j^{max} = 14 \cdot d^*_{j1}$ — maximal stock of product j at the end of any period (units).

Results. The AARC with real coefficients in decision rules

The AARC WORST-CASE (6.70)-(6.80) and the AARC SCENARIOS (6.81)-(6.93) models with real values of coefficients in the affine decision rules are considered.

Both models are MIP problems, which were implemented and solved using the IBM ILOG CPLEX Optimization Studio. Detailed information about the created models and the generated solutions is presented in Table 6.8.

The computational time for the AARC SCENARIOS model is less by about 7 minutes than for the AARC WORST-CASE model, but the accuracy tolerances (relative MIP gap between best node and best integer solution) also differ.

The numeric values of the objective functions were calculated for both AARC models on 100 generated demand scenarios. The optimal value of total costs was calculated for each demand scenario from the deterministic CLSP that describes an ideal case.

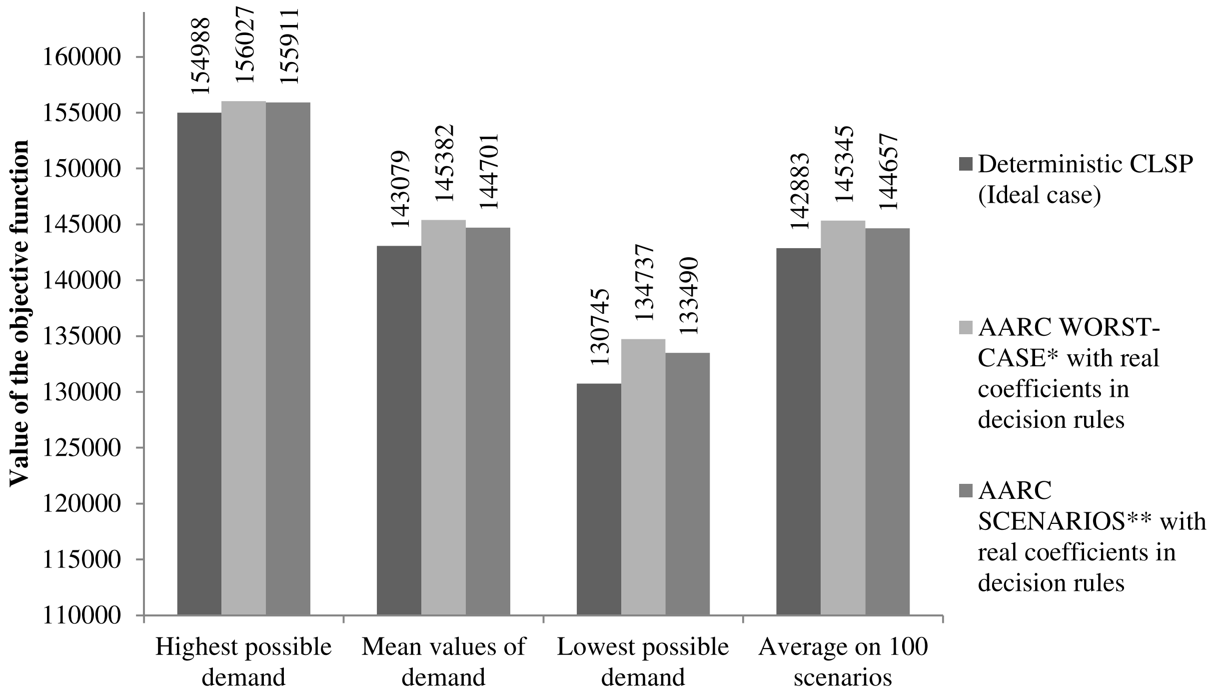
Table 6.8: Solving properties of the AARC WORST-CASE and AARC SCENARIOS models with real coefficients in the decision rules

	AARC WORST-CASE* with real coefficients in decision rules	AARC SCENARIOS** with real coefficients in decision rules
Number of MIP model matrix rows after MIP pre-solve	55050	54180
Number of MIP model matrix columns after MIP pre-solve	49380	48810
Number of MIP model non-zero matrix coefficients after MIP pre-solve	489130	468880
MIP model binary variables	1200	1200
MIP search method	Dynamic search	Dynamic search
Solving time	1838.4 sec	1398.2 sec
Accuracy tolerance	1,8%	2,9%

*AARC that optimizes the worst-case demand scenario,
**AARC that optimizes the weighted sum of several demand scenarios.

The values of the objective functions of the AARC WORST-CASE, the AARC SCENARIOS and the ideal case for three fixed demand scenarios and on average over 100 generated scenarios were compared, see Figure 6.14.

The objective values of the AARC SCENARIOS model were, on each fixed scenario and on average over 100 scenarios, closer to the optimal values than the objective values of the AARC WORST-CASE model. The absolute and relative gaps between the objectives of the AARC models and the optimal objectives are provided in Table 6.9. Even with positive accuracy tolerances and 10% uncertainty level of demand, the price of robustness was less than 3%.



*AARC that optimizes the worst-case demand scenario,

**AARC that optimizes the weighted sum of several demand scenarios.

Figure 6.14. Comparison between the AARC WORST-CASE, the AARC SCENARIOS (with real coefficients in the decision rules) and the optimal objective values on selected demand scenarios and on average over 100 demand scenarios

Table 6.9: Comparison between the AARC WORST-CASE, the AARC SCENARIOS solutions (with real coefficients in the decision rules) and the optimal solution

	AARC WORST-CASE* with real coefficients in decision rules	AARC SCENARIOS** with real coefficients in decision rules
Maximal absolute gap ($)	3786,78	3178,02
Maximal relative gap	2,9%	2,43%
Average relative gap (over 100 generated demand scenarios)	1,665%	1,51%

*AARC that optimizes the worst-case demand scenario,

**AARC that optimizes the weighted sum of several demand scenarios.

Results. The AARC with integer coefficients in decision rules

The AARC WORST-CASE (6.70)-(6.80) and the AARC SCENARIOS (6.81)-(6.93) models with integer coefficients in affine decision rules are considered.

Both models are MIP problems, which were implemented and solved using the IBM ILOG CPLEX Optimization Studio. The number of the MIP model matrix rows, columns and non-zero matrix coefficients after MIP pre-solve coincides with the numbers presented in Table 6.8. The only difference is that the newly constructed AARC WORST-CASE and AARC SCENARIOS models have additional 19800 integer variables, since the coefficients of the decision rules become integers, not reals.

Table 6.10: Solving properties of the AARC WORST-CASE and AARC SCENARIOS models with integer coefficients in decision rules

	AARC WORST-CASE* with integer coefficients in decision rules	AARC SCENARIOS** with integer coefficients in decision rules
MIP search method	Dynamic search	Dynamic search
Solving time	1949.3 sec	1697.8 sec
Accuracy tolerance	1,87%	2,9%

*AARC that optimizes the worst-case demand scenario,
**AARC that optimizes the weighted sum of several demand scenarios.

Both models were solved with nearly the same accuracy as in the case with real coefficients in the decision rules. The computational time increased insignificantly by switching to the integer coefficients: less than by 2 minutes for the AARC WORST-CASE model and by about 5 minutes for the AARC SCENARIOS model. Detailed information about the created models and the solutions is presented in Table 6.10.

Analogically to the case considered for standard affine decision rules (real coefficients), the numeric values of the objective functions were calculated for both AARC models based on 100 generated demand scenarios. The optimal value of the total costs was calculated for each demand scenario with the help of the deterministic CLSP that describes an ideal case.

A comparison of the AARC WORST-CASE, the AARC SCENARIOS and the optimal objective values shows a similar trend to the results indicated in Figure 6.14: the production plan provided by the AARC SCENARIOS model was closer to the optimal production plan. The price of robustness also changed insignificantly after switching to the integer coefficients in the affine decision rules, see Table 6.11. It increased by 0,15% for the AARC WORST-CASE model and decreased by 0,33% for the AARC SCENARIOS model in comparison with the models containing real coefficients in the decision rules.

Table 6.11: Comparison between the AARC WORST-CASE, the AARC SCENARIOS solutions (with integer coefficients in the decision rules) and the optimal solution

	AARC WORST-CASE* with integer coefficients in decision rules	AARC SCENARIOS** with integer coefficients in decision rules
Maximal absolute gap ($)	3992,16	2745
Maximal relative gap	3,05%	2,1%
Average relative gap (over 100 generated demand scenarios)	1,73%	1,24%
Accuracy tolerance	1,87%	2,9%

*AARC that optimizes the worst-case demand scenario,
**AARC that optimizes the weighted sum of several demand scenarios.

Additionally, a simulation which is the closest to the reality was done: the demand for product 1 took values from the set {1,2,3}, while the demand for other products took integer values from corresponding uncertainty interval: $[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*]$, $\forall j \in \{2 \dots M\}$, $t \in \{1 \dots N\}$.

The comparison of the integer demand simulation with the continuous demand simulation was done. The price of robustness for the AARC WORST-CASE model became insignificantly higher in case of integer demand simulation, see Table 6.11 and Table 6.12. The price of robustness for the AARC SCENARIOS model increased by 1,1% in the worst case and became by 0,2% higher than the analogous by the AARC WORST-CASE, on average of 100 scenarios.

Table 6.12: Comparison between the AARC WORST-CASE, AARC SCENARIOS solutions (with integer coefficients in the decision rules) and the optimal solution. Simulation with integer demand values

	AARC WORST-CASE* with integer coefficients in decision rules	AARC SCENARIOS** with integer coefficients in decision rules
Maximal absolute gap ($)	4182,6	4168,5
Maximal relative gap	3,21%	3,2%
Average relative gap (over 100 generated demand scenarios)	1,7%	1,9%
Accuracy tolerance	1,9%	3,7%

*AARC that optimizes the worst-case demand scenario,

**AARC that optimizes the weighted sum of several demand scenarios.

It is quite possible that the accepted accuracy tolerance influenced the obtained changes, since it was slightly higher for the integer demand simulation.

To compare results with the ones provided by probabilistic model, it was assumed that the demand is uniformly distributed on the uncertainty interval. Given this additional probabilistic information, expected values of demand were calculated (they equal to the nominal demand values) and plugged the initial CLSP model. Resulting deterministic MIP problem was solved with the accuracy tolerance 0.5%. Demand simulation showed that the inventory capacity constraints of the constructed model were typically violated: as large as 66 of the total of 300 constraints were violated on average over 100 generated demand scenarios. For the demand scenario with the highest possible demand values, the stock became negative 58 times (totally over 30 planning periods and 10 products). Moreover, proposed production plan does not allow to satisfy the demand completely, which is highly undesirable.

The influence of uncertainty level on the total costs value

To evaluate the influence of uncertainty level θ on the total costs value, the same instances of AARCs were solved for the uncertainty level of 5%, 10%, 20%, 30% and 50%.

The AARC WORST-CASE model with integrality restrictions on the decision rules was considered. The model was solved and evaluated based on the 100 generated demand scenarios. The demand was assumed to be a continuous variable, so it took real values from the corresponding uncertainty intervals. The results in Table 6.13 shows that even for the 50% demand uncertainty the maximum price of robustness reaches 11,21% and is only 3,28% on average.

Table 6.13: Percentage difference in costs between the AARC WORST-CASE solution (integer coefficients in the decision rules) and the optimal solution depending on the uncertainty level

Uncertainty level	AARC WORST-CASE* objective value (integer coefficients in decision rules) on average over 100 demand scenarios ($)	Optimal objective value on average over 100 demand scenarios ($)	Maximal relative gap	Average relative gap	Accuracy tolerance
5%	144121,5	142912,3	1,59%	0,85%	1,7%
10%	145344,8	142883,8	3,05%	1,73%	1,9%
20%	144307,6	142867,6	3,72%	1,02%	2%
30%	145141,6	143088,8	6,83%	1,46%	1,4%
50%	147419,8	142851,5	11,21%	3,28%	2,5%

*AARC that optimizes the worst-case demand scenario.

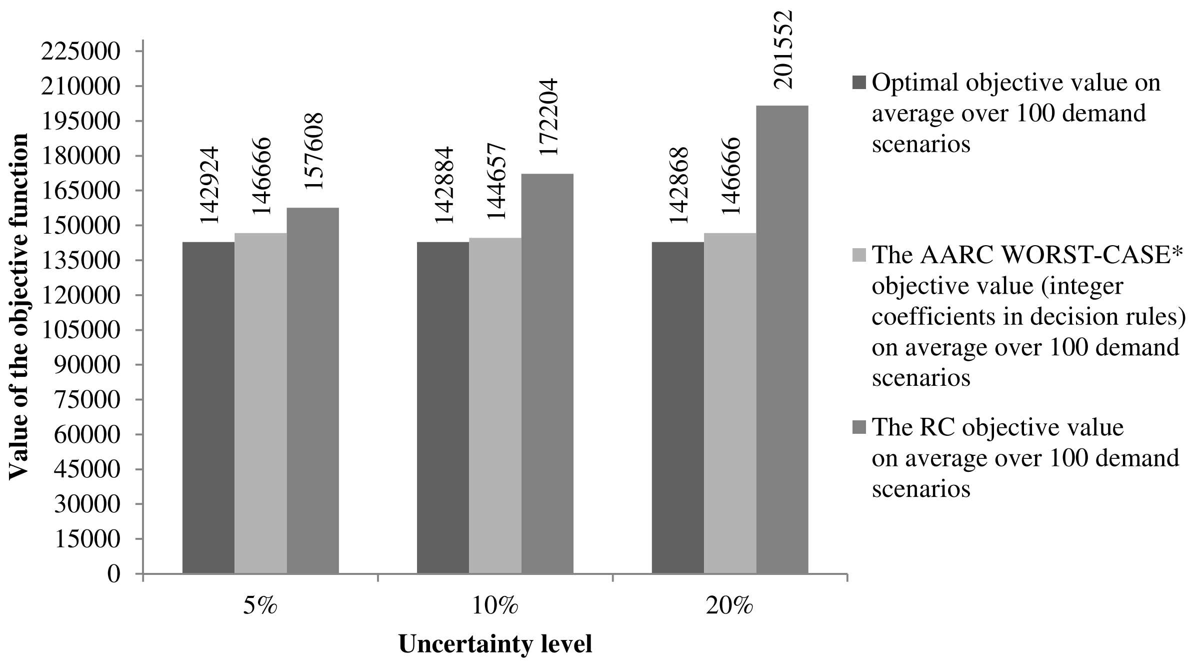
Obviously, a direct dependence exists between the uncertainty level of the market data and the price of robustness of the AARC model solution.

Adjustability advantage

In this section, the advantages of the AARC of the initial CLSP model are reflected over a non-adjustable RC.

The AARC WORST-CASE model (with integer coefficients in decision rules) and the non-adjustable RC model were tested based on 100 generated demand scenarios for several uncertainty levels. The RC model became already infeasible for 21% level of demand uncertainty, so the RC solution was only calculated for θ equal to 5%, 10% and 20%.

The absolute values of the optimal total costs and the total costs of the RC and the AARC solutions (on average over 100 demand scenarios) are shown in Figure 6.15. The production plan constructed by the RC has always the highest costs. The absolute gap between the RC and the optimal solutions varies from 14684$ to 58684$, depending on uncertainty level, on average over 100 demand scenarios.



*AARC that optimizes the worst-case demand scenario.

Figure 6.15. Comparison between the AARC WORST-CASE, the RC and the optimal objective values on average over 100 demand scenarios for different levels of demand uncertainty

The absolute and relative gaps between the total costs values of the RC solution and the optimal total costs are presented in Table 6.14. The maximal difference in costs exceeds 21% already for 5% of demand uncertainty and reaches 98% for θ=20%. The average relative gap varies from 10% to 41%. This significant difference in costs is caused by the higher conservativeness of the RC model and shows the adjustability advantage.

Table 6.14: Percentage difference in costs between the AARC WORST-CASE solution (integer coefficients in decision rules) and optimal solution, between the RC solution and the optimal solution depending on the uncertainty level

Uncertainty level	AARC WORST-CASE* with integer coefficients in decision rules		RC	
	Maximal relative gap	Average relative gap over 100 generated demand scenarios	Maximal relative gap	Average relative gap over 100 generated demand scenarios
5%	1,47%	0,92%	21,39%	10,28%
10%	3,05%	1,73%	44,5%	20,56%
20%	3,72%	1,02%	98,27%	41,27%

*AARC that optimizes the worst-case demand scenario.

6.4 Robust Optimization approach for the DLSP

6.4.1 Robust Counterpart (RC)

For the construction of the RC for the DLSP problem, the formulation of the uncertain DLSP mathematical model with several producing machines and several products (2.12)-(2.19) forms the basis.

To construct the RC, transition to the model with the certain objective function is necessary. In order to get it, the objective (2.12) is equivalently rewritten, introducing the extra variable F and the additional restriction:

$$min\, F$$

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}p_{ij}z_{ijt} + s_{ijt}\, max\left(0, z_{ijt} - z_{ij,t-1}\right)\right) + \sum_{t=1}^{N}\sum_{j=1}^{M} h_{jt}I_{jt} \leq F$$

However, the RC should be written in a solvable form. So analogically to the RC of the CLSP construction, each original constraint is replaced by the system of linear inequalities using the transformation (5.10).

The transformation (5.10) is directly applicable for all inequality constraints, while equality constraints (2.13), (2.14) of the initial DLSP should be first replaced equivalently by two inequalities. Functions $f_0(\boldsymbol{a})$ and $f_{ijt}(\boldsymbol{a})$, which are mentioned in the transformation (5.10), are formed for each constraint by combining the terms that respectively do or do not contain the uncertain demand d_{jt}. Introducing the additional variables p_{jt} if needed, each constraint is transformed to the system of linear inequalities.

Additionally, the decision variable I_{jt} is expressed in terms of $I_{j0}, p_{ij}, z_{ijt}, d_{jt}$ and is eliminated from the RC:

$$I_{jt} = I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} p_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}\,, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

The construction of the RC may be done not only for the symmetric interval uncertainty, but for any convex uncertainty set. For shortening, the following notation is used in this case:

$$d_{jt}^{min} = d_{jt}^{*} - \theta d_{jt}^{*}$$

$$d_{jt}^{max} = d_{jt}^{*} + \theta d_{jt}^{*}$$

Combining the system of inequalities described above together and augmenting the resulting system of linear constraints with our original objective to be minimized, the following model results:

$$min\, F$$

s.t.:

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}u_{ij}z_{ijt} + s_{ijt}\, max\left(0, z_{ijt} - z_{ij,t-1}\right)\right) + \sum_{t=1}^{N}\sum_{j=1}^{M} h_{jt}p_{jt} \le F$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{min} \le p_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{max} \le p_{jt}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

$$\sum_{j=1}^{M} z_{ijt} \le 1, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{min} \geq I_j^{min}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{max} \geq I_j^{min}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{min} \leq I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{max} \leq I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

$$z_{ijt} \in \{0,1\}, \quad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

Furthermore, the same argumentation as used for the construction of the RC for the uncertain CLSP problem is applied. Redundant constraints are eliminated, since decision variables z_{ijt} and parameters u_{ij}, d_{jt} are non-negative. Thereupon, the two first constraints can be combined, as there is no need for additional variable p_{jt} any more.

For the initial uncertain DLSP problem, the resulting RC is the following:

$$min\, F \tag{6.94}$$

s.t.:

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}u_{ij}z_{ijt} + s_{ijt}\, max\left(0, z_{ijt} - z_{ij,t-1}\right)\right) + \sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijr} - \sum_{r=1}^{t} d_{jt}^{min}\right)\right) \leq F \tag{6.95}$$

$$\sum_{j=1}^{M} z_{ijt} \leq 1, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{6.96}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{max} \geq I_j^{min}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.97}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}z_{ijt} - \sum_{r=1}^{t} d_{jt}^{min} \leq I_j^{max}, \qquad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.98}$$

$$z_{ijt} \in \{0,1\}, \quad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.99}$$

The created RC belongs to the class of Mixed-Integer Programming (MIP) problems and includes real, integer and binary variables. It can be solved using any appropriate optimization software. Due to existing binary and mixed integer variables, the tractability issues of the constructed model should be considered.

6.4.2 Affinely Adjustable Robust Counterpart (AARC)

The initial DLSP model (2.12)-(2.19) and it's RC (6.94)-(6.99) are considered. To model adjustability of the variables, the vector of decision variables $\boldsymbol{z}$ is allowed to depend on the prescribed amount $P_k d$ of the true demand d for every $k \leq n$:

$$z_{ijt} = X_k(P_k \boldsymbol{d}) = p_k + q_k^T P_k \boldsymbol{d},$$

where $P_1, \dots P_n$ are matrices given in advance, specifying the information base of the decisions z_{ijt}, and $X_k(\cdot)$ are affine decision rules. It is assumed that the manufacturer works with an online information base $B_t = \{1 \dots t\}$: at the beginning of the planning period t the actual values of demands $d_{j1} \dots d_{jt}$ are known. So, the decision variables z_{ijt} are replaced by the affine decision rules:

$$z_{ijt} = \pi_{ijt}^0 + \sum_{s \in B_t} \pi_{ijt}^s \, d_{js}$$

To make these transformations more specific, the symmetric demand uncertainty is considered, but the AARC can be constructed analogically for other types of uncertainty. The positive nominal demands d_{jt}^* are given in advance for all planning periods and a positive θ is the given uncertainty level. So $d_{jt}^{min} = d_{jt}^* - \theta d_{jt}^*$, $d_{jt}^{max} = d_{jt}^* + \theta d_{jt}^*$ and the uncertain demand belongs to the newly defined uncertainty interval:

$$d_{jt} \in \left[d_{jt}^* - \theta d_{jt}^*, d_{jt}^* + \theta d_{jt}^*, \right], \forall j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

Recalling the created RC (6.94)-(6.99) of the initial DLSP, the affine decision rules are plugged into the model, but the uncertain demand d_{jt} remains unchanged:

$$min\, F \tag{6.100}$$

s.t.:

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}u_{ij}\left(\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js}\right)\right.$$

$$\left.+\,s_{ijt}\,max\left(0,\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js}-\pi_{ij,t-1}^{0}-\sum_{s\in B_{t-1}}\pi_{ij,t-1}^{s}\,d_{js}\right)\right) \qquad (6.101)$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}(\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js})-\sum_{r=1}^{t}d_{jt}\right)\right)\leq F$$

$$\sum_{j=1}^{M}(\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js})\leq 1, \qquad \forall i\in\{1\ldots K\},t\in\{1\ldots N\} \qquad (6.102)$$

$$I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}(\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js})-\sum_{r=1}^{t}d_{jt}\geq I_j^{min}, \qquad \forall j\in\{1\ldots M\},t\in\{1\ldots N\} \quad (6.103)$$

$$I_{j0}+\sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}(\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js})-\sum_{r=1}^{t}d_{jt}\leq I_j^{max}, \qquad \forall j\in\{1\ldots M\},t\in\{1\ldots N\} \quad (6.104)$$

$$\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js}\in\{0;1\}, \quad \forall i\in\{1\ldots K\},j\in\{1\ldots M\},t\in\{1\ldots N\} \qquad (6.105)$$

The obvious modeling issue in the system above is that the decision variable z_{ijt} has a binary nature. Therefore, the affine function $\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js}$ should only take 0 or 1 values in the constraint (6.105).

The concept proposes the following steps to overcome this issue:

- leave z_{ij1} in the AARC model as a binary decision variable, not as the decision rule;
- let decision rules $\pi_{ijt}^{0}+\sum_{s\in B_t}\pi_{ijt}^{s}\,d_{js}$ for all other planning periods to take values within the interval [0,1];
- update data and generate a new solution at the end of each planning period.

By implementing the proposed approach, the following model is constructed:

$$min\, F$$

s.t.:

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\Bigg(c_{ijt}u_{ij}\,\pi_{ijt}^{0}+c_{ijt}u_{ij}\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js} + s_{ijt}\,max\left(0,\pi_{ijt}^{0}-\pi_{ij,t-1}^{0}+\sum_{s=1}^{t-1}\left(\pi_{ijt}^{s}-\pi_{ijt-1}^{s}\right)d_{js}+\ \pi_{ijt}^{t}d_{jt}\ \right)\Bigg)$$

$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0}+\ \sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}\pi_{ijr}^{0}+\sum_{s=1}^{t}\left(\sum_{i=1}^{K}\sum_{\substack{r\le t:\\ s\in[1\ldots r]}}u_{ij}\pi_{ijr}^{s}-1\right)d_{js}\right)\right)\le F$$

$$\sum_{j=1}^{M}\pi_{ijt}^{0}+\sum_{j=1}^{M}\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\le 1,\qquad \forall i\in\{1\ldots K\},t\in\{1\ldots N\}$$

$$I_{j0}+\ \sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}\pi_{ijr}^{0}+\sum_{s=1}^{t}\left(\sum_{i=1}^{K}\sum_{\substack{r\le t:\\ s\in[1\ldots r]}}u_{ij}\pi_{ijr}^{s}-1\right)d_{js}\ge I_{j}^{min},\qquad \forall j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$I_{j0}+\ \sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}\pi_{ijr}^{0}+\sum_{s=1}^{t}\left(\sum_{i=1}^{K}\sum_{\substack{r\le t:\\ s\in[1\ldots r]}}u_{ij}\pi_{ijr}^{s}-1\right)d_{js}\le I_{j}^{max},\qquad \forall j\in\{1\ldots M\},t\in\{1\ldots N\}$$

$$\pi_{ij1}^{0}\in\{0;1\},\ \ \forall i\in\{1\ldots K\},j\in\{1\ldots M\}$$

$$\pi_{ij1}^{s}=0,\ \ \forall i\in\{1\ldots K\},j\in\{1\ldots M\}$$

$$\pi_{ij0}^{0}=E_{ij0},\ \ \forall i\in\{1\ldots K\},j\in\{1\ldots M\}$$

$$\pi_{ij0}^{s}=0,\ \ \forall i\in\{1\ldots K\},j\in\{1\ldots M\}$$

$$0\le\ \pi_{ijt}^{0}+\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}\le 1,\ \ \forall i\in\{1\ldots K\},j\in\{1\ldots M\},t\in\{1\ldots N\}$$

The constructed model is updated and resolved at the end of each planning period, realizing a folding horizon.

To simplify the notation, additional variables $\alpha_{js}, \beta_{js}, \gamma_{ijt}^{s}, \xi_{jt}^{s}, \eta_{jt}^{s}, \delta_{ijt}$ are defined:

$$\alpha_{js} = \sum_{i=1}^{K}\sum_{t=s}^{N} c_{ijt}u_{ij}\pi_{ijt}^{s}, \quad \forall j \in \{1 \dots M\}, s \in \{1 \dots N\}$$

$$-\beta_{js} \le \alpha_{js} \le \beta_{js}, \quad \forall j \in \{1 \dots M\}, s \in \{1 \dots N\}$$

$$-\gamma_{ijt}^{s} \le \pi_{ijt}^{s} \le \gamma_{ijt}^{s}, \qquad \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\}, s \in \{1 \dots t\}$$

$$\sum_{i=1}^{K}\sum_{r=s}^{t} u_{ij}\pi_{ijr}^{s} - \xi_{jt}^{s} = 1, \quad \forall j \in \{1 \dots M\}, 1 \le s \le t \le N$$

$$-\eta_{jt}^{s} \le \xi_{jt}^{s} \le \eta_{jt}^{s}, \quad \forall j \in \{1 \dots M\}, 1 \le s \le t \le N$$

$$\delta_{ijt} = max\left(0, \pi_{ijt}^{0} - \pi_{ij,t-1}^{0} + \pi_{ijt}^{t}d_{jt}^{*} + \theta\gamma_{ijt}^{t}d_{jt}^{*} + \sum_{s=1}^{t-1}\left(\pi_{ijt}^{s} - \pi_{ijt-1}^{s}\right)\left(d_{js}^{*} + \theta d_{js}^{*}\right)\right),$$
$$\forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\}$$

Utilizing the equivalencies mentioned above, the final version of the AARC for the initial DLSP model (2.12)-(2.19) is constructed:

$$\min_{\boldsymbol{\pi},\boldsymbol{F},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\delta}} F \tag{6.106}$$

s.t.:

$$\sum_{i=1}^{K}\sum_{t=1}^{N}\sum_{j=1}^{M}\left(c_{ijt}u_{ij}\,\pi_{ijt}^{0} + s_{ijt}\delta_{ijt}\right) + \sum_{s=1}^{N}\sum_{j=1}^{M}\alpha_{js}d_{js}^{*} + \theta\sum_{s=1}^{N}\sum_{j=1}^{M}\beta_{js}d_{js}^{*}$$
$$+\sum_{t=1}^{N}\sum_{j=1}^{M}\left(h_{jt}\left(I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}\pi_{ijr}^{0} + \sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*} + \theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*}\right)\right) \le F \tag{6.107}$$

$$\sum_{j=1}^{M}\pi_{ijt}^{0} + \sum_{j=1}^{M}\sum_{s=1}^{t}\pi_{ijt}^{s}d_{js}^{*} + \theta\sum_{j=1}^{M}\sum_{s=1}^{t}\gamma_{ijt}^{s}d_{js}^{*} \le 1, \qquad \forall i \in \{1 \dots K\}, t \in \{1 \dots N\} \tag{6.108}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t}u_{ij}\pi_{ijr}^{0} + \sum_{s=1}^{t}\xi_{jt}^{s}\,d_{js}^{*} - \theta\sum_{s=1}^{t}\eta_{jt}^{s}\,d_{js}^{*} \ge I_{j}^{min},$$
$$\forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.109}$$

$$I_{j0} + \sum_{i=1}^{K}\sum_{r=1}^{t} u_{ij}\pi_{ijr}^{0} + \sum_{s=1}^{t} \xi_{jt}^{s}\, d_{js}^{*} + \theta \sum_{s=1}^{t} \eta_{jt}^{s}\, d_{js}^{*} \leq I_{j}^{max}, \quad \forall j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.110}$$

$$\pi_{ij1}^{0} \in \{0; 1\}, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\} \tag{6.111}$$

$$\pi_{ij1}^{\mathrm{s}} = 0, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\} \tag{6.112}$$

$$\pi_{ij0}^{0} = E_{ij0}, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\} \tag{6.113}$$

$$\pi_{ij0}^{\mathrm{s}} = 0, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\} \tag{6.114}$$

$$\pi_{ijt}^{0} + \sum_{s=1}^{t} \pi_{ijt}^{\mathrm{s}} d_{js}^{*} - \theta \sum_{s=1}^{t} \gamma_{ijt}^{\mathrm{s}} d_{js}^{*} \geq 0, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.115}$$

$$\pi_{ijt}^{0} + \sum_{s=1}^{t} \pi_{ijt}^{\mathrm{s}} d_{js}^{*} + \theta \sum_{s=1}^{t} \gamma_{ijt}^{\mathrm{s}} d_{js}^{*} \leq 1, \ \ \forall i \in \{1 \dots K\}, j \in \{1 \dots M\}, t \in \{1 \dots N\} \tag{6.116}$$

6.4.3 Computational example and simulation

In this section, the aim is to test the RO solution approach using a computational example. The production system parameters and market data indicated below are provided by the operating manufacturing company:

$j = 1 \dots M, M = 10$ — products,

$i = 1 \dots K, K = 10$ — production machines,

$t = 1 \dots N, N = 24$ — planning periods,

d_{jt}^{*} — nominal demand for product j in planning period t (units):

d_{1t}^{*}	d_{2t}^{*}	d_{3t}^{*}	d_{4t}^{*}	d_{5t}^{*}	d_{6t}^{*}	d_{7t}^{*}	d_{8t}^{*}	d_{9t}^{*}	d_{10t}^{*}
2	200	10	10	30	30	60	60	100	100

$\theta = 0{,}1$ — uncertainty level of demand (10%),

$u_{ij} = 240$ — production speed of machine i for product j (units per period),

c_{ijt} — production costs (per unit) for product j in normal working time slot of period t using production machine i ($):

c_{ijt} in normal working shift ($t < 16$)	c_{ijt} in overtime shift ($16 \leq t \leq 24$)
6	7,5

$h_{jt} = 0{,}6$ — holding costs for product j (per unit and per period) in period t ($),

$s_{ijt} = 5 \cdot c_{ijt}$ — setup costs for machine i in normal working time slot of period t, when producing product j ($),

$E_{ij0} = 0$ — binary variable describing the initial state of machine i; it is equal to 1 when machine i is installed to produce product j and 0 otherwise,

$I_{j0} = 2 \cdot d_{j1}^{*}$ — initial stock of product j (units),

$I_j^{min} = 2 \cdot d_{j1}^{*}$ — minimal stock of product j at the end of any period (units),

$I_j^{max} = 12 \cdot u_{ijt}$ — maximal stock of product j at the end of any period (units).

The DLSP is a small bucket model, so the planning period is relatively small. The considered production model includes 24 planning periods equal to one hour, and the production schedule is created for one day. The first 16 hours are contained within the normal working time slot, whereas the last 8 periods are contained within the overtime slot and, therefore, production is more costly then. Stock limits meet the requirements of the manufacturing company: the lower bound on stock is defined based on the safety stocks strategy, the upper bound on stock equals to the maximal warehouse capacity, which equals to a half-day production capacity.

Results. The AARC with real coefficients in decision rules

The AARC (6.106)-(6.116) of the initial DLSP is considered. The model is a MIP problem, which was implemented and solved using the IBM ILOG CPLEX Optimization Studio. Detailed information about the created model and the generated solution is presented in Table 6.15.

Table 6.15: Solving properties of the AARC of the DLSP model

Property	Value
Number of MIP model matrix rows after MIP pre-solve	79790
Number of MIP model matrix columns after MIP pre-solve	77850
Number of MIP model non-zero matrix coefficients after MIP pre-solve	758980
MIP model binary variables	4700
MIP search method	dynamic search
Solving time	258 sec
Accuracy tolerance	0%

The time of computations for the AARC of the DLSP model was about 7 times less than for the AARC WORST-CASE of the CLSP model, even though the model size and the number of binary variables is significantly higher for the AARC of the DLSP model. The obtained solution is optimal, since the value of the accuracy tolerance (relative MIP gap between best node and best integer solution) is zero.

The numeric values of the objective function were calculated for the AARC (6.106)-(6.116) based on 20 generated demand scenarios. The optimal value of the total costs was calculated for each demand scenario from the deterministic DLSP that describes an ideal case.

The values of the objective functions of the AARC were compared with the ideal case for three fixed demand scenarios and on average over 20 generated scenarios, see Figure 6.16.

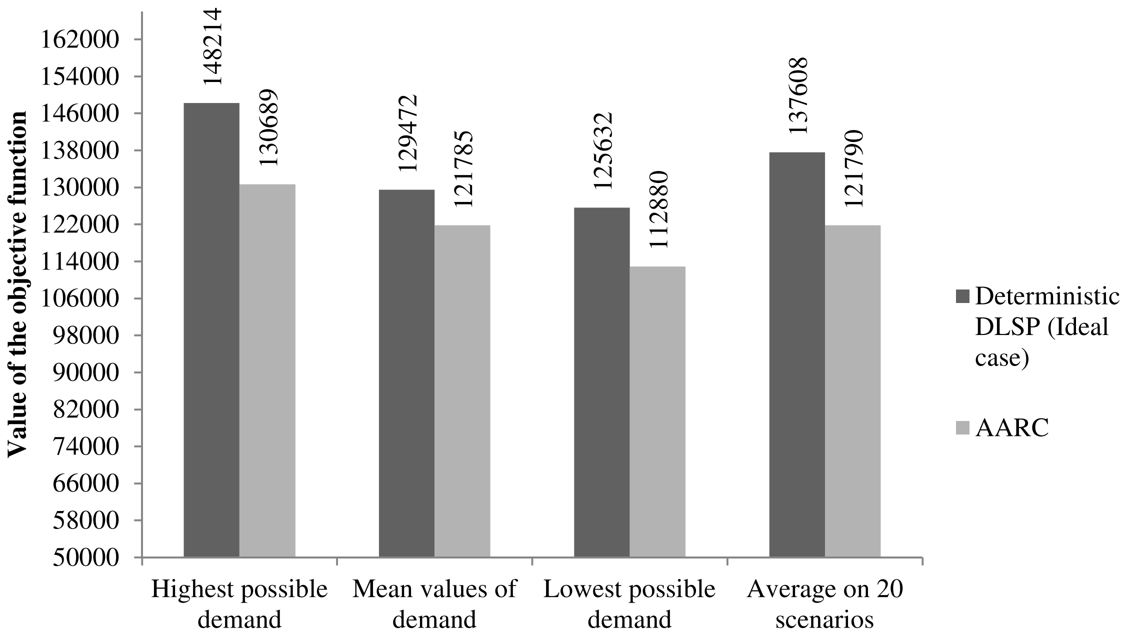


Figure 6.16. Comparison between the AARC of the DLSP and the optimal objective values on selected demand scenarios and on average over 20 demand scenarios

The objective values of the AARC model were better than the optimal values, in each fixed scenario and on average over 20 scenarios, which is of course nonsense. This is caused by the fact that the AARC production plan only provides the probabilities of productions for all planning periods except the first one. For example, the probability of the product j production on the machine i may be equal to 0,5 and, at the same time the probability of the product $j+1$ production on the same machine may be also equal to 0,5. So the model neglect that one machine produces only one product during one planning period or produces nothing.

The absolute and relative gaps between the objectives of the AARC model and the optimal objectives are provided in Table 6.16. Even with positive accuracy tolerances and 10% uncertainty level of demand, the price of robustness was less than 3%.

Table 6.16: Comparison between the AARC of the DLSP and the optimal solution

Property	Value
Maximal absolute gap ($)	-19935,1
Maximal relative gap	-14%
Average relative gap (over 100 generated demand scenarios)	-11,5%

The provided comparison is not completely fair, since it was proposed to resolve the AARC model at the end of each planning period. Here, the AARC solution provided at the very beginning of the planning period was compared with the optimal one.

The AARC with folding horizon

The simulation for the AARC model with a folding horizon was implemented utilizing the algorithm shown in Figure 5.3.

The values of the objective functions of the AARC with the folding horizon were compared with the ideal case for three fixed demand scenarios and on average over 20 generated scenarios, see Figure 6.17.

In the provided experiment, the objective values of the AARC model were worse than the optimal values on each fixed scenario and on average over 20 scenarios. The absolute and relative gaps between the objectives of the AARC with a folding horizon and the optimal objectives are provided in Table 6.17. Even though that the AARC with a folding horizon was solved with an accuracy tolerance equal to 2,2% and 10% uncertainty level of demand, the price of robustness was more than 10% on average.

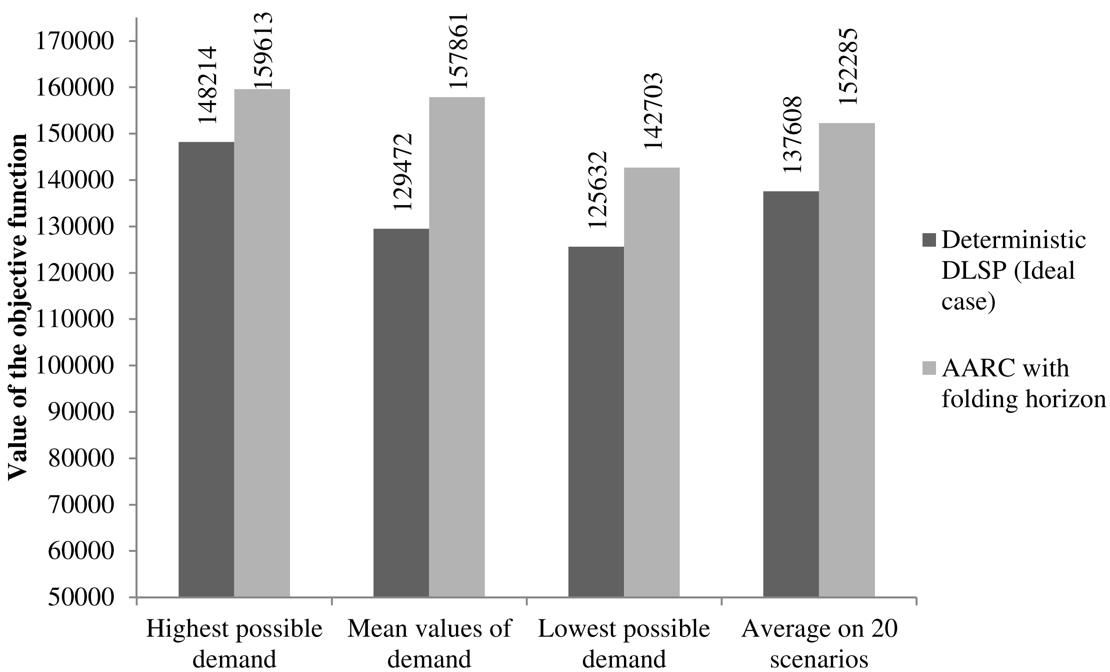


Figure 6.17. Comparison between the AARC of the DLSP with folding horizon and the optimal objective values on the selected demand scenarios and on average over 20 demand scenarios.

Table 6.17: Comparison between the AARC of the DLSP with folding horizon and the optimal solution

Property	Value
Maximal absolute gap ($)	28389
Maximal relative gap	21,9%
Average relative gap (over 20 generated demand scenarios)	10,7%
Accuracy tolerance	2,2%

The influence of uncertainty level on the total costs value

To evaluate the influence of uncertainty level θ on the total costs value, the same instances of AARCs were solved for the uncertainty level of 5%, 10% and 20%.

The AARC model with a folding horizon was considered. The model was solved and evaluated based on the 20 generated demand scenarios. The demand was assumed to be a continuous variable, so it took real values from the corresponding uncertainty intervals. Table 6.18 shows that for all levels of demand uncertainty the maximum price of robustness reaches 21% and is about 11% on average.

Table 6.18: Comparison of the AARC with folding horizon and the optimal objective values on generated demand scenarios depending on the uncertainty level

Uncertainty level	AARC objective value with folding horizon on average over 20 demand scenarios ($)	Optimal objective value on average over 20 demand scenarios ($)	Maximal relative gap	Average relative gap	Accuracy tolerance
5%	156495,8	139030,1	21,7%	12,6%	0%
10%	152285,2	137607,9	21,9%	10,7%	0%
20%	152350,8	138785,9	21,7%	9,9%	0%

*AARC that optimizes the worst-case demand scenario.

It can be admitted that there is no dependence between the uncertainty level of the market data and the price of robustness of the AARC model solution.

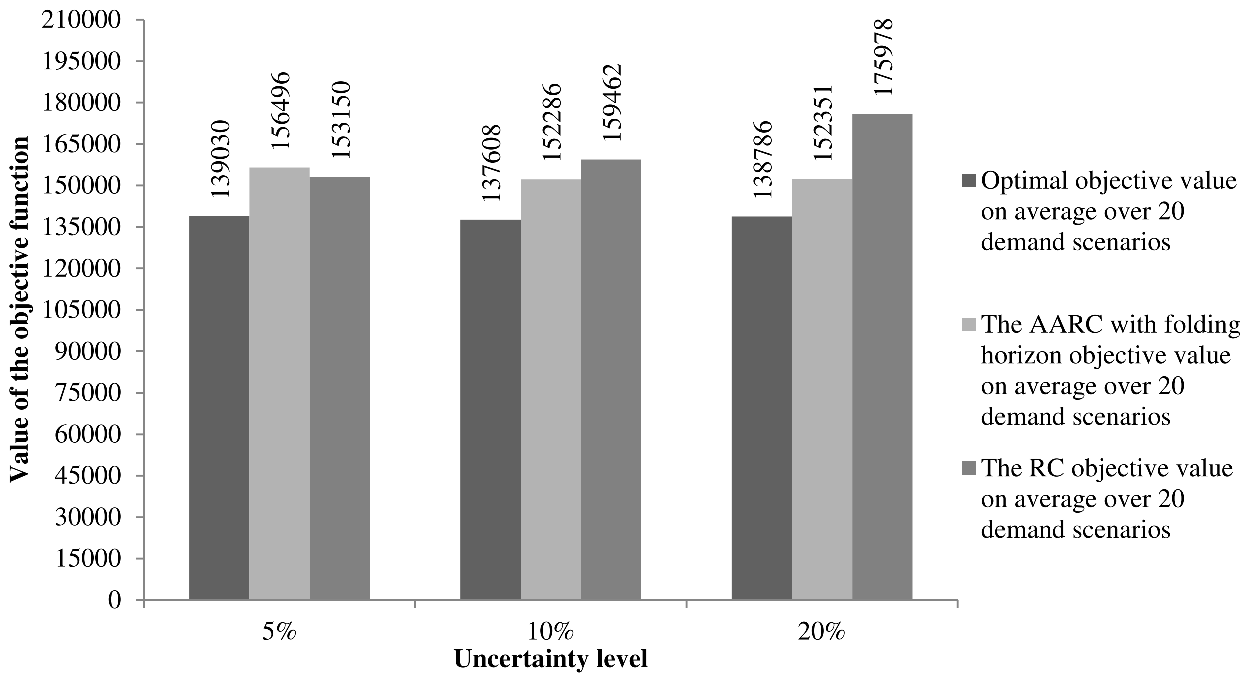
Adjustability advantage

In this section, the solutions of the AARC with a folding horizon are compared with the solutions of non-adjustable RC for different levels of demand uncertainty.

The AARC model with a folding horizon and the non-adjustable RC model were tested based on 20 generated demand scenarios. The RC and the AARC of the initial DLSP model became infeasible for 30% demand uncertainty, so solutions were calculated only for θ equal to 5%, 10% and 20%.

The absolute values of the optimal total costs and the total costs of the RC and the AARC solutions (on average over 20 demand scenarios) are shown in Figure 6.18. The production plan constructed by the RC has the highest costs for 10 and 20% of demand uncertainty level, but is better than the AARC solution for θ=5%.

The absolute gap between the RC and the optimal solutions varies from 14120$ to 37192$ on average over 20 demand scenarios, depending on the uncertainty level.



*AARC that optimizes the worst-case demand scenario.

Figure 6.18. Comparison between the AARC of the DLSP with folding horizon, the RC and the optimal objective values on average over 20 demand scenarios for different levels of demand uncertainty

The absolute and relative gaps between the total costs values of the RC solution and the optimal total costs are presented in Table 6.19. The maximal difference in costs exceeds 21% already for 5% demand uncertainty and reaches 73% for $\theta = 20\%$. The average relative gap varies from 10% to 27%.

Table 6.19: Percentage difference in costs between the AARC with folding horizon solution and the optimal solution, between the RC solution and the optimal solution depending on the uncertainty level

Uncertainty level	AARC with folding horizon		RC	
	Maximal relative gap	Average relative gap over 20 generated demand scenarios	Maximal relative gap	Average relative gap over 20 generated demand scenarios
5%	21,7%	12,6%	21,7%	10,3%
10%	21,9%	10,7%	35,5%	16,1%
20%	21,7%	9,9%	72,9%	27,4%

*AARC that optimizes the worst-case demand scenario.

7 Conclusions

Solving the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) with robustness guarantees is a challenging task, especially for cases with online information base.

The worst case demand scenario for the special case of uncertain CLSP (2.1)-(2.11) with one product, one machine and no setup costs was determined. A theorem that describes the worst case of the demand realization, and consequently defines the competitive ratio of any online algorithm, was formulated and proved. The theorem statement was also tested empirically. However, applicability of the presented analytical approach for determining the worst case demand scenario is limited and, therefore, the obtained results should be seen as a foundation for a better problem understanding. Further research is required in order to determine the worst case demand scenario for the uncertain DLSP and for more complex uncertain CLSP structures, see Table 6.2.

The analytical approach proposed in [25] was extended in the presented thesis. The sufficient condition for the existence of a finite competitive ratio and the formulas for its upper and lower bounds were deduced for two production planning models with additional capacity restrictions: perishable products with lost sales (6.3) and durable products with backlogging (6.10)-(6.13). The suggested method for deriving the upper and lower bounds for competitive ratio is applicable to the CLSP with backlogging and setup costs, but not to the production planning problems with different model structures, e.g. for DLSP (see Table 6.3). It should also be noted that the analytical approach for competitive ratio estimation is only applicable to strict online problems, meaning that actual demand values become known after the production process has finished. Additional market information, e.g. in production planning models with the rolling or folding horizon, is ignored. Moreover, the computational examples showed that the bounds for the competitive ratio do not exist if the production stock is always non-negative.

Finally, the presented work investigated the Robust Optimization (RO) approach for the uncertain CLSP (2.1)-(2.11) and DLSP (2.12)-(2.19) with interval uncertainty. In particular, the RC and the AARC models were constructed (sections 6.3, 6.4). The models were applied to a real-world planning problem and demonstrated competitive performance and high accuracy. The comparison with other state-of-the-art solutions was also implemented.

The demand scenario simulation was implemented for the constructed AARCs (6.70)-(6.80) and (6.81)-(6.93); both AARCs were tested using 100 randomly generated demand scenarios. The results of the simulation showed that the RO approach can be applied successfully for the considered uncertain CLSP, since the gap between the optimal and the generated solution is relatively small – about 2% on average of 100 generated demand scenarios (for the demand uncertainty level of 10%). The solution provided by AARC is closer to the optimal one than the solution produced by the probabilistic model, which assumed uniform demand distribution over the uncertainty intervals. It also has the advantage of being feasible for any demand scenario from the uncertainty set.

The results of the demand simulations also showed that the production plan created by the AARC (6.70)-(6.80) is less conservative (closer to the optimal) than one created by the AARC (6.81)-(6.93), so modeling with the weighted sum of demand scenarios in the objective works better than considering the worst-case scenario only. Additionally, the direct connection between the level of data uncertainty and the price of robustness was empirically observed. It was also shown that the adjustability of decision variables gives a distinct advantage, since the non-adjustable RC of the initial CLSP model becomes infeasible already for 21% of demand uncertainty.

The integrality of the decision variables in the initial uncertain CLSP model was achieved by restricting the coefficients of the AARC affine decision rules to integral values. This technique is novel and worked well for the presented CLSP problem, showing competitive performance to the conventional method of decision rules construction (usage of the real coefficients in the decision rules). The technique utilized in the presented research provides a production plan with integral decision variables and guarantees the feasibility of the created production plan for each possible demand scenario. It can also be used in other RO applications, where integrality is required.

Presented results of the computational experiments show that criticism of the conservatism of the RO approach is not always justified, especially for adjustable robust optimization.

The RC and the AARC of the uncertain DLSP problem were also constructed and analyzed. Integrality of the decision variables in the initial uncertain DLSP model was achieved in the AARC by restricting the affine decision rules to take values from the interval [0,1] and implementing folding planning horizon. The constructed AARC has the advantage of being less conservative than the corresponding RC for the DLSP due to the adjustability of the variables. However, is also has several disadvantages. First, the robustness of the AARC solution is not guaranteed since the DLSP "all or nothing" assumption can be violated. The manufacturer may not able to satisfy the customer's demand due to the insufficient production capacity, and, consequently, no strict upper bound of the total costs provided to the manufacturer.

The demand scenarios simulation was implemented for the constructed AARC (6.106)-(6.116) and the AARC was tested using 20 generated demand scenarios and folding horizon. The simulation results showed that the gap between the optimal, and the generated solution is about 11% on average over 20 generated demand scenarios (for the demand uncertainty level of 10%) and 22% in the worst case. Additionally, after several computational examples, it was observed that the level of data uncertainty does not influence the price of robustness: the average and maximal gap between the robust and the optimal solution did not change significantly. It was also shown that the AARC performs nearly as well as the RC when demand uncertainty level is 5%, while adjustability of decision variables provides an advantage for higher uncertainty levels.

From the theoretical point of view, the RO approach can be successfully applied in the field of production planning under demand uncertainty. A practical implementation of the considered RO techniques and analysis of the obtained results will be helpful and may be considered as one of the directions for further research. The considered robust approach for lot sizing decisions can be directly applied to the CLSP and DLSP problems with uncertain demand and can be implemented in the following steps:

1. Construct a mathematical model that corresponds to the production process and the market, formally describing the data uncertainty;
2. Construct the corresponding AARC;
3. Collect the production planning system and market data, numerically define uncertainty intervals (or mean values and uncertainty level) for the demand;
4. Solve the constructed AARC using the given model data;

5. Implement the obtained robust production plan, using new information about the market coming into the system;
6. Repeat steps 3-5 regularly in order to avoid human errors and account for changes in the system state[4].

Moreover, the considered robust approach is relevant in other applications of robust optimization. Research results are significant for the companies that are utilizing business intelligence on top of the risk management solutions or working with uncertain data.

Since additional probabilistic information may be useful in reducing the conservativeness of the robust optimization approach, the analysis of production planning models with demand forecasts is an important direction for further research.

To identify the performance of the RO solution, a variety of production planning models with different data structures can be considered in the context of further research.

Finally, the solution provided by the RO approach should be compared in detail with the solution provided by the SO approach based on the simulation of future demand scenarios.

[4] This step is optional, but is typically implemented by manufacturing companies.

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